Potential Estimates for Large Solutions of Semilinear Elliptic Equations

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Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday

1 Introduction

In this report we describe our recent existence results [26], [27] for two not quite standard problems for semilinear elliptic equations in general domains.

The first problem concerns *large solutions* to nonlinear elliptic equations in arbitrary bounded domains $\Omega \subset \mathbf{R}^n$, $n \geq 3$. These are solutions $u \in C^2_{\text{loc}}(\Omega)$ to the nonlinear problem

$$\begin{cases} \Delta u - |u|^{q-1}u = 0 & \text{in} \quad \Omega \\ u(x) \to +\infty & \text{when} \quad x \to \partial\Omega. \end{cases} \tag{1.1}$$

For the parameter q we always assume

$$q > 1.$$

Note that the strong maximum principle for elliptic equations implies that u from (1.1) satisfies

$$u > 0, \quad \Delta u - u^q = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

Hence without the loss of generality we can consider only positive solutions of (1.1).

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Let now $\mathbf{S}^n$ be the unit n-dimensional sphere with the standard metric g_0 induced by the embedding $\mathbf{S}^n \hookrightarrow \mathbf{R}^{n+1}$. Our second problem concerns finding *geometrically complete solutions* in arbitrary domains $\Omega \subset \mathbf{S}^n$, $n \geq 3$. These are solutions $u \in C^2_{\text{loc}}(\Omega)$ to the problem

$$\begin{cases} \Delta_{g_0} u - R(g_0)\, \dfrac{(n-2)}{4(n-1)}\, u - 1\, \dfrac{(n-2)}{4(n-1)}\, u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad \Omega \\ \hfill u > 0 & \text{in} \quad \Omega \\ \hfill u^{\frac{4}{n-2}} g_0 \text{ is complete metric} & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

Here Δ_{g_0} is the Laplace-Beltrami operator on $(\mathbf{S}^n, g_0)$, and

$$R(g_0) = n(n-1)$$

is the scalar curvature of g_0. Geometrically (1.3) means that in the domain Ω we conformaly deform the standard metric g_0 to the new metric

$$g = u^{\frac{4}{n-2}} g_0,$$

which is complete in Ω, and has the scalar curvature

$$R(g) \equiv -1.$$

By definition, the metric g in (1.3) is complete in Ω if for any semi-open curve $\gamma\colon [0,1) \to \Omega$ such that

$$\text{dist}_{g_0}(\gamma(t), \partial\Omega) \to 0 \quad \text{when} \quad t \to 1-0,$$

we have

$$\text{length}_g(\gamma) = \int_0^1 u^{\frac{2}{n-2}}(\gamma) \left(g_0(\gamma)_{ij} \dot{\gamma}^i \dot{\gamma}^j \right)^{1/2} dt = +\infty.$$

As compared to (1.1), (1.2), we have the particular case

$$q = (n+2)/(n-2)$$

in (1.3).

Thus instead of more common boundary value problems (Dirichlet, Neuman, and their nonlinear analogies) we impose in (1.1), (1.3) the pointwise blow-up condition, or the condition of the completeness of the corresponding metric. The crucial fact about the equation

$$\Delta u - |u|^{q-1} u = 0, \quad q > 1, \tag{1.4}$$

is that the elliptic comparison principle holds for its sub- and supersolutions. Equation (1.4) is the Euler–Lagrange equation for the functional

$$F(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx + \frac{1}{q+1}\int_\Omega |u|^{q+1}\,dx.$$

Another nice property of (1.4) is that the functional F is bounded from below, obviously $F \geq 0$. All these facts make the treatment of the traditional boundary value problems for (1.4) rather straightforward. In particular such effects for unbounded functionals as Pohozaev identities, quantisation of energy for the strong convergence, and so on, do not arise for the "minus sign" nonlinearity in (1.1)-(1.4).

The main result of our work is the necessary and sufficient conditions on the domains Ω for the solubility of (1.1), (1.3). Our criteria for solvability of (1.1), (1.3) use the language of potential theory. Let us recall a fundamental result from the classical potential theory for the Laplace equation. This is the Wiener test for solvability of the classical Dirichlet problem for harmonic functions. Wiener theorem states that in a bounded domain $D \subset \mathbf{R}^n$, $n \geq 3$, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad D \\ u = f & \text{on} \quad \partial D \end{cases}$$

is solvable for all boundary data data $f \in C(\partial D)$, if and only if $\mathbf{R}^n \setminus D$ is not thin. Formally the latter means that

$$\int_0^1 \frac{cap(B(x,\rho)\setminus D)}{cap(B(x,\rho))}\,\frac{d\rho}{\rho} = +\infty \quad \text{for any } x \in \partial D.$$

Here $cap(\cdot)$ is the classical (electrostatic) capacity, and the upper integration limit 1 can be replaced by any $\delta > 0$. Our main theorems state that (1.1), (1.3) admit a solution if and only if the corresponding Wiener-type tests with certain capacities hold. We describe the results for problem (1.1) in section 2, and for problem (1.3) in section 3.

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2 Solutions with pointwise blowup

Of the two basic questions on problem (1.1) in arbitrary domains Ω namely existence and uniqueness, our main result completely resolves the first. Theorem 2.1 states that the solubility of (1.1) is equivalent to the Wiener-type test with respect to a certain capacity. On the second question it is well known that the uniqueness for (1.1) fails in general domains [25], [19].

After the groundbreaking papers by Perkins Dynkin and LeGall solutions of (1.1), (1.2) attracted a lot of attention of probabilists. Currently this is a very active area of research on the interface between the theory of random processes, nonlinear partial differential equations, and analysis. We refer to the ICM reports by Perkins [45] and LeGall [30] for a survey of the progress in the field and bibliography, cf. also [29]. Recent monographs [11] [12], [14], and [31] are dedicated to different aspects of the theory. At the moment the probabilistic methods are limited to the case

$$1 < q \leq 2$$

in (1.1), (1.2). Our research was inspired by a recent result of Dhersin and LeGall [10]. They proved that the existence for the problem (1.1) for $q = 2$ is equivalent to a Wiener type criterion for $\Omega^c = \mathbf{R}^n \setminus \Omega$. This result is one of the milestones of the theory, cf. [30], [31]. The crucial idea of Dhersin and LeGall was to combine the classical potential theory with the sharp bounds on the hitting probability for the super-Brownian motion related to positive solutions of

$$\Delta u - u^2 = 0.$$

An open problem in this area was to extend the result to the full range $q > 1$, cf. e. g. [31]. Relying upon entirely analytic ideas we prove the Wiener test for solubility of (1.1) for all $q > 1$.

Large solutions (1.1) were initially studied by Loewner and Nirenberg [34], as well as in the earlier papers of Keller [23] and Osserman [44]. Loewner and Nirenberg considered the case $q = (n + 2)/(n - 2)$ arising in conformal differential geometry. They proved that in smooth domains Ω there exists the unique solution of (1.1). Later the questions of existence, uniqueness, and the rate of the boundary blow-up were investigated by many authors to which the bibliography is very extensive [54]. For example, Brezis and Veron [8] proved that singletons are regular boundary points for (1.1) if and only if

$$1 < q < \frac{n}{n-2}.$$

Aviles, Bandle, Essen, Finn, Marcus, McOwen, Veron, and others investigated the questions for domains bounded by non-smooth hypersurfaces or manifolds of lower dimensions, as well as for more general semilinear equations. Kondratiev and Nikishkin [25] discovered the non-uniqueness for (1.1), cf. also [19]. We refer to survey [43] and monograph [54], for further description and references. Additionally papers [18] and [53] contain the very recent results. However, up to this point, the analytic approach has not obtained the necessary and sufficient condition for the existence in (1.1).

The capacity suitable for problem (1.1) is defined as follows. Fix $x_0 \in \mathbf{R}^n$, $n \geq 3$. Let $K \subset \mathbf{R}^n$ be a compact subset of the ball $B(x_0, 3/2)$. For $1 < p < \infty$ define

$$\mathcal{C}_p(K) = \inf \left\{ \int_{B(x_0,2)} |D^2\varphi|^p \ : \ \varphi \in C_0^\infty(B(x_0,2)),\ \varphi|_K \geq 1 \right\}. \tag{2.1}$$

Following the axiomatic potential theory we extend $\mathcal{C}_p$ as an outer capacity to any set E, $\overline{E} \subset B(x_0, 3/2)$. Capacities defined with different x_0 are equivalent. Capacity $\mathcal{C}_p$ is essentially the Bessel capacity associated with the Sobolev space $W^{2,p}(\mathbf{R}^n)$. Such capacities were carefully investigated in the theory of nonlinear potentials. The theory originates in early works by Maz'ya and Serrin in the 1960s, later developed in the 1970-1980s in papers by Adams, Fuglede, Havin, Hedberg, Maz'ya, Meyers, and many others. Good references are monographs [4], [39], and [58] wherein the reader can also find a rich bibliography along with historical notes. Now our main result [26] on (1.1) is stated below.

Theorem 2.1. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a bounded domain, and let $q > 1$. The following statements are equivalent:*
(i) Problem (1.1) has a solution $u \in C^2_{\mathrm{loc}}(\Omega)$.
(ii) The set $\Omega^c = \mathbf{R}^n \setminus \Omega$ is not thin, that is

$$\int_0^1 \frac{\mathcal{C}_{q'}\left(\Omega^c \cap B(x,r)\right)}{r^{n-2}} \frac{dr}{r} = +\infty \quad \textit{for} \quad \textit{all} \quad x \in \Omega^c, \tag{2.2}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

For $q = 2$, Theorem 2.1 was proved in [10] using probabilistic methods. It is very likely that the proof from [10] can be generalised for $1 < q \leq 2$ using ideas from [13], Condition (2.2) and well-known properties of the capacity easily imply the solubility of (1.1) in specific classes of domains Ω.

Wiener proved his criterion for solubility of the Dirichlet problem for harmonic functions in the fundamental papers [55], [56]. Later Wiener tests for the Dirichlet problem for more general *linear* second-order (degenerately) elliptic and parabolic equations were proven in [33], [17], [7], [9], [15], [16]. Recently the first complete results were obtained for linear elliptic equations of higher order [41]. Seminal papers [38] and [20] started the research on Wiener regularity of the Dirichlet problem for *quasilinear* equations of the second order by proving the sufficiency of a Wiener type criterion. A recent paper [24] completed the investigation of the basic question by proving the necessity, cf. also earlier contribution [32]. Monographs [21] and [35] give a comprehensive exposition of these results. Trudinger and Wang [49] presented an alternative more general and concise approach to quasilinear equations of the second order. In [28], Wiener criterion was proved for Hessian equations. Hessian equations [50], [51], [52] are *fully nonlinear* (nonlinear on the second derivatives) elliptic equations. We refer to surveys [2], [40] for further description of this area and the bibliography. In connection with Theorem 2.1 we mention the following result. Consider the classical (finite data) Dirichlet problem

$$\begin{cases} \Delta u - |u|^{q-1}u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega, \end{cases}$$

$q > 1$. Adams and Heard [3], [1] proved that it is solvable for all $f \in C(\partial\Omega)$ if and only if the classical Wiener test from [55], [56] holds for Ω.

Capacity (2.1) has been used in the previous works on potential theory for semilinear equations. Baras and Pierre [6] used it to characterise removable singularities for solutions of (1.2). In [5], [22] capacity (2.1) was used to investigate a different class of semilinear equations. We also mention the continuing series of papers by Marcus and Veron [36], [37] on Riesz-Herglotz type effects for equation (1.2) and its parabolic counterpart, questions which are also under current active study from the probabilistic point of view by Dynkin, Kuznetsov, LeGall, Delmas, Dhersin, and others.

3 Geometrically complete solutions

The problem of characterisation of domains on the sphere, which admit a complete metric with constant scalar curvature conformal to the standard metric on the sphere, originates in the works of Loewner and Nirenberg [34], and Schoen and Yau [47]. We refer to problem #36 from Yau's problem list [57], [48], and to the survey by McOwen [43] for more detailed description. Analytically for a given domain $\Omega \subset \mathbf{S}^n$ and a given constant

$$R \in \{-1, 0, 1\}$$

one seeks a solution $u \in C^2_{\mathrm{loc}}(\Omega)$ to the problem

$$\begin{cases} \Delta_{g_0} u - \dfrac{n(n-2)}{4}\, u + R\, \dfrac{(n-2)}{4(n-1)}\, u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u^{\frac{4}{n-2}} g_0 \text{ is complete metric} & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

In our paper [27] we resolve problem (3.1) in the case of negative constant scalar curvature

$$R = -1.$$

In other words we find the necessary and sufficient condition on the domain Ω for the existence of a solution to (1.3). As it was mentioned in the introduction, our condition is inspired by the notion of thinness from potential theory. Let us remark that our work concerns the basic problem of existence for (1.3), although other questions about solutions of (1.3), (3.1) can be asked as well, e.g. [46], [42], [43].

Now we briefly describe the previous results on solvability of (1.3) and more generally (3.1). For more detailed description and references cf. McOwen's survey [43]. We will formulate the results only for $\mathbf{S}^n$ although some of them have been proved for arbitrary compact manifolds. The case $R = 1$ is regarded as the hardest among the three.

Set

$$K = \mathbf{S}^n \setminus \Omega.$$

General *necessary* conditions for solvability of (3.1) in terms of the Hausdorff measure of K were given by Schoen and Yau [47]. They are

$$\Lambda_{\frac{n-2}{2}+\varepsilon}(K) = 0 \ \forall \varepsilon > 0 \quad \text{for} \quad R = 1,$$
$$\Lambda_{\frac{n}{2}+\varepsilon}(K) = 0 \ \forall \varepsilon > 0 \quad \text{for} \quad R = 0.$$

Earlier Loewner and Nirenberg [34] found that a necessary condition for solvability of (1.3) is

$$\Lambda_{\frac{n-2}{2}}(K) = +\infty \quad \text{for} \quad R = -1.$$

In the converse direction up to the present moment the existence was established under much more restrictive *sufficient* conditions on K than those in terms of Hausdorff measure. In papers by Schoen, Mazzeo, and Pacard existence for (3.1) with $R = 1$ was proved if K is a disjoint union of points and smooth submanifolds of dimensions less or equal $(n-2)/2$. It is easy to show, that the statement holds in the case $R = 0$ as well. For (3.1) with $R = -1$ Veron, Aviles, and McOwen proved that the problem is solvable if K is a disjoint union of smooth submanifolds of dimensions strictly greater than $(n-2)/2$. There are further results by Finn for submanifolds with boundary, conical edges, and certain types of their unions. We refer to Finn [18] for the latest results and references.

Clearly there was still a substantial gap between such sufficient and necessary conditions.

The capacity suitable for problem (1.3) is defined for subsets $E \subset \mathbf{S}^n$, $n \geq 3$, satisfying

$$\mathrm{diam}_{g_0}(\overline{E}) < \pi/2.$$

Any such set lies in a hemisphere. After a rotation we can assume that it lies in the south hemisphere. Set

$$[g_0^{ij}] = g_0^{-1}.$$

Then for any compactum E we define

$$\begin{aligned} \mathcal{C}(E) &= \inf\left\{\int_{\mathbf{S}^n} \left|\nabla^2_{g_0}\varphi\right|_{g_0}^{\frac{n+2}{4}} \, dV_{g_0}\right\} \\ &= \inf\left\{\int_{\mathbf{S}^n} \left(g_0^{ij} g_0^{kl} (\nabla^2_{g_0}\varphi)_{ij} (\nabla^2_{g_0}\varphi)_{kl}\right)^{\frac{n+2}{8}} \, dV_{g_0}\right\}. \end{aligned} \tag{3.2}$$

Note that

$$\frac{1}{(n+2)/(n-2)} + \frac{1}{(n+2)/4} = 1.$$

In (3.2) symbols ∇_{g_0}, $|\cdot|_{g_0}$, dV_{g_0} stand respectively for the Riemannian connection, the norm with respect to the inner product of tensors, the Riemannian volume element of the metric g_0. The infimum in (3.2) is taken over the set

$$\Big\{\varphi \in C_0^\infty(\mathbf{S}^n): \quad \varphi \equiv 1 \text{ in a neighbourhood of } E \text{ and } \varphi|_{B(N,\pi/4)} \equiv 0\Big\},$$

where $B(N, \pi/4)$ is the ball in the metric g_0 of radius $\pi/4$ centered at the north pole N. Essentially, $\mathcal{C}(\cdot)$ is the Bessel capacity $\mathcal{C}_{(n+2)/4}(\cdot)$ for the

Sobolev space $W^{2,(n+2)/4}(\mathbf{R}^n)$ defined in the previous section. Now we state the main result [27] for problem (1.3).

Theorem 3.1. *Let $\Omega \subset \mathbf{S}^n$, $n \geq 3$, be an open set, $K = \mathbf{S}^n \setminus \Omega$. The following statements are equivalent:*
(i) There exists a complete metric in Ω with constant negative scalar curvature conformal to g_0.
(ii) The compactum K is not thin, that is

$$\int_0^1 \left(\frac{\mathcal{C}(B(x,\rho) \cap K)}{\mathcal{C}(B(x,\rho))} \right)^{\frac{2}{n-2}} \frac{d\rho}{\rho} = +\infty \quad \text{for all } x \in K.$$

In view of Theorem 3.1 it would now be interesting to clarify how the condition

$$\mathcal{C}(\mathbf{S}^n \setminus \Omega) = 0$$

relates to the solvability of (3.1) with $R = 0, 1$.

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