BIFURCATION OF LIMIT CYCLES FROM TWO FAMILIES OF CENTERS

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Abstract. We study the number of limit cycles that bifurcate from the periodic orbits of a center in two families of planar polynomial systems. One of these families has a global center. The other family is obtained by adding a straight line of critical points to the first one. The common point between both unperturbed families is that they can be integrated by using the Lyapunov polar coordinates. The study of the number of limit cycles bifurcating from the centers is done by considering the zeros of the associated Poincaré-Melnikov integrals. As a consequence of our study we provide quadratic lower bounds for the number of limit cycles surrounding an unique critical point in terms of the degree of the system.

1. Introduction and Main Results

A complete study of the real planar differential systems requires, from the qualitative point of view, determining the number and nature of critical points, the separatrix structure and the number and location of closed trajectories. The third question is still the main open problem in the qualitative theory of real planar differential systems.

In the study of two-dimensional systems,

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y, the determination of the number and position of limit cycles is known as the second part of the *Hilbert 16th problem* (see [H]). Let $\mathcal{H}_n$ denote the maximum possible number of limit cycles of system (1) when P and Q are arbitrary polynomials of degree at most n. The $\mathcal{H}_n$ are known as the Hilbert numbers, and it is still an open problem whether $\mathcal{H}_n$ is finite or not for each $n \geq 2$. To know about $\mathcal{H}_n$ one strategy is to control its lower bounds in

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terms of n. This can be done by studying particular families of polynomial differential systems of type (1).

In this paper we consider systems of the form

$$\frac{dx}{dt}=-V(x,y)\frac{\partial H(x,y)}{\partial y}+\varepsilon P(x,y), \quad \frac{dy}{dt}=V(x,y)\frac{\partial H(x,y)}{\partial x}+\varepsilon Q(x,y), \tag{2}$$

where V, H, P and Q are polynomials in two variables such that $\max(\partial V + \partial H - 1, \partial P, \partial Q) \leq n$ and ε is small enough. Here, given a polynomial $R(x,y)$, ∂R denotes its degree.

Consider the Poincaré-Melnikov or Abelian integral

$$I(h) = \int_{H=h} \frac{P(x,y)\,dy - Q(x,y)\,dx}{V(x,y)}, \tag{3}$$

defined on a region $D_{[h_0,h_1]} := \{H(x,y) = h \,:\, h_0 \leq h \leq h_1)\} \subset \mathbb{R}^2$ where V does not vanish and the Hamiltonian system associated to H has all their solutions periodic. It is well known that the number of simple zeros of $I(h)$ when $h \in [h_0, h_1]$ gives the number of limit cycles of (2) which bifurcate from these periodic orbits for ε small enough, see [A] or [R].

In order to motivate the concrete family of type (2) that we consider we give some preliminary results about the number of limit cycles for system (2) that have been obtained in previous papers by studying the number of zeros of the Abelian integral $I(h)$ given in (3):

(I) When $V \equiv 1$ and $H = (x^2 + y^2)/2$ at most $[\frac{n-1}{2}]$ limit cycles, see [GLV]. As usual, we use $[\,\cdot\,]$ to denote the integer part function.

(II) When $V = 1 + x$ and $H = (x^2 + y^2)/2$ at most n limit cycles, see [LPR].

(III) When $V \equiv 1$ and $H = \frac{1}{2k}x^{2k} + \frac{1}{2l}y^{2l}$ at most $\phi(2\left[\frac{n-1}{2}\right] + 1)$ limit cycles, where $\phi(n) = n^2/8 + n/2 - 5/8$, see [LZ]. In [LLLZ] it is also proved that the same number of limit cycle is obtained just for the case $l = 1$.

From the above results we have the following consequences:

- The effect of introducing the line critical points $1 + x = 0$ in the linear center dobles the number of limit cycle obtained from the Abelian integral (from $[\frac{n-1}{2}]$ to n).
- A good way to increase the number of zeroes is by degenerating the critical point at the origin (from $[\frac{n-1}{2}]$ to $\phi(2\left[\frac{n-1}{2}\right] + 1)$).

With the above information, one might think that, by introducing a straight line of singular points in a system with a degenerate critical point, we can double the number of limit cycles, obtaining an example with $2\phi(2\left[\frac{n-1}{2}\right] + 1) \simeq \frac{n^2}{4}$. Unfortunately, as we will prove in this paper, the introduction of the line of critical points for a system with degenerate origin just increases the number of limit cycle by $n/2+1$.

To be more precise in this paper we study the following family of differential equations,

$$\begin{cases} \dot{x} &= -y^{2l-1}(1+bx)+\varepsilon P_n(x,y), \\ \dot{y} &= x^{2k-1}(1+bx)+\varepsilon Q_n(x,y), \end{cases} \tag{4}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are real polynomials of degree at most n, $0<\varepsilon\ll 1$ and either

(i) $b=0$, or
(ii) $l=b=1$. (Notice that the case $b\neq 0$ can be reduced to the case $b=1$ just by a linear change of variables.)

Observe that case (i) has been already considered in [LZ], but we also include it in our study because the tools that we use provide a different proof of the results of [LZ].

Our approach is mainly based in the expression of the closed orbits of the unperturbed Hamiltonian in the generalized polar coordinates introduced by Lyapunov, see [L] or [BDST]. We recall these coordinates in Section 2 and also make some preliminary computations involving them.

In all the cases (I-II-III) the study of the number of zeros of $I(h)$ is reduced, trough a change of variables if necessary, to the study of a polynomial function. A main difference between the study of case (ii) and these previous results is that in our situation the function $I(h)$ is no more a polynomial function. This is the main difficulty of the paper.

In Section 3 we give new proofs of results (I-III) described above. Our main result is the following theorem, which is proved in Section 4.

Theorem 1.1. *Consider the family of polynomial vector fields of type (4) with $l=1$ and $b=1$, that is*

$$\begin{cases} \dot{x} &= -y(1+x)+\varepsilon P_n(x,y), \\ \dot{y} &= x^{2k-1}(1+x)+\varepsilon Q_n(x,y), \end{cases} \tag{5}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are real polynomials of degree at most $n\in\{2k,2k+1\}$. Then, for $0<\varepsilon\ll 1$, there are systems of this type with $\phi(n+1)$ limit cycles surrounding the origin, where $\phi(n)=n^2/8+n/2-5/8$.

In the proof of the above theorem we will see that the system giving rise to $\phi(n+1)$ limit cycles has $Q_n(x,y)\equiv 0$. Some comments about the case $Q_n(x,y)\not\equiv 0$ are made in Remark 4.6.

Notice that the above result gives the same number of limit cycles that the study of the case (III) if n is odd. On the other hand, when n is even Theorem 1.1 gives $\phi(n+1)-\phi(n-1)=\frac{n}{2}+1$ limit cycles more than the number obtained by studying the case (III).

2. Preliminary results

Let us introduce some notation and preliminary results which will be useful in the proof of the main results.

We write the functions $P_n(x,y)$ and $Q_n(x,y)$ of system (4) as follows:

$$P_n(x,y)=\sum_{0\le i+j\le n} a_{i,j}x^iy^j, \qquad Q_n(x,y)=\sum_{0\le i+j\le n} b_{i,j}x^iy^j,$$

and a periodic orbit of the unperturbed system for $\varepsilon=0$ as

$$H(x,y)=\frac{1}{2k}x^{2k}+\frac{1}{2l}y^{2l}=h, \qquad h>0.$$

We recall the (ρ,θ,l,k)-generalized polar coordinates introduced by Lyapunov. They are $x=\rho^l\mathrm{Cs}(\theta)$, $y=\rho^k\mathrm{Sn}(\theta)$ where $z(\theta):=\mathrm{Cs}(\theta)$ and $w(\theta):=\mathrm{Sn}(\theta)$ are the solution of the Cauchy problem:

$$\dot z=-w^{2l-1}, \qquad \dot w=z^{2k-1}, \qquad z(0)=\sqrt[2k]{\frac{1}{l}}, \qquad w(0)=0,$$

(see [L] and [BDST], for instance). Let us call $T_{l,k}$ the period of the functions $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$. It is well known the following property,

$$l\mathrm{Cs}^{2k}(\theta)+k\mathrm{Sn}^{2l}(\theta)=1. \tag{6}$$

The study of the Abelian integrals coming from (3) and the use of the Lyapunov polar coordinates will lead us to study integrals of the form

$${}^{l,k}\mathcal{F}_b^{m,n}(\rho):=\int_0^{T_{l,k}}\frac{\mathrm{Cs}^m(\theta)\mathrm{Sn}^n(\theta)}{1+b\rho^l\mathrm{Cs}(\theta)}\,d\theta. \tag{7}$$

When in the above function some of the parameters is fixed during some reasoning it is usually omitted. In the following lemma we collect some of its properties.

Lemma 2.1. *Consider the function ${}^{l,k}\mathcal{F}_b^{m,n}(\rho)$ given in (7). The following properties hold:*

(i) ${}^{l,k}\mathcal{F}_b^{m,2n+1}(\rho)\equiv 0$, *for* $b\in\{0,1\}$.

(ii) For $b=0$, ${}^{l,k}\mathcal{F}_0^{m,n}(\rho)$ *does not depend on* ρ *and* ${}^{l,k}\mathcal{F}_0^{2m+1,n}(\rho)\equiv 0$.

(iii) For the cases not covered in the above two items we have that

$${}^{l,k}\mathcal{F}_b^{m,n}(\rho)=\frac{2}{k^{n-1+1/(2l)}}\int_{-1}^{1}\frac{u^m(1-lu^{2k})^{n-1+1/(2l)}}{1+b\rho^l u}du.$$

(iv) For short, fixing $l=1$ *and* k, *denote* $\mathcal{F}^m(\rho):={}^{1,k}\mathcal{F}_1^{m,0}(\rho)$, *and* $K_m={}^{1,k}\mathcal{F}_0^{m,0}(\rho)=\int_0^{T_{1,k}}Cs^m(\theta)\,d\theta$. *Then the following recurrences hold:*

$$\mathcal{F}_1^{m,2j}(\rho)=\frac{1}{k^j}\sum_{s=0}^{j}\binom{j}{s}(-1)^s\mathcal{F}_1^{m+2ks,0}(\rho)=\frac{1}{k^j}\sum_{s=0}^{j}\binom{j}{s}(-1)^s\mathcal{F}^{m+2ks}(\rho),$$

$$\mathcal{F}^m(\rho) = \frac{\mathcal{F}^0(\rho) - \sum_{l=0}^{[\frac{m-1}{2}]} K_{2l}\rho^{2l}}{(-b\rho)^m}.$$

(v) *The function $\mathcal{F}^m(\rho)$ is no a rational function of the variable ρ. In particular*

$$ {}^{1,1}\mathcal{F}^{0,0}(\rho) = \frac{2\pi}{\sqrt{1-\rho^2}}.$$

(vi) *The function $\mathcal{F}^0(\rho)$ is an analytical function of the variable $\rho \in (0,1)$.*

Proof. (i-ii) These results follow by using the symmetries of the function $\mathrm{Sn}(\theta)$ and $\mathrm{Cs}(\theta)$. Part (iii) is proved by performing the change of variables $u = \mathrm{Cs}(\theta)$ in the integral expression.

(iv) The first expression follows by using equality (6). The second recurrent expression can be proved by induction, by noticing that

$$\mathcal{F}^m(\rho) + b\rho\mathcal{F}^{m+1}(\rho) = K_m.$$

(v) By using (iii) with $b = 1$, we have the following expression for $\mathcal{F}^0$,

$$\mathcal{F}^0(\rho) = 2\sqrt{k}\int_{-1}^{1} \frac{du}{(1+\rho u)\sqrt{1-u^{2k}}}.$$

When $k = 1$ the above expression can be explicitly computed giving $\frac{2\pi}{\sqrt{1-\rho^2}}$. In general consider the function $F_k(\rho) = \int_{-1}^{1} \frac{du}{(1+\rho u)\sqrt{1-u^{2k}}}$, for $k \in \mathbb{N}$ and $0 \le \rho < 1$. In order to prove that it is not a rational function in the variable ρ we will prove that

$$\lim_{\rho\to 1^-} F_k(\rho)(1-\rho) = 0. \tag{8}$$

Observe that if (8) is proved the result follows. This is due to the following facts: The $\lim_{\rho\to 1^-} F_k(\rho) = +\infty$. So if F_k where a rational function the point $\rho = 1$ would be a pole of F_k. Therefore it should exists a $m \in \mathbb{N}$ (the order of the pole) such that

$$\lim_{\rho\to 1} F_k(\rho)(1-\rho)^m \neq 0,$$

but the above inequality is in contradiction with (8).

To prove (8) let us write $F_k(\rho) = \int_{-1}^{0} \frac{du}{(1+\rho u)\sqrt{1-u^{2k}}} + \int_{0}^{1} \frac{du}{(1+\rho u)\sqrt{1-u^{2k}}}$. For $-1 < u < 0$ and $0 \le \rho < 1$, we have that $\frac{1}{(1+\rho u)\sqrt{1-u^{2k}}} < \frac{1}{(1+\rho u)\sqrt{1+u}}$. Hence,

$$\begin{aligned}\int_{-1}^{0} \frac{du}{(1+\rho u)\sqrt{1-u^{2k}}} &< \int_{-1}^{0} \frac{du}{(1+\rho u)\sqrt{1+u}} \\ &= \frac{2}{\sqrt{\rho(1-\rho)}} \arctan\left(\sqrt{\frac{\rho}{1-\rho}}\right).\end{aligned}$$

Therefore, for $\rho > \frac{1}{2}$, we can write,

$$F_k(\rho)(1-\rho) < \frac{\sqrt{2}\pi}{\sqrt{1-\rho}}(1-\rho) + \int_0^1 \frac{du}{(1+\rho u)\sqrt{1-u^{2k}}}(1-\rho).$$

Taking, in both sides, limit when $\rho \to 1^-$ we get that $\lim_{\rho\to 1^-} F_k(\rho)(1-\rho) = 0$, because the integral between 0 and 1 is finite. Hence this part of the lemma follows.

(vi) Let us consider $\rho_0 \in (0,1)$. From the expression of $\mathcal{F}^0(\rho)$ obtained in the proof of former part (v) it is easy to get

$$\mathcal{F}^0(\rho) = 2\sqrt{k}\sum_{n\geq 0}(-1)^n(\rho_0+\lambda)^n \int_{-1}^1 \frac{u^n}{\sqrt{1-u^{2k}}}\,du,$$

where $\rho = \rho_0 + \lambda$, $\lambda \in (-\varepsilon, \varepsilon)$ and $\varepsilon = \frac{1}{2}\min\{|\rho_0 - 1|, \rho_0\}$. In this situation last series converges. □

Lemma 2.2. *Consider the expression of K_m given by Lemma 2.1.(iv). Then for any natural numbers i and j,*

$$\sum_{s=0}^{j}(-1)^s \begin{pmatrix} j \\ s \end{pmatrix} K_{2ks+2i} > 0. \tag{9}$$

Proof. Let $(1-u^{2k})^j = \sum_{s=0}^{j}\binom{j}{s}(-1)^s u^{2ks}$, be. By multiplying both sides of the above equality by $\frac{2\sqrt{k}u^{2i}}{\sqrt{1-u^{2k}}}$ we get that

$$\frac{2\sqrt{k}u^{2i}}{\sqrt{1-u^{2k}}}(1-u^{2k})^j = \sum_{s=0}^{j}\binom{j}{s}(-1)^s 2\sqrt{k}\frac{u^{2ks+2i}}{\sqrt{1-u^{2k}}}.$$

By integrating in both sides of the above formula for u between -1 and 1, and taking into account that $K_{2ks+2i} = \int_0^{T_{1,k}} \mathrm{Cs}^{2ks+2i}(\theta)\,d\theta = 2\sqrt{k}$ $\int_{-1}^{1}\frac{u^{2ks+2i}}{\sqrt{1-u^{2k}}}\,du$ the result follows. □

3. A new study of cases (I-III)

This section is devoted to give a different and shorter proof of the results of [GLV, LLLZ, LZ] presented in the introduction. More specifically we study system (4) when $b = 0$. It writes as

$$\begin{cases} \dot{x} &= -y^{2l-1} + \varepsilon P_n(x,y), \\ \dot{y} &= x^{2k-1} + \varepsilon Q_n(x,y), \end{cases} \tag{10}$$

where $P_n(x,y) = \sum_{0\leq i+j\leq n} a_{i,j}x^i y^j$ and $Q_n(x,y) = \sum_{0\leq i+j\leq n} b_{i,j}x^i y^j$. We prove the following theorem.

Theorem 3.1. *Consider the family of polynomial vector fields of type (10) and set* $N = \max\{2l-1, 2k-1, n\}$. *Then the maximum number of isolated zeroes (taking into account their multiplicity) of the Abelian integral (3) associated to it is* $\phi(2\left[\frac{n-1}{2}\right]+1)$, *where* $\phi(n) = n^2/8 + n/2 - 5/8$. *Furthermore, this maximum value can be taken with simple zeros for systems such that* $N = n$, *either* l *or* k *are* $[\frac{n+1}{2}]$ *and* $gcd(l,k) = 1$.

Proof. The Abelian integral (3) associated to the system (4) and expressed in the Lyapunov generalized polar coordinates writes as

$$I(\rho) = \rho^k \sum_{0\le i+j\le n} a_{i,j}\, {}^{l,k}\mathcal{F}_0^{2k-1+i,j} \rho^{il+jk} + \rho^l \sum_{0\le i+j\le n} b_{i,j}\, {}^{l,k}\mathcal{F}_0^{i,2l-1+j} \rho^{il+jk},$$

where $P_n(x,y) = \sum_{0\le i+j\le n} a_{i,j}x^iy^j$ and $Q_n(x,y) = \sum_{0\le i+j\le n} b_{i,j}x^iy^j$. Note that I is parameterized by ρ, where $h = \frac{\rho^{2kl}}{2kl}$. By using (i) and (ii) of Lemma 2.1 we have that the above expression can be reduced to

$$I(\rho) = \sum_{1\le 2i+1+2j+1\le n+1} \left[a_{2i+1,2j}\, {}^{l,k}\mathcal{F}_0^{2k+2i,2j} - b_{2i,2j+1}\, {}^{l,k}\mathcal{F}_0^{2i,2l+2j}\right] \times \times \rho^{(2i+1)l+(2j+1)k}. \quad (11)$$

Note that in the above expression $0 \le i,j$ and $0 \le i+j \le [(n-1)/2]$. Therefore it has at most $([(n-1)/2]+1)[(n-1)/2]/2$ summands. Let us prove that under our hypotheses all the powers of ρ are different: Assume that there are two couples of indices (i,j) and (i',j') such that $(2i+1)l + (2j+1)k = (2i'+1)l + (2j'+1)k$. Then $(i-i')l = (j'-j)k$. Since l and k are coprime and $\max(k,l) = [(n+1)/2]$ we get that $i = i'$ and $j = j'$.

Hence $I(\rho)$ is a polynomial with $([(n-1)/2]+1)[(n-1)/2]/2$ different monomials with arbitrary coefficients. By the Descartes Theorem (see [BZ], for instance) it is clear that this function has at most $([(n-1)/2]+1)$ $[(n-1)/2]/2 - 1 = \phi(2\left[\frac{n-1}{2}\right]+1)$ real roots (taking into account their multiplicity) and that this number of zeros can be attached. Therefore the theorem is proved. □

Remark 3.2. *From the proof of the above theorem the following consequences can be easily deduced:*

(1) The maximum number of real roots of $I(\rho) = 0$ *does not increase if, instead of taking* n *an odd number, we take* $n+1$.
(2) The maximum number of real roots of $I(\rho)$ *can also be taken when either* $P_n \equiv 0$ *or* $Q_n \equiv 0$.
(3) When $k = l$ *the maximum number of isolated zeroes of* $I(\rho)$ *is just* $[(n-1)/2]$.

Remark 3.3. *The above proof, restricted to a special case of system (10), gives the following result: Consider*

$$\begin{cases} \dot{x} &= -y^{2l-1} + \varepsilon R_{2l-1}(x,y), \\ \dot{y} &= x^{2k-1} + \varepsilon S_{2k-1}(x,y), \end{cases}$$

where $k \neq l$, and R_{2l-1} and S_{2k-1} are real homogeneous polynomials of degrees $2l-1$ and $2k-1$ respectively. Then the maximum number of limit cycles obtained by studying its associated Abelian integral is $l+k-1$. Of course when $k = l$ no limit cycles are obtained, because the above system is homogeneous. This result has already been proved in [CGM].

4. Proof of Theorem 1.1

In next lemma we obtain an expression of the Abelian integral associated with system studied in Theorem 1.1. We recall here the expression of the system,

$$\begin{cases} \dot{x} &= -y(1+x) + \varepsilon P_n(x,y), \\ \dot{y} &= x^{2k-1}(1+x) + \varepsilon Q_n(x,y). \end{cases} \tag{12}$$

Lemma 4.1. *By using $(\rho, \theta, 1, k)$-generalized polar coordinates, the Abelian integral associated with system (12), $I(\rho)$, can be expressed as*

$$I(\rho) = \sum_{0 \le i+2j \le n} a_{i,2j}\, f_{i,2j}(\rho) + \sum_{0 \le i+2j+1 \le n} b_{i,2j+1}\, g_{i,2j+1}(\rho), \tag{13}$$

where $f_{i,2j}(\rho)$, for $i \ge 0$, and $g_{i,2j+1}(\rho)$, for $i \ge 1$, are given by

$$f_{i,2j}(\rho) = (-1)^{i-1} \frac{1}{k^j} \sum_{s=0}^{j} (-1)^s \binom{j}{s} \times$$
$$\times \left(\mathcal{F}^0(\rho) - \sum_{l=0}^{(1+s)k-1+[i/2]} K_{2l}\rho^{2l} \right) \rho^{2jk-(1+2s)k+1}, \tag{14}$$

$$g_{i,2j+1}(\rho) = (-1)^i \frac{1}{k^{j+1}} \sum_{s=0}^{j+1} (-1)^s \binom{j+1}{s} \times$$
$$\times \left(\mathcal{F}^0(\rho) - \sum_{l=0}^{ks+[\frac{i-1}{2}]} K_{2l}\rho^{2l} \right) \rho^{(2j+1)k-2sk+1}, \tag{15}$$

and

$$g_{0,2j+1}(\rho) = \frac{1}{k^{j+1}} \sum_{s=1}^{j+1} (-1)^s \binom{j+1}{s} \left(\mathcal{F}^0(\rho) - \sum_{l=0}^{ks-1} K_{2l}\rho^{2l} \right) \rho^{(2j+1)k-2sk+1} + \frac{1}{k^{j+1}} \mathcal{F}^0(\rho)\rho^{(2j+1)k+1}. \quad (16)$$

Proof. Making the same kind of computations than in the proof of Theorem 3.1 we get that

$$I(\rho) = \rho^k \sum_{0\le i+j\le n} a_{i,j}\, {}^{1,k}\mathcal{F}_1^{2k-1+i,j}(\rho)\rho^{i+jk} + \rho \sum_{0\le i+j\le n} b_{i,j}\, {}^{1,k}\mathcal{F}_1^{i,1+j}(\rho)\rho^{i+jk}.$$

Using Lemma 2.1.(i) we get that the only nonzero terms in the former sum having $a_{i,j}$ (resp. $b_{i,j}$) as a factor have j as an even (resp. odd) number. Therefore

$$I(\rho) = \sum_{0\le i+2j\le n} a_{i,2j}\, f_{i,2j}(\rho) + \sum_{0\le i+2j+1\le n} b_{i,2j+1}\, g_{i,2j+1}(\rho),$$

where

$$f_{i,2j}(\rho) = {}^{1,k}\,\mathcal{F}_1^{2k-1+i,2j}(\rho)\rho^{i+2jk+k},$$

and

$$g_{i,2j+1}(\rho) = {}^{1,k}\,\mathcal{F}_1^{i,2j+2}(\rho)\rho^{i+(2j+1)k+1}.$$

By applying the two recurrences given in Lemma 2.1.(iv) the result follows. □

To study the number of zeroes of the Abelian Integral $I(\rho)$ associated with system (12), we give next technical result. By way of notation we write,

$$f_{i,2j}(\rho) = f_{i,2j}^{1,n}(\rho)\mathcal{F}^0(\rho) + f_{i,2j}^{2,n}(\rho), \qquad 0 \le i+2j \le n, \quad (17)$$

and

$$g_{i,2j+1}(\rho) = g_{i,2j+1}^{1,n}(\rho)\mathcal{F}^0(\rho) + g_{i,2j+1}^{2,n}(\rho), \qquad 0 \le i+2j+1 \le n, \quad (18)$$

where $f_{i,2j}^{l,n}(\rho)$ and $\rho^{k-1}g_{i,2j}^{l,n}(\rho)$, $l = 1,2$, are real polynomials in the variable ρ.

Lemma 4.2. *Let the Abelian integral $I(\rho)$ associated with system (12) be given by (13). Let $f_{i,2j}(\rho)$ (resp. $g_{i,2j+1}(\rho)$) be given by the expression (17) (resp. (18)). The following holds:*

(i) $f_{2i+1,2j}(\rho) = -f_{2i,2j}(\rho)$, *(resp.* $g_{2i,2j+1}(\rho) = -g_{2i-1,2j+1}(\rho)$,*) for all i and j for which both functions are defined.*

(ii) The function $f_{0,0}(\rho)$ has no zeros in $(0,1)$. In fact $f_{0,0}(\rho)|_{(0,1)} < 0$.

Proof. The proof of (i) is trivial. To prove (ii) recall that $f_{0,0}(\rho) = \rho^k \left({}^{1,k}\mathcal{F}_1^{2k-1,0} \right) = \rho^k \int_0^{T_{1,k}} \frac{\mathrm{Cs}^{2k-1}(\theta)}{1+\rho \mathrm{Cs}(\theta)}\, d\theta$. Since $f_{0,0}(0) = 0$ and by using the fact that $\frac{d}{d\rho} \int_0^{T_{1,k}} \frac{\mathrm{Cs}^{2k-1}(\theta)}{1+\rho \mathrm{Cs}(\theta)}\, d\theta = -\int_0^{T_{1,k}} \frac{\mathrm{Cs}^{2k}(\theta)}{(1+\rho \mathrm{Cs}(\theta))^2}\, d\theta < 0$, for all $\rho \in (0,1)$, the proof is ended. □

Because previous lemma, next result is useful to obtain the independency of the functions $f_{2i,2j}(\rho)$, $g_{2i+1,2j+1}(\rho)$ and $g_{0,2j+1}(\rho)$. In order to prove it, let us consider the Abelian integral $I(\rho)$ associated with system (12) given by (13). From expression (17) we define

$$f^{1,n}(\rho) = \sum_{(i,j)\in\mathcal{T}} a_{2i,2j} f^{1,n}_{2i,2j}(\rho), \tag{19}$$

$$g^{1,n}(\rho) = \sum_{(i,j)\in\mathcal{S}} b_{2i+1,2j+1} g^{1,n}_{2i+1,2j+1}(\rho) + \sum_{j=0}^{[\frac{n-1}{2}]} b_{0,2j+1} g^{1,n}_{0,2j+1}(\rho), \tag{20}$$

and

$$f^{2,n}(\rho) = \sum_{(i,j)\in\mathcal{T}} a_{2i,2j} f^{2,n}_{2i,2j}(\rho), \tag{21}$$

$$g^{2,n}(\rho) = \sum_{(i,j)\in\mathcal{S}} b_{2i+1,2j+1} g^{2,n}_{2i+1,2j+1}(\rho) + \sum_{j=0}^{[\frac{n-1}{2}]} b_{0,2j+1} g^{2,n}_{0,2j+1}(\rho), \tag{22}$$

where $\mathcal{T} = \{(i,j)\,|\,i,j \in \mathbb{Z},\, 0 \le i,j \le [\frac{n}{2}],\, 0 \le 2i+2j \le n\}$ and $\mathcal{S} = \{(i,j)\,|\,i,j \in \mathbb{Z},\, 0 \le i,j \le [\frac{n}{2}]-1,\, 0 \le 2i+2j+2 \le n\}$.

Lemma 4.3. *Let us consider the expression of $f^{1,n}(\rho)$ and $g^{1,n}(\rho)$ given by (19) and (20) and suppose $n = 2k$. Then,*

$$f^{1,n}(\rho)+g^{1,n}(\rho) = -\rho \left[\sum_{p=0}^{\frac{n}{2}} \rho^{2kp-k} \sum_{s=\max\{0,-p+1\}}^{\frac{n}{2}-p} \frac{(-1)^s}{k^{p+s}} \binom{p+s}{s} C_{p,s} + \rho^{-k} C_{0,0} \right], \tag{23}$$

where $C_{p,s} = \sum_{i=0}^{\frac{n}{2}-p-s} a_{2i,2(p+s)} + b_{2i+1,2(p+s)-1} - b_{0,2(p+s)-1}$ if $p+s \neq 0$, and $C_{0,0} = \sum_{i=0}^{\frac{n}{2}} a_{2i,0}$.

Proof. The proof follows by observing that in the expression (23), each monomial in ρ^{2kp-k} corresponds with the gathering of all monomials in $\rho^{2jk-(1+2s)k}$ with $j - s = p$, in the expressions (14), (15) and (16) having $\mathcal{F}^0(\rho)$ as a common factor. More specifically, if we put $j - s = p$ in the expression (14) (resp. (15) and (16)) then $0 \le p \le \frac{n}{2}$ (resp. $-1 \le p \le \frac{n}{2} - 1$) and, fixed one of these values of p, then $0 \le s \le \frac{n}{2} - p$ (resp. $\max\{0, -p\} \le s \le \frac{n}{2} - p - 1$). □

Lemma 4.4. *Consider the functions $f_{i,2j}(\rho)$ and $g_{i,2j+1}(\rho)$, introduced in Lemma 4.1.*

(i) If $n \leq 2k$, the set

$$\{f_{2i,2j}(\rho)\}_{0\leq 2i+2j\leq n}\bigcup\{g_{2i+1,2j+1}(\rho)\}_{0\leq 2i+2j+2\leq n}\bigcup\{g_{0,2j+1}(\rho)\}_{0\leq 2j+1\leq n},$$

has at most $\frac{n^2+10n}{8}$ functions linearly independent.

(ii) If $n \leq 2k$, the set of $\frac{n^2+6n+8}{8}$ functions, $\{f_{2i,2j}(\rho)\}_{0\leq 2i+2j\leq n}$, are linearly independent.

Proof. Let us prove part (i). Suppose $n = 2k$. By using expressions (19), (20), (21) and (22) we get

$$\sum_{0\leq 2i+2j\leq n} a_{2i,2j}\, f_{2i,2j}(\rho) + \sum_{0\leq 2i+2j+2\leq n} b_{2i+1,2j+1}\, g_{2i+1,2j+1}(\rho) + \sum_{0\leq 2j+1\leq n} b_{0,2j+1}\, g_{0,2j+1}(\rho) = = f^{2,n}(\rho)+g^{2,n}(\rho)+\mathcal{F}^0(\rho)(f^{1,n}(\rho)+g^{1,n}(\rho)) = 0. \tag{24}$$

Using the fact that $\mathcal{F}^0(\rho)$ is no a rational function (Lemma 2.1.(v)) we get $f^{2,n}+g^{2,n}\equiv f^{1,n}+g^{1,n}\equiv 0$. Hence, because of the relations $C_{p-1,s+1} = C_{p,s}$, $p = 1,\dots,\frac{n}{2}$, $s = 0,\dots,\frac{n}{2}-p$, from Lemma 4.3, the fact that $f^{1,n}+g^{1,n}\equiv 0$ gives the following $\frac{n}{2}+1$ relations

$$\sum_{i=0}^{\frac{n}{2}-p} a_{2i,2p}+b_{2i+1,2p-1}-b_{0,2p-1} = 0, \qquad \sum_{i=0}^{\frac{n}{2}} a_{2i,0} = 0, \tag{25}$$

where $1\leq p\leq \frac{n}{2}$. Now let us argue about function $f^{2,n}(\rho)+g^{2,n}(\rho)$. From expressions (14), (15), (16) we have that

$$\begin{aligned}
&f^{2,n}(\rho)+g^{2,n}(\rho) = \\
&= \rho\sum_{j=0}^{\frac{n}{2}}\frac{1}{k^j}\sum_{i=0}^{\frac{n}{2}-j} a_{2i,2j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}\rho^{2jk-(1+2s)k}\left(\sum_{l=0}^{ks-1}K_{2l}\rho^{2l}+\sum_{l=ks}^{(1+s)k-1+i}K_{2l}\rho^{2l}\right)\\
&+\rho\sum_{j=1}^{\frac{n}{2}}\frac{1}{k^j}\sum_{i=0}^{\frac{n}{2}-j} b_{2i+1,2j-1}\sum_{s=0}^{j}(-1)^s\binom{j}{s}\rho^{2jk-(1+2s)k}\left(\sum_{l=0}^{ks-1}K_{2l}\rho^{2l}+\sum_{l=ks}^{ks+i}K_{2l}\rho^{2l}\right)\\
&-\rho\sum_{j=1}^{\frac{n}{2}}\frac{1}{k^j}b_{0,2j-1}\sum_{s=1}^{j}(-1)^s\binom{j}{s}\rho^{2jk-(1+2s)k}\left(\sum_{l=0}^{ks-1}K_{2l}\rho^{2l}+\sum_{l=ks}^{(1+s)k-1}K_{2l}\rho^{2l}\right).
\end{aligned}$$

Using elementary transformations and by imposing the $\frac{n}{2}+1$ relations given by (25), we get

$$\begin{aligned} f^{2,n}(\rho)+g^{2,n}(\rho) &= \rho \sum_{i=0}^{\frac{n}{2}} a_{2i,0}\,\rho^{-k} \sum_{l=k}^{k-1+i} K_{2l}\rho^{2l} \\ &+ \rho \sum_{j=1}^{\frac{n}{2}} \frac{1}{k^j} \sum_{i=0}^{\frac{n}{2}-j} a_{2i,2j} \sum_{s=0}^{j} (-1)^s \binom{j}{s} \rho^{2jk-(1+2s)k} \sum_{l=ks}^{(1+s)k-1+i} K_{2l}\rho^{2l} \\ &+ \rho \sum_{j=1}^{\frac{n}{2}} \frac{1}{k^j} \sum_{i=0}^{\frac{n}{2}-j} b_{2i+1,2j-1} \sum_{s=0}^{j} (-1)^s \binom{j}{s} \rho^{2jk-(1+2s)k} \sum_{l=ks}^{ks+i} K_{2l}\rho^{2l}. \end{aligned}$$

In fact, we can rewrite $f^{2,n}(\rho)+g^{2,n}(\rho)$ in the following equivalent way,

$$\begin{aligned} & f^{2,n}(\rho)+g^{2,n}(\rho) = \\ & = \rho \sum_{l=0}^{\frac{n}{2}-1} \Bigg[K_{2(k+l)} \sum_{i=0}^{\frac{n}{2}-l-1} a_{2(i+l+1),0} \\ & + \frac{1}{k} \sum_{s=0}^{1} (-1)^s K_{2(ks+l)} \left(\sum_{i=0}^{\frac{n}{2}-1} a_{2i,2} + \sum_{i=0}^{\frac{n}{2}-1-l} b_{2(i+l)+1,1} \right) \Bigg] \rho^{k+2l} \\ & + \rho \sum_{j=1}^{\frac{n}{2}-1} \frac{1}{k^j} \sum_{l=0}^{\frac{n}{2}-j-1} \Bigg[\sum_{s=0}^{j} (-1)^s \binom{j}{s} K_{2((1+s)k+l)} \sum_{i=0}^{\frac{n}{2}-j-l-1} a_{2(l+1+i),2j} \\ & + \frac{1}{k} \sum_{s=0}^{j+1} (-1)^s \binom{j+1}{s} K_{2(ks+l)} \sum_{i=0}^{\frac{n}{2}-j-1} a_{2i,2(j+1)} \\ & + \frac{1}{k} \sum_{s=0}^{j+1} (-1)^s \binom{j+1}{s} K_{2(ks+l)} \sum_{i=0}^{\frac{n}{2}-j-l-1} b_{2(i+l)+1,2j+1} \Bigg] \rho^{2jk+k+2l} \\ & + \frac{1}{k} \sum_{l=\frac{n}{2}-j}^{\frac{n}{2}-1} \Bigg[\sum_{s=0}^{j+1} (-1)^s \binom{j+1}{s} K_{2(ks+l)} \sum_{i=0}^{\frac{n}{2}-j-1} a_{2i,2(j+1)} \Bigg] \rho^{2jk+k+2l}. \end{aligned} \tag{26}$$

From last expression it is easy to check that all powers of ρ in $f^{2,n}(\rho)+g^{2,n}(\rho)$ are different if $n \le 2k$. By imposing $f^{2,n}(\rho)+g^{2,n}(\rho)\equiv 0$ and by using Lemma 2.2, we get $\frac{n^2+6n-8}{8}$ linear relations on the coefficients of the expression (13).

Hence, the set of $\frac{n^2+10n}{8}$ conditions given by $\{f^{1,n}(\rho) + g^{1,n}(\rho) \equiv 0, f^{2,n}(\rho) + g^{2,n}(\rho) \equiv 0\}$ on the $\frac{n^2+6n+4}{4}$ unknowns,

$$\{a_{2i,2j}\}_{0\le 2i+2j\le n} \bigcup \{b_{2i+1,2j+1}\}_{0\le 2i+2j+2\le n} \bigcup \{b_{0,2j+1}\}_{0\le 2j+1\le n},$$

give us a homogeneous system of linear equations. Hence part (i) follows.

To prove part (ii), let us suppose $n = 2k$. From expressions (25) and (26), by taking all the coefficients $\{b_{2i+1,2j+1}\}_{0\le 2i+2j+2\le n} \bigcup \{b_{0,2j+1}\}_{0\le 2j+1\le n}$ zero and by using Lemma 2.2, we conclude that the set of conditions given by $\{f^{1,n}(\rho) \equiv 0, f^{2,n}(\rho) \equiv 0\}$ on the unknowns $\{a_{2i,2j}\}_{0\le 2i+2j\le n}$ is the following system of linear equations

$$\text{(i)} \quad \sum_{i=0}^{\frac{n}{2}-p} a_{2i,2p} = 0, \qquad \text{(ii)} \quad \sum_{i=0}^{\frac{n}{2}-j-l-1} a_{2(l+1+i),2j} = 0, \tag{27}$$

where $0 \le p \le \frac{n}{2}$, $j \in \{0, \ldots, \frac{n}{2} - 1\}$ and $l \in \{0, \ldots, \frac{n}{2} - j - 1\}$. For an arbitrary but fixed value of j, the $\frac{n}{2} - j$ relations given by (27)-(ii) imply $a_{2(l+1),2j} = 0$, for $l \in \{0, \ldots, \frac{n}{2} - j - 1\}$. Hence, $a_{2i,2j} = 0$, for $i \in \{1, \ldots, \frac{n}{2} - j\}$ and $j \in \{0, \ldots, \frac{n}{2} - 1\}$. Furthermore, from expression (27)-(i), we have $a_{0,2j} = 0$ for $j \in \{0, \ldots, \frac{n}{2}\}$. In other words, by imposing the set of conditions given by (27), we obtain a homogeneous system of linear equations which has the trivial solution as the unique solution. Hence, if $n = 2k$ the functions $\{f_{2i,2j}\}_{(i,j)\in\mathcal{T}}$ are linearly independent. □

The following lemma will be useful in the proof of Theorem 1.1.

Lemma 4.5. *Consider $p+1$ linearly independent functions $f_i : U \subset \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \ldots, p$.*

(i) Given p arbitrary values $x_i \in U$, $i = 1, 2, \ldots, p$ there exist $p + 1$ constants C_i, $i = 0, 1, \ldots, p$ such that

$$f(x) := \sum_{i=0}^{p} C_i f_i(x) \tag{28}$$

is not the zero function and $f(x_i) = 0$ for $i = 1, 2, \ldots, p$.

(ii) Furthermore, if all f_i are analytical functions on U and there exists $j \in \{0, 1, \ldots, p\}$ such that $f_j|_U$ has constant sign, it is possible to get an f given by (28), such that it has at least p simple zeroes in U.

Proof. From expression (28), by imposing $f(x_i) = 0$ for these p arbitrary values x_i, we get an homogeneous linear system in the variables C_i, $i = 0, 1, \ldots, p$. This system has solutions different from the zero solution. Furthermore the independence of the functions f_i proves that the function f is not the zero function.

To prove (ii), from part (i) and using the fact that f is analytic in U, we can consider a compact subset K of U on which f has exactly p zeroes. Let us call them x_i, $i = 1, \ldots, p$. On the subset K let us call p_1 the number of zeroes of f on which f has a local minimum, p_2 the number of zeroes of f on which f has a local maximum, p_3 the number of zeroes of f on which f' vanishes but where f has neither a local minimum nor a local maximum and p_4 the number of zeroes of f on which f has a simple zero. Then $p = p_1 + p_2 + p_3 + p_4$. Let us suppose, for instance, $p_1 \geq p_2$ and x_1 a multiple zero of even order m of f, where f has a local minimum. Hence, $f(x) = a(x - x_1)^m + O((x - x_1)^{m+1})$, where $a > 0$ for all x in a open subset, $V \subset K$. From the hypothesis, there exist $j \in \{0, 1, \ldots, p\}$ such that $f_j|_U > 0$. Then, for all $x \in V$,

$$f_j(x) = \sum_{i=0}^{m} b_i(x - x_1)^i + O((x - x_1)^{m+1})),$$

where $b_0 > 0$.

Let us define

$$f_\varepsilon(x) = f(x) - \varepsilon f_j(x),$$

for all $x \in V$. About the zeroes of f_ε, for a small enough $\varepsilon > 0$, we will prove that for each zero of f at which f has a local minimum we get two simple zeroes of f_ε, in the neighbourhood V. In fact, the function $F(\varepsilon, z) = f_\varepsilon(z + x_1)$ is an analytical function such that $F(0, 0) = 0$ and $\frac{\partial F}{\partial \varepsilon}(0, 0) \neq 0$. Hence, from the Implicit Function Theorem, in a neighbourhood of $(0, 0)$ we can write $\varepsilon = \varepsilon(z) = \frac{a}{b_0} z^m + O(z^{m+1})$ in such a way that $F(\varepsilon(z), z) = 0$ on this neighbourhood. From the equality $\varepsilon = \varepsilon(z)$ we obtain $z = z(\varepsilon) = \pm \sqrt[m]{\frac{b_0}{a}}\, \varepsilon^{\frac{1}{m}} + O(\varepsilon^\alpha)$, $\alpha < \frac{1}{m}$, $\alpha \in \mathbb{R}$, what assure two simple zeroes in V.

For the multiple zeroes of f of odd order, using the same technique, it is possible to prove that we get one simple zero of f_ε for a small enough $\varepsilon > 0$.

As a consequence, using the fact that $p_1 \geq p_2$, the number of simple zeroes of f_ε in U is, at least, p. □

Proof of Theorem 1.1. Lemma 4.1 gives the expression (13), of the Abelian integral, $I(\rho)$, associated to system (12). Let us consider the case $n = 2k$ (the case n odd follows with similar arguments). Consider also $Q_n(x, y) \equiv 0$ and all $a_{2i+1,2j} = 0$. From Lemma 4.2.(i), we can write

$$I(\rho) = \sum_{0 \leq 2i+2j \leq n} a_{2i,2j}\, f_{2i,2j}(\rho). \tag{29}$$

Recall that from Lemma 4.4.(ii), the set of $p + 1 = \frac{n^2+6n+8}{8}$ functions $f_{2i,2j}(\rho)$, for $0 \leq 2i + 2j \leq n$ are linearly independent.

From Lemma 4.5.(i), we can fix p arbitrary values x_i, $x_i \in (0,1)$, $i = 1, 2, \ldots, p$, for which there exist values $a_{2i,2j}$, $0 \leq 2i+2j \leq n$, such that $I(\rho)$ has these p zeroes and it is not the zero function. If not all these zeroes are simple, then using the fact that the functions $f_{2i,2j}(\rho)$ are analytical (see Lemma 2.1.(vi)) and the fact that $f_{0,0}(\rho)|_{(0,1)} < 0$ (see Lemma 4.2.(ii)), from Lemma 4.5.(ii) we can get new values $a_{2i,2j}$, $0 \leq 2i+2j \leq n$, for which $I(\rho)$ has at least p simple zeroes in $(0,1)$. As a consequence (see for instance [R]), we obtain $p = \phi(n+1)$ limit cycles for system (12), as we wanted to prove. □

Remark 4.6. *(i) To get the number of limit cycles stated in Theorem 1.1 we just have considered the case* $Q_n(x,y) \equiv 0$. *If we take* $Q_n(x,y) \not\equiv 0$, *and for instance* n *even, by using Lemma 4.4.(i) we get that the maximum number of limit cycles for system (12) that we could produce by using the method developed in this paper is* $-1 + (n^2 + 10n)/8$. *Notice that if this number were attained we would have obtained just* $1 + n/2$ *limit cycles more that in the case* $Q_n(x,y) \equiv 0$. *Since the possible increase of the number of limit cycles obtained is so small we have decided do not study this problem here.*

(ii) In Theorem 1.1 we have not been able to give an upper bound for the number of zeroes of $I(\rho)$. *Nevertheless, if we just consider the case* $k = 1$, *with an arbitrary* n, *by using Lemma 2.1.(v) we have that the function* $\mathcal{F}^0(\rho)$ *can be given explicitly. Then, by making similar computations to the ones done to prove Theorem 1.1, we get a compact expression of* $I(\rho)$. *By using the change of variables* $\sqrt{1-\rho^2} = R$ *the study of the zeros of* $I(\rho)$ *can be reduced to the study of the zeros of a polynomial function in the variable* R. *In particular it can be proved that* $I(\rho)$ *has at most* n *zeros and that this number of zeros can be attached. This result has been already obtained in* [LPR].

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