On the homotopy type of p-completions of infra-nilmanifolds

Gemma Bastardas and An Descheemaeker*

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Abstract

Infra-nilmanifolds are compact $K(G,1)$-manifolds with G a torsion-free, finitely generated, virtually nilpotent group. Motivated by previous results of various authors on p-completions of $K(G,1)$-spaces with G a finite or a nilpotent group, we study the homotopy type of p-completions of infra-nilmanifolds, for any prime p. We prove that the p-completion of an infra-nilmanifold is a virtually nilpotent space which is either aspherical or has infinitely many nonzero homotopy groups. The same is true for p-localization. Moreover, we show by means of examples that rationalizations of infra-nilmanifolds may be elliptic or hyperbolic.

1 Introduction

In [15], McGibbon and Neisendorfer used Miller's solution of the Sullivan conjecture to prove that, for a simply connected space X and a prime p such that $\tilde{H}_k(X;\mathbb{Z}/p)$ is nonzero for at least one and at most a finite number of values of k, infinitely many homotopy groups of X contain a subgroup of order p. Although this theorem was already improved in [18] and [23] by relaxing the 1-connectivity condition, we here provide yet another generalization, holding for certain nilpotent spaces. This version serves our purpose: for a fixed prime p, understand the homotopy type of the p-completion, and on second hand of the p-localization, of classifying spaces of finitely

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generated virtually nilpotent groups (i.e. extensions of a finitely generated nilpotent group by a finite group).

Our interest in the topic originated from the fact that for the simplest cases, results were known. Indeed, if $X = K(N, 1)$ with N a finitely generated nilpotent group, then it follows from [3] that the p-completion $X_p^\wedge$ is aspherical with fundamental group the p-profinite completion of N. For classifying spaces of finite groups, the p-completion is either aspherical or has infinitely many nonzero homotopy groups [14], all being finite p-groups [3, VII.4.3]. Analogous results hold for p-localization [4], [6].

As for the p-completion and p-localization of virtually nilpotent aspherical spaces in general, no such complete determinations of the homotopy type were established up till now. However, the area is not entirely unexplored. For instance, in [8], criteria are developed to check preservation of asphericity under p-localization for aspherical spaces having a torsion-free, finitely generated, virtually nilpotent group, i.e. infra-nilmanifolds. Several examples are computed, invoking the problem to determine the homotopy type in case asphericity is not preserved. May there, as in the finite case, arise an infinite number of nonzero homotopy groups? Are the resulting spaces again finitely covered by nilpotent spaces? To these questions, already raised in [5], we find an answer in the present paper.

Inspiring source for our approach is the main theorem in [9], which implies that for each virtually nilpotent space X with $\pi_1(X)$ finitely generated and having a nilpotent p-localization $X_{(p)}$, the arithmetic square obtained by rationalizing the natural map $X_{(p)} \to X_p^\wedge$ is a homotopy fibre square. In other words, information on $X_{(p)}$ can be retrieved from the p-completion $X_p^\wedge$, from the rationalization $X_{(0)}$ and from certain coherence data.

We apply this idea to infra-nilmanifolds. We study the number of homotopy groups of their p-completion and use this to determine the quantity of homotopy groups of their p-localization. Both p-completions and p-localizations are virtually nilpotent spaces which are either aspherical or have infinitely many nonzero homotopy groups (see Sections 3 and 4). Furthermore, we illustrate how finiteness of the higher homotopy groups of the p-localization can be studied by rationalizing. More explicitly, in Section 5, we describe an infinite family of infra-nilmanifolds which are rationally homotopy equivalent to a sphere. If X is of this kind and p is a prime number such that $X_{(p)}$ is nilpotent, this implies that $\pi_k(X_{(p)})$ is finite for k large enough. However, we show that there also exist infra-nilmanifolds whose p-localization is nilpotent and has infinitely many infinite homotopy groups.

2 Preliminaries

A group is virtually nilpotent if and only if it contains a nilpotent normal subgroup of finite index. Hence, each virtually nilpotent group G fits into a group extension $1 \to \mathrm{Fitt}(G) \to G \to G/_{\mathrm{Fitt}(G)} \to 1$, where the Fitting subgroup $\mathrm{Fitt}(G)$ of G is maximal nilpotent amongst all normal subgroups of G and the Fitting quotient $G/_{\mathrm{Fitt}(G)}$ is a finite group. If $\mathrm{Fitt}(G)$ is finitely generated, torsion–free, maximal nilpotent, of Hirsch length n and of finite index in G, then G is an n-dimensional almost-crystallographic group. If G is moreover torsion–free, then G is called an almost-Bieberbach group. Almost-crystallographic (almost-Bieberbach) groups are natural generalizations of the crystallographic (Bieberbach) groups [7].

If a nilpotent group G acts on a group N, then we say that this action is nilpotent, or equivalently, that G acts nilpotently on N, if $N \rtimes G$ is a nilpotent group. A space X (by which we mean a connected CW-complex) is nilpotent if and only if $\pi_1(X)$ is nilpotent and acts nilpotently on the higher homotopy groups. If for each $k \geq 2$, $\pi_1(X)$ has a nilpotent normal subgroup of finite index acting nilpotently on $\pi_k(X)$, then X is called a virtually nilpotent space. Interesting examples are the classifying spaces of the almost-Bieberbach groups, called infra-nilmanifolds. In case of Bieberbach groups these spaces are flat Riemannian manifolds.

Fix a prime p. We denote the p-profinite completion of a group G by $G^{\wedge}_p$. We say that G is p-local if nth-roots exist and are unique for all integers n coprime to p (write $n \in p'$). The p-localization of G is given by a p-local group $G_{(p)}$ and a homomorphism $l_G\colon G \to G_{(p)}$ which is initial among all homomorphisms from G into p-local groups [19]. If X is a space, then we write $X^{\wedge}_p$ for its Bousfield–Kan p-completion [3]. In case $H_k(X; \mathbb{Z}/p)$ is finite for all k, then $X^{\wedge}_p$ coincides with Sullivan's p-profinite completion of X [22] and $\pi_1(X^{\wedge}_p) \cong \pi_1(X)^{\wedge}_p$ [2], [3]. We say that X is p-local if and only if, for all $k \geq 2$, $\pi_k(X) \rtimes \pi_1(X)$ is a p-local group. For every space X, there exists a unique (up to homotopy) p-local space $X_{(p)}$ and a map $l_X : X \to X_{(p)}$ which is initial (in the homotopy category) among all maps from X into p-local spaces. The induced map $\pi_1(X) \to \pi_1(X_{(p)})$ is p-localization in the category of groups [6].

A map $f\colon X \to Y$ is called a p-equivalence if and only if $X_{(p)} \simeq Y_{(p)}$. Each p-equivalence induces isomorphisms on homology with trivial $\mathbb{Z}/p$- or $\mathbb{Z}_{(p)}$-coefficients [6]. Since the Bousfield–Kan p-completion functor turns $H_*(-; \mathbb{Z}/p)$-equivalences into homotopy equivalences, this implies that $X^{\wedge}_p \simeq (X_{(p)})^{\wedge}_p$. Also, if we denote the homological localization of a space X with respect to the ring $\mathbb{Z}_{(p)}$ by $X_{H\mathbb{Z}_{(p)}}$ [1], then $X^{\wedge}_p \simeq (X_{H\mathbb{Z}_{(p)}})^{\wedge}_p$.

3 On the number of homotopy groups

The object of this section is to study the homotopy type of p-completed and p-localized classifying spaces of finitely generated virtually nilpotent groups. The finite case already shows that spaces of these forms need not be aspherical. Indeed, since p-completion and p-localization coincide for $K(F,1)$-spaces with F finite, [4] tells us that $K(F,1)^\wedge_p$ is aspherical if and only if F is p-nilpotent (i.e., the subgroup $i_{p'}(F)$ of F which is generated by all p'-torsion elements is a p'-torsion group). Combining this with [14, Th. 1.1.4] we obtain:

Proposition 3.1 *If $X = K(F,1)$ with F a finite group, then $X^\wedge_p$ is either $K(F^\wedge_p,1)$ or it has infinitely many nonzero homotopy groups.* □

If one considers classifying spaces of finitely generated nilpotent groups, then the situation is different. There, p-completion and p-localization do not coincide in general. However, it is well known that both preserve asphericity [3, Prop. 2.6], [6]. This result together with Proposition 3.1 lead to ask whether the p-completion, respectively p-localization, of $K(G,1)$-spaces with G a finitely generated, virtually nilpotent group is again aspherical or has infinitely many nonzero homotopy groups. We here prove the truth of this assertion in case G is an almost-Bieberbach group (Theorem 3.5) and in case G is an almost-crystallographic group having a p-nilpotent Fitting quotient (Theorem 3.7).

Fundamental in our reasoning is the following theorem, which generalizes the Serre Conjecture Theorem of McGibbon–Neisendorfer [15, Th. 1] to a certain class of nilpotent spaces. Its proof proceeds along the same lines as the one in [15], and is based on Miller's theorem [16, Th. C], saying that for nilpotent spaces X with bounded mod p homology, the space of pointed maps from $K(\mathbb{Z}/p,1)$ to X is weakly contractible. Another essential ingredient is the notion of homotopy groups with $\mathbb{Z}/p$-coefficients. For a space X, a prime p and integers $k \geq 2$, these are defined as $\pi_k(X;\mathbb{Z}/p) = [S^{k-1} \cup_p e^k, X]$. If $k > 2$, then $\pi_k(X;\mathbb{Z}/p)$ is a group, related to the ordinary homotopy groups of X by a short exact sequence

$$\pi_k(X) \otimes \mathbb{Z}/p \rightarrowtail \pi_k(X;\mathbb{Z}/p) \twoheadrightarrow \mathrm{Tor}(\pi_{k-1}(X), \mathbb{Z}/p). \qquad (*)$$

If $k = 2$, then $(*)$ is an exact sequence of pointed sets where $\mathrm{Tor}(\pi_1(X), \mathbb{Z}/p)$ is defined as the set of p-torsion elements in $\pi_1(X)$. Exactness should be interpreted as follows: the preimages of Tor are orbits of the $\pi_2(X)$-action on $\pi_2(X;\mathbb{Z}/p)$. Further details can be found in [17].

Theorem 3.2 *Fix a prime number p. Let X be a p-local nilpotent space whose higher homotopy groups are finitely generated $\mathbb{Z}_{(p)}$-modules. If*

(1) $\pi_k(X;\mathbb{Z}/p) \neq 0$ for some $k > 1$ and

(2) $H_k(X;\mathbb{Z}/p) = 0$ for all k sufficiently large,

then for infinitely many values of k, $\pi_k(X)$ contains a subgroup of order p.

Proof. Assume $\pi_k(X) \neq 0$ for at most a finite number of values of k. From the short exact sequence $(*)$, it follows that $\pi_k(X;\mathbb{Z}/p)$ is nontrivial for only finitely many k. By (1), there exists a largest integer m, greater than 1, such that $\pi_m(X;\mathbb{Z}/p) \neq 0$. Thus, $\pi_k(X) \otimes \mathbb{Z}/p = 0$ if $k > m$ and $\mathrm{Tor}(\pi_k(X), \mathbb{Z}/p) = 0$ if $k \geq m$. Since the higher homotopy groups of X are finitely generated $\mathbb{Z}_{(p)}$-modules, this implies that $\pi_k(X) = 0$ for all $k > m$.

Suppose first that $\pi_m(X) \otimes \mathbb{Z}/p \neq 0$. Since $\mathrm{Tor}(\pi_m(X), \mathbb{Z}/p) = 0$, the p-local group $\pi_m(X)$ contains $\mathbb{Z}_{(p)}$ as a direct summand. Hence, there exists an essential map $f_2 : K(\mathbb{Z}_{(p)}, 2) \to K(\pi_m(X), 2)$. The composition

$$g_2 : K(\mathbb{Z}/p, 1) \to K(\pi_m(X), 2)$$

of f_2 with a generator of $H^2(K(\mathbb{Z}/p, 1); \mathbb{Z}_{(p)})$ is also essential. We want to lift g_2 up to the Postnikov tower of the universal cover $\Omega^{m-2}X\langle 1\rangle$ of $\Omega^{m-2}X$ to a map

$$g_\infty : K(\mathbb{Z}/p, 1) \to \Omega^{m-2}X\langle 1\rangle.$$

The obstructions to this lifting vanish since $\pi_k(X) = 0$ and consequently $H^*(\mathbb{Z}/p; \pi_k(X)) = 0$ if $k > m$. Hence, g_∞ exists. One easily checks that g_∞ and its composition with the covering projection back into $\Omega^{m-2}X$ are essential. This last assertion contradicts Miller's theorem.

Now assume $\pi_m(X) \otimes \mathbb{Z}/p = 0$. Since $\mathrm{Tor}(\pi_m(X), \mathbb{Z}/p) = 0$ and $\pi_m(X)$ is a finitely generated $\mathbb{Z}_{(p)}$-module, it follows that $\pi_m(X) = 0$. Consequently, by $(*)$, $\mathrm{Tor}(\pi_{m-1}(X), \mathbb{Z}/p) \neq 0$. Thus, $\mathbb{Z}/p$ is a subgroup of $\pi_{m-1}(X)$, which provides us with an essential map

$$f_1 : K(\mathbb{Z}/p, 1) \to K(\pi_{m-1}(X), 1).$$

We want to lift this map up to the Postnikov tower of $\Omega^{m-2}X$ to a map

$$f_\infty : K(\mathbb{Z}/p, 1) \to \Omega^{m-2}X.$$

The obstructions to this lifting are in the twisted groups $H^*(\mathbb{Z}/p, \pi_k(X))$ where $k \geq m$. These groups are zero, because $\pi_k(X) = 0$ for all $k \geq m$. Hence, there exists an essential map f_∞, which contradicts Miller's theorem.

Finally suppose $\mathrm{Tor}(\pi_k(X), \mathbb{Z}/p) \neq 0$ for only a finite number of values of k. Then there exists an integer m large enough such that for all $q > 0$, $\mathrm{Tor}(\pi_{m+q}(X), \mathbb{Z}/p) \cong \mathrm{Tor}(\pi_q(\Omega^m X), \mathbb{Z}/p) = 0$, and $\pi_{m+2}(X) \otimes \mathbb{Z}/p \cong \pi_2(\Omega^m X) \otimes \mathbb{Z}/p \neq 0$. Since $\pi_2(\Omega^m X)$ is a finitely generated $\mathbb{Z}_{(p)}$-module, this implies the existence of an injection $\mathbb{Z}_{(p)} \rightarrowtail \pi_2(\Omega^m X)$, and hence of an essential map

$$f_2 \colon K(\mathbb{Z}_{(p)}, 2) \to K(\pi_2(\Omega^m X), 2).$$

We want to lift f_2 up to the Postnikov tower of $\Omega^m X\langle 1\rangle$ to an essential map

$$f_\infty \colon K(\mathbb{Z}_{(p)}, 2) \to \Omega^m X\langle 1\rangle.$$

At the nth-stage we have the following diagram:

$$\begin{array}{ccccc} & & P_{n+1}\Omega^m X\langle 1\rangle & & \\ & & \downarrow & & \\ K(\mathbb{Z}_{(p)}, 2) & \xrightarrow{f_n} & P_n\Omega^m X\langle 1\rangle & \xrightarrow{k} & K(\pi_{n+1}(\Omega^m X\langle 1\rangle), n+2), \end{array}$$

where f_n denotes some lift of f_2. Since $\Omega^m X\langle 1\rangle$ is an H-space, the k-invariant and hence $k \circ f_n$ is taken to zero under rationalization. Furthermore, since $\pi_{n+1}(\Omega^m X\langle 1\rangle)$ is torsion-free, it embeds into $\pi_{n+1}(\Omega^m X\langle 1\rangle)\otimes\mathbb{Q}$ and the induced map

$$H^*(K(\mathbb{Z}_{(p)}, 2); \pi_{n+1}(\Omega^m X\langle 1\rangle)) \to H^*(K(\mathbb{Z}_{(p)}, 2); \pi_{n+1}(\Omega^m X\langle 1\rangle) \otimes \mathbb{Q})$$

is injective. This implies that the map $k \circ f_n$ represents the zero class in $H^*(K(\mathbb{Z}_{(p)}, 2); \pi_{n+1}(\Omega^m X\langle 1\rangle))$, or equivalently that f_{n+1}, the next lift of f_2, exists. Hence, also f_∞ exists. Its composition with a generator of $H^2(K(\mathbb{Z}/p, 1); \mathbb{Z}_{(p)})$ and with the covering projection $\Omega^m X\langle 1\rangle \to \Omega^m X$ is essential. This contradicts Miller's theorem. □

In addition to the above theorem, we need two more lemmas in order to prove our main results.

Lemma 3.3 *If $Y \to X \to K(Q, 1)$ is a fibration with Q a finite p-group, then $Y_p^\wedge \to X_p^\wedge \to K(Q, 1)$ is also a fibration.*

Proof. Recall from [3, p. 215] that every action of a finite p-group on a $\mathbb{Z}/p$-module is nilpotent. Then apply [3, II.5.1]. □

In case of p-localization, the analogue of Lemma 3.3 was obtained in [5]:

Lemma 3.4 *If $Y \to X \to K(Q, 1)$ is a fibration with Q a finite p-group, then $Y_{(p)} \to X_{(p)} \to K(Q, 1)$ is also a fibration.* □

We are now ready to prove

Theorem 3.5 *If $X = K(G,1)$ is an infra-nilmanifold, then $X^\wedge_p$ is either $K(G^\wedge_p, 1)$ or it has infinitely many nonzero homotopy groups.*

PROOF. Consider the Fitting extension $N \rightarrowtail G \twoheadrightarrow Q$ of G. Denote T for the preimage of $i_{p'}(Q)$ in G and let Y be $K(T,1)$. Then $Y \to X \to K(Q^\wedge_p, 1)$ is a fibration and because of Lemma 3.3, it is enough to study the homotopy type of $Y^\wedge_p$. This on its turn is determined by the one of $Y_{H\mathbb{Z}_{(p)}}$. Indeed, $Y^\wedge_p \simeq (Y_{H\mathbb{Z}_{(p)}})^\wedge_p$ and since $Y_{H\mathbb{Z}_{(p)}}$ is a nilpotent space [9, Lemma 5.2], there exists a short exact sequence

$$\mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_k(Y_{H\mathbb{Z}_{(p)}})) \rightarrowtail \pi_k((Y_{H\mathbb{Z}_{(p)}})^\wedge_p) \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{k-1}(Y_{H\mathbb{Z}_{(p)}})),$$

for every $k \geq 1$ [3, VI.5.1].

Since $Y_{H\mathbb{Z}_{(p)}}$ is moreover p-local and the class of nilpotent groups H such that $H_{(p)}$ is the p-localization of a finitely generated nilpotent group is a Serre class [11], it follows from [13, Th. II.2.16] that, for all $k \geq 1$, the groups $\pi_k(Y_{H\mathbb{Z}_{(p)}})$ are finitely generated as p-local groups. This implies that for all $k \geq 1$, $\pi_k(Y^\wedge_p) \cong \pi_k((Y_{H\mathbb{Z}_{(p)}})^\wedge_p) \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_k(Y_{H\mathbb{Z}_{(p)}}))$. Therefore, if $Y^\wedge_p$ is not aspherical, then there exists an integer $k > 1$ such that $\pi_k(Y_{H\mathbb{Z}_{(p)}})$ and, because of $(*)$, $\pi_k(Y_{H\mathbb{Z}_{(p)}}; \mathbb{Z}/p)$ are nonzero. Since Y is finite dimensional, $H_k(Y_{H\mathbb{Z}_{(p)}}; \mathbb{Z}/p) \cong H_k(Y; \mathbb{Z}/p) = 0$ for all k sufficiently large. Thus, $Y_{H\mathbb{Z}_{(p)}}$ satisfies the conditions of Theorem 3.2. Hence, $Y_{H\mathbb{Z}_{(p)}}$ and $Y^\wedge_p$ have infinitely many nonzero homotopy groups. □

As indicated in the introduction, for a virtually nilpotent space X having a finitely generated fundamental group and a nilpotent p-localization, information on the homotopy type of $X_{(p)}$ can be recovered from the one of $X^\wedge_p$. The following corollary nicely illustrates this.

Corollary 3.6 *Let $X = K(G,1)$ be an infra-nilmanifold and fix a prime number p such that $X_{(p)}$ is nilpotent. Then $X_{(p)}$ is either $K(G_{(p)}, 1)$ or it has infinitely many nonzero homotopy groups.*

PROOF. Since $X_{(p)}$ is nilpotent and $X^\wedge_p \simeq (X_{(p)})^\wedge_p$, one shows as in the proof of the previous theorem that $\pi_k(X^\wedge_p) \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_k(X_{(p)}))$, for all $k \geq 1$. To conclude, use Theorem 3.5. □

Of course, the above results raise the question what happens for $K(G,1)$-spaces where G is finitely generated virtually nilpotent and not torsion-free. At present, we can only answer this in case G is an almost-crystallographic group with p-nilpotent Fitting quotient. The homological dimension of $\mathrm{Fitt}(G)$ then equals the dimension of G. Since the composition of restriction and corestriction is multiplication by the order of the Fitting quotient, it follows that $H_k(G;\mathbb{Z}/p) = 0$ for all k sufficiently large. Proceeding analogously as in the proofs of Theorem 3.5 and Corollary 3.6 we obtain:

Theorem 3.7 *Let G be an almost-crystallographic group and $X = K(G,1)$. If p is a prime number such that the Fitting quotient of G is p-nilpotent, then $X^\wedge_p$ is either $K(G^\wedge_p,1)$ or it has infinitely many nonzero homotopy groups.* □

Corollary 3.8 *Let G be an almost-crystallographic group and $X=K(G,1)$. If p is a prime number such that the Fitting quotient of G is p-nilpotent and such that $X_{(p)}$ is nilpotent, then $X_{(p)}$ is either $K(G_{(p)},1)$ or it has infinitely many nonzero homotopy groups.* □

Corollaries 3.6 and 3.8 invite us to look for conditions under which the p-localization of a virtually nilpotent space is nilpotent. The next section is designed to answer this. In fact, improvements of these corollaries are given by Theorems 4.3 and 4.4.

4 Preservation of virtual nilpotence

The following theorem is of the utmost relevance:

Theorem 4.1 *Let $X = K(G,1)$ be a virtually nilpotent aspherical space with Fitting quotient Q. Denote $Y = K(T,1)$ with T the preimage of $i_{p'}(Q)$ in G. Then $Y_{(p)}$ is nilpotent.*

Proof. The idea of the proof is extracted from [9] and is to show that Y is p-equivalent to a nilpotent space.

Consider the Fitting extension $N \rightarrowtail T \to i_{p'}(Q)$ of T. Applying fibrewise localization to the fibration $K(N,1) \to Y \to K(i_{p'}(Q),1)$ yields a homotopy fibre sequence $K(N_{(p)},1) \to Y' \to K(i_{p'}(Q),1)$ such that the map $Y \to Y'$ is a p-equivalence.

Let $x_1,\ldots,x_r$ be the generators of $i_{p'}(Q)$. For every i, $1 \le i \le r$, we have an extension

$$N_{(p)} \rightarrowtail \pi^{-1}(\langle x_i\rangle) \stackrel{\pi}{\twoheadrightarrow} \langle x_i\rangle,$$

where $\langle x_i\rangle$ denotes the subgroup of $i_{p'}(Q)$ generated by x_i, and π the induced map $\pi_1(Y') \to i_{p'}(Q)$. By [21, Prop. 5.9], these sequences split.

Write s_i for the corresponding section and $y_i = s_i(x_i)$, $1 \leq i \leq r$. Let S be the normal subgroup of $\pi_1(Y')$ generated by $y_1, \ldots, y_r$. Denote by $K(S,1)_{(p)} \to Y'' \to K(\pi_1(Y')/S, 1)$ the homotopy fibre sequence obtained by applying fibrewise localization to $K(S,1) \to Y' \to K(\pi_1(Y')/S,1)$. Since $Y'_{(p)} \simeq Y''_{(p)}$, it is enough to show that Y'' is nilpotent to complete our proof.

To do so, first note that $K(S,1)_{(p)}$ is 1-connected (S is generated by p'-torsion elements). Moreover, $\pi_1(Y'') \cong \pi_1(Y')/S$ is nilpotent and acts nilpotently on $H_n(K(S,1)_{(p)}) \cong H_n(K(S,1)_{(p)}; \mathbb{Z}_{(p)}) \cong H_n(K(S,1); \mathbb{Z}_{(p)})$ for all n. Indeed, since the homomorphism $N_{(p)} \to \pi_1(Y')/S$ is surjective, the first assertion is true and $\pi_1(Y')$ maps onto $i_{p'}(Q) \times \pi_1(Y')/S$. Then, by the Nilpotent Action Lemma in [9], $\pi_1(Y')/S$ acts nilpotently on all homology groups $H_n(K(S,1); \mathbb{Z}_{(p)})$. This ensures that Y'' is nilpotent [13]. □

The same result holds for p-completion, since $Y^{\wedge}_p \simeq (Y_{H\mathbb{Z}_{(p)}})^{\wedge}_p$ and $Y_{H\mathbb{Z}_{(p)}}$ is nilpotent [9]. In case of rationalization, an analogous proof as above shows that $Y_{(0)} \simeq X_{(0)}$ is nilpotent. Combining these observations with Lemma 3.3, Lemma 3.4 and Theorem 4.1, we obtain

Theorem 4.2 *The p-completion and the p-localization of a virtually nilpotent aspherical space are again virtually nilpotent. The rationalization of a virtually nilpotent aspherical space is nilpotent.* □

Thanks to Theorem 4.1, the nilpotency condition on the space $X_{(p)}$ in Corollaries 3.6 and 3.8 can now be removed. This yields

Theorem 4.3 *Let $X = K(G,1)$ be an infra-nilmanifold. Then $X_{(p)}$ is either $K(G_{(p)},1)$ or it has infinitely many nonzero homotopy groups.* □

Theorem 4.4 *Let G be an almost-crystallographic group and $X = K(G,1)$. If p is a prime such that the Fitting quotient of G is p-nilpotent, then $X_{(p)}$ is either $K(G_{(p)},1)$ or it has infinitely many nonzero homotopy groups.* □

To close this section we display two examples illustrating our results.

Example 4.5 Let $X = K(G, 1)$ be the orientable infra-nilmanifold with fundamental group

$$G = \langle a, b, c, \alpha \mid \begin{array}{llll} [b,a] = c^2 & [c,a] = 1 & [c,b] = 1 & \alpha^2 = a \\ \alpha a = a\alpha & \alpha b = b^{-1}\alpha c^{-1} & \alpha c = c^{-1}\alpha & \end{array} \rangle.$$

By Theorem 7.2.9 in [8], the space $X_{(p)}$ is aspherical if and only if the Fitting quotient of G, which here is $\mathbb{Z}/2$, is p-torsion. Hence, the space $X_{(p)}$ is $K(G_{(p)}, 1)$ if and only if $p = 2$. If $p \neq 2$, then Theorem 4.3 implies that $X_{(p)}$ has infinitely many nonzero homotopy groups. Moreover, because of Theorem 4.1, $X_{(p)}$ is a nilpotent space.

Example 4.6 Let $X = K(G, 1)$ with G the crystallographic group presented as

$$G = \langle a, b, \alpha \mid [b, a] = 1 \quad \alpha a = ab\alpha \quad \alpha b = a^{-1}\alpha \quad \alpha^6 = 1\rangle.$$

If p is a prime different from 2 or 3, then $X_{(p)}$ is 1-connected [8, p. 63]. As $H_2(X)$ has rank 1, we infer from Theorem 4.4 that $X_{(p)}$ has infinitely many nonzero homotopy groups. If $p = 2$ or $p = 3$, then $G_{(p)} \cong \mathbb{Z}/p$ [8, p. 63]. Denote $Y = K(T, 1)$ with T the preimage of $i_{p'}(\mathbb{Z}/6)$ in G. Then $Y_{(p)}$ is 1-connected since $\pi_1(Y_{(p)}) \rightarrowtail G_{(p)} \twoheadrightarrow (\mathbb{Z}/6)_{(p)}$ is exact and $(\mathbb{Z}/6)_{(p)} \cong \mathbb{Z}/p$. Since $H_2(Y)$ has rank 1, Theorem 4.4 implies that $Y_{(p)}$ and hence $X_{(p)}$ have infinitely many nonzero homotopy groups.

5 Homotopy types of rationalizations

The previous sections already give us a fairly good idea of the homotopy type of the p-completion and the p-localization of infra-nilmanifolds. In fact, the effect of applying these functors is really similar.

In case of p-localization however, more can be said. Indeed, recall from the introduction that, if the p-localization of a space X is nilpotent, then the rationalization $X_{(0)}$ should reveal information on $X_{(p)}$. More concretely, since $(X_{(p)})_{(0)} \simeq X_{(0)}$, there holds for all $k \in \mathbb{N}$ that $\pi_k(X_{(0)}) \cong (\pi_k(X_{(p)}))_{(0)} \cong \pi_k(X_{(p)}) \otimes \mathbb{Q}$. If moreover X is an infra-nilmanifold and $k \geq 2$, this implies that $\pi_k(X_{(p)})$ is a finite abelian p-group if and only if $\pi_k(X_{(0)}) = 0$ (since the higher homotopy groups of $X_{(p)}$ are finitely generated $\mathbb{Z}_{(p)}$-modules). The use of this is illustrated by the following examples:

Example 5.1 Let $X = K(G,1)$ be the Hantzsche–Wendt manifold. Its fundamental group G is presented as follows:

$$G = \langle a,b,c,\alpha,\beta \,|\, \begin{array}{lll} [b,a]=1 & [c,a]=1 & [c,b]=1 \\ \alpha a = a^{-1}\alpha & \alpha b = b^{-1}\alpha & \alpha c = c\alpha \\ \beta a = a^{-1}\beta & \beta b = b\beta & \beta c = c^{-1}\beta \\ \alpha^2 = c & \beta^2 = b & \beta\alpha = a^{-1}bc^{-1}\alpha\beta \;\rangle. \end{array}$$

If p is an odd prime, then $X_{(p)}$ is 1-connected [8, p. 64] and hence $\pi_n(X_{(0)}) \cong \pi_n(X_{(p)})_{(0)}$. Since X is 3-dimensional and orientable, Poincaré duality can be applied to show that $H_0(X_{(0)}) \cong H_3(X_{(0)}) \cong \mathbb{Q}$ and $H_n(X_{(0)}) = 0$ for all $n \neq 0, 3$. This means that $X_{(0)}$ has the same homology groups as $S^3_{(0)}$. Using the Hurewicz homomorphism, it is now easy to see that $X_{(0)}$ and $S^3_{(0)}$ are homotopy equivalent. Hence, we can conclude that for each $k > 3$, $\pi_k(X_{(p)})$ is a finite abelian p-group.

From [20] it follows that in each odd dimension there exist Hantzsche–Wendt-like compact flat Riemannian manifolds, i.e., having the rational homotopy type of a sphere. So we have an infinite class of infra-nilmanifolds whose p-localization has finite homotopy groups from a certain point onwards. Is it true that for each infra-nilmanifold X there exists an integer N_X such that for every prime p and for all $k > N_X$, $\pi_k(X_{(p)})$ is a finite p-group? The following example shows that the answer is negative.

Example 5.2 Let $X = K(G,1)$ be the non-orientable flat manifold with G presented as follows

$$G = \langle a,b,c,d,\alpha,\beta \,|\, \begin{array}{lll} \alpha^2 = c & \beta^2 = d & \alpha\beta = b^{-1}cd^{-1}\beta\alpha \\ \alpha a = a^{-1}\alpha & \alpha b = b^{-1}\alpha & \alpha c = c\alpha \\ \alpha d = d^{-1}\alpha & \beta a = a^{-1}\beta & \beta b = b^{-1}\beta \\ \beta c = c^{-1}\beta & \beta d = d\beta & \end{array} \rangle.$$

If p is an odd prime, then $X_{(p)}$ is 1-connected [8], and hence $\pi_n(X_{(0)}) \cong \pi_n(X_{(p)})_{(0)}$. Since X is 4-dimensional, $H^n(X_{(0)}) \cong H^n(X_{(0)}; \mathbb{Q}) = 0$ for all $n \geq 4$. Moreover, $H^3(X_{(0)}) \cong \mathbb{Q} \oplus \mathbb{Q}$ [7, p. 141], and $H^2(X_{(0)}) \cong \mathbb{Q}$ because the Euler characteristic of an infra-nilmanifold is zero [7, p. 134]. In fact, computing the cohomology products shows that the cohomology algebra of $X_{(0)}$ coincides with the one of $S^2_{(0)} \vee S^3_{(0)} \vee S^3_{(0)}$, which is a rationally hyperbolic space [10]. By [12], we obtain that $X_{(0)}$ and $S^2_{(0)} \vee S^3_{(0)} \vee S^3_{(0)}$ have the same minimal model and hence that $X_{(0)} \simeq S^2_{(0)} \vee S^3_{(0)} \vee S^3_{(0)}$. This implies that $X_{(p)}$ has infinitely many infinite homotopy groups.

From the rational dichotomy theorem in [10], it follows that a nilpotent space X with $\dim H_*(X;\mathbb{Q}) < \infty$ is either elliptic ($\dim(\pi_*(X) \otimes \mathbb{Q}) < \infty$) or hyperbolic ($\dim(\pi_*(X) \otimes \mathbb{Q}) = \infty$). This applies to the rationalization $X_{(0)}$ of any infra-nilmanifold X since $X_{(0)}$ is nilpotent (Theorem 4.2) and $H_*(X_{(0)};\mathbb{Q}) \cong H_*(X;\mathbb{Q})$. Moreover, because of Examples 5.1 and 5.2 we know that infra-nilmanifolds of both types (rationally elliptic and rationally hyperbolic) arise. Our next goal is to find criteria to decide of which type they are. However, this study goes beyond the scope of this paper and will be undertaken in the sequel.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
E-mail address: gemmab@mat.uab.es

Centre de Recerca Matemàtica, Apartat 50, E-08193 Bellaterra, Spain
E-mail address: adesche@crm.es