

# Equivalence and Hölder-Sobolev Regularity of Solutions for a Class of Non Autonomous Stochastic Partial Differential Equations

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## Abstract

In this article we investigate a class of non autonomous, semi-linear, stochastic partial differential equations defined on a smooth, bounded, convex domain of  $\mathbb{R}^d$  and driven by an infinite-dimensional noise; this noise is colored relative to the space variable and white relative to the time variable. Under an appropriate integrability condition regarding the covariance operator of the associated Wiener process, we introduce three notions of solution for them and prove their indistinguishability. We then prove the existence, the uniqueness and the pointwise boundedness of the moments, along with the spatial Sobolev regularity and the joint space-time Hölder regularity of such solutions. Moreover, we show how to weaken some requirements regarding the covariance operator in order to generalize the notions we alluded to above by introducing a fourth type of solution, whose existence and regularity properties we also analyze in detail. Our results represent a preliminary step toward the analysis of the support and the smoothness properties of the corresponding laws.

## 1 Introduction and Outline

Solutions to certain stochastic partial differential equations may be considered as random variables taking their values in a suitable functional space. As such, their laws are identified with probability measures on that space and it thus becomes natural to investigate the support and the smoothness properties of these laws. Recently, many works have been devoted to

these questions both in the hyperbolic and in the parabolic case, particularly when the solutions are jointly space-time Hölder continuous (see, for instance, [3], [37], [40], [41]). In this case, the functional space in question is typically a Banach space of Hölder continuous functions defined on some part of Euclidean space. In the works mentioned above, the deterministic part of the equations is autonomous; moreover, the driving noise is either white relative to both the space- and the time-variable, or colored relative to the space variable and white relative to the time variable; in addition, the space variable may vary over the entire Euclidean space  $\mathbb{R}^d$ , which makes tools from Fourier analysis readily applicable to investigate existence and regularity questions as these relate to the spatial correlations of the noise. One notable exception to this is the paper [3], in which the authors prove a support theorem in a Banach space of Hölder continuous functions for the law of the solution to a one-dimensional, autonomous, semilinear, initial-Neumann boundary value problem driven by a space-time white noise and defined on a *bounded* interval. In this case, the authors' analysis relies heavily on the existence of the corresponding parabolic Green's function and on very refined estimates for it.

In this article we investigate the indistinguishability and the joint space-time Hölder continuity properties of solutions to a class of non autonomous, semilinear, stochastic partial differential equations as a preliminary step toward the analysis of the support and the smoothness properties of their laws, this analysis being deferred to a separate publication. As we shall see, the complication in this case will stem from the fact that the equations are non autonomous, defined on a bounded domain of  $\mathbb{R}^d$  where  $d$  is arbitrary, and from the fact that there are a number of *a priori* non equivalent possibilities to define a notion of solution for them as is the case for deterministic partial differential equations. We can define the class of problems we shall investigate in the following way (here and below, we use the standard notations for the usual Banach spaces of differentiable functions, of Hölder continuous functions and of Lebesgue integrable functions defined on regions of Euclidean space): for  $d \in \mathbb{N}^+$  let  $D \subset \mathbb{R}^d$  be open, bounded, convex and assume that the boundary  $\partial D$  is of class  $\mathcal{C}^{2+\alpha}$  for some  $\alpha \in (0, 1)$  (see, for instance, [17], [18], [31] and [47] for a definition of this and related concepts). Let  $C$  be a linear, self-adjoint, positive, non degenerate trace-class operator in  $L^2(D)$ ; this implies that  $C$  is an integral transform whose generating kernel we denote by  $\kappa$ . In the sequel we write  $(e_j)_{j \in \mathbb{N}^+}$  for an orthonormal basis of  $L^2(D)$  consisting of eigenfunctions of the operator  $C$  and  $(\lambda_j)_{j \in \mathbb{N}^+}$  for the sequence of the corresponding eigenvalues. Let  $(W(\cdot, t))_{t \in \mathbb{R}_0^+}$  be an  $L^2(D)$ -valued Wiener process defined on a complete stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_0^+}, \mathbb{P})$ , starting at the origin and

having the covariance operator  $tC$ . Recall that this means  $(W(\cdot, t))_{\mathbb{R}_0^+}$  has independent Gaussian increments  $W(\cdot, s+t) - W(\cdot, t)$  of average zero and covariance operator  $sC$  for all  $s, t \in \mathbb{R}_0^+$ , as well as continuous trajectories almost surely. Moreover, writing  $(\cdot, \cdot)_2$  for the usual scalar product in  $L^2(D)$  we have

$$\begin{aligned} \mathbb{E}((W(\cdot, s), v)_2(W(\cdot, t), \hat{v})_2) &= (s \wedge t)(Cv, \hat{v})_2 \\ &= (s \wedge t) \int_{D \times D} dx dy \kappa(x, y) v(x) \hat{v}(y) \end{aligned} \quad (1)$$

for all  $s, t \in \mathbb{R}_0^+$  and all  $v, \hat{v} \in L^2(D)$ ; we also assume that  $(W(\cdot, t))_{t \in \mathbb{R}_0^+}$  is  $(\mathcal{F}_t)_{t \in \mathbb{R}_0^+}$ -adapted and that the increments  $W(\cdot, s+t) - W(\cdot, t)$  are  $\mathcal{F}_t$ -independent for each  $s, t \in \mathbb{R}_0^+$ . Finally, there is another important property of the Wiener process  $(W(\cdot, t))_{\mathbb{R}_0^+}$  that we shall invoke below, namely, its Fourier decomposition

$$W(\cdot, t) = \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} e_j(\cdot) B_j(t) \quad (2)$$

in  $L^2(D)$  where  $((B_j(t))_{t \in \mathbb{R}_0^+})_{j \in \mathbb{N}^+}$  denotes a sequence of one-dimensional, independent, standard Brownian motions (see, for instance, [14]).

Let  $T \in \mathbb{R}^+$  and let us consider the following class of real, parabolic, Itô initial-boundary value problems:

$$\begin{aligned} du(x, t) &= (\operatorname{div}(k(x, t) \nabla u(x, t)) + g(u(x, t))) dt + h(u(x, t)) W(x, dt), \\ &\quad (x, t) \in D \times (0, T], \\ u(x, 0) &= \varphi(x), \quad x \in \overline{D}, \\ \frac{\partial u(x, t)}{\partial n(k)} &= 0, \quad (x, t) \in \partial D \times (0, T]. \end{aligned} \quad (3)$$

In the preceding equations, the function  $k$  is matrix-valued and the last relation stands for the conormal derivative of  $u$  relative to  $k$ ; moreover, we denote by  $n$  the unit outer normal vector to  $\partial D$  and we assume that the functions  $k$  and  $n$  satisfy the following hypothesis:

(K) The entries of  $k$  satisfy the symmetry relation  $k_{i,j}(\cdot) = k_{j,i}(\cdot)$  for every  $i, j \in \{1, \dots, d\}$ . Moreover, there exists a constant  $\beta \in (\frac{1}{2}, 1]$  such that  $k_{i,j} \in \mathcal{C}^{\alpha, \beta}(\overline{D} \times [0, T])$  for each  $i, j$  and, in addition, we have  $k_{i,j,x_l} := \frac{\partial k_{i,j}}{\partial x_l} \in \mathcal{C}^{\alpha, \frac{\alpha}{2}}(\overline{D} \times [0, T])$  for each  $i, j, l$ ; furthermore, there exists a positive constant  $\underline{k}$  such that the inequality  $\underline{k} |a|^2 \leq (k(x, t)a, a)_{\mathbb{R}^d}$  holds for all

$a \in \mathbb{R}^d$  and all  $(x, t) \in \overline{D} \times [0, T]$ , where  $|\cdot|$  and  $(\cdot, \cdot)_{\mathbb{R}^d}$  denote the Euclidean norm and the Euclidean scalar product in  $\mathbb{R}^d$ , respectively. Finally, we have  $(x, t) \mapsto \sum_{i=1}^d k_{i,j}(x, t)n_i(x) \in \mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}(\partial D \times [0, T])$  for each  $j$  and the conormal vector-field  $(x, t) \mapsto n(k)(x, t) := k(x, t)n$  is outward pointing, nowhere tangent to  $\partial D$  for every  $t \in [0, T]$ .

Regarding the drift-nonlinearity  $g$ , the noise-nonlinearity  $h$  and the initial condition  $\varphi$  we have the following hypotheses, respectively:

(L) The functions  $g, h : \mathbb{R} \mapsto \mathbb{R}$  are Lipschitz continuous.

(I) We have  $\varphi \in \mathcal{C}^{2+\alpha}(\overline{D})$ ; moreover,  $\varphi$  satisfies the conormal boundary condition relative to  $k$ .

Relations (3) define a class of non autonomous, semilinear, stochastic initial-boundary value problems driven by an infinite-dimensional noise which depends on both the space variable  $x$  and the time variable  $t$ . By virtue of (1), this noise is colored with respect to  $x$  and white with respect to  $t$ , all properties of its spatial correlations being completely encoded in the generating kernel  $\kappa$ .

Problems of the form (3) that involve a spatially colored noise are quite relevant to the mathematical analysis of a variety of physical processes in which the scale of the spatial correlations of the noise is much larger than that of its time correlations; particular cases of them as well as their deterministic counterparts have been used over the years to model, for instance, certain migration phenomena in population dynamics and population genetics (see, for instance, [4], [5], [9], [10], [11], [23], [49] and their references). Furthermore, there are many possible ways to define a notion of solution for them and it is not *a priori* evident to know which notions lead to indistinguishable, jointly space-time Hölder continuous processes. Accordingly, we shall organize the remaining part of this article in the following way: in Section 2, we state and discuss our main results concerning indistinguishability, existence, uniqueness and Hölder regularity, after having introduced four notions of solution; two of these are variational notions while the third and fourth one involve a family of evolution operators defined through the deterministic, parabolic Green's function associated with the principal part of (3), whose existence and regularity properties are ensured by Hypotheses (K) and (I); after having proved the equivalence of the first three notions in Section 3, we use the properties of the Green's function to prove the existence, the uniqueness, the pointwise boundedness of the moments and the joint space-time Hölder continuity of such solutions, as the fundamental heat kernel estimates for the Green's function turn out to be the most appropriate tools that allow us to do so. Our proof of these properties also shows that those solutions exhibit Sobolev regularity in the space vari-

able, and in fact brings about the equivalence between two theories hitherto unrelated for models as general as (3), namely, the variational theory developed in ([30], [42]) and the Green's function theory initiated in [50]. In Section 3, we also show how to weaken some requirements concerning the covariance operator  $C$  in order to prove the existence and the regularity of a solution of the fourth type, and establish an analogy between those weakened requirements and the so-called *spectral measure conditions* that have been introduced recently to analyze some classes of *autonomous* stochastic partial differential equations defined on the whole of  $\mathbb{R}^d$  (see, for instance, [12], [25], [33], [44], [46] and their references). Finally, we refer the reader to [45] for a short announcement of the above and related results, and to [29], [35], [36], [39] and their references for other recent results about existence, uniqueness and regularity proved by completely different methods.

## 2 Statement and Discussion of the Main Results

In the remaining part of this article we write  $\|\cdot\|_s$  for  $L^s(D)$ -norms,  $\|\cdot\|_{1,2}$  for the norm in the usual Sobolev space  $H^1(D)$  of functions on  $D$  and  $\mathcal{C}([0, T]; L^2(D))$  for the space of all continuous mappings from the interval  $[0, T]$  into  $L^2(D)$  endowed with the uniform topology. We write  $c$  for all irrelevant, positive constants that occur in the various estimates unless we specify the constants otherwise. The first notion we introduce is that of a variational solution tested with functions that depend only on the space variable. In addition to (K), (L) and (I) above, this requires the following hypothesis regarding the basis  $(e_j)_{j \in \mathbb{N}^+}$  and the eigenvalues  $(\lambda_j)_{j \in \mathbb{N}^+}$  of the operator  $C$ :

(C) We have  $e_j \in L^\infty(D)$  for each  $j$  and

$$\sum_{j=1}^{+\infty} \lambda_j \|e_j\|_\infty^2 < +\infty. \quad (4)$$

Since we can rewrite the eigenvalue equation  $Ce_j = \lambda_j e_j$  as

$$e_j(x) = \frac{1}{\lambda_j} \int_D dy \kappa(x, y) e_j(y)$$

for almost every  $x \in D$ , and since  $\|e_j\|_2 = 1$  for each  $j$ , we can easily infer from the preceding relation and from Schwarz inequality that  $e_j \in L^\infty(D)$

for each  $j$  if we impose, for instance, the integrability condition

$$x \mapsto \int_D dy |\kappa(x, y)|^2 \in L^\infty(D).$$

In this context, we remark that Hypothesis (C) defines a restricted set of trace-class covariance operators since Condition (4) implies  $\sum_{j=1}^{+\infty} \lambda_j := \text{Tr} C < +\infty$  by virtue of the existence of the continuous embedding  $L^\infty(D) \rightarrow L^2(D)$ .

**Definition 1.** We say that the  $L^2(D)$ -valued random field  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is a *variational solution of the first kind* to Problem (3) if the following conditions hold:

- (1)  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  is progressively measurable on  $[0, T] \times \Omega$ .
- (2) We have  $u_\varphi^1 \in L^2((0, T) \times \Omega; H^1(D)) \cap L^2(\Omega; \mathcal{C}([0, T]; L^2(D)))$  and consequently

$$\begin{aligned} & \mathbb{E} \int_0^T d\tau \|u_\varphi^1(\cdot, \tau)\|_{1,2}^2 \\ &= \mathbb{E} \int_0^T d\tau \left( \|u_\varphi^1(\cdot, \tau)\|_2^2 + \|\nabla u_\varphi^1(\cdot, \tau)\|_2^2 \right) < +\infty \end{aligned} \quad (5)$$

as well as

$$\mathbb{E} \sup_{t \in [0, T]} \|u_\varphi^1(\cdot, t)\|_2^2 < +\infty. \quad (6)$$

- (3) The integral relation

$$\begin{aligned} \int_D dx v(x) u_\varphi^1(x, t) &= \int_D dx v(x) \varphi(x) \\ &\quad - \int_0^t d\tau \int_D dx (\nabla v(x), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\ &\quad + \int_0^t d\tau \int_D dx v(x) g(u_\varphi^1(x, \tau)) \\ &\quad + \int_0^t \int_D dx v(x) h(u_\varphi^1(x, \tau)) W(x, d\tau) \end{aligned} \quad (7)$$

holds a.s. for every  $v \in H^1(D)$  and every  $t \in [0, T]$ , where we have defined the stochastic integral by

$$\int_0^t \int_D dx v(x) h(u_\varphi^1(x, \tau)) W(x, d\tau) := \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v, h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau).$$

From the preceding definition and from the above hypotheses, we easily infer that each term in Equation (7) is well defined and finite a.s.; in particular, our definition of the stochastic integral with respect to  $(W(\cdot, t))_{t \in \mathbb{R}_0^+}$  as an infinite sum of one-dimensional, independent Itô integrals is based on the Fourier decomposition (2) and represents a real-valued, square integrable random variable. In order to see this we invoke successively the isometry property of Itô's integral, Schwarz inequality, Hölder's inequality between  $L^1(D)$  and  $L^\infty(D)$  along with Hypothesis (L) to obtain

$$\begin{aligned} & \mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v, h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \right|^2 \\ & \leq \|v\|_2^2 \sum_{j=1}^{+\infty} \lambda_j \mathbb{E} \int_0^t d\tau \int_D dx |h(u_\varphi^1(x, \tau))|^2 |e_j(x)|^2 \\ & \leq c \|v\|_2^2 \left( \sum_{j=1}^{+\infty} \lambda_j \|e_j\|_\infty^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \|u_\varphi^1(\cdot, t)\|_2^2 \right) < +\infty \end{aligned} \quad (8)$$

as a consequence of Hypothesis (C) and Relation (6).

Variational solutions such as  $u_\varphi^1$  have been used in a number of situations (see, for instance, [4], [9], [10], [11]) and the proof of their existence and their uniqueness for problems such as (3) can be traced to rather standard monotonicity and compactness arguments ([30], [42], [43]). Relation (7), however, does not seem to be suitable for the investigation of the joint Hölder continuity properties of  $u_\varphi^1$  as it only defines this random field implicitly. A preliminary step toward getting an explicit relation for variational solutions in terms of the Green's function associated with the principal part of (3) can consist in testing them with functions that depend on *both* the space and the time variable. For every  $t \in (0, T]$ , let us write  $H^1(D \times (0, t))$  for the Sobolev space of all real-valued functions  $v \in L^2(D \times (0, t))$  that possess distributional derivatives  $v_{x_j} \in L^2(D \times (0, t))$  for every  $j \in \{1, \dots, d\}$ , along with a distributional time-derivative  $v_\tau \in L^2(D \times (0, t))$ . We denote the norm of  $H^1(D \times (0, t))$  by

$$\begin{aligned} \|v\|_{1,2,t}^2 &= \int_{D \times (0,t)} dx d\tau |v(x, \tau)|^2 + \sum_{j=1}^d \int_{D \times (0,t)} dx d\tau |v_{x_j}(x, \tau)|^2 \\ &+ \int_{D \times (0,t)} dx d\tau |v_\tau(x, \tau)|^2. \end{aligned} \quad (9)$$

The following definition requires exactly the same four hypotheses as above.

**Definition 2.** We say that the  $L^2(D)$ -valued random field  $(u_\varphi^2(\cdot, t))_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is a *variational solution of the second kind* to Problem (3) if the first two conditions of Definition 1 hold, and if the integral relation

$$\begin{aligned} \int_D dx v(x, t) u_\varphi^2(x, t) &= \int_D dx v(x, 0) \varphi(x) + \int_0^t d\tau \int_D dx v_\tau(x, \tau) u_\varphi^2(x, \tau) \\ &\quad - \int_0^t d\tau \int_D dx (\nabla v(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \\ &\quad + \int_0^t d\tau \int_D dx v(x, \tau) g(u_\varphi^2(x, \tau)) \\ &\quad + \int_0^t \int_D dx v(x, \tau) h(u_\varphi^2(x, \tau)) W(x, d\tau) \end{aligned} \quad (10)$$

holds a.s. for every  $v \in H^1(D \times (0, t))$  and every  $t \in [0, T]$ , where  $x \mapsto v(x, 0) \in L^2(D)$  and  $x \mapsto v(x, t) \in L^2(D)$  denote the Sobolev traces of  $v$  on  $D$  and  $D \times \{\tau \in \mathbb{R} : \tau = t\}$ , respectively, and where we have defined the stochastic integral as in Definition 1.

Again, we see that every term in Equation (10) is well defined and finite a.s., and that the structure of (10) is identical to that of (7) up to the appearance of the term that involves the partial derivative  $v_\tau$ .

It turns out that these two notions of solution are equivalent, which, together with the remark following (8), immediately implies the existence and the uniqueness of a variational solution of the second kind to (3); more precisely we have the following result whose complete proof we give in Section “active@prefix ~“active@char~ 3.

**Theorem 1.** *Assume that the above hypotheses hold; then, an  $L^2(D)$ -valued random field is a variational solution of the first kind to (3) if, and only if, it is a variational solution of the second kind; in fact, there exists a unique variational solution of the second kind to (3) and we have  $u_\varphi^1(\cdot, t) = u_\varphi^2(\cdot, t)$  a.s. as equalities in  $L^2(D)$  for every  $t \in [0, T]$ .*

We can actually prove Theorem 1 under much weaker conditions concerning the regularity of  $k$  and  $\varphi$ , but we shall refrain from doing so in view of the fact that Hypotheses (K) and (I) are crucial regarding the formulation of the variational solutions in terms of the Green’s function  $G$  associated with the principal part of (3). Recall that under Hypotheses (K) and (I), the function  $G : \overline{D} \times [0, T] \times \overline{D} \times [0, T] \setminus \{s, t \in [0, T] : s \geq t\} \mapsto \mathbb{R}$  is continuous,

twice continuously differentiable in  $x$ , once continuously differentiable in  $t$  and satisfies the fundamental heat kernel estimates

$$|\partial_x^\mu \partial_t^\nu G(x, t; y, s)| \leq c(t-s)^{-\frac{d+|\mu|+2\nu}{2}} \exp\left[-c\frac{|x-y|^2}{t-s}\right] \quad (11)$$

where  $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}^d$ ,  $\nu \in \mathbb{N}$  and  $|\mu| + 2\nu \leq 2$  with  $|\mu| = \sum_{j=1}^d \mu_j$  (see, for instance, [17]). This allows us to define the following notion of mild solution to Problem (3).

**Definition 3.** We say that the  $L^2(D)$ -valued random field  $(u_\varphi^3(\cdot, t))_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is a *mild solution* to Problem (3) if the first two conditions of Definition 1 hold, and if the relation

$$\begin{aligned} u_\varphi^3(\cdot, t) &= \int_D dy G(\cdot, t; y, 0) \varphi(y) + \int_0^t d\tau \int_D dy G(\cdot, t; y, \tau) g(u_\varphi^3(y, \tau)) \\ &\quad + \int_0^t \int_D dy G(\cdot, t; y, \tau) h(u_\varphi^3(y, \tau)) W(y, d\tau) \end{aligned} \quad (12)$$

holds a.s. for every  $t \in [0, T]$  as an equality in  $L^2(D)$ , where for  $t = 0$  we have  $\int_D dy G(\cdot, 0; y, 0) \varphi(y) := \lim_{t \searrow 0} \int_D dy G(\cdot, t; y, 0) \varphi(y) = \varphi(\cdot)$  and where we have defined the stochastic integral as above.

The proof that each term on the right-hand side of (12) defines an  $L^2(D)$ -valued function a.s. is complicated by the existence of the singularity on the time-diagonal in  $G$ ; for the first term the statement follows from the fact that  $\varphi$  is bounded and from (11) for  $\mu = \nu = 0$ , since the right-hand side of (11) then extends to a Gaussian measure on  $\mathbb{R}^d$ , a fact that we shall use often in the sequel and refer to as the Gaussian property of  $G$ . For the remaining part of the argument we restrict ourselves to the analysis of the stochastic term; owing to the isometry property of Itô's integral, Hypotheses (C), (L) and the Gaussian property we just alluded to, we first have

$$\begin{aligned} &\mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (G(x, t; \cdot, \tau), h(u_\varphi^3(\cdot, \tau)) e_j)_2 B_j(d\tau) \right|^2 \\ &\leq \left( \sum_{j=1}^{+\infty} \lambda_j \|e_j\|_\infty^2 \right) \mathbb{E} \int_0^t d\tau \left( \int_D dy |G(x, t; y, \tau) h(u_\varphi^3(y, \tau))| \right)^2 \\ &\leq c \mathbb{E} \left( 1 + \int_0^t d\tau \left( \int_D dy |G(x, t; y, \tau)| |u_\varphi^3(y, \tau)| \right)^2 \right) \end{aligned}$$

$$\leq c\mathbb{E}\left(1 + \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| |u_\varphi^3(y, \tau)|^2\right) \quad (13)$$

for every  $x \in \overline{D}$ , where we obtained the very last estimate by applying Schwarz inequality relative to the finite measure  $dy |G(x, t; y, \tau)|$  on  $D$  in order to control the singularity of  $G$ . We then integrate both sides of (13) with respect to  $x$ ; through repeated applications of Fubini's theorem and by using the Gaussian property once again along with (6) we obtain

$$\begin{aligned} & \mathbb{E} \int_D dx \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (G(x, t; \cdot, \tau), h(u_\varphi^3(\cdot, \tau))e_j)_2 B_j(d\tau) \right|^2 \\ & \leq c \left( 1 + \sup_{\tau \in [0, T]} \mathbb{E} \|u_\varphi^3(\cdot, \tau)\|_2^2 \right) < +\infty, \end{aligned}$$

which proves that  $x \mapsto \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (G(x, t; \cdot, \tau), h(u_\varphi^3(\cdot, \tau))e_j)_2 B_j(d\tau) \in L^2(D)$  a.s..

Over the years, there have been several results in various contexts that establish relationships between different kinds of variational solutions and their mild formulations, both in the deterministic and in the stochastic case (see, for instance, [2], [8], [14], [16], [22], [32], [50]). In particular, the case of semilinear, non autonomous, stochastic evolution equations driven by semimartingales has been analyzed in [32] from a very abstract viewpoint. However, none of the above works has dealt with stochastic reaction-diffusion equations such as (3). Moreover, following [50], several notions of mild solutions that involve Green's functions, Green's distributions or more general semi-group arguments have been used to investigate the existence and the regularity properties of solutions to several classes of hyperbolic and parabolic stochastic partial differential equations (see, for instance, [6], [7], [12], [13], [14], [20], [44], [46] and their references). In this perspective, we next state a result which, together with Theorem 1, establishes the existence and the uniqueness of a mild solution to (3).

**Theorem 2.** *Assume that the above hypotheses hold; then, an  $L^2(D)$ -valued random field is a variational solution of the second kind to (3) if, and only if, it is a mild solution; in fact, there exists a unique mild solution to (3) and we have  $u_\varphi^2(\cdot, t) = u_\varphi^3(\cdot, t)$  a.s. as equalities in  $L^2(D)$  for every  $t \in [0, T]$ .*

As a consequence of Theorems 1 and 2, which prove the equivalence of the above three definitions, it is from now on legitimate to call *solution* to

Problem (3) an  $L^2(D)$ -valued random field  $(u_\varphi(\cdot, t))_{t \in [0, T]}$  that solves (3) in the sense of *any* of the three notions we have introduced. It turns out that such a solution enjoys several important boundedness and regularity properties, as stated in the following result.

**Theorem 3.** *Assume that the above hypotheses hold; then there exists a unique solution to Problem (3) such that  $x \mapsto u_\varphi(x, t) \in H^1(D)$  a.s. for every  $t \in [0, T]$ , which satisfies the relation*

$$\sup_{(x, t) \in D \times [0, T]} \mathbb{E} |u_\varphi(x, t)|^r < +\infty \quad (14)$$

for every  $r \in [1, +\infty)$ . Moreover, there is a version of  $(u_\varphi(x, t))_{(x, t) \in D \times [0, T]}$  such that  $u_\varphi(\cdot, \cdot) \in C^{\beta_1, \beta_2}(D \times [0, T])$  a.s. for every  $\beta_1 \in (0, \alpha)$  and every  $\beta_2 \in \left(0, \frac{\alpha}{2} \wedge \frac{2}{d+2}\right)$ .

In the preceding statement we remark that both  $\beta_1$  and  $\beta_2$  are independent of the exponent  $\beta$  of Hypothesis (K); moreover,  $\beta_1$  is always independent of  $d$ , whereas  $\beta_2$  depends explicitly on the dimension but only for  $d \geq 3$ ; we shall see that the latter phenomenon is inherent in the presence of the stochastic term in (3).

As testified by the many references we have quoted in this article, a significant part of the recent literature on stochastic partial differential equations is based on notions of mild solution which differ from ours in that they do not have a built-in requirement for  $H^1(D)$ -regularity. In order to investigate this point in detail, we conclude this section by introducing a fourth type of solution for (3); we also state two existence and regularity theorems for it which hold under conditions weaker than (C); the first of these is the following:

(C<sup>d</sup>) There exists  $s \in (d, +\infty)$  such that  $e_j \in L^s(D)$  for each  $j$  and

$$\sum_{j=1}^{+\infty} \lambda_j \|e_j\|_s^2 < +\infty. \quad (15)$$

We remark that Hypothesis (C) implies Hypothesis (C<sup>d</sup>).

**Definition 4.** We say that the real-valued random field  $(u_\varphi^4(x, t))_{(x, t) \in D \times [0, T]}$  defined on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is a *strong solution* to Problem (3) if the following conditions hold:

- (1)  $u_\varphi^4$  is progressively measurable on  $D \times [0, T] \times \Omega$ .

(2) We have  $\sup_{(x,t) \in D \times [0,T]} \mathbb{E} |u_\varphi^4(x,t)|^r < +\infty$  for every  $r \in [1, +\infty)$ .

(3) The relation

$$\begin{aligned} u_\varphi^4(x,t) &= \int_D dy G(x,t;y,0) \varphi(y) + \int_0^t d\tau \int_D dy G(x,t;y,\tau) g(u_\varphi^4(y,\tau)) \\ &\quad + \int_0^t \int_D dy G(x,t;y,\tau) h(u_\varphi^4(y,\tau)) W(y, d\tau) \end{aligned} \quad (16)$$

holds a.s. for every  $(x,t) \in D \times [0,T]$ , where  $G$  satisfies the same properties as in Definition 3.

We note the change of viewpoint in the preceding definition: we consider  $u_\varphi^4$  along with each term on the right-hand side of (16) as real-valued random fields indexed by  $(x,t) \in D \times [0,T]$ , and no longer as random fields taking values in some functional space; furthermore, we assume the boundedness of the moments from the outset. From the preceding definition, it is then immediate that the first two terms on the right-hand side of (16) are finite a.s.. The same is true for the stochastic term by virtue of  $(C^d)$ ; in order to see this let  $s^* \in \left(1, \frac{d}{d-2}\right)$  be the dual exponent of  $\frac{s}{2}$ ; then, by using the isometry property of Itô's integral, Hypothesis (L), Schwarz inequality relative to the measure  $dy |G(x,t;y,\tau)|$  on  $D$ , the Gaussian property, (2) of Definition 4 and Hölder's inequality, we obtain

$$\begin{aligned} &\mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (G(x,t; \cdot, \tau) h(u_\varphi^4(\cdot, \tau)), e_j)_2 B_j(d\tau) \right|^2 \\ &\leq c \sum_{j=1}^{+\infty} \lambda_j \mathbb{E} \int_0^t d\tau \left( \int_D dy |G(x,t;y,\tau) e_j(y)| (1 + |u_\varphi^4(y,\tau)|) \right)^2 \\ &\leq c \sum_{j=1}^{+\infty} \lambda_j \int_0^t d\tau \int_D dy |G(x,t;y,\tau)| |e_j(y)|^2 \\ &\leq c \left( \sum_{j=1}^{+\infty} \lambda_j \|e_j\|_s^2 \right) \int_0^t d\tau \left( \int_D dy |G(x,t;y,\tau)|^{s^*} \right)^{\frac{1}{s^*}} \\ &\leq c \int_0^t d\tau (t-\tau)^{-\frac{d}{2} + \frac{d}{2s^*}} \left( \int_D dy (t-\tau)^{-\frac{d}{2}} \exp \left[ -c \frac{|x-y|^2}{t-\tau} \right] \right)^{\frac{1}{s^*}} \\ &\leq c \int_0^t d\tau (t-\tau)^{-\frac{d}{2} + \frac{d}{2s^*}} < +\infty \end{aligned} \quad (17)$$

since  $1 - \frac{d}{2} + \frac{d}{2s^*} > 0$ .

Whereas Hypotheses (K), (L), (I) and  $(C^d)$  allow us to prove the existence of a unique strong solution to (3), they do not suffice to imply the existence of a Hölder continuous version; for this we need to strengthen  $(C^d)$  in the following way:

$(C_\eta^d)$  There exist  $\eta \in (0, \frac{1}{2})$ ,  $s \in (\frac{d}{1-2\eta}, +\infty)$  such that  $e_j \in L^s(D)$  for each  $j$  and

$$\sum_{j=1}^{+\infty} \lambda_j \|e_j\|_s^2 < +\infty. \quad (18)$$

Our next result is then the following.

**Theorem 4.** *Assume that Hypotheses (K), (L), (I) and  $(C^d)$  hold; then, there exists a unique strong solution  $(u_\varphi^4(x, t))_{(x,t) \in D \times [0, T]}$  to (3). Moreover, if Hypothesis  $(C_\eta^d)$  holds, there is a version of  $(u_\varphi^4(x, t))_{(x,t) \in D \times [0, T]}$  such that  $u_\varphi^4(\cdot, \cdot) \in \mathcal{C}^{\gamma_1, \gamma_2}(D \times [0, T])$  a.s. for every  $\gamma_1 \in (0, \alpha)$  and every  $\gamma_2 \in (0, \frac{\alpha}{2} \wedge \frac{2}{d+2} \wedge \eta)$ .*

Finally, we note that we can weaken  $(C^d)$  and  $(C_\eta^d)$  even further by introducing the following two hypotheses, which now relate the covariance operator of the Wiener process to the differential operator in the principal part of “active@prefix ~ “active@char~ (3).

(H) We have

$$\sup_{(x,t) \in D \times [0, T]} \int_0^t d\tau \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy |G(x, t; y, \tau | e_j(y)) \right)^2 < +\infty. \quad (19)$$

$(H_\eta)$  There exists  $\eta \in (0, \frac{1}{2})$  such that

$$\sup_{(x,t) \in D \times [0, T]} \int_0^t d\tau (t - \tau)^{-2\eta} \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy |G(x, t; y, \tau | e_j(y)) \right)^2 < +\infty. \quad (20)$$

Indeed, in the next section we show that  $(C^d)$  implies (H), that  $(C_\eta^d)$  implies  $(H_\eta)$  and that we can still prove the existence and the Hölder regularity of a strong solution to (3) under Hypotheses (19) and (20); however, this is at the expense of having to assume  $\kappa(x, y) \geq 0$  for almost all  $x, y \in D$ ; in fact, under this additional restriction we notice that the third term on the right-hand side of (16) is still finite a.s.: from the isometry property of Itô’s

integral, Parseval's relation relative to the orthonormal basis  $(e_j)_{j \in \mathbb{N}^+}$  and the self-adjointness of  $C$ , we get

$$\begin{aligned}
& \mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (G(x, t; \cdot, \tau) h(u_\varphi^4(\cdot, \tau)), e_j)_2 B_j(d\tau) \right|^2 \\
& \leq \mathbb{E} \int_0^t d\tau \sum_{j=1}^{+\infty} \lambda_j (G(x, t; \cdot, \tau) h(u_\varphi^4(\cdot, \tau)), e_j)_2^2 \\
& = \mathbb{E} \int_0^t d\tau \sum_{j=1}^{+\infty} \left( C^{\frac{1}{2}} G(x, t; \cdot, \tau) h(u_\varphi^4(\cdot, \tau)), e_j \right)_2^2 \\
& = \mathbb{E} \int_0^t d\tau (CG(x, t; \cdot, \tau) h(u_\varphi^4(\cdot, \tau)), G(x, t; \cdot, \tau) h(u_\varphi^4(\cdot, \tau)))_2 \\
& \leq c \int_0^t d\tau \int_{D \times D} dy dz |G(x, t; y, \tau)| \times \kappa(y, z) \times |G(x, t; z, \tau)| \\
& \quad \times \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u_\varphi^4(y, \tau)|^2 \right) < +\infty \tag{21}
\end{aligned}$$

by virtue of Hypothesis (L), Schwarz inequality applied to the expectation functional, (2) of Definition 4 and the fact that we have

$$\begin{aligned}
& \int_0^t d\tau \int_{D \times D} dy dz |G(x, t; y, \tau)| \times \kappa(y, z) \times |G(x, t; z, \tau)| \\
& = \int_0^t d\tau \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy |G(x, t; y, \tau) e_j(y)| \right)^2 < +\infty \tag{22}
\end{aligned}$$

because of (19). The last result of this section is then the following.

**Theorem 5.** *Assume that Hypotheses (K), (L), (I), (H) hold and that  $\kappa(x, y) \geq 0$  for almost all  $x, y \in D$ ; then, there exists a unique strong solution to (3). Moreover, if Hypothesis  $(H_\eta)$  holds, there is a version  $(u_\varphi^4(x, t))_{(x, t) \in D \times [0, T]}$  of this solution such that  $u_\varphi^4(\cdot, \cdot) \in \mathcal{C}^{\gamma_1, \gamma_2}(D \times [0, T])$  a.s. for every  $\gamma_1 \in (0, \alpha)$  and every  $\gamma_2 \in \left(0, \frac{\alpha}{2} \wedge \frac{2}{d+2} \wedge \eta\right)$ .*

We devote the remaining part of this article to proving the above five theorems; in particular, we show that it is precisely conditions (19) and (20) that play a similar rôle in our analysis of (3) as the spectral measure conditions we referred to at the very end of Section 1 play in the recent works we quoted there.

### 3 Proof of the Main Results

We begin by observing that every variational solution of the second kind to Problem (3) is trivially a variational solution of the first kind. Therefore, we can reduce the proof of Theorem 1 to that of the converse statement. Let  $p : D \times [0, T] \mapsto \mathbb{R}$  be a polynomial in  $x$  and  $t$ , that is a finite sum of the form  $p(x, t) = \sum_{\mu, \nu} c_{\mu, \nu} x^\mu t^\nu$  where  $c_{\mu, \nu} \in \mathbb{R}$ , where  $\mu$  and  $\nu$  have the same meaning as in the preceding section, and where  $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \dots x_d^{\mu_d}$  for  $x = (x_1, \dots, x_d)$ . Our first auxiliary result toward the proof of Theorem 1 is the following.

**Proposition 1.** *Assume that the same hypotheses as in Theorem 1 hold and let  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  be a variational solution of the first kind to Problem (3). Then the integral relation*

$$\begin{aligned} \int_D dx p(x, t) u_\varphi^1(x, t) &= \int_D dx p(x, 0) \varphi(x) + \int_0^t d\tau \int_D dx p_\tau(x, \tau) u_\varphi^1(x, \tau) \\ &\quad - \int_0^t d\tau \int_D dx (\nabla p(x, \tau), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\ &\quad + \int_0^t d\tau \int_D dx p(x, \tau) g(u_\varphi^1(x, \tau)) \\ &\quad + \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (p(\cdot, \tau), h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \end{aligned} \quad (23)$$

holds a.s. for every polynomial  $p$  and every  $t \in [0, T]$ .

The proof of the preceding proposition relies on several lemmas. Let us first introduce the anisotropic Sobolev space  $H^{1,0}(D \times (0, T))$  of all real-valued functions  $v \in L^2(D \times (0, T))$  that possess distributional derivatives  $v_{x_j} \in L^2(D \times (0, T))$  for every  $j \in \{1, \dots, d\}$ , whose norm we denote by

$$\|v\|_{1,2,T;0}^2 = \int_{D \times (0,T)} dx d\tau |v(x, \tau)|^2 + \sum_{j=1}^d \int_{D \times (0,T)} dx d\tau |v_{x_j}(x, \tau)|^2. \quad (24)$$

While the  $H^1(D \times (0, t))$ 's are the basic spaces of test functions for variational solutions of the second kind,  $H^{1,0}(D \times (0, T))$  is the fundamental space in which the random field  $u_\varphi^1$  lives since Relation (5) immediately implies that  $u_\varphi^1(\cdot, \cdot) \in H^{1,0}(D \times (0, T))$  a.s.. The preceding remark first leads to the following integrability properties, whose proofs are elementary and therefore omitted.

**Lemma 1.** *Assume that the same hypotheses as in Theorem 1 hold and let  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  be as in Proposition 1. Then we have*

$$(x, \tau) \mapsto (\nabla v(x), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \in L^1(D \times (0, T))$$

and

$$(x, \tau) \mapsto v(x) g(u_\varphi^1(x, \tau)) \in L^1(D \times (0, T))$$

a.s. for every  $v \in H^1(D)$ .

The preceding lemma now leads to the following identity.

**Lemma 2 .** *Assume that the same hypotheses as in Theorem 1 hold and let  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  be as in Proposition 1. Then, for any real-valued function  $\chi \in C^1([0, T])$  satisfying  $\chi(0) = 0$ , the identity*

$$\begin{aligned} & \int_0^t d\tau \chi'(\tau) \int_D dx v(x) u_\varphi^1(x, \tau) \\ &= \chi(t) \int_D dx v(x) u_\varphi^1(x, \tau) \\ &+ \int_0^t d\tau \chi(\tau) \int_D dx (\nabla v(x), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\ &- \int_0^t d\tau \chi(\tau) \int_D dx v(x) g(u_\varphi^1(x, \tau)) \\ &- \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t \chi(\tau) (v, h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \end{aligned} \quad (25)$$

holds a.s. for every  $v \in H^1(D)$  and every  $t \in [0, T]$ .

**Proof.** We may assume  $t > 0$  and then start out from Relation (7) at  $t = \sigma$ , multiply both sides by  $\chi'(\sigma)$  and integrate with respect to  $\sigma$  on the interval  $(0, t)$ ; we obtain

$$\begin{aligned} & \int_0^t d\sigma \chi'(\sigma) \int_D dx v(x) u_\varphi^1(x, \sigma) \\ &= \chi(t) \int_D dx v(x) \varphi(x) \\ &- \int_0^t d\sigma \chi'(\sigma) \int_0^\sigma d\tau \int_D dx (\nabla v(x), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \end{aligned}$$

$$\begin{aligned}
& + \int_0^t d\sigma \chi'(\sigma) \int_0^\sigma d\tau \int_D dx v(x) g(u_\varphi^1(x, \tau)) \\
& + \int_0^t d\sigma \chi'(\sigma) \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^\sigma (v, h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \quad (26)
\end{aligned}$$

a.s. for every  $v \in H^1(D)$  and every  $t \in [0, T]$ . Owing to the result of Lemma 1 we may then integrate by parts the second, third and fourth terms on the right-hand side of (26); in this way, by invoking Itô's formula to handle the stochastic term and by taking into account the fact that  $\chi$  is non-random and satisfies  $\chi(0) = 0$  we get

$$\begin{aligned}
& \int_0^t d\tau \chi'(\tau) \int_D dx v(x) u_\varphi^1(x, \tau) \\
& = \chi(t) \int_D dx v(x) \varphi(x) \\
& - \chi(t) \int_0^t d\tau \int_D dx (\nabla v(x), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\
& + \int_0^t d\tau \chi(\tau) \int_D dx (\nabla v(x), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\
& + \chi(t) \int_0^t d\tau \int_D dx v(x) g(u_\varphi^1(x, \tau)) \\
& - \int_0^t d\tau \chi(\tau) \int_D dx v(x) g(u_\varphi^1(x, \tau)) \\
& + \chi(t) \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v, h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \\
& - \int_0^t \chi(\tau) \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} (v, h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \quad (27)
\end{aligned}$$

a.s. for every  $v \in H^1(D)$  and every  $t \in [0, T]$ . We then group together all terms containing  $\chi(t)$  and use Relation (7) once again to obtain (25). ■

The preceding considerations now allow us to prove Relation (23).

**Proof of Proposition 1.** We first split the polynomial  $p$  as

$$\begin{aligned} p(x, t) &= p(x, 0) + p^*(x, t) \\ &:= \sum_{\mu} c_{\mu,0} x^{\mu} + \sum_{\substack{\mu, \nu \\ \nu \neq 0}} c_{\mu, \nu} x^{\mu} t^{\nu} \end{aligned} \quad (28)$$

and we observe that  $x \mapsto p(x, 0) \in H^1(D)$  since  $D$  is bounded. We may then choose  $v(\cdot) = p(\cdot, 0)$  in Relation (7), so that we have

$$\begin{aligned} \int_D dx p(x, t) u_{\varphi}^1(x, t) &= \int_D dx p(x, 0) u_{\varphi}^1(x, t) \\ &+ \int_D dx p^*(x, t) u_{\varphi}^1(x, t) \\ &= \int_D dx p(x, 0) \varphi(x) \\ &- \int_0^t d\tau \int_D dx (\nabla p(x, 0), k(x, \tau) \nabla u_{\varphi}^1(x, \tau))_{\mathbb{R}^d} \\ &+ \int_0^t d\tau \int_D dx p(x, 0) g(u_{\varphi}^1(x, \tau)) \\ &+ \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (p(\cdot, 0), h(u_{\varphi}^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \\ &+ \int_D dx p^*(x, t) u_{\varphi}^1(x, t) \end{aligned} \quad (29)$$

a.s. for every  $t \in [0, T]$ . In order to deal with the last term of the preceding expression, we consider the term that contains the partial derivative  $p_{\tau}$  in (23), which we rewrite as

$$\begin{aligned} &\int_0^t d\tau \int_D dx p_{\tau}(x, \tau) u_{\varphi}^1(x, \tau) \\ &= \int_0^t d\tau \int_D dx p_{\tau}^*(x, \tau) u_{\varphi}^1(x, \tau) \\ &= \sum_{\substack{\mu, \nu \\ \nu \neq 0}} c_{\mu, \nu} \int_0^t d\tau (\nu \tau^{\nu-1}) \int_D dx x^{\mu} u_{\varphi}^1(x, \tau). \end{aligned} \quad (30)$$

The next, crucial observation is that the integral contribution in the very last term of (30) is exactly equal to the left-hand side of (25) when we

choose  $\tau \mapsto \chi(\tau) = \tau^\nu$  and  $x \mapsto v(x) = x^\mu$  there. Since these two functions obviously satisfy the hypotheses of Lemma 2, we may then rewrite the very last term of (30) by means of Relation (25) for these choices of  $\chi$  and  $v$ . By substituting the resulting expression in (30) and by resumming over  $\mu$  and  $\nu$  we obtain

$$\begin{aligned}
\int_D dx p^*(x, t) u_\varphi^1(x, t) &= \int_0^t d\tau \int_D dx p_\tau^*(x, \tau) u_\varphi^1(x, \tau) \\
&\quad - \int_0^t d\tau \int_D dx (\nabla p^*(x, \tau), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\
&\quad + \int_0^t d\tau \int_D dx p^*(x, \tau) g(u_\varphi^1(x, \tau)) \\
&\quad + \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (p^*(\cdot, \tau), h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \quad (31)
\end{aligned}$$

a.s. for every  $t \in [0, T]$ . We finally replace the last term on the right-hand side of (29) by the right-hand side of (31) and group together all terms of the resulting expression by means of Relation (28). ■

We can now easily extend the validity of Relation (23) by means of Weierstrass' approximation theorem. Indeed, for every  $t \in (0, T]$  let  $\mathcal{C}^1(\overline{D} \times [0, t])$  be the space of all real, once continuously differentiable functions  $v$  defined on  $\overline{D} \times [0, t]$ , endowed with the  $\mathcal{C}^1$ -topology induced by the norm

$$\begin{aligned}
\|v\|_{\mathcal{C}^1, t} &= \max_{(x, \tau) \in \overline{D} \times [0, t]} |v(x, \tau)| + \sum_{j=1}^d \max_{(x, \tau) \in \overline{D} \times [0, t]} |v_{x_j}(x, \tau)| \\
&\quad + \max_{(x, \tau) \in \overline{D} \times [0, t]} |v_\tau(x, \tau)|. \quad (32)
\end{aligned}$$

We then have the following result.

**Proposition 2.** *Assume that the same hypotheses as in Theorem 1 hold and let  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  be a variational solution of the first kind to Problem (3). Then the integral relation*

$$\begin{aligned}
\int_D dx v(x, t) u_\varphi^1(x, t) &= \int_D dx v(x, 0) \varphi(x) + \int_0^t d\tau \int_D dx v_\tau(x, \tau) u_\varphi^1(x, \tau) \\
&\quad - \int_0^t d\tau \int_D dx (\nabla v(x, \tau), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d}
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t d\tau \int_D dx v(x, \tau) g(u_\varphi^1(x, \tau)) \\
& + \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v(\cdot, \tau), h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \quad (33)
\end{aligned}$$

holds a.s. for every  $v \in \mathcal{C}^1(\overline{D} \times [0, t])$  and every  $t \in [0, T]$ .

**Proof.** Relation (33) clearly holds for  $t = 0$ , so that we may assume  $t > 0$ . Let  $v \in \mathcal{C}^1(\overline{D} \times [0, t])$ ; on the one hand, by the classic Weierstrass approximation theorem, there exists a sequence of polynomials  $(p_n)_{n \in \mathbb{N}^+}$  such that the estimate

$$\|v - p_n\|_{\mathcal{C}^1, t} < \frac{1}{n} \quad (34)$$

holds for every  $n \in \mathbb{N}^+$  (see, for instance, [28]). On the other hand, we have

$$\begin{aligned}
\int_D dx p_n(x, t) u_\varphi^1(x, t) & = \int_D dx p_n(x, 0) \varphi(x) + \int_0^t d\tau \int_D dx p_{n, \tau}(x, \tau) u_\varphi^1(x, \tau) \\
& - \int_0^t d\tau \int_D dx (\nabla p_n(x, \tau), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\
& + \int_0^t d\tau \int_D dx p_n(x, \tau) g(u_\varphi^1(x, \tau)) \\
& + \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (p_n(\cdot, \tau), h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \quad (35)
\end{aligned}$$

a.s. for every  $n \in \mathbb{N}^+$  and every  $t \in [0, T]$  by the statement of Proposition 1. We now show that Relations (34) and (35) imply Relation (33). The convergence of the terms of the first line in (35) toward the corresponding terms of (33) as  $n \rightarrow +\infty$  is trivial. As for the gradient term we have, owing to Schwarz inequality in  $\mathbb{R}^d$ , Relation (34), the boundedness of the coefficients  $k_{i,j}$  on  $\overline{D} \times [0, T]$  and the definition of the norm (24), the sequence of estimates

$$\begin{aligned}
& \int_0^t d\tau \int_D dx |(\nabla v(x, \tau) - \nabla p_n(x, \tau), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d}| \\
& \leq \int_0^t d\tau \int_D dx |\nabla v(x, \tau) - \nabla p_n(x, \tau)| |k(x, \tau) \nabla u_\varphi^1(x, \tau)|
\end{aligned}$$

$$\begin{aligned}
&\leq c \|v - p_n\|_{\mathcal{C}^1, t} \sum_{i,j=1}^d \int_0^t d\tau \int_D dx \left| k_{i,j}(x, \tau) u_{\varphi, x_j}^1(x, \tau) \right| \\
&\leq \frac{c}{n} \|u_{\varphi}^1(\cdot, \cdot)\|_{1,2,T;0} \rightarrow 0
\end{aligned} \tag{36}$$

a.s. as  $n \rightarrow +\infty$ . In a similar way we have

$$\begin{aligned}
&\int_0^t d\tau \int_D dx |v(x, \tau) - p_n(x, \tau)| |g(u_{\varphi}^1(x, \tau))| \\
&\leq \frac{c}{n} \left(1 + \|u_{\varphi}^1(\cdot, \cdot)\|_{1,2,T;0}\right) \rightarrow 0
\end{aligned} \tag{37}$$

a.s. as  $n \rightarrow +\infty$  since  $g$  is Lipschitz continuous. It remains to investigate the convergence of the stochastic integrals in (35). More specifically, we wish to show that

$$\lim_{l \rightarrow +\infty} \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v(\cdot, \tau) - p_{n_l}(\cdot, \tau), h(u_{\varphi}^1(\cdot, \tau))e_j)_2 B_j(d\tau) = 0 \tag{38}$$

a.s. for every  $t \in [0, T]$  along a suitable subsequence of polynomials  $(p_{n_l})_{l \in \mathbb{N}^+}$ . In order to achieve this it is sufficient to prove that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v(\cdot, \tau) - p_n(\cdot, \tau), h(u_{\varphi}^1(\cdot, \tau))e_j)_2 B_j(d\tau) \right|^2 = 0 \tag{39}$$

for every  $t \in [0, T]$ . Using successively the isometry property of Itô's integral, the definition of the norm (32), Schwarz inequality and the fact that  $h$  is Lipschitz continuous we obtain

$$\begin{aligned}
&\mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v(\cdot, \tau) - p_n(\cdot, \tau), h(u_{\varphi}^1(\cdot, \tau))e_j)_2 B_j(d\tau) \right|^2 \\
&\leq \sum_{j=1}^{+\infty} \lambda_j \mathbb{E} \int_0^t d\tau |(v(\cdot, \tau) - p_n(\cdot, \tau), h(u_{\varphi}^1(\cdot, \tau))e_j)_2|^2 \\
&\leq c \|v - p_n\|_{\mathcal{C}^1, t}^2 \sum_{j=1}^{+\infty} \lambda_j \mathbb{E} \int_0^T d\tau \int_D dx (1 + |u_{\varphi}^1(x, \tau)|^2) \\
&\leq \frac{c}{n^2} \text{Tr } C \left(1 + \mathbb{E} \int_0^T d\tau \|u_{\varphi}^1(\cdot, \tau)\|_2^2\right) \rightarrow 0
\end{aligned}$$

as  $n \rightarrow +\infty$ , because of (34), (5) and the fact that  $\|e_j\|_2 = 1$  for each  $j \in \mathbb{N}^+$ . This proves (39) and hence (38), so that the above remarks along with Relations (36), (37), and (38) prove Relation (33). ■

The above considerations now lead to the following.

**Proof of Theorem 1.** Let  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  be a variational solution of the first kind, fix  $t > 0$  and let  $v \in H^1(D \times (0, t))$ ; since the base  $\partial D$  of the cylinder  $D \times (0, t)$  is smooth, there exists a sequence  $(v_n)_{n \in \mathbb{N}^+} \subset \mathcal{C}^1(\bar{D} \times [0, t])$  such that the estimate

$$\|v - v_n\|_{1,2,t} \leq \frac{1}{n} \quad (40)$$

holds for every  $n \in \mathbb{N}^+$  (see, for instance, [38]). Furthermore, we have

$$\begin{aligned} \int_D dx v_n(x, t) u_\varphi^1(x, t) &= \int_D dx v_n(x, 0) \varphi(x) + \int_0^t d\tau \int_D dx v_{n,\tau}(x, \tau) u_\varphi^1(x, \tau) \\ &\quad - \int_0^t d\tau \int_D dx (\nabla v_n(x, \tau), k(x, \tau) \nabla u_\varphi^1(x, \tau))_{\mathbb{R}^d} \\ &\quad + \int_0^t d\tau \int_D dx v_n(x, \tau) g(u_\varphi^1(x, \tau)) \\ &\quad + \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v_n(\cdot, \tau), h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \end{aligned} \quad (41)$$

a.s. for every  $n \in \mathbb{N}^+$  and every  $t \in [0, T]$ , by the statement of Proposition “active@prefix ~“active@char~ 2. We can now ensure the convergence of each term of the first line in (41) toward the corresponding term in (33) by means of standard Sobolev trace-inequalities, while we can handle the third and fourth term on the right-hand side of (41) exactly as we did in the proof of Proposition 2. Regarding the convergence of the stochastic integrals, we have to argue slightly differently than we did to establish Relation (38) in order to retrieve the appropriate norm; in fact, it is sufficient to proceed exactly as we did to establish Relation (8); this gives

$$\begin{aligned} &\mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v(\cdot, \tau) - v_n(\cdot, \tau), h(u_\varphi^1(\cdot, \tau)) e_j)_2 B_j(d\tau) \right|^2 \\ &\leq c \|v - v_n\|_{1,2,t}^2 \left( \sum_{j=1}^{+\infty} \lambda_j \|e_j\|_\infty^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \|u_\varphi^1(\cdot, t)\|_2^2 \right) \end{aligned}$$

$$\leq \frac{c}{n^2} \left( \sum_{j=1}^{+\infty} \lambda_j \|e_j\|_\infty^2 \right) \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \|u_\varphi^1(\cdot, t)\|_2^2 \right) \rightarrow 0$$

as  $n \rightarrow +\infty$  by virtue of (4), (6) and (40). This proves that an appropriate subsequence of the stochastic integrals in (41) converges to the stochastic integral in (33) a.s. for each  $t \in [0, T]$ , thereby completing the proof of Relation (33) for  $v \in H^1(D \times (0, t))$ ; from this and the standard existence and uniqueness results for variational solutions of the first kind ([30], [42]), we can conclude that there exists a unique variational solution of the second kind to (3) such that  $u_\varphi^1(\cdot, t) = u_\varphi^2(\cdot, t)$  a.s. as equalities in  $L^2(D)$  for every  $t \in [0, T]$ . ■

We now turn to the proof of Theorem 2, which will require one preparatory result. Let  $q: [0, T] \times H^1(D) \times H^1(D) \mapsto \mathbb{R}$  be the symmetric quadratic form defined by

$$q(t; v, \hat{v}) = \int_D dx (k(x, t) \nabla v(x), \nabla \hat{v}(x))_{\mathbb{R}^d}$$

and set  $q(t; v) = q(t; v, v)$ . From this definition and Hypothesis (K), we infer in particular that the Hölder continuity estimate

$$|q(s; v) - q(t; v)| \leq c |s - t|^\beta q(t; v) \quad (42)$$

holds for all  $s, t \in [0, T]$  and every  $v \in H^1(D)$ , where  $\beta \in (\frac{1}{2}, 1]$ . From (42), the uniform ellipticity of (3) and the general theory of linear parabolic equations (see, for instance, [1], [26], [34], [48]), we conclude that there exists a two-parameter family of evolution operators  $U(t; s)_{0 \leq s \leq t \leq T}$  in  $L^2(D)$  associated with the principal part of (3) given by

$$U(t, s)v = \begin{cases} v & \text{if } s = t \\ \int_D dy G(\cdot, t; y, s)v(y) & \text{if } s < t \end{cases} \quad (43)$$

where  $G$  is the Green's function that enters Relation (12). We also infer from the first representation theorem for forms [27], or from the general considerations of [34], that there exists a self-adjoint, positive realization  $A(t) = -\operatorname{div}(k(\cdot, t)\nabla)$  of the elliptic partial differential operator with conormal boundary conditions in the principal part of (3); this operator generates the family  $U(t; s)_{0 \leq s \leq t \leq T}$ , and its self-adjointness domain in  $L^2(D)$  is

$$\mathcal{D}(A(t)) = \{v \in H^1(D) : A(t)v \in L^2(D), (A(t)v, \hat{v})_2 = q(t; v, \hat{v})\} \quad (44)$$

for every  $\hat{v} \in H^1(D)$ . We note that the self-adjointness of  $A(t)$  implies the self-adjointness of each one of the  $U(t, s)$  (see, for instance, [1], [48]), which in turn implies that the Green's function  $G$  is symmetric in its space variables, a fact we shall use frequently in the sequel. The preparatory result we alluded to above is central to the proof of Theorem 2; it shows that we can cancel out two terms in Relation (10) provided we choose an appropriate class of test functions there, which we construct from the operators  $U(t; s)_{0 \leq s \leq t \leq T}$ . We write  $\mathcal{C}_0^2(D)$  for the space of all real-valued, twice continuously differentiable functions with compact support in  $D$ .

**Lemma 3.** *Assume that the same hypotheses as in Theorem 2 hold and let  $(u_\varphi^2(\cdot, t))_{t \in [0, T]}$  be a variational solution of the second kind to Problem (3). For every  $v \in \mathcal{C}_0^2(D)$ , define  $v^t(\cdot, s) = U(t, s)v$  for all  $s, t \in [0, T]$  such that  $s \leq t$ . Then  $v^t \in H^1(D \times (0, t))$  for every  $t \in (0, T]$  and the relation*

$$\begin{aligned} \int_D dx v(x) u_\varphi^2(x, t) &= \int_D dx v^t(x, 0) \varphi(x) + \int_0^t d\tau \int_D dx v^t(x, \tau) g(u_\varphi^2(x, \tau)) \\ &\quad + \sum_{j=1}^{+\infty} \lambda_j^{\frac{1}{2}} \int_0^t (v^t(\cdot, \tau), h(u_\varphi^2(\cdot, \tau)) e_j)_2 B_j(d\tau) \end{aligned} \quad (45)$$

holds a.s. for every  $t \in [0, T]$ .

**Proof.** The symmetry of  $G$  and Relation (43) imply that

$$v^t(x, s) = \begin{cases} v(x) & \text{if } s = t \\ \int_D dy G(y, t; x, s) v(y) & \text{if } s < t \end{cases} \quad (46)$$

for every  $x \in \bar{D}$ . Moreover, for  $s < t$  the function  $G$  is twice continuously differentiable in  $x$ , once continuously differentiable in  $s$  and is a classical solution to the boundary-value problem

$$\begin{aligned} G_s(y, t; x, s) &= -\operatorname{div}(k(x, s) \nabla_x G(y, t; x, s)), (x, s) \in D \times (0, T], \\ \frac{\partial G(y, t; x, s)}{\partial n(k)} &= 0, (x, s) \in \partial D \times (0, T] \end{aligned} \quad (47)$$

(see, for instance, [17] or [19]). From these considerations, the fact that  $G$  satisfies the heat-kernel estimates (11) and from Gauss' divergence theorem, we easily infer that

$$\nabla v^t(x, s) = \int_D dy \nabla_x G(y, t; x, s) v(y) \quad (48)$$

along with

$$\begin{aligned} v_s^t(x, s) &= \int_D dy G_s(y, t; x, s) v(y) \\ &= - \int_D dy G(y, t; x, s) \operatorname{div}(k(y, s) \nabla v(y)) \end{aligned} \quad (49)$$

and we have  $v^t \in H^1(D \times (0, t))$ ; we may then choose  $v^t$  as a test function in (10), which shows that (45) holds if, and only if, the relation

$$\begin{aligned} &\int_0^t d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) \\ &= \int_0^t d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \end{aligned} \quad (50)$$

holds a.s. for every  $t \in [0, T]$ . In order to prove (50) we assume  $t > 0$ , choose  $\epsilon > 0$  sufficiently small and first show that we have

$$\begin{aligned} &\int_0^{t-\epsilon} d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) \\ &= \int_0^{t-\epsilon} d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \end{aligned} \quad (51)$$

a.s.. From Relation (11) and for a fixed  $\tau \in [0, t - \epsilon]$  we first have  $(x, y) \mapsto G_\tau(y, t; x, \tau) v(y) u_\varphi^2(x, \tau) \in L^1(D \times D)$  a.s. as a consequence of the integrability properties of  $v$  and  $u_\varphi^2(\cdot, \tau)$ . Therefore, by invoking successively the first equality in (49), the first equation in (47) and Fubini's theorem, we obtain

$$\begin{aligned} &\int_0^{t-\epsilon} d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) \\ &= - \int_0^{t-\epsilon} d\tau \int_D dy \left( \int_D dx \operatorname{div}(k(x, \tau) \nabla_x G(y, t; x, \tau)) u_\varphi^2(x, \tau) \right) v(y) \end{aligned} \quad (52)$$

a.s.. Furthermore, Relation (11) also implies that  $\sum_{j=1}^d k_{i,j}(\cdot, t) \times G_{x_j}(\cdot, t; y, s) \in H^1(D)$  for every  $i \in \{1, \dots, d\}$  since the  $k_{i,j}$ 's and the  $k_{i,j,x_l}$ 's are bounded from above on  $D \times (0, T)$  because of Hypothesis (K). This property along with the fact that  $u_\varphi^2(\cdot, \tau) \in H^1(D)$  a.s. allows us to transform the integral between the parentheses of (52) by using Gauss' divergence theorem, the second equation in (47) along with the fact that  $k(x, \tau)$  is a

symmetric matrix; we get

$$\begin{aligned}
& \int_0^{t-\epsilon} d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) \\
&= \int_0^{t-\epsilon} d\tau \int_D dy \int_D dx (\nabla_x G(y, t; x, \tau) v(y), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \\
&= \int_0^{t-\epsilon} d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d}
\end{aligned}$$

a.s. as a consequence of (48) and Fubini's theorem, which is the desired assertion. It remains to investigate the limit  $\epsilon \rightarrow 0$  in (51). Regarding the convergence of the left-hand side of that expression we have successively

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) - \int_0^{t-\epsilon} d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) \right| \\
& \leq c_t \mathbb{E} \int_{t-\epsilon}^t d\tau \|u_\varphi^2(\cdot, \tau)\|_1 \\
& \leq c_t \int_{t-\epsilon}^t d\tau \mathbb{E}^{\frac{1}{2}} \left( \|u_\varphi^2(\cdot, \tau)\|_2^2 \right) \\
& \leq c_t \epsilon \mathbb{E}^{\frac{1}{2}} \left( \sup_{\tau \in [0, T]} \|u_\varphi^2(\cdot, \tau)\|_2^2 \right) < +\infty
\end{aligned}$$

by virtue of the boundedness of  $v_\tau^t$  in  $D \times (0, t)$ ; as for the corresponding estimate of the right-hand side of (51) we get

$$\begin{aligned}
& \mathbb{E} \left| \int_{t-\epsilon}^t d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \right| \\
& \leq c \sum_{j=1}^d \left( \int_{t-\epsilon}^t d\tau \int_D dx |v_{x_j}^t(x, \tau)|^2 \right)^{\frac{1}{2}} \\
& \quad \times \mathbb{E}^{\frac{1}{2}} \left( \int_0^T d\tau \|u_\varphi^2(\cdot, \tau)\|_{1,2}^2 \right)
\end{aligned}$$

as a consequence of Schwarz inequality and (5), so that the preceding expression goes to zero as  $\epsilon \rightarrow 0$  by virtue of the absolute continuity of the Lebesgue integral in the second to last factor. Therefore, there exists a sequence  $(\epsilon_n)_{n \in \mathbb{N}^+} \subset \mathbb{R}^+$  converging to zero such that the two relations

$$\lim_{n \rightarrow +\infty} \int_0^{t-\epsilon_n} d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) = \int_0^t d\tau \int_D dx v_\tau^t(x, \tau) u_\varphi^2(x, \tau) \quad (53)$$

and

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_0^{t-\epsilon_n} d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \\ &= \int_0^t d\tau \int_D dx (\nabla v^t(x, \tau), k(x, \tau) \nabla u_\varphi^2(x, \tau))_{\mathbb{R}^d} \end{aligned} \quad (54)$$

hold a.s.. It is now plain that (51), (53) and (54) imply (50). ■

The preceding considerations now lead to the following.

**Proof of Theorem 2.** By substituting (46) into (45), by applying the deterministic and stochastic versions of Fubini's theorem to the resulting expression and by regrouping terms we get

$$\begin{aligned} & \int_D dx v(x) \left( u_\varphi^2(x, t) - \int_D dy G(x, t; y, 0) \varphi(y) \right) \\ &= \int_D dx v(x) \int_0^t d\tau \int_D dy G(x, t; y, \tau) g(u_\varphi^2(y, \tau)) \\ &+ \int_D dx v(x) \int_0^t \int_D dy G(x, t; y, \tau) h(u_\varphi^2(y, \tau)) W(y, d\tau) \end{aligned}$$

a.s. for every  $v \in \mathcal{C}_0^2(D)$  and every  $t \in [0, T]$ ; from this, we infer that

$$\begin{aligned} & u_\varphi^2(\cdot, t) - \int_D dy G(\cdot, t; y, 0) \varphi(y) - \int_0^t d\tau \int_D dy G(\cdot, t; y, \tau) g(u_\varphi^2(y, \tau)) \\ &- \int_0^t \int_D dy G(\cdot, t; y, \tau) h(u_\varphi^2(y, \tau)) W(y, d\tau) \end{aligned}$$

is orthogonal to  $\mathcal{C}_0^2(D)$  a.s. for every  $t \in [0, T]$ , so that  $(u_\varphi^2(\cdot, t))_{t \in [0, T]}$  is a mild solution to (3) since  $\mathcal{C}_0^2(D)$  is dense in  $L^2(D)$ . Conversely, let  $(u_\varphi^3(\cdot, t))_{t \in [0, T]}$  be a mild solution to (3); then, both  $(u_\varphi^2(\cdot, t))_{t \in [0, T]}$  and  $(u_\varphi^3(\cdot, t))_{t \in [0, T]}$  satisfy (12), so that we get

$$\begin{aligned} & \mathbb{E} |u_\varphi^2(x, t) - u_\varphi^3(x, t)|^2 \\ & \leq c \mathbb{E} \left| \int_0^t d\tau \int_D dy G(x, t; y, \tau) (g(u_\varphi^2(y, \tau)) - g(u_\varphi^3(y, \tau))) \right|^2 \\ & + c \mathbb{E} \left| \int_0^t \int_D dy G(x, t; y, \tau) (h(u_\varphi^2(y, \tau)) - h(u_\varphi^3(y, \tau))) W(y, d\tau) \right|^2 \\ & \leq c \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| \mathbb{E} |u_\varphi^2(y, \tau) - u_\varphi^3(y, \tau)|^2 \end{aligned}$$

by using techniques similar to the ones above. By integrating the preceding inequality with respect to  $x$ , by applying Fubini's theorem and by invoking the Gaussian property for  $G$ , we obtain

$$\begin{aligned} & \mathbb{E} \|u_\varphi^2(\cdot, t) - u_\varphi^3(\cdot, t)\|_2^2 \\ & \leq c \int_0^t d\tau \int_D dy \mathbb{E} |u_\varphi^2(y, \tau) - u_\varphi^3(y, \tau)|^2 \\ & = c \int_0^t d\tau \mathbb{E} \|u_\varphi^2(\cdot, \tau) - u_\varphi^3(\cdot, \tau)\|_2^2 \end{aligned} \quad (55)$$

for every  $t \in [0, T]$ . We now notice that  $\tau \mapsto \mathbb{E} \|u_\varphi^2(\cdot, \tau) - u_\varphi^3(\cdot, \tau)\|_2^2 \in L^1((0, t))$  by virtue of (6), so that from (55) and Gronwall's inequality we can conclude that  $u_\varphi^2(\cdot, t) = u_\varphi^3(\cdot, t)$  a.s. in  $L^2(D)$  for every  $t \in [0, T]$  since  $u_\varphi^2, u_\varphi^3 \in L^2(\Omega; \mathcal{C}([0, T]; L^2(D)))$ . Therefore, every mild solution to (3) is a variational solution of the second kind, and there exists a unique such mild solution to (3). ■

We now turn to the proof of Theorem 3, which will require several preparatory results as it is not *a priori* evident that the above random fields should also satisfy (14) along with joint Hölder regularity properties in  $(x, t)$ ; in fact, we will need quite a few additional arguments to show that there exists a version of  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  with these properties. In our next result we prove the existence of a progressively measurable, real-valued process  $(u_\varphi(x, t))_{(x, t) \in D \times [0, T]}$  that satisfies (12) along with (14), through a suitable fixed point argument. From now on we take  $r \in [2, +\infty)$  without restricting the generality, and define  $\mathcal{B}_r$  as the real Banach space consisting of all real-valued (equivalence classes of) processes  $u$  indexed by  $(x, t) \in D \times [0, T]$ , progressively measurable on  $D \times [0, T] \times \Omega$ , endowed with the usual pointwise operations and the norm

$$u \mapsto \left( \sup_{(x, t) \in D \times [0, T]} \mathbb{E} |u(x, t)|^r \right)^{\frac{1}{r}} < +\infty. \quad (56)$$

Let  $M_\varphi : \mathcal{B}_r \rightarrow \mathcal{B}_r$  be the map induced by (12), that is

$$\begin{aligned} M_\varphi u(x, t) &= \int_D dy G(x, t; y, 0) \varphi(y) + \int_0^t d\tau \int_D dy G(x, t; y, \tau) g(u(y, \tau)) \\ & \quad + \int_0^t \int_D dy G(x, t; y, \tau) h(u(y, \tau)) W(y, d\tau) \end{aligned} \quad (57)$$

a.s.. Regarding  $M_\varphi$  we have the following result.

**Proposition 3.** *Assume that the same hypotheses as in Theorem “active@prefix ~“active@char~ 3 hold; then  $M_\varphi$  possesses a unique fixed point  $u_\varphi$  in  $\mathcal{B}_r$  for every  $r \in [2, +\infty)$ .*

**Proof.** We begin by showing that  $M_\varphi$  is indeed well defined on  $\mathcal{B}_r$ . Owing to the boundedness of  $\varphi$  and the Gaussian property for  $G$  we first get

$$\begin{aligned} & \mathbb{E} |M_\varphi u(x, t)|^r \\ & \leq c \left( 1 + \mathbb{E} \left| \int_0^t d\tau \int_D dy G(x, t; y, \tau) g(u(y, \tau)) \right|^r \right. \\ & \quad \left. + \mathbb{E} \left| \int_0^t \int_D dy G(x, t; y, \tau) h(u(y, \tau)) W(y, d\tau) \right|^r \right). \end{aligned} \quad (58)$$

Furthermore, as a consequence of Hypothesis (L), the Gaussian property and Hölder’s inequality relative to the finite measure  $dyd\tau |G(x, t; y, \tau)|$  on  $D \times (0, t)$ , we can estimate the first expectation in (58) as

$$\begin{aligned} & \mathbb{E} \left| \int_0^t d\tau \int_D dy G(x, t; y, \tau) g(u(y, \tau)) \right|^r \\ & \leq c \left( 1 + \mathbb{E} \left( \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| |u(y, \tau)| \right)^r \right) \\ & \leq c \left( 1 + \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| \mathbb{E} |u(y, \tau)|^r \right) \\ & \leq c \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u(y, \tau)|^r \right) < +\infty \end{aligned} \quad (59)$$

since  $u \in \mathcal{B}_r$ . In order to obtain a similar estimate for the second expectation in (58), we invoke successively the definition of the stochastic integral, Burkholder’s inequality and Hölder’s inequality relative to the measure  $d\tau$  on the interval  $(0, t)$ ; we get

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \int_D dy G(x, t; y, \tau) h(u(y, \tau)) W(y, d\tau) \right|^r \\ & \leq c \mathbb{E} \left| \int_0^t d\tau \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy G(x, t; y, \tau) h(u(y, \tau)) e_j(y) \right)^2 \right|^{\frac{r}{2}} \\ & \leq c \mathbb{E} \int_0^t d\tau \left( \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy G(x, t; y, \tau) h(u(y, \tau)) e_j(y) \right)^2 \right)^{\frac{r}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq c\mathbb{E} \int_0^t d\tau \left( 1 + \int_D dy |G(x, t; y, \tau)| |u(y, \tau)|^r \right) \\
&\leq c \left( 1 + \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| \mathbb{E} |u(y, \tau)|^r \right) \\
&\leq c \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u(y, \tau)|^r \right) < +\infty \tag{60}
\end{aligned}$$

where we have also used Hypotheses (L) and (C) along with Hölder's inequality relative to the finite measure  $dy |G(x, t; y, \tau)|$  on  $D$ . From (58), (59) and (60) we infer that  $\sup_{(x, t) \in D \times [0, T]} \mathbb{E} |M_\varphi u(x, t)|^r < +\infty$ , so that  $M_\varphi u \in \mathcal{B}_r$ . Now let  $u, u^* \in \mathcal{B}_r$ ; then, from (57) we have

$$\begin{aligned}
&M_\varphi u(x, t) - M_\varphi u^*(x, t) \\
&= \int_0^t d\tau \int_D dy G(x, t; y, \tau) (g(u(y, \tau)) - g(u^*(y, \tau))) \\
&+ \int_0^t d\tau \int_D dy G(x, t; y, \tau) (h(u(y, \tau)) - h(u^*(y, \tau))) W(y, d\tau)
\end{aligned}$$

a.s., so that the Lipschitz properties of  $g$  and  $h$  along with arguments entirely similar to those leading to (59) and (60) give

$$\begin{aligned}
&\mathbb{E} |M_\varphi u(x, t) - M_\varphi u^*(x, t)|^r \\
&\leq c \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| \mathbb{E} |u(y, \tau) - u^*(y, \tau)|^r \\
&\leq c \int_0^t d\tau \sup_{y \in D} \mathbb{E} |u(y, \tau) - u^*(y, \tau)|^r
\end{aligned}$$

for every  $(x, t) \in D \times [0, T]$ . The preceding relation along with standard considerations now show that the  $N^{\text{th}}$  iterate  $M_\varphi^{(N)}$  of  $M_\varphi$  is a contraction in  $\mathcal{B}_r$  for  $N$  sufficiently large. ■

It is worth stressing the fact that the preceding construction does *not* imply  $u_\varphi$  should exhibit any Sobolev regularity in  $x$  or any continuity in  $t$  as  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  does, so that the preceding result does *not yet* prove that  $u_\varphi$  is a solution to (3); in fact, thus far the variables  $(x, t) \in D \times [0, T]$  merely index  $u_\varphi$  but we shall show below that  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  and  $(u_\varphi(x, t))_{(x, t) \in D \times [0, T]}$  are actually indistinguishable; for the time being we prove a series of results that will lead us to the existence of a jointly Hölder continuous version of  $u_\varphi$ . For this we also use Relation (12), each term of which we investigate separately; we begin with the following proposition,

which is an immediate consequence of the theory developed in [17] as the first term of (12) is a classical solution to (3) when  $g = h = 0$ .

**Proposition 4.** *Assume that Hypotheses (K) and (I) hold; then we have  $(x, t) \mapsto \int_D dy G(x, t; y, 0) \varphi(y) \in C^{\alpha, \frac{\alpha}{2}}(\bar{D} \times [0, T])$ .*

We next turn to the analysis of the second term in (12) for which we have the following result.

**Proposition 5.** *Assume that the same hypotheses as in Theorem 3 hold and let  $(u_\varphi(x, t))_{(x,t) \in D \times [0, T]}$  be the random field of Proposition 3; then we have*

$$\mathbb{E} \left| \int_0^t d\tau \int_D dy (G(x_1, t; y, \tau) - G(x_2, t; y, \tau)) g(u_\varphi(y, \tau)) \right|^r \leq c |x_1 - x_2|^r \quad (61)$$

for all  $x_1, x_2 \in D$  uniformly in  $t \in [0, T]$ , along with

$$\mathbb{E} \left| \int_0^{t_1} d\tau \int_D dy G(x, t_1; y, \tau) g(u_\varphi(y, \tau)) - \int_0^{t_2} d\tau \int_D dy G(x, t_2; y, \tau) g(u_\varphi(y, \tau)) \right|^r \leq c |t_1 - t_2|^{\gamma r} \quad (62)$$

for all  $t_1, t_2 \in [0, T]$  uniformly in  $x \in D$ , for every  $\gamma \in (0, 1)$  and every  $r \in [2, +\infty)$ .

**Proof.** Since  $D$  is convex, we first invoke the mean-value theorem for  $G$  along with estimate (11) and Hypothesis (L); since the measure  $dy d\tau (t - \tau)^{-\frac{d+1}{2}} \exp\left[-c \frac{|x^* - y|^2}{t - \tau}\right]$  is finite on  $D \times (0, t)$  by virtue of the Gaussian property, we obtain

$$\begin{aligned} & \mathbb{E} \left| \int_0^t d\tau \int_D dy (G(x_1, t; y, \tau) - G(x_2, t; y, \tau)) g(u_\varphi(y, \tau)) \right|^r \\ & \leq c \mathbb{E} \left( \int_0^t d\tau \int_D dy (t - \tau)^{-\frac{d+1}{2}} \exp\left[-c \frac{|x^* - y|^2}{t - \tau}\right] (1 + |u_\varphi(y, \tau)|) \right)^r |x_1 - x_2|^r \\ & \leq c \mathbb{E} \left( 1 + \int_0^t d\tau \int_D dy (t - \tau)^{-\frac{d+1}{2}} \exp\left[-c \frac{|x^* - y|^2}{t - \tau}\right] |u_\varphi(y, \tau)| \right)^r |x_1 - x_2|^r \end{aligned}$$

where  $x^*$  belongs to the segment connecting  $x_1$  and  $x_2$ ; consequently, by applying Hölder's inequality for this measure to the last integral and by taking (56) into account, we get Relation (61). Without restricting the generality we now choose  $\sigma > 0$  sufficiently small and set  $t_1 = t + \sigma$  and  $t_2 = t$  in order to prove (62); for the left-hand side of (62) we first get the upper bounds

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t d\tau \int_D dy (G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)) g(u_\varphi(y, \tau)) \right|^r \\
& \quad + \mathbb{E} \left| \int_t^{t+\sigma} d\tau \int_D dy G(x, t + \sigma; y, \tau) g(u_\varphi(y, \tau)) \right|^r \\
& \leq \mathbb{E} \left| \int_0^t d\tau \int_D dy (G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)) g(u_\varphi(y, \tau)) \right|^r \\
& \quad + c \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u_\varphi(y, \tau)|^r \right) \sigma^r \tag{63}
\end{aligned}$$

where we have used the Gaussian property for  $G$ , Hypothesis (L) along with (56) to obtain the last inequality. It remains to estimate the integral involving the time-increment of  $G$ ; the first part of the argument is essentially similar to what we just did and we obtain

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t d\tau \int_D dy (G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)) g(u_\varphi(y, \tau)) \right|^r \\
& \leq c \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u_\varphi(y, \tau)|^r \right) \\
& \quad \times \left( \int_0^t d\tau \int_D dy |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)| \right)^r. \tag{64}
\end{aligned}$$

But we now have to proceed differently than we did to establish (61) in order to control the singularity on the time-diagonal of  $G$ . Let  $\gamma \in \left( \frac{d}{d+2}, 1 \right)$ ; then, by invoking successively the mean-value theorem, (11) and the Gaussian property, we can estimate the last integral in (64) as

$$\begin{aligned}
& \int_0^t d\tau \int_D dy |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)| \\
& \leq c \left( \int_0^t d\tau \int_D dy (|G(x, t + \sigma; y, \tau)| + |G(x, t; y, \tau)|)^{1-\gamma} |G_{t^*}(x, t^*; y, \tau)|^\gamma \right) \sigma^\gamma \\
& \leq c \left( \int_0^t d\tau (t - \tau)^{-\frac{d}{2}(1-\gamma)} (t^* - \tau)^{-\frac{d+2}{2}\gamma + \frac{d}{2}} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \int_D dy (t^* - \tau)^{-\frac{d}{2}} \exp \left[ -c \frac{|x - y|^2}{t^* - \tau} \right] \sigma^\gamma \\
& \leq c \left( \int_0^t d\tau (t - \tau)^{-\frac{d}{2}(1-\gamma)} (t^* - \tau)^{-\frac{d+2}{2}\gamma + \frac{d}{2}} \right) \sigma^\gamma \\
& \leq c \left( \int_0^t d\tau (t - \tau)^{-\gamma} \right) \sigma^\gamma \leq c\sigma^\gamma
\end{aligned}$$

since  $\sigma > 0$ ,  $t^* \in (t, t + \sigma)$ ,  $-\frac{d+2}{2}\gamma + \frac{d}{2} < 0$  and  $-\frac{d}{2}(1-\gamma) - \frac{d+2}{2}\gamma + \frac{d}{2} = -\gamma$ . The substitution of the preceding estimate into (64) along with (63) lead to Relation (62) for every  $\gamma \in \left(\frac{d}{d+2}, 1\right)$ , and *a fortiori* for every  $\gamma \in (0, 1)$ . ■

Finally, regarding the third term in (12) we have the following result in which the dimension  $d$  *does* impose a restriction on one of the estimates for the first time.

**Proposition 6.** *Assume that the same hypotheses as in Theorem 3 hold and let  $(u_\varphi(x, t))_{(x,t) \in D \times [0, T]}$  be the random field of Proposition 3; then we have*

$$\begin{aligned}
\mathbb{E} \left| \int_0^t \int_D dy (G(x_1, t; y, \tau) - G(x_2, t; y, \tau)) h(u_\varphi(y, \tau)) W(y, d\tau) \right|^r \\
\leq c |x_1 - x_2|^{\gamma r}
\end{aligned} \tag{65}$$

for all  $x_1, x_2 \in D$  uniformly in  $t \in [0, T]$ , for every  $\gamma \in (0, 1)$  and every  $r \in [2, +\infty)$ ; moreover, we have

$$\begin{aligned}
\mathbb{E} \left| \int_0^{t_1} \int_D dy G(x, t_1; y, \tau) h(u_\varphi(y, \tau)) W(y, d\tau) \right. \\
\left. - \int_0^{t_2} \int_D dy G(x, t_2; y, \tau) h(u_\varphi(y, \tau)) W(y, d\tau) \right|^r \\
\leq c |t_1 - t_2|^{\gamma r}
\end{aligned} \tag{66}$$

for all  $t_1, t_2 \in [0, T]$  uniformly in  $x \in D$ , for every  $\gamma \in \left(0, \frac{1}{2} \wedge \frac{2}{d+2}\right)$  and every  $r \in [2, +\infty)$ .

The proof of this proposition is much more elaborate than that of Proposition 5 and relies on two lemmas; it is based on an extension of the so-called *factorization method*, which was originally introduced in [15] to deal with regularity questions concerning autonomous, linear stochastic partial differential equations; the method provides a way to express the stochastic

integral in (12) by means of an auxiliary random field whose moments are uniformly bounded in some sense. For every  $\delta \in (0, \frac{1}{2})$ , we define  $Y_\delta$  by

$$Y_\delta(x, t) = \int_0^t (t - \tau)^{-\delta} \int_D dy G(x, t; y, \tau) h(u_\varphi(y, \tau)) W(y, d\tau) \quad (67)$$

for every  $x \in D$  and every  $t \in [0, T]$ ; as before, we can easily prove that this expression is well-defined and finite a.s.; in fact, this is a trivial consequence of the boundedness property we just alluded to, which we describe in the following result.

**Lemma 4.** *Assume that the same hypotheses as in Theorem 3 hold; then we have*

$$\sup_{(x,t) \in D \times [0,T]} \mathbb{E} |Y_\delta(x, t)|^r < +\infty \quad (68)$$

for every  $r \in [2, +\infty)$ .

**Proof.** We show that (68) is a direct consequence of (56) for  $u_\varphi$ . Owing to Burkholder's inequality and to Hypotheses (C) and (L), we first get the estimates

$$\begin{aligned} \mathbb{E} |Y_\delta(x, t)|^r &\leq c \left( \sum_{j=1}^{+\infty} \lambda_j \|e_j\|_\infty^2 \right)^{\frac{r}{2}} \\ &\times \mathbb{E} \left( \int_0^t d\tau (t - \tau)^{-2\delta} \left( \int_D dy |G(x, t; y, \tau) h(u_\varphi(y, \tau))| \right)^2 \right)^{\frac{r}{2}} \\ &\leq c \mathbb{E} \left( \int_0^t d\tau (t - \tau)^{-2\delta} \left( \int_D dy |G(x, t; y, \tau)| (1 + |u_\varphi(y, \tau)|) \right)^2 \right)^{\frac{r}{2}}. \end{aligned}$$

Furthermore, the measure  $d\tau (t - \tau)^{-2\delta}$  is finite on  $(0, t)$  since we have  $\delta \in (0, \frac{1}{2})$ ; by applying Hölder's inequality for this measure to the last integral and by invoking the Gaussian property of  $G$ , we obtain

$$\begin{aligned} \mathbb{E} |Y_\delta(x, t)|^r &\leq c \mathbb{E} \int_0^t d\tau (t - \tau)^{-2\delta} \left( \int_D dy |G(x, t; y, \tau)| (1 + |u_\varphi(y, \tau)|) \right)^r \\ &\leq c \mathbb{E} \int_0^t d\tau (t - \tau)^{-2\delta} \left( 1 + \int_D dy |G(x, t; y, \tau)| |u_\varphi(y, \tau)|^r \right) \\ &\leq c \left( 1 + \sup_{(y,\tau) \in D \times [0,T]} \mathbb{E} |u_\varphi(y, \tau)|^r \right) < +\infty \end{aligned}$$

uniformly in  $(x, t) \in D \times [0, T]$ , where we used Hölder's inequality relative to the finite measure  $dy |G(x, t; y, \tau)|$  on  $D$  along with Relation (56) for  $u_\varphi$ . ■

Property (68) along with the following relation between the stochastic integral in (12) and  $Y_\delta$  will be crucial to our proof of Proposition 6.

**Lemma 5.** *Assume that the same hypotheses as in Theorem 3 hold; then the relation*

$$\begin{aligned} & \int_0^t \int_D dy G(x, t; y, \tau) h(u_\varphi(y, \tau)) W(y, d\tau) \\ &= \frac{\sin(\delta\pi)}{\pi} \int_0^t d\tau (t - \tau)^{\delta-1} \int_D dy G(x, t; y, \tau) Y_\delta(y, \tau) \end{aligned} \quad (69)$$

holds a.s. for every  $\delta \in (0, \frac{1}{2})$  and every  $(x, t) \in D \times [0, T]$ .

**Proof.** Let  $\sigma, \tau, t \in [0, T]$  such that  $\sigma < \tau < t$ ; then the evolution operators (43) satisfy the fundamental property  $U(t, \tau)U(\tau, \sigma) = U(t, \sigma)$ ; equivalently, we have

$$G(x, t; z, \sigma) = \int_D dy G(x, t; y, \tau) G(y, \tau; z, \sigma) \quad (70)$$

for the corresponding Green's function, for all  $x, z \in D$ . Relation (69) then follows from the substitution of (67) into (69), the deterministic and stochastic versions of Fubini's theorem, Relation (70) and the identity  $\int_\sigma^t d\tau (t - \tau)^{\delta-1} \times (\tau - \sigma)^{-\delta} = \frac{\pi}{\sin(\delta\pi)}$ . ■

The preceding results now lead to the following.

**Proof of Proposition 6.** In order to prove (65), we have to argue differently than we did to prove (61) because of the singular factor  $(t - \tau)^{\delta-1}$  in (69); owing to the Gaussian property for  $G$ , we first notice that the measure  $dy d\tau (t - \tau)^{\delta-1} |G(x_1, t; y, \tau) - G(x_2, t; y, \tau)|$  is finite on  $D \times (0, t)$ ; then, by using successively (69), Hölder's inequality relative to this measure along with (68) we get

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \int_D dy (G(x_1, t; y, \tau) - G(x_2, t; y, \tau)) h(u_\varphi(y, \tau)) W(y, d\tau) \right|^r \\ & \leq c \left( \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |Y_\delta(y, \tau)|^r \right) \\ & \quad \times \left( \int_0^t d\tau (t - \tau)^{\delta-1} \int_D dy |G(x_1, t; y, \tau) - G(x_2, t; y, \tau)| \right)^r. \end{aligned} \quad (71)$$

Let  $\gamma \in (0, 1)$ ; by using the Gaussian property we can estimate the last integral in the preceding expression as

$$\begin{aligned}
& \int_0^t d\tau (t - \tau)^{\delta-1} \int_D dy |G(x_1, t; y, \tau) - G(x_2, t; y, \tau)| \\
& \leq \int_0^t d\tau (t - \tau)^{\delta-1} \\
& \times \int_D dy (|G(x_1, t; y, \tau)| + |G(x_2, t; y, \tau)|)^{1-\gamma} |G(x_1, t; y, \tau) - G(x_2, t; y, \tau)|^\gamma \\
& \leq c \int_0^t d\tau (t - \tau)^{\delta-1-\frac{d}{2}(1-\gamma)} \int_D dy |G(x_1, t; y, \tau) - G(x_2, t; y, \tau)|^\gamma. \quad (72)
\end{aligned}$$

In order to control the space-increment of  $G$  in the last line of (72) we now choose  $\delta \in (\frac{\gamma}{2}, \frac{1}{2})$ , and then use successively the mean-value theorem along with (11) and the Gaussian property; writing again  $x^*$  for a point on the segment between  $x_1$  and  $x_2$  we obtain from (72) the estimates

$$\begin{aligned}
& \int_0^t d\tau (t - \tau)^{\delta-1} \int_D dy |G(x_1, t; y, \tau) - G(x_2, t; y, \tau)| \\
& \leq c \int_0^t d\tau (t - \tau)^{\delta-1-\frac{d}{2}(1-\gamma)} \\
& \times \int_D dy (t - \tau)^{-\frac{d+1}{2}\gamma} \exp \left[ -c \frac{|x^* - y|^2}{t - \tau} \right] |x_1 - x_2|^\gamma \\
& = c \left( \int_0^t d\tau (t - \tau)^{\delta-1-\frac{\gamma}{2}} \int_D dy (t - \tau)^{-\frac{d}{2}} \exp \left[ -c \frac{|x^* - y|^2}{t - \tau} \right] \right) |x_1 - x_2|^\gamma \\
& \leq c \left( \int_0^t d\tau (t - \tau)^{\delta-1-\frac{\gamma}{2}} \right) |x_1 - x_2|^\gamma \leq c |x_1 - x_2|^\gamma
\end{aligned}$$

since  $\delta - \frac{\gamma}{2} > 0$ . The substitution of the preceding estimate into (71) proves (65).

We now show that (66) holds by choosing again  $\sigma > 0$  sufficiently small,  $t_1 = t + \sigma$  and  $t_2 = t$ ; owing to (69) we can first bound the left-hand side of (66) from above by

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy G(x, t + \sigma; y, \tau) Y_\delta(y, \tau) \right. \\
& \quad \left. - \int_0^t d\tau (t - \tau)^{\delta-1} \int_D dy G(x, t; y, \tau) Y_\delta(y, \tau) \right|^r
\end{aligned}$$

$$+ \mathbb{E} \left| \int_t^{t+\sigma} d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy G(x, t + \sigma; y, \tau) Y_\delta(y, \tau) \right|^r \quad (73)$$

and we proceed by investigating each term of (73) separately. Regarding the second term, we notice by arguing as before that the measure  $dy d\tau (t + \sigma - \tau)^{\delta-1} |G(x, t + \sigma; y, \tau)|$  is finite on  $D \times (t, t + \sigma)$ ; then, letting  $\gamma \in (0, \frac{1}{2})$ ,  $\delta \in (\gamma, \frac{1}{2})$  and using Hölder's inequality relative to this measure along with (68) and the Gaussian property, we obtain

$$\begin{aligned} & \mathbb{E} \left| \int_t^{t+\sigma} d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy G(x, t + \sigma; y, \tau) Y_\delta(y, \tau) \right|^r \\ & \leq \left( \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |Y_\delta(y, \tau)|^r \right) \\ & \quad \times \left( \int_t^{t+\sigma} d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy |G(x, t + \sigma; y, \tau)| \right)^r \\ & \leq c \left( \int_t^{t+\sigma} d\tau (t + \sigma - \tau)^{\delta-1} \right)^r \leq c\sigma^{\delta r} \leq c\sigma^{\gamma r}. \end{aligned} \quad (74)$$

The analysis of the first term in (73) is more complicated; the first part of the argument is similar to what we just did to derive the first inequality in (74); this remark and the Gaussian property lead to

$$\begin{aligned} & \mathbb{E} \left| \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy G(x, t + \sigma; y, \tau) Y_\delta(y, \tau) \right. \\ & \quad \left. - \int_0^t d\tau (t - \tau)^{\delta-1} \int_D dy G(x, t; y, \tau) Y_\delta(y, \tau) \right|^r \\ & \leq \left( \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |Y_\delta(y, \tau)|^r \right) \\ & \quad \times \left( \int_0^t d\tau \int_D dy |(t + \sigma - \tau)^{\delta-1} G(x, t + \sigma; y, \tau) - (t - \tau)^{\delta-1} G(x, t; y, \tau)| \right)^r \\ & \leq c \left( \int_0^t d\tau |(t + \sigma - \tau)^{\delta-1} - (t - \tau)^{\delta-1}| \right)^r \\ & \quad + c \left( \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)| \right)^r. \end{aligned} \quad (75)$$

On the one hand, in order to control the increment of the line before last in (75), let us choose  $\gamma \in (0, \frac{1}{2})$  and  $\delta \in (\gamma, \frac{1}{2})$  again; from the mean-value

theorem we get

$$\begin{aligned}
& \int_0^t d\tau |(t + \sigma - \tau)^{\delta-1} - (t - \tau)^{\delta-1}| \\
& \leq c \int_0^t d\tau (t - \tau)^{(\delta-1)(1-\gamma)} |(t + \sigma - \tau)^{\delta-1} - (t - \tau)^{\delta-1}|^\gamma \\
& \leq c \left( \int_0^t d\tau (t - \tau)^{(\delta-1)(1-\gamma) + (\delta-2)\gamma} \right) \sigma^\gamma \leq c\sigma^\gamma
\end{aligned} \tag{76}$$

since  $(\delta - 1)(1 - \gamma) + (\delta - 2)\gamma = \delta - \gamma - 1 > -1$ . On the other hand, in order to control the increment of the last line of (75) we first invoke the Gaussian property; we obtain

$$\begin{aligned}
& \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)| \\
& \leq c \int_0^t d\tau (t + \sigma - \tau)^{\delta-1 - \frac{d}{2}(1-\gamma)} \\
& \quad \times \int_D dy \exp \left[ -c \frac{|x - y|^2}{t + \sigma - \tau} \right] |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)|^\gamma \\
& + c \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} (t - \tau)^{-\frac{d}{2}(1-\gamma)} \\
& \quad \times \int_D dy \exp \left[ -c \frac{|x - y|^2}{t - \tau} \right] |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)|^\gamma.
\end{aligned}$$

We then apply the mean-value theorem to the last two time-increments of  $G$  along with (11); this leads to

$$\begin{aligned}
& \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)| \\
& \leq c \int_0^t d\tau (t + \sigma - \tau)^{\delta-1 - \frac{d}{2}(1-\gamma)} (t^* - \tau)^{-\frac{d+2}{2}\gamma} \\
& \quad \times \int_D dy \exp \left[ -c \frac{|x - y|^2}{t + \sigma - \tau} \right] \sigma^\gamma \\
& + c \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} (t - \tau)^{-\frac{d}{2}(1-\gamma)} (t^* - \tau)^{-\frac{d+2}{2}\gamma} \\
& \quad \times \int_D dy \exp \left[ -c \frac{|x - y|^2}{t - \tau} \right] \sigma^\gamma.
\end{aligned}$$

To go further, we have to impose an additional restriction on  $\gamma$  by choosing  $\gamma \in \left(0, \frac{2}{d+2}\right)$ ; we then take  $\delta \in \left(\gamma, \frac{2}{d+2}\right)$  and use the Gaussian property once again; from the preceding estimates we get

$$\begin{aligned}
& \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} \int_D dy |G(x, t + \sigma; y, \tau) - G(x, t; y, \tau)| \\
& \leq c \left( \int_0^t d\tau (t + \sigma - \tau)^{\delta-1 + \frac{d}{2}\gamma} (t^* - \tau)^{-\frac{d+2}{2}\gamma} \right) \sigma^\gamma \\
& \quad + c \left( \int_0^t d\tau (t + \sigma - \tau)^{\delta-1} (t - \tau)^{\frac{d}{2}\gamma} (t^* - \tau)^{-\frac{d+2}{2}\gamma} \right) \sigma^\gamma \\
& \leq c \left( \int_0^t d\tau (t - \tau)^{\delta-1 + \frac{d}{2}\gamma - \frac{d+2}{2}\gamma} \right) \sigma^\gamma \\
& = c \left( \int_0^t d\tau (t - \tau)^{\delta-1-\gamma} \right) \sigma^\gamma \leq c\sigma^\gamma \tag{77}
\end{aligned}$$

since  $\sigma > 0$ ,  $t^* \in (t, t + \sigma)$ ,  $\delta - 1 + \frac{d}{2}\gamma < \frac{d+2}{2}\delta - 1 < 0$  and  $\delta - 1 - \gamma > -1$ . It is now clear that if  $\gamma \in \left(0, \frac{1}{2} \wedge \frac{2}{d+2}\right)$ , then the substitution of (76) and (77) into (75) along with (74) imply Relation (66). ■

The preceding considerations now lead to the following.

**Proof of Theorem 3.** We first show that the random fields  $(u_\varphi^1(\cdot, t))_{t \in [0, T]}$  and  $(u_\varphi(\cdot, t))_{t \in [0, T]}$  are indistinguishable, which will immediately imply the first two properties of the theorem. On the one hand, from our construction of  $u_\varphi$  in the proof of Proposition 3 we have  $u_\varphi(\cdot, t) \in L^2(D)$  a.s. for every  $t \in [0, T]$ . On the other hand, from Theorems 1, 2 and Proposition 3 we infer that both random fields satisfy (12); therefore, arguing *verbatim* as in the proof of Theorem 2 by means of Gronwall's inequality, we may conclude that  $u_\varphi^1(\cdot, t) = u_\varphi(\cdot, t)$  a.s. as an inequality in  $L^2(D)$  for every  $t \in [0, T]$  since  $u_\varphi^1 \in L^2(\Omega; \mathcal{C}([0, T]; L^2(D)))$ . This proves that the random field  $(u_\varphi(\cdot, t))_{t \in [0, T]}$  is a mild solution to (3) which exhibits  $H^1(D)$ -regularity in the space variable and satisfies (14). In order to prove that there exists a jointly Hölder continuous version of  $(u_\varphi(\cdot, t))_{t \in [0, T]}$ , it is sufficient to invoke Propositions 4, 5 and 6; indeed, these propositions together with a multi-dimensional version of Kolmogorov's continuity theorem (see, for instance, [24]) imply the result. ■

We now turn to the proofs of Theorems 4 and 5, which rest on the following preliminary result and its proof.

**Lemma 6.**  $(C^d)$  implies  $(H)$  while  $(C_\eta^d)$  implies  $(H_\eta)$ .

**Proof.** We can argue exactly as in the proof of (17) to get

$$\begin{aligned}
& \int_0^t d\tau \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy |G(x, t; y, \tau) e_j(y)| \right)^2 \\
& \leq c \sum_{j=1}^{+\infty} \lambda_j \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| |e_j(y)|^2 \\
& \leq c \int_0^t d\tau (t - \tau)^{-\frac{d}{2} + \frac{d}{2s^*}} \leq c < +\infty
\end{aligned} \tag{78}$$

uniformly in  $(x, t) \in D \times [0, T]$ , which proves the first assertion of the lemma; similarly we have

$$\begin{aligned}
& \sum_{j=1}^{+\infty} \lambda_j \int_0^t d\tau (t - \tau)^{-2\eta} \int_D dy |G(x, t; y, \tau)| |e_j(y)|^2 \\
& \leq c \int_0^t d\tau (t - \tau)^{-2\eta - \frac{d}{2} + \frac{d}{2s^*}} \leq c < +\infty
\end{aligned} \tag{79}$$

since  $s \in \left( \frac{d}{1-2\eta}; +\infty \right)$  implies  $1 - 2\eta - \frac{d}{2} + \frac{d}{2s^*} > 0$ , which proves the second assertion. ■

We then have the following.

**Proof of Theorem 4.** In order to prove the first statement of the theorem, it is sufficient to show that the conclusion of Proposition 3 holds; since estimates (58) and (59) are still valid, we need only show that an estimate of the form (60) remains true. By virtue of (78) we first note that the measure  $\lambda_j dy d\tau |G(x, t; y, \tau)| |e_j(y)|^2$  is finite on  $\mathbb{N}^+ \times D \times (0, t)$  uniformly in  $(x, t) \in D \times [0, T]$ ; consequently, by using successively Burkholder's inequality, Hypothesis (L) along with Hölder's inequality relative to  $\lambda_j dy d\tau |G(x, t; y, \tau)| |e_j(y)|^2$  on  $\mathbb{N}^+ \times D \times (0, t)$  we get

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t \int_D dy G(x, t; y, \tau) h(u(y, \tau)) W(y, d\tau) \right|^r \\
& \leq c \mathbb{E} \left| \sum_{j=1}^{+\infty} \lambda_j \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| |e_j(y)|^2 \left( 1 + |u(y, \tau)|^2 \right) \right|^{\frac{r}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{j=1}^{+\infty} \lambda_j \int_0^t d\tau \int_D dy |G(x, t; y, \tau)| |e_j(y)|^2 (1 + \mathbb{E} |u(y, \tau)|^r) \\
&\leq c \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u(y, \tau)|^r \right) < +\infty. \tag{80}
\end{aligned}$$

As for the proof of the second statement, it is sufficient to show that the conclusion of Proposition 6 holds true for  $(u_\varphi^4(x, t))_{(x, t) \in D \times [0, T]}$  since Propositions 4 and 5 remain unchanged; for this we need only prove that the conclusion of Lemma 4 is valid for the auxiliary random field  $Y_\eta$  defined by

$$Y_\eta(x, t) = \int_0^t (t - \tau)^{-\eta} \int_D dy G(x, t; y, \tau) h(u_\varphi^4(y, \tau)) W(y, d\tau).$$

But this is immediate, for the measure  $\lambda_j dy d\tau (t - \tau)^{-2\eta} |G(x, t; y, \tau)| |e_j(y)|^2$  is finite on  $\mathbb{N}^+ \times D \times (0, t)$  uniformly in  $(x, t) \in D \times [0, T]$  as a consequence of (79); we can then argue exactly as in the proof of (80) to get

$$\mathbb{E} |Y_\eta(x, t)|^r \leq c \left( 1 + \sup_{(y, \tau) \in D \times [0, T]} \mathbb{E} |u(y, \tau)|^r \right) < +\infty \tag{81}$$

as desired. ■

Finally, we have the following.

**Proof of Theorem 5.** From the preceding considerations it is now clear that inequalities of the form (80) and (81) are the only estimates we need; in order to derive such estimates under the hypotheses of the theorem, the only concrete changes lie in the choice of the respective measures to which we apply Hölder's inequality. In the case of (80) we choose the finite measure  $d\tau dy dz |G(x, t; y, \tau)| \times \kappa(y, z) \times |G(x, t; z, \tau)|$  on  $(0, t) \times D \times D$  (see, for instance, Relations (21) and (22)), while in the case of (81) we choose  $d\tau dy dz (t - \tau)^{-2\eta} \times |G(x, t; y, \tau)| \times \kappa(y, z) \times |G(x, t; z, \tau)|$ , which is also finite by virtue of  $(H_\eta)$ ; the remaining part of the proof is essentially the same as that of Theorem 4 and is thereby omitted. ■

We conclude this article by comparing briefly the rôle played by  $(C^d)$  and  $(C_\eta^d)$  with that of the spectral measure conditions we alluded to at the very end of Section 1. Assume that the generating kernel  $\kappa : D \times D \mapsto \mathbb{R}^+$  admits a translation-invariant extension  $\kappa^* : \mathbb{R}^d \mapsto \mathbb{R}$  in such a way that the measure  $dy \kappa^*(y)$  be positive-definite and tempered on  $\mathbb{R}^d$ , and write  $m$

for the spectral measure associated with  $dy\kappa^*(y)$  (see, for instance, [21]). We impose the following two conditions on  $m$ :

(M<sup>d</sup>) We have

$$\int_{\mathbb{R}^d} \frac{m(d\zeta)}{1 + |\zeta|^2} < +\infty. \quad (82)$$

(M<sub>η</sub><sup>d</sup>) There exists  $\eta \in (0, \frac{1}{2})$  such that

$$\int_{\mathbb{R}^d} \frac{m(d\zeta)}{(1 + |\zeta|^2)^{1-2\eta}} < +\infty. \quad (83)$$

Condition (M<sup>d</sup>) has appeared, for instance, in [12], [25] and [44] while (M<sub>η</sub><sup>d</sup>) has been introduced in [33] and [46]; a comparison of our arguments with the methods of proof used in these references suggests that the analogy between (C<sup>d</sup>), (C<sub>η</sub><sup>d</sup>) and (M<sup>d</sup>), (M<sub>η</sub><sup>d</sup>) is best illustrated if we can show that (M<sup>d</sup>) implies (H) while (M<sub>η</sub><sup>d</sup>) implies (H<sub>η</sub>). Regarding the proof of the first statement our starting point is Relation (22); by using successively the translation invariance of  $\kappa^*$ , the Gaussian property, the definition of  $m$  along with (82) we obtain

$$\begin{aligned} & \int_0^t d\tau \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy |G(x, t; y, \tau) e_j(y)| \right)^2 \\ & \leq c \int_0^t d\tau \int_{\mathbb{R}^d} m(d\zeta) \exp \left[ -c(t - \tau) |\zeta|^2 \right] \\ & = c \int_{\mathbb{R}^d} m(d\zeta) \frac{1 - \exp \left[ -ct |\zeta|^2 \right]}{|\zeta|^2} \leq c \int_{\mathbb{R}^d} \frac{m(d\zeta)}{1 + |\zeta|^2} < +\infty, \end{aligned}$$

which is (H). Starting in a similar way for the proof of (H<sub>η</sub>) and writing  $\Gamma$  for Euler's Gamma function we get

$$\begin{aligned} & \int_0^t d\tau (t - \tau)^{-2\eta} \sum_{j=1}^{+\infty} \lambda_j \left( \int_D dy |G(x, t; y, \tau) e_j(y)| \right)^2 \\ & \leq c \int_0^t d\tau \tau^{-2\eta} \int_{\{\zeta \in \mathbb{R}^d: |\zeta| < 1\}} m(d\zeta) \exp \left[ -c\tau |\zeta|^2 \right] \\ & + c \int_0^t d\tau \tau^{-2\eta} \int_{\{\zeta \in \mathbb{R}^d: |\zeta| \geq 1\}} m(d\zeta) \exp \left[ -c\tau |\zeta|^2 \right] \end{aligned}$$

$$\begin{aligned} &\leq c \int_0^t d\tau \tau^{-2\eta} \int_{\{\zeta \in \mathbb{R}^d: |\zeta| < 1\}} m(d\zeta) \exp[-c\tau |\zeta|^2] \\ &+ c\Gamma(1 - 2\eta) \int_{\{\zeta \in \mathbb{R}^d: |\zeta| \geq 1\}} \frac{m(d\zeta)}{(1 + |\zeta|^2)^{1-2\eta}} < +\infty \end{aligned}$$

by virtue of the properties of  $m$ , (83) and the fact that  $1 - 2\eta > 0$ . It is natural to conclude, therefore, that  $(C^d)$  and  $(C_\eta^d)$  play the same rôle in our analysis of the strong solutions to (3) as  $(M^d)$  and  $(M_\eta^d)$  have played in the above references. In this sense, they may be considered as natural substitutes for  $(M^d)$  and  $(M_\eta^d)$  when  $\kappa$  is *not* translation-invariant.

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