# COMPUTATION OF MILNOR NUMBERS AND CRITICAL VALUES IN AFFINE SPACE AND AT INFINITY

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Abstract. We describe a program for Singular that enables to computes topological objects associated to a complex polynomial map of $n \geqslant 2$ variables with isolated singularities. These objects are: the affine critical values, the affine Milnor numbers for all irregular fibers, the critical values at infinity, and the Milnor numbers at infinity for all irregular fibers.

## 1. Introduction and mathematical background

Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map, $n \geqslant 2$. By a result of Thom [Th] there is a minimal *set of critical values* $\mathcal{B}$ of point of $\mathbb{C}$ such that $f : f^{-1}(\mathbb{C} \setminus \mathcal{B}) \longrightarrow \mathbb{C} \setminus \mathcal{B}$ is a fibration.

1.1. **Affine singularities.** We suppose that *affine singularities are isolated i.e.* that the set $\{x \in \mathbb{C}^n \mid \operatorname{grad}_f x = 0\}$ is a finite set. Let $\mu_c$ be the sum of the local Milnor numbers at the points of $f^{-1}(c)$. Let

$$\mathcal{B}_{\mathit{aff}} = \big\{c \mid \mu_c > 0\big\} \quad \text{and} \quad \mu = \sum_{c \in \mathbb{C}} \mu_c$$

be the *affine critical values* and the *affine Milnor number*.

1.2. **Singularities at infinity.** See [Br]. Let $d$ be the degree of $f : \mathbb{C}^n \longrightarrow \mathbb{C}$, let $f = f^d + f^{d-1} + \cdots + f^0$ where $f^j$ is homogeneous of degree $j$. Let $\bar{f}(x,z)$ (with $x = (x_1, \ldots, x_n)$) be the homogenization of $f$ with the new variable $z$: $\bar{f}(x,z) = f^d(x) + f^{d-1}(x)z + \cdots + f^0(x)z^d$. Let

$$X = \big\{((x:z),t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x,z) - cz^d = 0\big\}\,.$$

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Let $\mathcal{H}_\infty$ be the hyperplane at infinity of $\mathbb{P}^n$ defined by $(z=0)$. The singular locus of $X$ has the form $\Sigma \times \mathbb{C}$ where

$$\Sigma = \left\{ (x:0) \mid \frac{\partial f^d}{\partial x_1} = \cdots = \frac{\partial f^d}{\partial x_n} = f^{d-1} = 0 \right\} \subset \mathcal{H}_\infty.$$

We suppose that $f$ has *isolated singularities at infinity* that is to say that $\Sigma$ is finite. This is always true for $n=2$. For a point $(x:0) \in \mathcal{H}_\infty$, assume, for example, that $x = (x_1, \ldots, x_{n-1}, 1)$ and set $\check{x} = (x_1, \ldots, x_{n-1})$ and

$$F_c(\check{x}, z) = \bar{f}(x_1, \ldots, x_{n-1}, 1) - cz^d.$$

Let $\mu_{\check{x}}(F_c)$ be the local Milnor number of $F_c$ at the point $(\check{x}, 0)$. If $(x:0) \in \Sigma$ then $\mu_{\check{x}}(F_c) > 0$. For a generic $s$, $\mu_{\check{x}}(F_s) = \nu_{\check{x}}$, and for finitely many $c$, $\mu_{\check{x}}(F_c) > \nu_{\check{x}}$. We set $\lambda_{c,\check{x}} = \mu_{\check{x}}(F_c) - \nu_{\check{x}}$, $\lambda_c = \sum_{(x:0)\in\Sigma} \lambda_{c,\check{x}}$. Let

$$\mathcal{B}_\infty = \left\{ c \in \mathbb{C} \mid \lambda_c > 0 \right\} \quad \text{and} \quad \lambda = \sum_{c\in\mathbb{C}} \lambda_c$$

be the *critical values at infinity* and the *Milnor number at infinity.* We can now describe the set of critical values $\mathcal{B}$ as follows (see [HL] and [Pa]):

$$\mathcal{B} = \mathcal{B}_{\mathit{aff}} \cup \mathcal{B}_\infty.$$

Moreover by [HL] and [ST] for all $c \in \mathbb{C}$, $f^{-1}(c)$ has the homotopy type of a wedge of $\mu + \lambda - \mu_c - \lambda_c$ spheres of real dimension $n-1$.

This paper describes a computer program that enables to calculate all the objects defined above: $\mathcal{B}_{\mathit{aff}}$, $\mu$, $\mu_c$ for $c \in \mathcal{B}_{\mathit{aff}}$ ; $\mathcal{B}_\infty$, $\lambda$, $\lambda_c$, for $c \in \mathcal{B}_\infty$. This program is written for Singular, [GPS]. It is based on polar curves and on the article of D. Siersma and M. Tibăr, [ST]. For polynomials in two variables ($n=2$) a program in Maple has been written by G. Bailly-Maître, [BM], based on a discriminant formula of Hà H.V., [Ha].
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## 2. Simple use of the program

The following example shows how to use the program once you have started Singular. We have to load the librabry `critic.lib`, then we set the ring, with $n+1$ variables, the last variables will able to have the critical values (as the zeroes of a polynomial) in return. Let $f(x,y) = x(xy-1)$, (Broughton's example).

```
LIB "critic.lib";
ring r = 0, (x,y,t), dp;
poly f = x*(xy-1);
crit(f);
```

The result is the following:

```
Polynomial : x2y-x
Affine critical values are the roots of 1
Affine Milnor number : 0
Critical values at infinity are the roots of t
Milnor number at infinity : 1
Details of critical values at infinity :
  t   1
```

This shows, that there is no affine critical value and that 0 (as the root of the polynomial $t$) is the only critical values at infinity, with Milnor number at infinity equal to 1.

## 3. Milnor numbers and critical values in affine space

3.1. **Milnor number.** The computation of the affine Milnor number $\mu$ is easy and well-known (see [GPS] for example). Let $f \in \mathbb{C}[x_1, \dots, x_n]$. Let $J$ be the Jacobian ideal of the partial derivative $(\partial f/\partial x_i)_i$. Then $\mu$ is the vector space dimension (over $\mathbb{C}$) of a Groebner basis (or standard basis) of the quotient $\mathbb{C}[x_1, \dots, x_n]/J$. In Singular it could be written:

```
ring r = 0, (x(1..n)), dp;
poly f = ...;
ideal J = jacob(f);
int mu = vdim(std(J));
```

3.2. **Critical values.** We add a new variable $t$. We consider the variety

$$C = \left\{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid f(x) - t = 0 \ \text{ and } \ \operatorname{grad}_f x = 0\right\}.$$

The critical values are the projection of $C$ on the second factor: $\mathcal{B}_{\mathit{aff}} = \mathrm{pr}_2(C)$. The computation is as follows :

```
ring r = 0, (x(1..n),t), dp;
poly f = ...;    // polynomial in x(1),...,x(n)
ideal C = f-t, jacob(f);
poly Baff = eliminate(C,x(1)x(2)..x(n))[1];
```

Here `Baff` is a polynomial in $\mathbb{C}[t]$ whose roots are the set $\mathcal{B}_{\mathit{aff}}$.

3.3. **Milnor number of a fiber.** Set $c \in \mathbb{C}$. We would like to compute $\mu_c$ the sum of the Milnor numbers of the points of $f^{-1}(c)$. Let $J$ be the Jacobian ideal of $f$ and set $x$ a critical point. We denote by $J_x$ the localization of $J$ at $x$. Let $I_x = (t-c, J_x)$, the dimension of $I_x$ is equal to the Milnor number of $f$ at $x$. For $k \geqslant 1$ we consider $K_x^k = ((f-t)^k, I_x)$. Then $f(x) = c$ if and only if $K_x$ has non-zero dimension (as a vector space). Moreover if $f(x) = c$ then, by the Nullstellensatz, $(f-t)^k$ is in $I_x$ for a sufficiently large $k$. For such a $k$, the dimension of $K_x$ is the Milnor number at $x$ if $f(x) = c$, and it is 0 otherwise. Such a $k$ is less or equal to the Milnor number at $x$, but $k$ can often be chosen much less. The minimal $k$ is the first integer such that the vector space dimension of $K_x^k$ is equal to the one of $K_x^{k+1}$. We make a procedure `stab` that computes this minimal $k$. Then we proceed as follows:

```
ring r = 0, (x(1..n),t), dp;
number c = ...;
poly f = ...;
ideal I = t-c, jacob(f);
int k = stab(I,f-t);
ideal K = (f-t)^k, I;
int muc = vdim(std(K));
```

## 4. Milnor numbers and critical values at infinity

We now give the computation of the objects at infinity. We will suppose that $f$ has isolated singularities at infinity, this can be verified with a procedure `isisolatedinf`. In fact computations are valid for a larger class of polynomials but it can not be computed if $f$ belongs to this class. The algorithm is based on the article of D. Siersma and M. Tibăr, [ST], that gives critical values at infinity and Milnor numbers at infinity with the help of polar curves.

4.1. **Working space.** We will work in $\mathbb{P}^n \times \mathbb{C}$, with the homogeneous coordinates of $\mathbb{P}^n$: $(x_1 : \cdots : x_n : z)$; we still need $t$ which is a parameter or a variable depending on the context.
We recall that

$$X = \left\{((x:z),t) \in \mathbb{P}^n \times \mathbb{C} \mid \bar{f}(x,z) - tz^d = 0\right\}.$$

The part at infinity of $X$ is $X_\infty = X \cap (\mathcal{H}_\infty \times \mathbb{C})$:

$$X_\infty = \left\{((x:0),t) \in \mathbb{P}^n \times \mathbb{C} \mid f^d(x) = 0\right\}.$$

Where $f = f^d + f^{d-1} + \cdots$ is the decomposition in homogeneous polynomials.
In Singular, we write:

```
ring r = 0, (x(1..n),z,t), dp;
poly f = ...;
// Homogenization : here t is a parameter not a variable
poly fH = homog(f,z)-t*z^deg(f);
ideal X = fH;
ideal Xinf = z, fH;
```

4.2. **Polar curve.** Let $k$ be in $\{1,\dots,n\}$. The polar curve $\mathcal{P}$ is the critical locus of the map $\phi\colon \mathbb{C}^n \longrightarrow \mathbb{C}^2$ defined for $x = (x_1,\dots,x_n)$ by $\phi(x) = (f(x), x_k)$:

$$\mathcal{P} = \left\{ x \in \mathbb{C}^n \mid \frac{\partial f}{\partial x_i}(x) = 0, \forall i \neq k \right\}.$$

We suppose that $\mathcal{P}$ is a curve or is void. A procedure `iscoordgeneric` enables to verify it for all $k$. We call $\mathcal{P}_H$ the projective closure of $\mathcal{P}$. This curve intersects the hyperplane at infinity $\mathcal{H}_\infty$ in finitely many points.

```
ideal P = diff(f,x(1)),..., diff(f,x(k-1)), diff(f,x(k)),...;
ideal PH = homog(P,z);
```

The former objects can be viewed in $X$, we will also denote by $\mathcal{P}_H$, the set $(\mathcal{P}_H \times \mathbb{C}) \cap X$. In the chart $x_k = 1$, we denote the curve $\mathcal{P}_H$ by $\bar{\mathcal{C}}$. The "real" polar curve $\mathcal{C}$ in this chart is the closure of $\bar{\mathcal{C}} \setminus X_\infty$. The procedure `sat` from the library `elim.lib` computes, in term of ideals, exactly this.

```
ideal Cbar = x(k)-1, PH, X;
ideal C = sat(Cbar,Xinf)[1];
```

4.3. **Critical values at infinity.** We need the following result of [ST]. A value $c$ is a critical values at infinity if and only there is coordinate $x_k$ and a point $(x:0,t)$ in $X_\infty$ (with $x_k \neq 0$) such that $(x:0,t) \in \mathcal{C}$. That is to say $\mathcal{B}_\infty$ is the projection of $\mathcal{C}_\infty = X_\infty \cap \mathcal{C}$ on the space of parameters $t \in \mathbb{C}$. Then the critical values are computed with:

```
ideal Cinf = z, C;
poly Binf = eliminate(Cinf,x(1)x(2)..x(n)z)[1];
```

The set of critical values at infinity are the roots of the polynomial `Binf`, which belongs to $\mathbb{C}[t]$.

4.4. **Milnor numbers at infinity.** Actually the results in [ST] are more precise. For a fixed $t$, let $X_t = \{(x:z,t) \in X\}$, this is a projective model for the fiber $f^{-1}(t)$. The Milnor number at infinity at a point $(x:0,t) \in \mathcal{C}_\infty$ is given by the intersection number (in $X$) of $\mathcal{C}$ with $X_t$ at $(x:0,t)$.

So, for $c \in \mathcal{B}_\infty$, the Milnor number at infinity $\lambda_c$, is equal to the sum of all intersection numbers of $X_c$ and $\mathcal{C}$ in $X_\infty$. In fact we have cheated a bit because we also have to add what happens in the other charts, *i.e.* for $x_k = 0$.

We compute an ideal $I$ which correspond to $X_c \cap \mathcal{C}$, then we only deals with points at infinity by intersecting it this set with $z^k = 0$. As in the computation of $\mu_c$ we have to choose a sufficiently large $k$ with our procedure `stab`.

```
number c = ...;
ideal Xc = t-c, X;
ideal I = Xc, C;
int k = stab(I,z);
ideal K = z^k, I;
lambdac = vdim(std(K));
```

Once we have computed $\lambda_c$ for all $c \in \mathcal{B}_\infty$, we have $\lambda = \sum_{c\in\mathcal{B}_\infty} \lambda_c$.

## 5. Examples

You have to load the library with the command:

```
LIB "critic.lib";
```

5.1. **Briançon polynomial.** The following code gives critical values at infinity of Briançon polynomial (in fact this the usual polynomial by a factor 3).

```
ring r = 0, (x,y,t), dp;
poly s = xy+1;
poly p = x*s+1;
poly f = 3*y*p^3+3*p^2*s-5*p*s-s;
crit(f);
```

The result is:

```
Affine critical values are the roots of 1
Affine Milnor number : 0
Critical values at infinity are the roots of 3t2+16t
Milnor number at infinity : 4
Details of critical values at infinity :
  t       1
  3t+16   3
```

5.2. **An example of Deligne.** The critical values are given as the roots of a polynomial. When this polynomial is not of degree 1, all its roots are critical values, and the associated Milnor number is the sum of Milnor numbers over all the roots.
For example the following polynomial of Deligne has 3 critical values, each one having a affine Milnor number equal to 1.

```
poly f = xy2+x2+y;
crit(f);

Affine critical values are the roots of 64t3-27
Affine Milnor number : 3
Details of affine critical values :
```

```
  4t-3          1
  16t2+12t+9    2
Critical values at infinity are the roots of 1
Milnor number at infinity : 0
```

5.3. **Parameters.** SINGULAR enables parameters, but computation can be quiet tedious. Please notice that you have to carefully check the results for the critical parameters.

```
ring r = (0,a,b), (x,y,t), dp;
poly f = x*(x3y2+ax+1)+b;
crit(f);

Affine critical values are the roots of (-4a)*t+(4ab-1)
Affine Milnor number : 1
Critical values at infinity are the roots of t+(-b)
Milnor number at infinity : 2
```

5.4. **More variables.** Let $f(a,b,c,d) = a + a^4b + b^2c^3 + d^5$ be the example of Choudary-Dimca, [CD] and [ACD]. This polynomial has isolated singularities at infinity. The only singularity is a singularity at infinity for the critical value 0. Let's check it.

```
ring r = 0, (a,b,c,d,T), dp;
poly f = a+a4b+b2c3+d5;
crit(f);

Affine critical values are the roots of 1
Affine Milnor number : 0
Critical values at infinity are the roots of T
Milnor number at infinity : 8
```

## References

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