THE GEOMETRY OF QUADRATIC DIFFERENTIAL SYSTEMS WITH A WEAK FOCUS OF THIRD ORDER[1]

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Abstract. In this article we determine the global geometry of the planar quadratic differential systems with a weak focus of third order. This class plays a significant role in the context of Hilbert's 16th problem. Indeed, all examples of quadratic differential systems with the maximum number attained so far for such systems, of four limit cycles, were obtained by perturbing a system in this family. We use the algebro-geometric concepts of divisor and zero-cycle to encode global properties of the systems and give structure to this class. We give a theorem of topological classification of such systems in terms of an integer–valued invariant $\mathcal{I}(S)$. According to the possible values taken by $\mathcal{I}(S)$ in this family we obtain a total of 18 topologically distinct phase portraits. We then sum up the specific global geometrical properties for this class of quadratic differential systems.

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1 Introduction

The complete characterization of the phase portraits for planar quadratic vector fields is not known and attempting to classify these systems, which occur rather often in applications, is quite a complex task. This family of systems depends on twelve parameters but due to the group action of affine transformations and positive time rescaling, the class ultimately depends on five parameters. Bifurcation diagrams were constructed for some algebraic and semialgebraic subsets of this class, for example [4, 5, 7, 9, 23, 30, 38, 36, 47]. Except for [30, 38, 47], the classiffications of systems in the previously mentioned literature were done in terms of local charts and inequalities on the coefficients of the systems written in these charts. Since

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it is expected that quadratic vector fields will yield more than two thousand phase portraits, tools for organizing this maze of phase portraits need to be introduced. We would also want to see that classifications be done not in terms of coordinate charts as in previous works but in more intrinsic terms in order to reveal the geometry of the systems.

In this work we continue the more conceptual study initiated in [30, 31, 39, 38] for families of quadratic differential systems. We aim at defining a strategy and at introducing some new tools to pursue classifications of classes of low degree polynomial systems in more intrinsic ways. For an initial study of the geometrical structure of the whole class of quadratic differential systems we refer to the recent article of Roussarie and Schlomiuk [35].

The behaviour of all the phase curves of a planar vector field is contained in its phase portrait. The phase portraits are classified by using the topological equivalence [45]. Just as in topology, homotopy groups could be used to distinguish topological spaces, we need to have invariants to distinguish between phase portraits. A complete set of invariants was given by Markus for C^1 systems on the plane (cf. [26], see also [28]). He associated to every C^1 system its separatrix configuration and he showed that two C^1 systems are topologically equivalent if and only if their corresponding separatrix configurations are topologically equivalent. The separatrix configuration is formed by all the separatrices of the system, together with an orbit for each open connected component of the complement in the plane of the union of all separatrices. Each of these components is called a *canonical region*. The separatrix configuration could still be rather complex. We thus need criteria to distinguish between separatrix configurations. We associate to every system a number of geometric global invariants which will help us to distinguish between separatrix configurations. These geometric global invariants will be constructed from two kinds of objects: zero–cycles of the affine or projective plane, real or complex, and divisors corresponding to the line at infinity. We associate to these divisors and zero–cycles, integer–valued invariants. These encode globally information on the systems. We illustrate these ideas by applying them to the particular class of all quadratic systems with a weak focus of third order. We focus our attention on this class which we consider as a case study , but we aim at the more general problem for all quadratic systems.

Due to the concerns mentioned above, this work is very different from most previous studies in this subject and it also constitutes a substantial improvement of the results obtained in [3, 5, 2] on quadratic systems with a weak focus of third order.

Since a weak focus may produce up to three limit cycles in a quadratic perturbation of the system, the study of this class is of interest in itself in the context of Hilbert's 16th problem. This problem asks for the maximum $H(m)$ (the Hilbert number) of the numbers of limit cycles which a real polynomial vector field defined by polynomials of maximum degree m could have [19]. The nature of the problem is global in two ways: we are interested in the whole class of polynomial vector fields of a maximum fixed degree m and at the same time in the behavior on the whole plane of phase curves of each individual member of this family. The problem is still unsolved, even for the case $m = 2$. The most important result of a global nature is the individual finiteness theorem proved independently in the late 1980's by Ecalle [15] and Il'yashenko [20]. This theorem says that each fixed polynomial vector field has a finite number of limit cycles. This result was stated as a theorem by Dulac [12] in 1923 but the proof Dulac gave was flawed. If we consider the family of all quadratic vector fields, it is not even known that $H(2)$ is a finite number. There is a program under way (cf.[14]) to prove this. The program proposes to show that all the 121 graphics listed in [14] which appear in this class, have finite cyclicity (cf. [33]). Since in one hundred years since Hilbert stated the problem, no example was found where we can prove that more than four limit cycles exist, it is not only conjectured that $H(2)$ is finite but also that $H(2) = 4$.

All known examples of quadratic vector fields having four limit cycles were obtained by perturbing a quadratic vector field with a weak focus of third order. This also motivated us to do a geometric study of quadratic vector fields with a weak focus of third order.

Assuming the proof of the finiteness of $H(2)$ were done, we would still know very little about the geometry of the systems in this class or of the class itself. For this reason together with the finiteness part it is necessary to advance the knowledge of this class and the present work is a step in this direction.

The work is organized as follows: In Section 2 we give an informal overview of our results and of closely related literature. In Section 3 we describe the normal form which helps us to calculate the necessary ingredients for obtaining the results. The normal forms are an essential tool for performing calculations. We want to stress however, that this is a tool used only as an intermediate step. In the final part of the article we break away from the normal form and state results which are independent of the particular normal forms which could be used.

For the study of real planar polynomial vector fields two real compactifications are used. In Section 4 we describe, for a planar real polynomial vector field, its Poincaré compactification on the 2–dimensional sphere. In Section 5 we introduce two foliations with singularities associated to the polynomial vector fields which are respectively on the real and on the complex projective planes. Although a different compactification than the one introduced in Section 4, the real foliation with singularities on the real projective plane is closely related to the one defined by the Poincaré compactification on the sphere introduced in Section 4. The two polynomials defining a real planar polynomial vector field, also define a complex vector field when the variables range over $\mathbf{C}$. To this complex vector field on $\mathbf{C}^2$ we can associate a compactification by passing to the foliation with singularities on the complex projective plane $\mathbf{CP}^2$. This foliation turns out to be important for the study of the real vector field. In fact, the first step in the classification theorem we give in Section 10 is provided by information about this foliation.

In Section 6 we associate intersection numbers to the singularities of the complex foliation with singularities associated to the real vector field. In Section 7 we introduce some of the main global concepts which encode the global information about singularities of both the real and complex foliations with singularities. These concepts, of an algebro–geometric nature, are zero–cycles of the plane and divisors on the line at infinity. These are then applied in Section 8 to the specific study of the class $QW3$ of quadratic systems with a weak focus of third order. We calculate for the class $QW3$ these zero–cycles and divisors.

In Section 9 we list the basic theorems from the literature, and in particular the properties of quadratic systems or of $QW3$, which are used in Section 10. We thus mention two important results: the nonexistence of a limit cycle surrounding a weak focus of third order (cf. [22]) and a recently obtained important result affirming that if a quadratic system has limit cycles around two foci, then around one of them we cannot have more than just one limit cycle (cf. [49]). The main result obtained in this Section is the bifurcation diagram for the class $QW3$.

In Section 10 we first introduce integer–valued global invariants collected in the invariant we denote by $\mathcal{I}$ which classify completely, up to topological equivalence, the systems in the class $QW3$. We state and prove the theorems which describe the global geometry of the systems in $QW3$. The Section 11 is reserved for concluding comments, confrontation with other results in the literature about the class $QW3$ and a program for future work.

2 Informal outline of results and brief overview of the closely related literature

The literature on quadratic systems is vast, with numerous studies on subfamilies of this class. With the exception of some articles (for example [30, 47, 39, 38] and other contributions by K.S. Sibirsky or by N. Vulpe), in these studies normal forms are chosen for the specific subclass under study and then phase portraits and bifurcation diagrams are computed with respect to the specific normal forms. The results are stated by giving long lists of inequalities on the coefficients of the normal forms and for each such inequality, producing the corresponding phase portrait either local, around a singularity or global. This line of work follows the program stated by Coppel in his nice short article which appeared in 1966 (cf. [10]). There Coppel wrote: *Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with limited success.* We now know that algebraic inequalities would not be sufficient; analytic as well as non–analytic ones (cf. references [34] and [13] pages 118–119) need to be taken into consideration. But even if we take into account such inequalities, the program which was a good starting point in the 60's, no longer befits our present goals.

In [30, 31, 39, 38] a different view was taken and global geometric invariants were used for studying and classifying families of quadratic systems. In [30, 38] the family of Hamiltonian quadratic systems with a center was considered.

In this work we go further and construct simple invariants for a non–integrable class of systems, the class $QW3$ and we obtain a topological classification of the systems in $QW3$. Two of the features of planar polynomial vector fields are: the singularities and the limit cycles.

We consider first the singularities, finite and infinite (these last ones, obtained by using the compactifications), they are at most seven when we count them in the real (or complex) projective planes. Some singularities are composite or multiple, arising from collision of neighboring points which are singularities in a perturbation of the system. In this case the total number of distinct singularities is at most six. We use the algebro–geometric notions of *zero–cycle* (cf.[16]) on both the real and the complex projective planes and *divisor* [16] on the line at infinity to encode the multiplicities of all the singular points finite or infinite. But encoding globally the multiplicities of the singularities is not sufficient. We also need to encode globally the information on the topology of the phase portrait around each one of the singularities. This is done by using in Proposition 16 (see Section 8), the zero–cycle DI (defined by us in Section 7), encoding globally the indices of singularities in the real projective plane of the compactified systems. The types of all these zero–cycles and divisors are invariant with respect to the real affine group action. Theorems 19 and 23 show that for a large part, call it here A, of the parameter space, the phase portraits are uniquely determined (up to topological equivalence) by the values of four integer valued invariants whose computation results from the values of the above divisors and zero–cycles and thus, in this region A the bifurcation diagram is obtained and it is shown to be part of a real algebraic curve. For any value in A we have a phase portrait which turns out to have no limit cycle and no graphic (or polycycle) (cf.[14]).

We next consider the complement B of A in the parameter space. It is in this part only that limit cycles could occur. In this part B, the four–tuple of these integer–valued invariants takes just two values for which these invariants are insufficient for distinguishing among the phase portraits. For these cases new concepts are needed and we set out to introduce them in Section 10. It is in this region that a connected bifurcation curve G, part algebraic, part very likely non–algebraic, occurs on the diagram of Figure 3. On this curve G the systems do not have limit cycles but on this curve and only on it they have a unique graphic (polycycle) (cf.[14]) with two singularities, both at infinity, and two path curves, one of them part of the line at infinity. There are three distinct topological phase portraits on this curve. The existence of such a curve is determined by qualitative arguments. Numerical computations determine the position of its non–algebraic part with as high accuracy as we wish.

We have thus isolated the region with limit cycles. We have three distinct phase portraits in this region which is bounded by the curve G and by the line $a = 0$ on which we have differential systems which are symmetric and have centers (more precisely systems with two centers located on $a = 0$ on the diagram of Figure 3). This region is split in three parts: a region with only one singular point at infinity, another with three singular points at infinity, separated by a bifurcation curve on which two of the points at infinity collide. We finally obtain in Figure 2 and its enlargement Figure 3 the bifurcation diagram for the whole parameter space. Unlike other studies done in the literature (cf. [3], [4] and [2]), we construct the bifurcation diagram in just on picture, the real projective plane viewed on the disc with opposite points on its boundary identified. Our methods allowed us also to correct statements made in the above mentioned literature as we indicate in Section 11. We obtain an intrinsic topological classification by using a sequence of five integer-values invariants which convey to us the main geometric characteristics of the systems in this class.

As far as we know, this is the first work where specific global concepts are used to encode global geometric information on the whole plane as well as at infinity, about a family of quadratic (or more general polynomial) systems for the purpose of topologically classifying this family. We note here the mixed character of the methods used for the study of these real polynomial systems: complex as well as real, algebraic or algebro-geometric, analytical as well and numerical ones.

3 Quadratic vector fields with a weak focus of third order

A singular point p of a planar vector field X in $\mathbf{R}^2$ is a *linear center* if the eigenvalues of its linear part, $DX(p)$, are imaginary numbers, i.e. $\pm\beta i$ with $\beta \in \mathbf{R} \setminus \{0\}$. We say that p is a *center* of X if there exists a neighborhood U of p such that all orbits of X in $U \setminus \{p\}$ are periodic. It is known that a linear center p is either a center, or a focus. In this last case p is called a *weak focus*. We recall that p is a *strong focus* if the eigenvalues of its linear part are of the form $\alpha \pm \beta i$ with $\alpha\beta \neq 0$.

The next result is due to Shi [42, 43]. This lemma is better stated in [37] where the rings involved are explicitly written, and where a proof is also given. For the sake of completeness we give below the statement of this lemma.

Lemma 1. *Consider the polynomial system:*

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = p(x,y) = -y + p_2(x,y) + \cdots + p_m(x,y) \ , \\ \frac{dy}{dt} &= \dot{y} = q(x,y) = x + q_2(x,y) + \cdots + q_m(x,y) \ , \end{aligned} \tag{1}$$

where

$$p_i(x,y) = \sum_{j=0}^{i} a_{ij} x^{i-j} y^j \ , \qquad q_i(x,y) = \sum_{j=0}^{i} b_{ij} x^{i-j} y^j \ .$$

Then there exists a formal power series $F \in \mathbf{Q}[a_{20}, \dots, b_{0l}][[x,y]]$,

$$F = \frac{1}{2}(x^2 + y^2) + F_3(x,y) + F_4(x,y) + \cdots \ ,$$

and there exists polynomials $V_1, \dots, V_i, \dots \in \mathbf{Q}[a_{20}, \dots, b_{0l}]$ *such that*

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} p + \frac{\partial F}{\partial y} q = \sum_{i=1}^{\infty} V_i (x^2 + y^2)^{i+1} \ .$$

The quantities V_i are not uniquely determined. For each i there is an infinite number of possibilities for a V_i. But according to a result also proven by Shi, all such V_i's are in the same coset modulo the ideal generated by

$V_1, \ldots, V_{i-1}$ in the ring $\mathbf{Q}[a_{20}, \ldots, b_{02}]$. From the work of Poincaré [32] it follows that system (1) has a center at the origin if and only if $V_i = 0$ for all i. By Hilbert's basis theorem, the ideal $I =< V_1, \ldots, V_i, \ldots >$ has a finite basis. It follows from the work of Bautin [6] that for quadratic systems ($m = 2$) this ideal is determined by the values of V_i with $i \leq 3$. A quadratic system having a weak focus can always be written into the form

$$\begin{aligned} \dot{x} &= -y + lx^2 + rxy + ny^2 \ , \\ \dot{y} &= x + ax^2 + bxy \, . \end{aligned}$$

The above implies that $V_1 = V_2 = V_3 = 0$ if and only if $V_i = 0$ for all i and the origin is a center. Calculations of Li [21] yield that $V_1 = L_1$, $V_2 = L_2$ (mod V_1), $V_3 = L_3$ (mod V_1, V_2) where

$$\begin{aligned} L_1 &= r(l+n) - a(b+2l) \ , \\ L_2 &= ra(5a-r)[(l+n)^2(n+b) - a^2(b+2l+n)] \ , \\ L_3 &= ra^2[2a^2 + n(l+2n)][(l+n)^2(n+b) - a^2(b+2l+n)] \ . \end{aligned}$$

We say that the origin of the above system is

(i) a *weak focus of first order* if $L_1 \neq 0$;

(ii) a *weak focus of second order* if $L_1 = 0$ and $L_2 \neq 0$; and

(iii) a *weak focus of third order* if $L_1 = L_2 = 0$ and $L_3 \neq 0$.

The next result follows easily. For more details see [21].

Corollary 2. *A planar quadratic vector field with a weak focus of third order may be written in the following form where the weak focus is placed at the origin:*

$$\begin{aligned} \dot{x} &= p(x,y) = -y + lx^2 + 5axy + ny^2 \ , \\ \dot{y} &= q(x,y) = x + ax^2 + (3l+5n)xy \ , \end{aligned} \tag{2}$$

with $L_3 = 5a^3[2a^2 + n(l+2n)][3(l+n)^2(l+2n) - a^2(5l+6n)] \neq 0$.

Systems (2) depend on the parameter $\lambda = (a, l, n) \in \mathbf{R}^3$. We consider systems (2) which are nonlinear, i.e. $\lambda = (a, l, n) \neq 0$. In this case system (2) can be rescaled, therefore the parameter space needed is actually the real projective plane $\mathbf{RP}(2)$ and not $\mathbf{R}^3$.

4 Poincaré compactification of a polynomial vector field

Let X be a polynomial vector field of degree m. The *Poincaré compactified vector field* $p(X)$ *corresponding to* X is the analytic vector field induced on $\mathbf{S}^2$ as follows (see, for instance [17]). We denote by $\mathbf{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbf{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ the *Poincaré sphere* and by $T_y\mathbf{S}^2$ the tangent space to $\mathbf{S}^2$ at point y. We denote by $H^+ = \{y \in \mathbf{S}^2 : y_3 > 0\}$ and by $H^- = \{y \in \mathbf{S}^2 : y_3 < 0\}$. Consider the central projections $f_+ : \mathbf{R^2} = \mathbf{T_{(0,0,1)}S^2} \to \mathbf{H}^+$ and $f_- : \mathbf{R^2} \to \mathbf{H}^-$ defined by $f_+(y) = (y_1, y_2, 1)/\Delta(y)$ and $f_-(y) = -(y_1, y_2, 1)/\Delta(y)$, where $\Delta(y) = ({y_1}^2 + {y_2}^2 + 1)^{1/2}$. Denote by X' the vector field defined on $H^+ \cup H^- = \mathbf{S}^2 \setminus \mathbf{S}^1$ where $\mathbf{S}^1 = \{y \in \mathbf{S}^2 : y_3 = 0\}$ by $Df_+(y)X(y)$ on H^+ and by $Df_-(y)X(y)$ on H^-. Clearly $\mathbf{S}^1$ means the infinity of $\mathbf{R}^2$. Then $p(X)$ is the only analytic extension of $y_3^{m-1}X'$ to $\mathbf{S}^2$. The projection of the closed northern hemisphere H^+ of $\mathbf{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \to (y_1, y_2)$ is called *the Poincaré disc.* We compute the expression of $p(X)$ by using the local charts $U_i = \{y \in \mathbf{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbf{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$, and the diffeomorphisms $F_i : U_i \to \mathbf{R}^2$ and $G_i : V_i \to \mathbf{R}^2$ which are the inverses of convenient central projections. We denote by $z = (z_1, z_2)$ the value of $F_i(y) = (y_j, y_k)/y_i$ or $G_i(y) = (y_j, y_k)/y_i$ for any $i = 1, 2, 3$ with $j < k$ and $i \neq j, k$, so z represents different things according to the local chart under consideration. Now some easy computations give for $p(X)$ the following expressions

$$z_2^m \cdot \Delta(z)^{-(m-1)} \left[q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 p\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 p\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right)\right] \quad \text{in } U_1,$$

$$z_2^m \cdot \Delta(z)^{-(m-1)} \left[p\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)\right] \quad \text{in } U_2,$$

$$\Delta(z)^{-(m-1)} \left[p(z_1, z_2), q(z_1, z_2)\right] \quad \text{in } U_3.$$

Changing the reparametrization of the solutions of the vector fields $p(X)$ in there follows we will omit the factor $\Delta(z)$ in these local expressions. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{m-1}$.

A singular point q of $p(X)$ is called an *infinite* (respectively *finite*) singular point if $q \in \mathbf{S}^1$ (respectively $q \in \mathbf{S}^2 \setminus \mathbf{S}^1$). So the infinite singular points $(z_1, 0)$ are given by the equations:

$$\begin{aligned} F(z_1) &= q_m(1, z_1) - z_1 p_m(1, z_1) \quad \text{in} \quad U_1, \\ G(z_1) &= p_m(z_1, 1) - z_1 q_m(z_1, 1) \quad \text{in} \quad U_2, \end{aligned}$$

where p_m and q_m are the homogeneous parts of degree m of p and q, respectively. Eventually, we will use the equivalent notation $F(x,y) = xq_m(x,y) - yp_m(x,y)$ and $G(x,y) = yp_m(x,y) - xq_m(x,y)$, respectively.

5 The complex (respectively real) foliation with singularities on $\mathbf{CP^2}$ (respectively $\mathbf{RP^2}$) associated to a planar real polynomial vector field

Ideas in this section go back to Darboux's work [11]. Let $p(x,y)$ and $q(x,y)$ be polynomials with real coefficients. We consider for the vector field

$$p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} \ , \tag{3}$$

or equivalently for the differential system

$$\begin{aligned} \frac{dx}{dt} &= p(x,y) \ , \\ \frac{dy}{dt} &= q(x,y) \ , \end{aligned} \tag{4}$$

the associated differential 1–form

$$\omega_1 = q(x,y)dx - p(x,y)dy \ ,$$

and the differential equation

$$\omega_1 = 0 \ . \tag{5}$$

Clearly equation (5) defines a foliation with singularties on $\mathbf{C}^2$. The affine plane is compactified on the complex projective space $\mathbf{CP}^2 = (\mathbf{C}^3 \setminus \{0\})/\sim$, where $(X,Y,Z) \sim (X',Y',Z')$ if and only if $(X,Y,Z) = \lambda(X',Y',Z')$ for some complex $\lambda \neq 0$. The equivalence class of (X,Y,Z) will be denoted by $[X:Y:Z]$.

The foliation defined by equation (5) on $\mathbf{C}^2$ can be extended to a singular foliation on $\mathbf{CP}^2$ and the 1–form ω_1 can be extended to a meromorphic 1–form on $\mathbf{CP}^2$. This 1–form can be described by a single 1–form on $\mathbf{C}^3$ given by

$$A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ = 0 \ ,$$

for which

$$A(X,Y,Z)X + B(X,Y,Z)Y + C(X,Y,Z)Z = 0 \ ,$$

where A, B and C are homogeneous polynomials. Indeed, consider the map

$$i : \mathbf{C}^3 \setminus \{(X,Y,Z) : Z \neq 0\} \to \mathbf{C}^2 \ ,$$

given by $i(X,Y,Z) = (X/Z, Y/Z)$ and suppose that $\max\{\deg(p), \deg(q)\} = m > 0$. Since $x = X/Z$ and $y = Y/Z$ we have:

$$dx = \frac{ZdX - XdZ}{Z^2} \ , \qquad dy = \frac{ZdY - YdZ}{Z^2} \ ,$$

the pull–back form $i^*(\omega_1)$ has poles at $Z = 0$ and the equation (5) can be written as

$$i^*(\omega_1) = q\left(\frac{X}{Z}, \frac{Y}{Z}\right)\frac{ZdX - XdZ}{Z^2} - p\left(\frac{X}{Z}, \frac{Y}{Z}\right)\frac{ZdY - YdZ}{Z^2} = 0 \ .$$

Then the 1–form $\omega = Z^{m+2} i^*(\omega_1)$ in $\mathbf{C}^3 \setminus \{Z \neq 0\}$ has homogeneous polynomial coefficients of degree $m+1$, and for $Z \neq 0$ the equations $\omega = 0$ and $i^*(\omega_1) = 0$ have the same solutions. The differential equation $\omega = 0$ can be written as

$$\omega = A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ = 0 \ , \tag{6}$$

where

$$\begin{aligned} A(X,Y,Z) &= ZQ(X,Y,Z) \ , \\ B(X,Y,Z) &= -ZP(X,Y,Z) \ , \\ C(X,Y,Z) &= YP(X,Y,Z) - XQ(X,Y,Z) \ , \end{aligned}$$

and

$$P(X,Y,Z) = Z^m p\left(\frac{X}{Z}, \frac{Y}{Z}\right) \ , \qquad Q(X,Y,Z) = Z^m q\left(\frac{X}{Z}, \frac{Y}{Z}\right) \ ,$$

are homogeneous polynomials in X, Y, Z obtained from p and q. Clearly A, B and C are homogeneous polynomials of degree $m+1$ in the variables X, Y, Z and

$$A(X,Y,Z)X + B(X,Y,Z)Y + C(X,Y,Z)Z = 0 \ .$$

To study the foliation with singularities defined by the differential equation (6) subject to the above condition in the neighborhood of the line $Z = 0$, we consider the two charts of $\mathbf{CP}^2$:

$$(u,z) = \left(\frac{Y}{X}, \frac{Z}{X}\right) \ , \ X \neq 0 \ , \quad \text{and} \quad (v,w) = \left(\frac{X}{Y}, \frac{Z}{Y}\right) \ , \ Y \neq 0 \ ,$$

covering this line. We note that in the intersection of the charts $(x, y) = (X/Z, Y/Z)$ and (u, z) (respectively (v, w)) we have the change of coordinates $x = 1/z$, $y = u/z$ (respectively $x = v/w$, $y = 1/w$). Except in the neighborhood of the point $[0 : 1 : 0]$, the foliation defined by equation (6) in the neighbourhood of the line $Z = 0$ is topologically equivalent to the foliation defined by the phase curves of the vector field associated with the system

$$\frac{du}{dt} = uP(1, u, z) - Q(1, u, z) = C(1, u, z) \ , \qquad \frac{dz}{dt} = zP(1, u, z) \ . \quad (7)$$

Here, we say that two vector fields X_1 and X_2 defined on open subsets U_1 and U_2 respectively, are *topologically equivalent* if there exists a homeomorphism $h : U_1 \to U_2$ which carries orbits of X_1 into orbits of X_2 preserving or reversing the orientation (cf. [45]).

With the exception of a neighborhood of the point $[1 : 0 : 0]$ the foliation defined by (6) in the neighbourhood of the line $Z = 0$ is topologically equivalent to the foliation defined by the integral curves of the vector field associated with the system

$$\frac{dv}{dt} = vQ(v, 1, w) - P(v, 1, w) = -C(v, 1, w) \ , \qquad \frac{dw}{dt} = wP(v, 1, w) \ . \quad (8)$$

We remark that in a similar way we can associate a real foliation with singularities on $\mathbf{RP}^2$ to a real planar polynomial vector field.

6 Intersection numbers

In this section we recall the notion of intersection number of two algebraic curves at a point of $\mathbf{CP}^2$. This definition will be needed in the next section where using the notion of divisor we shall introduce a number of affine invariants for the class of quadratic systems with a weak focus of third order.

An *affine algebraic curve* on $\mathbf{C}^2$ is a set $\{(x, y) \in \mathbf{C}^2 : f(x, y) = 0\}$ where $f(x, y)$ is a non–zero polynomial in the variables x and y with coeffiecients in $\mathbf{C}$. We shall usually simply write $f(x, y) = 0$ (or $f = 0$) for the curve defined by f. Let $a = (x_0, y_0)$ be a point. Without loss of generality we can assume that $a = (0, 0)$. Then suppose that the expression of $f(x, y)$ is

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \ldots + f_n(x, y) \, ,$$

where $0 \leq m \leq n$ and $f_j(x, y)$ denotes a homogeneous polynomial of degree j in the variables x and y for $j = m, \ldots, n$, with f_m different from the zero polynomial. We say that $m = m_a(f)$ is the *multiplicity* of the curve $f = 0$ at the point a. This notion is well defined; it does not depend on the particular chart. If $m = 0$ then the point a does not belong to the curve $f = 0$. If $m = 1$ we say that a is a *simple* point for the curve $f = 0$. If $m > 1$ we say that a is a *multiple* point or a *singular* point of the curve.

Suppose that for $m > 0$ we have that $f_m = \prod_{i=1}^{s} l_i^{r_i}$ where the l_i are distinct linear factors. We call *tangent lines* to $f = 0$ at the point a, the lines defined by $l_i = 0$, and r_i is the *multiplicity* of the tangent line $l_i = 0$ at a. For $m > 1$ we say that a is an *ordinary multiple* point if the multiplicity of all tangents at a is 1, otherwise we say that a is a *non–ordinary multiple point*.

Consider two affine algebraic curves defined by $f = 0$ and $g = 0$. We say that these curves intersect *properly* at $a \in \mathbf{C}^2$, if the curves have no common component which pass through a. We say that $f = 0$ and $g = 0$ intersect *tranversally* at a if a is a simple point of $f = 0$ and of $g = 0$, and the tangent to $f = 0$ at a is distinct from the tangent to $g = 0$ at a. The proof of the following result can be found in [16], a key point in this proof is the fact that $\mathbf{C}$ is an algebraically closed field.

Intersection Number Theorem *There exists a unique intersection number $I_a(f, g)$ defined for any two affine algebraic curves $f = 0$ and $g = 0$ and for any point a of $\mathbf{C}^2$ satisfying the following properties:*

(i) *$I_a(f, g)$ is a non–negative integer for any f, g and a, when $f = 0$ and $g = 0$ intersect properly at a. $I_a(f, g) = \infty$ if $f = 0$ and $g = 0$ do not intersect properly at a.*

(ii) *$I_a(f, g) = 0$ if and only if a is not a common point to $f = 0$ and $g = 0$. $I_a(f, g)$ depends only on the components of $f = 0$ and $g = 0$ which pass through a.*

(iii) *If T is an affine change of coordinates and $T(a) = b$, then $I_b(f \circ T, g \circ T) = I_a(f, g)$.*

(iv) *$I_a(f, g) = I_a(g, f)$.*

(v) *$I_a(f, g) \geq m_a(f)m_a(g)$ and the equality holds if and only if $f = 0$ and $g = 0$ do not have common tangents at a.*

(vi) *$I_a(f, g_1g_2) = I_a(f, g_1) + I_a(f, g_2)$.*

(vii) *$I_a(f, g) = I_a(f, g + hf)$ for any $h \in \mathbf{C}[x, y]$.*

Furthermore, the intersection number satisfies

$$I_a(f,g) = \dim_{\mathbf{C}} \mathbf{O}_a/(f,g) \ ,$$

where $\mathbf{O}_a$ *is the local ring of the affine complex plane* $\mathbf{A}^2(\mathbf{C}) = \mathbf{C}^2$ *at* a*; i.e.* $\mathbf{O}_a$ *is the ring of rational functions* $r(x,y)/s(x,y)$ *which are defined at* a*, i.e.* $s(a) \neq 0$.

In our case, since the polynomial differential systems are quadratic, the intersection numbers $I_a(p,q)$ for p, q as in (2), at the singular points a in $\mathbf{C}^2$ can be computed easily by using the axioms.

A *projective algebraic curve* of $\mathbf{CP}^2$ is a set $\{[X : Y : Z] \in \mathbf{CP}^2 : F(X,Y,Z) = 0\}$ where $F(X,Y,Z)$ is a homogeneous polynomial of degree n in the variables X, Y and Z with coefficients in $\mathbf{C}$. We shall usually simply write $F(X,Y,Z) = 0$ (or $F = 0$) for the curve defined by F. If $A = [X : Y : Z]$ and for instance $Z \neq 0$, then we define the *multiplicity* $m_A(F)$ of the curve $F = 0$ at the point A as $m_a(f)$ where $f = F(x,y,1)$ and $a = (X/Z, Y/Z)$. It is known that the multiplicity is independent on the choice of a local projective chart (in this last case the chart corresponds to $Z \neq 0$), and of a projective change of variables; for more details see [16].

If $W = [X : Y : Z]$ and for instance $Z \neq 0$, then we define $I_W(F,G) = I_w(f,g)$ where $f = F(x,y,1)$, $g = G(x,y,1)$ and $w = (X/Z, Y/Z)$. It is known that $I_W(F,G)$ is independent of the choice of a local projective chart, and of a projective change of variables, see again [16].

Clearly the above concept of intersection multiplicity extends to that of intersection multiplicity of several curves at a point of the projective plane. In particular we will be interested in the way that the projective curves $A = 0$, $B = 0$ and $C = 0$ intersect and hence in the values of

$$I_a(A,B,C) = \dim_{\mathbf{C}} \mathbf{O}_a/(A,B,C) \ .$$

Here $\mathbf{O}_a$ is the local ring at a of the complex projective plane (for more information see [16]).

From Pal and Schlomiuk [31] we have that if a is a finite or infinite singular point of system (2) and A, B and C are defined as in (6), then $I_a(P,Q)$, $I_a(C,Z)$ and $I_a(A,B,C)$ are invariant with respect to affine transformations and

$$I_a(A,B,C) = \begin{cases} I_a(P,Q) = I_a(p,q) & \text{if } a \text{ is finite} \ , \\ I_a(P,Q) + I_a(C,Z) & \text{if } a \text{ is infinite} \ . \end{cases} \tag{9}$$

7 Zero–cycles and divisors

In this section we shall use the algebro–geometric notions of zero–cycle and divisor for the purpose of classifying systems (2). We recall the definitions of these notions.

Let V be an irreducible algebraic variety over a field K. A *cycle of dimension* r or r*–cycle* on V is a formal sum

$$\sum_W n_W W \ ,$$

where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbf{Z}$, and only a finite number of n_W are non–zero. An $(n-1)$–cycle is called a *divisor*. These notions, important in algebraic geometry [18], were used in the context of planar quadratic systems by Pal and Schlomiuk [31, 39], and they turn out to be very helpful in our context as we indicate below.

By using Section 5, to the differential system (2) we can associate a singular foliation on $\mathbf{CP}^2$ described by the equations

$$\begin{array}{rcl} \omega & = & A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ = 0 \ , \\ & & A(X,Y,Z)X + B(X,Y,Z)Y + C(X,Y,Z)Z = 0 \ , \end{array} \tag{10}$$

where

$$\begin{array}{rcl} A(X,Y,Z) & = & ZQ(X,Y,Z) = Z(XZ + aX^2 + (3l+5n)XY) \ , \\ B(X,Y,Z) & = & -ZP(X,Y,Z) = -Z(-YZ + lX^2 + 5aXY + nY^2) \ , \\ C(X,Y,Z) & = & YP(X,Y,Z) - XQ(X,Y,Z) \\ & = & -aX^3 - (2l+5n)X^2Y - X^2Z + 5aXY^2 - Y^2Z + nY^3 \ . \end{array}$$

We notice that this foliation has always an algebraic leaf, the straight line $Z = 0$. The singular points of this foliation are the solutions of the following system:

$$A(X,Y,Z) = 0 \ , \qquad B(X,Y,Z) = 0 \ , \qquad C(X,Y,Z) = 0 \ .$$

We consider a real polynomial differential system

$$\dot{x} = f(x,y) \ , \qquad \dot{y} = g(x,y) \ , \tag{11}$$

with f and g relatively prime polynomials in $\mathbf{C}[x,y]$ with $\max(\deg(f), \deg(g)) = m$. To this system we can associate several zero–cycles and divisors:

(i) Two zero–cycles which encode the information regarding the intersection numbers of the real projective curves $F = 0$ and $G = 0$ in $\mathbf{CP}^2$:

$$D_{\mathbf{K}}(F,G) = \sum_{W \in \mathbf{KP}^2} I_W(F,G)W \ ,$$

with $\mathbf{K}=\mathbf{C}$ or $\mathbf{K}=\mathbf{R}$, where

$$F(X,Y,Z)=Z^m f\left(\frac{X}{Z},\frac{Y}{Z}\right)\ , \qquad G(X,Y,Z)=Z^m g\left(\frac{X}{Z},\frac{Y}{Z}\right)\ .$$

We note that when $W\in\mathbf{RP}^2$ (respectively $W\in\mathbf{CP}^2$) the sum runs only on the real (respectively complex) intersection points of $F=0$ and $G=0$. Although the sum may run on $\mathbf{RP}^2$ in case $K=\mathbf{R}$, $I_W(F,G)$ is computed over $\mathbf{C}$.

(ii) Two divisors on the line at infinity $Z=0$ which encode the multiplicities of the intersection points of $F=0$ with $G=0$ in $\mathbf{KP}^2$ which lie on $Z=0$:

$$D_{\mathbf{K}}(F,G;Z)=\sum_{W\in\{Z=0\}\cap\mathbf{KP}^2} I_W(F,G)W\ .$$

(iii) Two zero–cycles which encode the information regarding the intersection numbers of the algebraic affine curves $f=0$ and $g=0$ in $\mathbf{K}^2$:

$$D_{\mathbf{K}}(f,g)=D_{\mathbf{K}}(F,G)-D_{\mathbf{K}}(F,G;Z)\ .$$

(iv) A divisor which encodes the multiplicities of complex (respectively real) singular points at infinity which count how many complex (respectively real) singular points at infinity will bifurcate at infinity in a complex (respectively real) perturbation of the system:

$$D_{\mathbf{C}}(C,Z)=\sum_{W\in\{Z=0\}\cap\mathbf{KP}^2} I_W(C,Z)W\ ,$$

where $C(X,Y,Z)=YF(X,Y,Z)-XG(X,Y,Z)$ and Z does not divide C.

(v) A zero–cycle which encodes the information regarding the intersection numbers of all the complex (respectively real) singularities W of the foliation in $\mathbf{CP}^2$ (respectively $\mathbf{RP}^2$) associated to a system (11):

$$D_{\mathbf{C}}=D_{\mathbf{C}}(f,g)+D_{\mathbf{C}}(F,G;Z)+D_{\mathbf{C}}(C,Z)=\sum_{W\in\mathbf{KP}^2} I_W(A,B,C)W\ ,$$

where Z does not divide C. In this last equality we have used (9).

The foliation with singularities in $\mathbf{RP}^2$ associated to the system (11) can be obtained by identifying the diametrically opposite trajectories of the

Poincaré compactification of system (11) and disregarding the orientation on the orbits. Then we can define for every isolated singular point W of the foliation in $\mathbf{RP}^2$ its *topological index*, $i(W)$, as the topological index of one of the two diametrically opposed singular points of the Poincaré compactification of system (11) which after the identification give the point W. A singular point is called *elementary* if at least one of its eigenvalues is not zero. We shall see in Proposition 7 that all finite and infinite singular points of a quadratic system having a weak focus of third order are elementary. It is well known that the indices of elementary singular points are -1, 0 or 1. For more details about the topological index see for instance [1] and [27].

(vi) We introduce a new zero–cycle DI, which will encode the topological indices of all the isolated singular points W of the foliation in $\mathbf{RP}^2$ associated to system (11) as follows:

$$DI = \sum_{W \in \mathbf{RP}^2} i(W)W \ .$$

where $i(W)$ is as defined above.

8 Zero–cycles and divisors for system (2) and the bifurcation curves

Since a system (2) corresponding to the parameter $\lambda = (a, l, n)$ with $a < 0$ is topologically equivalent to system (2) corresponding to the parameter $\lambda' = (-a, l, n)$ through the symmetry $(x, y, t) \to (-x, y, -t)$, we only consider the parameters $[a : l : n] \in \mathbf{RP}^2$ with $a \geq 0$ and $n \geq 0$. We can identify this subspace of the parameter space with the quarter of the 2–dimensional sphere $a^2 + l^2 + n^2 = 1$ of $\mathbf{R}^3 = \{(a, l, n) : a, b, c \in \mathbf{R}\}$ having $a \geq 0$ and $n \geq 0$. We can view the bifurcation diagram in the disc $\{(a, l) : a^2 + l^2 \leq 1\}$ via vertical projection. However to obtain a better picture, distinguishing more clearly the bifurcation curves, we prefer to project this quarter of sphere on the plane $n = 0$ as follows. To every point p on this quarter of sphere we associate the intersection point p' of the plane $n = 0$ with the line joining p with the point $(0, 0, -3/4)$. In this way we obtain that the parameter space of points $[a : l : n] \in \mathbf{RP}^2$, images of (a, l, n) with $a \geq 0$, $n \geq 0$ can be identified with the half–disc $D = \{(a, l) : a^2 + l^2 \leq 1$ and $a \geq 0\}$.

We apply the notions of the preceeding section to our family of real differential systems (2). Our final goal is to give a clear, geometrical classification of this class. For the real differential systems (2) with nonzero L_3

we can associate six families of zero–cycles or divisors on $Z = 0$ introduced in Section 7 which we shall also index by λ where $\lambda = [a : l : n] \in \mathbf{RP}(2)$. We shall first use the family of zero–cycles $D_{\lambda,\mathbf{C}}(p,q)$ and $D_{\lambda,\mathbf{R}}(p,q)$.

We define

$$\begin{aligned} \Omega &= \text{Resultant}(p_2(x,1), q_2(x,1), x) \\ &= n[l(3l+5n)^2 - 3a^2(5l+8n)] = n\overline{\Omega} \ . \end{aligned}$$

We remark that it is not possible to have $n = 0$ and $\overline{\Omega} = 0$ for a system (2), for otherwise $L_3 = 0$ and hence the origin is a center, not a weak focus of third order.

The differential equation (10) can have at most seven singular points in $\mathbf{CP}^2$ (see [11]), but only four (counted with multiplicities) come from the intersection of the curves $P = 0$ and $Q = 0$, namely, the origin $q_1 = [0 : 0 : 1]$ which is the weak focus of third order, the point $q_2 = [0 : 1 : n]$, and in general two additional points q_3 and q_4. If $\overline{\Omega} \neq 0$ then q_3 and q_4 are given by

$$[3a(2l+3n) + (3l+5n)\sqrt{3\delta} : 3a^2 - l(3l+5n) - a\sqrt{3\delta} : \overline{\Omega}] \ ,$$
$$[3a(2l+3n) - (3l+5n)\sqrt{3\delta} : 3a^2 - l(3l+5n) + a\sqrt{3\delta} : \overline{\Omega}] \ .$$

They are real if $\delta > 0$, or complex if $\delta < 0$, where

$$\delta = 3a^2 - l(l+2n) \ .$$

Of course, $q_3 = q_4$ if $\delta = 0$. If $\overline{\Omega} = 0$, then one of the singularities q_3 and q_4 is finite and the other $[3l + 5n : -a : 0]$ is an infinite singularity. When $\overline{\Omega} = 0$ and $L_3 \neq 0$ we have that the finite singularity is

$$[6a^2 - 3l^2 - 5ln : -al : a(2l(3l+5n) - 6a^2] \ .$$

We note that δ is a factor of the resultant of $R(y)$ with $R'(y)$, where $R(y)$ is the resultant of $p(x,y)$ and $q(x,y)$ with respect to x.

Viewed in the half disc D, the real part of the algebraic curve $\overline{\Omega} = 0$ restricted to D has three components that we denote by $\overline{\Omega}_i$, see Figure 1. The point $[0 : 1 : n]$ satisfies both $Z + aX + (3l + 5n)Y = 0$ and $P = 0$ whenever $l+2n = 0$ in which case q_2 coincides with one of the two points q_3, q_4. The line $l+2n = 0$ cuts $\overline{\Omega}_3$, and determines on $\overline{\Omega}_3$ two open components $\overline{\Omega}_{31}$ and $\overline{\Omega}_{32}$, as it is indicated in the mentioned figure.

Proposition 3. *For all values of the parameter $\lambda \in \mathbf{RP}^2$ the divisors $D_{\lambda,\mathbf{C}}(P,Q;Z)$ and $D_{\lambda,\mathbf{R}}(P,Q;Z)$ for systems (2) with $L_3 \neq 0$ are well defined. Moreover,*

$$D_{\lambda,\mathbf{C}}(P,Q;Z) = \begin{cases} 0 & \text{if } \Omega(\lambda) \neq 0 \ , \\ [3l+5n : -a : 0] & \text{if } \overline{\Omega}(\lambda) = 0 \ , \\ [0 : 1 : 0] & \text{if } n = 0 \ . \end{cases}$$

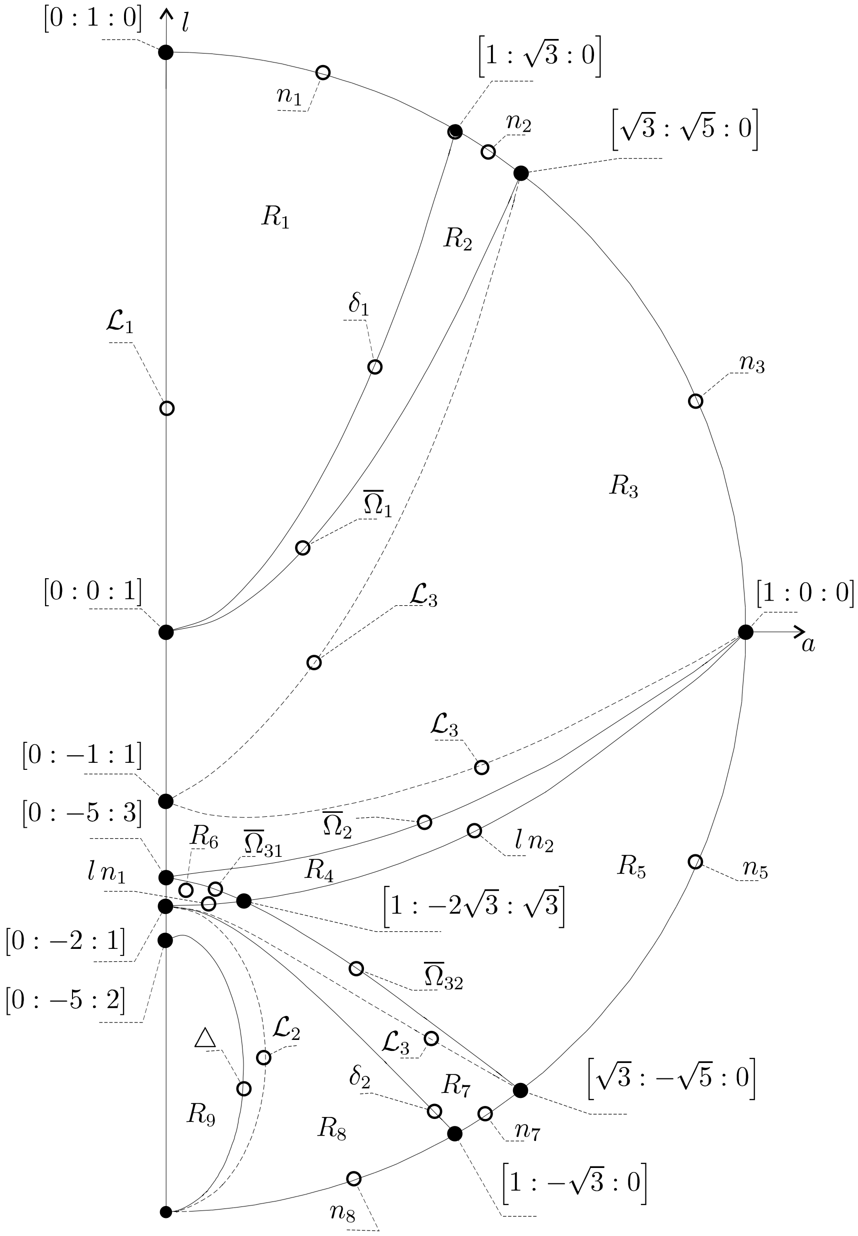

Figure 1: Bifurcation curves.

in this equality the points $[3l+5n:-a:0]$ and $[0:1:0]$ are always in the real part of the complex line at infinity. For this reason the expression of $D_{\lambda,\mathbf{R}}(P,Q;Z)$ is the same as the one indicated for $D_{\lambda,\mathbf{C}}(P,Q;Z)$.

Proof: The proposition follows easily from the expression of q_i for $i = 1,\ldots,4$, and the definition of intersection numbers. ∎

We note that in Proposition 3 when $n=0$ it follows that $\Omega \neq 0$, otherwise $L_3 = 0$. We denote by $\mathcal{L}$ the algebraic curve in $\mathbf{RP}^2$ formed by the points λ such that $L_3 = 0$.

Corollary 4. *The trace in $\mathbf{RP}^2 \setminus \mathcal{L}$ of the complex projective curve $n\overline{\Omega} = 0$ is a saddle–node bifurcation curve of infinite singularities for system (2). These saddle–nodes points at infinity appear from a collision between finite and infinite singularities.*

Proof: By using the classification of the planar singular points which have exactly one eigenvalue zero, it follows easily that the singular point q_3 when $n \neq 0$ and $\overline{\Omega} = 0$, or the singular point q_2 when $n = 0$, is a saddle–node. From the definition of the points q_i's, it is clear that the above two points are finite points on $\Omega \neq 0$ that reach the infinity on $\Omega = 0$. ∎

We denote by $c(x) = C(x,1,0)$; i.e. $c(x) = p_2(x,1) - xq_2(x,1)$, where p_2 and q_2 are the homogeneous parts of degree 2 of p and q, respectively. Let Δ be the discriminant of the cubic polynomial $c(x)$. We introduce the notation:

$$\begin{aligned}\Delta &= 125a^4 + a^2(25l^2 + 170ln + 262n^2) + n(2l+5n)^3 \\ &= \tfrac{1}{4a}\text{Resultant}(c(x), c'(x), x)\ .\end{aligned}$$

We note that it is only necessary to do the resultant with respect to x, because when a is not zero the system (2) has no infinite singular points $[X:Y:Z]$ with $Y=0$.

The complex projective curve $\delta = 0$ restricted to $\mathbf{R}^2$ has two branches which we denote by δ_i, see Figure 1. The curve $\overline{\Omega} = 0$ determines on the straight line $l + 2n = 0$ viewed on D two half–lines ln_1 and ln_2, as it is indicated in Figure 1. The algebraic curve $n\overline{\Omega}\delta(l+2n)\Delta = 0$ divides the part corresponding to the half–disc D into 9 open connected components R_i, see Figure 1. In the following discussion we omit from these components the points of D such that $L_3 = 0$. We denote by ∂R_i the boundary of R_i in D. The interior of $\partial R_i \cap \{n = 0\}$ with the topology of the straight line $n = 0$ viewed in D is denoted by n_i; consequently $n_i \subset \partial R_i$, see Figure 1. In this figure the curve $\Delta = 0$ is denoted simply by Δ.

Proposition 5. *For all values of the parameter $\lambda \in \mathbf{RP}^2$ the zero–cycles $D_{\lambda,\mathbf{C}}(P,Q)$ and $D_{\lambda,\mathbf{R}}(P,Q)$ for systems (2) with $L_3 \neq 0$ are well defined and they are given by*

$$D_{\lambda,\mathbf{C}}(P,Q) = \begin{cases} q_1+q_2+q_3+q_4 & \text{if } \delta(l+2n) \neq 0 \ , \\ q_1+q_2+2q_3 & \text{if } \delta = 0 \ , \\ q_1+2q_2+q_3 & \text{if } l+2n=0 \ . \end{cases}$$

and

$$D_{\lambda,\mathbf{R}}(P,Q) = \begin{cases} q_1+q_2+q_3+q_4 & \text{if } \delta > 0 \text{ and } l+2n \neq 0 \ , \\ q_1+q_2 & \text{if } \delta < 0 \text{ and } l+2n \neq 0 \ , \\ q_1+q_2+2q_3 & \text{if } \delta = 0 \ , \\ q_1+2q_2+q_3 & \text{if } l+2n=0 \ , \\ q_1 & \text{in } n_1 \cup n_8 \ . \end{cases}$$

Proof: Again the proposition follows easily from the expression of q_i for $i = 1, \ldots, 4$, and the definition of intersection numbers. ∎

Corollary 6. *The trace in $\mathbf{RP}^2 \setminus \mathcal{L}$ of the complex projective curve $\delta(l+2n) = 0$ is a saddle–node bifurcation of finite singularities for system (2).*

Proof: Using the classification of the planar singular points which have exactly one eigenvalue zero (see for instance [1]), it follows easily that the singular point q_3 when $\delta = 0$, or the singular point q_2 when $l + 2n = 0$, is a saddle–node. ∎

Let $p_i = [r_i : 1 : 0]$ where r_i are the roots of the polynomial $c(x)$, ordered in increasing values. From Section 4 we note that the points p_i are the singular points on the line of infinity of system (2), or simply the *infinite singular points* of system (2). Moreover, since $a \neq 0$ (see (2)), all infinite singular points of (10) are of the form $p_i = [r_i : 1 : 0]$.

Proposition 7. *For all values of the parameter $\lambda \in \mathbf{RP}^2$ the divisors $D_{\lambda,\mathbf{C}}(C,Z)$ and $D_{\lambda,\mathbf{R}}(C,Z)$ for systems (2) with $L_3 \neq 0$ are well defined and they are given by*

$$D_{\lambda,\mathbf{C}}(C,Z) = \begin{cases} p_1+p_2+p_3 & \text{if } \Delta \neq 0 \ , \\ 2p_1+p_2 & \text{if } \Delta = 0 \ . \end{cases}$$

and

$$D_{\lambda,\mathbf{R}}(C,Z) = \begin{cases} p_1+p_2+p_3 & \text{if } \Delta > 0 \ , \\ p_1 & \text{if } \Delta < 0 \ , \\ 2p_1+p_2 & \text{if } \Delta = 0 \ . \end{cases}$$

Proof: Since $c(x)$ is a real cubic polynomial, it follows that the sum of the multiplicities $I_W(C, Z)$ of all infinite singular points is 3. Since Δ is the discriminant of $c(x)$, then the proposition follows easily. We note that when $\Delta = 0$, the polynomial $c(x)$ has either one simple and one double real roots, or a triple real root. It is easy to see that this last case does not occur for our system (2). ∎

Corollary 8. *The trace in* $\mathbf{RP}^2 \setminus \mathcal{L}$ *of the complex projective curve* $\Delta = 0$ *is a saddle–node bifurcation curve of infinite singularities for system (2) when a is not zero. These saddle–nodes at infinity arise from the coalescence of two infinite singularities.*

Proof: It follows easily that the singular point p_1 when $\Delta = 0$ is a saddle–node. From the definition of the points p_i's, it is clear that the infinite singular points p_1 and p_2 collide when $\Delta = 0$. ∎

We can sum up the results of Corollaries 4, 6 and 8 as follows:

Theorem 9. *The trace in* $\mathbf{RP}^2 \setminus \mathcal{L}$ *of the complex projective curve* $\mathcal{C} = \{\lambda \in \mathbf{CP}^2 : n\overline{\Omega}\delta(l + 2n)\Delta = 0\}$ *is the saddle–node bifurcation curve for the systems (2) with weak focus of third order at the origin, see Figure 1. Furthermore,*

(a) the curve $n\overline{\Omega} = 0$ *corresponds to infinite saddle–nodes arising from the collision of one finite singularity with one infinite one;*

(b) the curve $\delta(l+2n) = 0$ *corresponds to finite saddle–nodes arising from the collision of two finite singularities;*

(c) the curve $\Delta = 0$ *corresponds to infinite saddle–nodes arising from the collision of two infinite singularities.*

The only real singularities of the curve $\mathcal{C}$ in $D \setminus \mathcal{L}$ are the three points: $\{[1 : \pm\sqrt{3} : 0]\} = \{\delta = 0\} \cap \{n = 0\}$ and $[1 : -2\sqrt{3} : \sqrt{3}] = \{\overline{\Omega} = 0\} \cap \{l + 2n = 0\}$. We denote by *Sing*$(\mathcal{C})$, the set formed by these three points.

We remark that the types of the expressions of the zero–cycles and divisors given in Propositions 3, 5 and 7 are intrinsic, i.e. independent of the normal form (2) chosen for the quadratic systems having a weak focus of third order. Indeed, this is due to the fact that the coefficients of the zero–cycles and divisors of these propositions are intersection numbers and they are affine invariants. We can sum up the results related with the complex singularities of system (2) in Propositions 3, 5 and 7 as follows:

Proposition 10. *For all values of the parameter* $\lambda \in \mathbf{RP}^2$ *the zero–cycle* $D_{\lambda,\mathbf{C}}$ *for a system (2) with* $L_3 \neq 0$ *is well defined and it is equal to*

$q_1 + q_2 + q_3 + q_4$	$+p_1 + p_2 + p_3$	*if* $\lambda \in R_1 \cup \cdots \cup R_9$,
$q_1 + q_3 + q_4$	$+2p_1 + p_2 + p_3$	*if* $\lambda \in n_1 \cup n_2 \cup n_3 \cup n_5 \cup n_7 \cup n_8$, *(*$q_2 = p_1$*)* ,
$q_1 + q_2 + 2q_3$	$+p_1 + p_2 + p_3$	*if* $\lambda \in \delta_1 \cup \delta_2$,
$q_1 + q_2 + q_4$	$+p_1 + 2p_2 + p_3$	*if* $\lambda \in \overline{\Omega}_1$*, (*$q_3 = p_2$*)* ,
$q_1 + q_2 + q_3$	$+2p_1 + p_2 + p_3$	*if* $\lambda \in \overline{\Omega}_2$*, (*$q_4 = p_1$*)* ,
$q_1 + q_2 + q_4$	$+p_1 + p_2 + 2p_3$	*if* $\lambda \in \overline{\Omega}_{31} \cup \overline{\Omega}_{32}$*, (*$q_3 = p_3$*)* ,
$q_1 + 2q_2 + q_3$	$+p_1 + p_2 + p_3$	*if* $\lambda \in ln_1 \cup \mathrm{ln}_2$,
$q_1 + q_2 + q_3 + q_4$	$+2p_1 + p_3$	*if* $\lambda \in \Delta$,
$q_1 + 2q_3$	$+2p_1 + p_2 + p_3$	*if* $\lambda = [1 : \pm\sqrt{3} : 0]$*, (*$q_2 = p_1$*)* ,
$q_1 + 2q_2$	$+p_1 + p_2 + 2p_3$	*if* $\lambda \in [1 : -2\sqrt{3} : \sqrt{3}]$*, (*$q_3 = p_3$*)* .

In the statement of Proposition 10 we idenfity the regions R_i and the curves n_i, δ_i, etc of D with their counterparts in $\mathbf{RP}^2$.

In Proposition 10 we really only described $D_{\lambda,\mathbf{C}}$ corresponding to the half–disc D introduced at the beginning of this section, because using symmetries we can obtained immediately its description for all $\lambda \in \mathbf{RP}^2$.

We now begin to introduce the basic integer–valued invariants which we shall use.

We call the *degree* of a zero–cycle of $\mathbf{C}^2$, (respectively $\mathbf{R}^2$, $\mathbf{CP}^2$, $\mathbf{RP}^2$) or of a divisor on a curve the sum of its coefficients. If J is a divisor or a zero–cycle we shall denote by $\deg(J)$, its degree.

We shall denote by $\mathbf{N_C}(S)$ the number of distinct complex singular points of the complex foliation with singularities (10) associated to a real planar quadratic system S. We shall denote by $\mathbf{N}_{\mathbf{R},f}(S)$, $\mathbf{N}_{\mathbf{R},\infty}(S)$ the number of finite, respectively infinite, distinct real singular points of the system S, by DI_f the finite part of the zero–cycle DI.

Proposition 11. *The degrees* $\deg(D_{\lambda,\mathbf{R}})$ *and* $\deg(DI_f)$ *are invariant with respect to real affine transformations. (*$\deg(D_{\lambda,\mathbf{C}}) = 7$, $\deg(DI) = 1$*)*

Proof: Since all the coefficients of $D_{\lambda,\mathbf{C}}$, $D_{\lambda,\mathbf{R}}$, DI and DI_f are invariants, the proposition follows. ∎

Our first classification result follows easily from Proposition 10.

Corollary 12. *For systems* S *in the class of quadratic systems with a weak focus of third order, the values taken by the function* $\mathbf{N_C}(S)$ *are* 7, 6 *and* 5.

For the normal form (2) we have for points corresponding to D that:

$$\mathbf{N_C}(\lambda) = \begin{cases} 7 & \textit{iff } \lambda \notin \mathcal{C} \ , \\ 6 & \textit{iff } \lambda \in \mathcal{C} \setminus \, Sing(\mathcal{C}) \ , \\ 5 & \textit{iff } \lambda \in Sing(\mathcal{C}) \ . \end{cases}$$

Following our strategy, we must now refine the above crude classification. For this we need to look at the real singularities. We can sum up the results related with the real singularities of systems (2) in Propositions 3, 5 and 7 as follows:

Proposition 13. *For all values of the parameter $\lambda \in \mathbf{RP}^2$ the zero–cycle $D_{\lambda,\mathbf{R}}$ for a system (2) with $L_3 \neq 0$ is well defined and for points corresponding to D it is equal to*

$$\begin{array}{lll}
q_1+q_2 & +p_1+p_2+p_3 & \textit{if } \lambda \in R_1 \ , \\
q_1+q_2+q_3+q_4 & +p_1+p_2+p_3 & \textit{if } \lambda \in R_2 \cup \cdots \cup R_7 \ , \\
q_1+q_2 & +p_1+p_2+p_3 & \textit{if } \lambda \in R_8 \ , \\
q_1+q_2 & +p_3 & \textit{if } \lambda \in R_9 \ , \\
q_1 & +2p_1+p_2+p_3 & \textit{if } \lambda \in n_1 \cup n_8, \ (q_2=p_1) \ , \\
q_1+q_3+q_4 & +2p_1+p_2+p_3 & \textit{if } \lambda \in n_2 \cup n_3 \cup n_5 \cup n_7 \, , \\
 & & (q_2=p_1) \ , \\
q_1+q_2+2q_3 & +p_1+p_2+p_3 & \textit{if } \lambda \in \delta_1 \cup \delta_2 \ , \\
q_1+q_2+q_4 & +p_1+2p_2+p_3 & \textit{if } \lambda \in \overline{\Omega}_1, \ (q_3=p_2) \ , \\
q_1+q_2+q_3 & +2p_1+p_2+p_3 & \textit{if } \lambda \in \overline{\Omega}_2, \ (q_4=p_1) \ , \\
q_1+q_2+q_4 & +p_1+p_2+2p_3 & \textit{if } \lambda \in \overline{\Omega}_{31} \cup \overline{\Omega}_{32}, \ (q_3=p_3) \ , \\
q_1+2q_2+q_3 & +p_1+p_2+p_3 & \textit{if } \lambda \in ln_1 \cup \ln_2 \ , \\
q_1+q_2 & +2p_1+p_3 & \textit{if } \lambda \in \Delta \ , \\
q_1+2q_3 & +2p_1+p_2+p_3 & \textit{if } \lambda = [1 : \pm\sqrt{3} : 0], \ (q_2=p_1) \ , \\
q_1+2q_2 & +p_1+p_2+2p_3 & \textit{if } \lambda \in [1 : -2\sqrt{3} : \sqrt{3}], \ (q_3=p_3) \ .
\end{array}$$

The next result follows easily from Proposition 13.

Corollary 14. *For systems S in the class of quadratic systems with a weak focus of third order, $\mathbf{N}_{\mathbf{R},f}(S)$ takes all the values from 1 to 4 and $\mathbf{N}_{\mathbf{R},\infty}(S)$ takes all the values from 1 to 3. For the normal form (2) we have for points corresponding to D that:*

$$\mathbf{N}_{\mathbf{R},f}(\lambda) = \begin{cases} 4 & \textit{iff } \lambda \in R_2 \cup \cdots \cup R_7 \ , \\ 3 & \textit{iff } \lambda \in n_2 \cup n_3 \cup n_5 \cup n_7 \cup \delta_1 \cup \delta_2 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_{31} \\ & \quad \cup \overline{\Omega}_{32} \cup ln_1 \cup \ln_2 \ , \\ 2 & \textit{iff } \lambda \in R_1 \cup R_8 \cup R_9 \cup \Delta \cup Sing(\mathcal{L}) \ , \\ 1 & \textit{iff } \lambda \in n_1 \cup n_8 \ ; \end{cases}$$

and

$$\mathbf{N}_{\mathbf{R},\infty}(\lambda) = \begin{cases} 1 & \textit{iff } \lambda \in R_9 \ , \\ 2 & \textit{iff } \lambda \in \Delta \ , \\ 3 & \textit{otherwise} \ . \end{cases}$$

The next result is about the nature of the singularities of system (2).

Proposition 15. *All finite and infinite singular points of a quadratic system with a weak focus of third order are elementary.*

Proof: We look at the normal form (2). Then it is easy to check that the linear part of the finite and infinite singular points have either determinant different from zero, or trace different from zero. So the proposition follows. ■

We recall that the local phase portraits of elementary singular points are well known and consequently also their topological indices which could only be −1, 0 or 1, see for instance [1].

Proposition 16. *For all values of the parameter $\lambda \in \mathbf{RP}^2$ the divisor DI_λ for a system (2) with $L_3 \neq 0$ is well defined and for points corresponding to D it is equal to*

$1q_1 - 1q_2$	$+1p_1 - 1p_2 + 1p_3$	*if* $\lambda \in R_1$,
$1q_1 - 1q_2 + 1q_3 - 1q_4$	$+1p_1 - 1p_2 + 1p_3$	*if* $\lambda \in R_2$,
$1q_1 - 1q_2 - 1q_3 - 1q_4$	$+1p_1 + 1p_2 + 1p_3$	*if* $\lambda \in R_3$,
$1q_1 - 1q_2 - 1q_3 + 1q_4$	$-1p_1 + 1p_2 + 1p_3$	*if* $\lambda \in R_4$,
$1q_1 + 1q_2 - 1q_3 - 1q_4$	$-1p_1 + 1p_2 + 1p_3$	*if* $\lambda \in R_5$,
$1q_1 - 1q_2 + 1q_3 + 1q_4$	$-1p_1 + 1p_2 - 1p_3$	*if* $\lambda \in R_6$,
$1q_1 + 1q_2 + 1q_3 - 1q_4$	$-1p_1 + 1p_2 - 1p_3$	*if* $\lambda \in R_7$,
$1q_1 + 1q_2$	$-1p_1 + 1p_2 - 1p_3$	*if* $\lambda \in R_8$,
$1q_1 + 1q_2$	$-1p_1$	*if* $\lambda \in R_9$,
$1q_1$	$+0p_1 - 1p_2 + 1p_3$	*if* $\lambda \in n_1,\ q_2 = p_1$,
$1q_1 + 1q_3 - 1q_4$	$+0p_1 - 1p_2 + 1p_3$	*if* $\lambda \in n_2 \cup n_7$, $q_2 = p_1$,
$1q_1 - 1q_3 - 1q_4$	$+0p_1 + 1p_2 + 1p_3$	*if* $\lambda \in n_3 \cup n_5$, $q_2 = p_1$,
$1q_1$	$+0p_1 + 1p_2 - 1p_3$	*if* $\lambda \in n_8$, $q_2 = p_1$,
$1q_1 - 1q_2 + 0q_3$	$+1p_1 - 1p_2 + 1p_3$	*if* $\lambda \in \delta_1$,
$1q_1 + 1q_2 + 0q_3$	$-1p_1 + 1p_2 - 1p_3$	*if* $\lambda \in \delta_2$,
$1q_1 - 1q_2 - 1q_4$	$+1p_1 + 0p_2 + 1p_3$	*if* $\lambda \in \overline{\Omega}_1,\ q_3 = p_2$,

$$
\begin{array}{lll}
1q_1 - 1q_2 - 1q_3 & +0p_1 + 1p_2 + 1p_3 & \textit{if } \lambda \in \overline{\Omega}_2,\ q_4 = p_1\ , \\
1q_1 - 1q_2 + 1q_4 & -1p_1 + 1p_2 + 0p_3 & \textit{if } \lambda \in \overline{\Omega}_{31},\ q_3 = p_3\ , \\
1q_1 + 1q_2 - 1q_4 & -1p_1 + 1p_2 + 0p_3 & \textit{if } \lambda \in \overline{\Omega}_{32},\ q_3 = p_3\ , \\
1q_1 + 0q_2 + 1q_3 & -1p_1 + 1p_2 - 1p_3 & \textit{if } \lambda \in ln_1\ , \\
1q_1 + 0q_2 - 1q_3 & -1p_1 + 1p_2 + 1p_3 & \textit{if } \lambda \in ln_2\ , \\
1q_1 + 1q_2 & +0p_1 - 1p_2 & \textit{if } \lambda \in \Delta\ , \\
1q_1 + 0q_3 & +0p_1 - 1p_2 + 1p_3 & \textit{if } \lambda \in [1 : \pm\sqrt{3} : 0]\,, \\
 & & q_2 = p_1\ , \\
1q_1 + 0q_2 & -1p_1 + 1p_2 + 0p_3 & \textit{if } \lambda \in [1 : -2\sqrt{3} : \sqrt{3}]\,, \\
 & & q_3 = p_3\ .
\end{array}
$$

We note that in Proposition 16 for the saddle–nodes points q_i or p_i we wrote $0q_i$ or $0p_i$, respectively, in order to draw attention to the fact that we have these singularities having topological index equal to zero. We observe that although the above values for DI_λ are computed for the canonical form (2), the types of these divisors are affine invariants. By the type of a divisor DI_λ we mean the sequence of the following numbers: the number of finite singularities; the number of infinite singularities; the numbers of finite singular points W of index respectively equal to -1, 0, 1; the numbers of infinite singularities W of index successively equal to -1, 0, 1.

We note that according with the Poincaré–Hopf Theorem (see for instance [27]) the sum of the indices of all singularities of the foliation in $\mathbf{RP}^2$ associated to system (2) is $\deg(DI) = 1$. Then, from Proposition 16 it follows immediately the next result.

Corollary 17. *For systems S in the class of quadratic systems with a weak focus of third order the values taken by the function $\deg(DI_f)(S)$ are -2, -1, 0, 1 and 2. For the normal form (2) we have for points corresponding to D that:*

$$
\deg\left[(DI_f)(\lambda)\right] = \begin{cases} -2 & \textit{iff } \lambda \in R_3\ , \\ -1 & \textit{iff } \lambda \in n_3 \cup n_5 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2\ , \\ 0 & \textit{iff } \lambda \in R_1 \cup R_2 \cup R_4 \cup R_5 \cup \delta_1 \cup ln_2\ , \\ 1 & \textit{iff } \lambda \in n_1 \cup n_2 \cup n_7 \cup n_8 \cup \overline{\Omega}_{31} \cup \overline{\Omega}_{32} \cup Sing(\mathcal{L}), \\ 2 & \textit{iff } \lambda \in R_6 \cup R_7 \cup R_8 \cup R_9 \cup \delta_2 \cup ln_1 \cup \Delta\ . \end{cases}
$$

We note that $\deg\left[(DI_f)(\lambda)\right]$ is a kind of global topological index measuring the relative number of finite saddles versus antisaddles.

9 The bifurcation diagram of the systems with a weak focus of third order

9.1 Basic properties of quadratic systems and specific properties of systems in $QW3$.

We list below results which will play a role in the study of the global phase portraits of real planar quadratic systems (2) having a weak focus of third order.

The following results hold for any quadratic system:

(i) A straight line either has at most two (finite) contact points with a quadratic system (which include the singular points), or it is formed by trajectories of the system; see Lemma 11.1 of [48]. We recall that by definition a *contact point* of a straight line L is a point of L where the vector field has the same direction as L, or it is zero.

(ii) If a straight line passing through two real finite singular points q_1 and q_2 of a quadratic system is not formed by trajectories, then it is divided by these two singular points in three segments $\overline{\infty q_1}$, $\overline{q_1 q_2}$ and $\overline{q_2 \infty}$ such that the trajectories cross $\overline{\infty q_1}$ and $\overline{q_2 \infty}$ in one direction, and they cross $\overline{q_1 q_2}$ in the opposite direction; see Lemma 11.4 of [48].

(iii) The straight line connecting a real finite singular point and a pair of real opposite infinite singular points in the Poincaré compactification of a system (2) is either formed by trajectories or is a straight line without (finite) contact points except at that finite singular point; see Lemma 11.5 of [48].

(iv) If a quadratic system has a limit cycle, then it surrounds a unique singular point, and this point is a focus; see [10].

(v) If in a quadratic system the separatrix of an infinite saddle connects with the separatrix of the diametrically opposite infinite saddle, then this separatrix is an invariant straight line; see [46].

(vi) If a quadratic system has a center, then it is integrable; i.e. there exists a nonconstant analytic first integral defined in the whole real plane except perhaps on some invariant algebraic curve; see [47], [25] and [36].

Finally we state here an important result on general quadratic systems obtained by Zhang Pingguang [49].

(vii) If there are limit cycles surrounding two foci of a quadratic system, then around one of the foci there is at most one limit cycle.

We shall also need the following specific result for quadratic systems having a weak focus of third order at the origin:

(viii) There are no limit cycles of systems (2) surrounding the weak focus of third order, see [22].

We shall now use these properties for the study of the class $QW3$.

From (iv) it follows that the limit cycles of a system (2) may only be around the singular points $(0,0)$ and $(0,1/n)$ because they are the unique possible foci of system (2) for convenient values of the parameters. Therefore, from (viii), the limit cycles of a system (2) can only be around the singular point $(0,1/n)$ in case they exist, and only when this singular point is a focus. Now, from (vii), since by perturbing conveniently a weak focus of third order it is possible to produce three limit cycles surrounding the origin, it follows that for those systems (2) which have two foci, if they have limit cycles around $(0,1/n)$, then they must have exactly one limit cycle.

From (vii) when systems (2) have a center, then we know their global phase portraits. The information about the global phase portraits of quadratic systems (2) with a center is instrumental in determining the global phase portraits of systems (2) nearby.

Modulo existence of more than one limit cycle surrounding a focus at $(0,1/n)$, the phase portraits of quadratic systems (2) were obtained by J.C. Artés in his 1984 Master's Thesis. This Master's Thesis [3] was supervised by J. Llibre and was published by the Universitat Autònoma de Barcelona. At the time of the publication in 1984, the result (vii) was not yet proved. Interest shown by some mathematicians in the work [3] determined the authors to write a new version and to publish it in a journal (cf [5]) in order to facilitate a wider access.

The phase portraits in [5] are classified in terms of inequalities on the coefficients appearing in the normal form (2). No concern was shown for obtaining intrinsic results.

In this work we are interested in the global intrinsic geometry of the systems. Also unlike the bifurcation diagram done in [5], we construct here just one picture containing all the information regarding the phase portraits. This is possible because unlike [5] or [3] we use here the real projective plane, the adequate parameter space for the problem. Our picture is obtained by using the half–disc D to represent half of the real projective plane λ, the rest is obtained by symmetry arguments.

The systems (2) are of two types:

- centers, which correspond to values of $\lambda = [a : l : n]$ for which $L_3 = 0$; or
- weak foci, which correspond to values of λ for which $L_3 \neq 0$.

The phase portraits of quadratic systems with centers are known (cf. [36, 47]). We place these phase portraits on the bifurcation curves which contain centers. These are the components of the curve $\mathcal{L}$ or $L_3 = 0$, i.e.:

- the line $\mathcal{L}_1$ defined by $a = 0$ (the diameter in Figure 2),
- the conic $\mathcal{L}_2$ (a parabola in the affine part $n \neq 0$) defined by $2a^2 + n(l + 2n) = 0$, and
- the singular cubic $\mathcal{L}_3$ defined by $3(l + n)^2(l + 2n) - a^2(5l + 6n) = 0$ with a nodal singularity at the point $[0 : -1 : 1]$.

We remark that the phase portraits with centers placed on the curves $\mathcal{L}_i$, $i = 1, 2, 3$ will play a very important role in determining the phase portraits of the quadratic systems having a weak focus of third order in the neighborhood of the curves $\mathcal{L}_i$'s.

Since for us the important object is the parameter space $\mathbf{RP}^2$, although we picture our diagram on the half–disc D, we shall mark the points on D by their corresponding points in $\mathbf{RP}^2$ given in homogeneous coordinates. Thus, for instance, the point (a, l) of D will be denoted by $[a : l : \sqrt{1 - a^2 - l^2}]$.

Systems (2) with $a = 0$ are given by

$$\dot{x} = -y + lx^2 + ny^2 \ , \qquad \dot{y} = x + (3l + 5n)xy \ .$$

These are the systems (4.10) in [36] with $b = -l$, $d = -n$ and $A = 3l + 5n = -3b - 5d$. So the line $a = 0$ is the line $A + 3b + 5d = 0$ which could be traced in Figure 5 of [36] with affine part ($d \neq 0$) in Figure 2 of [36]. We place on our Figure 2, and on $\mathcal{L}_1$ ($a = 0$) the corresponding phase portraits. We remark that the changes in phase portraits on $a = 0$ are given by the changes in the invariant algebraic curves which such systems have; namely, the straight line $(3l + 5n)y + 1 = 0$ and the conic $-l(2l + 5n)(l + 5n)x^2 - 2(l+3n) - 4l(l+3n)y + ln(l+5n)y^2 = 0$. In a similar way we put on Figures 2 and 3 all the phase portraits having a center.

9.2 Phase portraits of $QW3$ and basic features of the bifurcation diagram

We highlight below the type of arguments needed for obtaining the phase portraits on the Poincaré disc of the quadratic systems with a weak focus

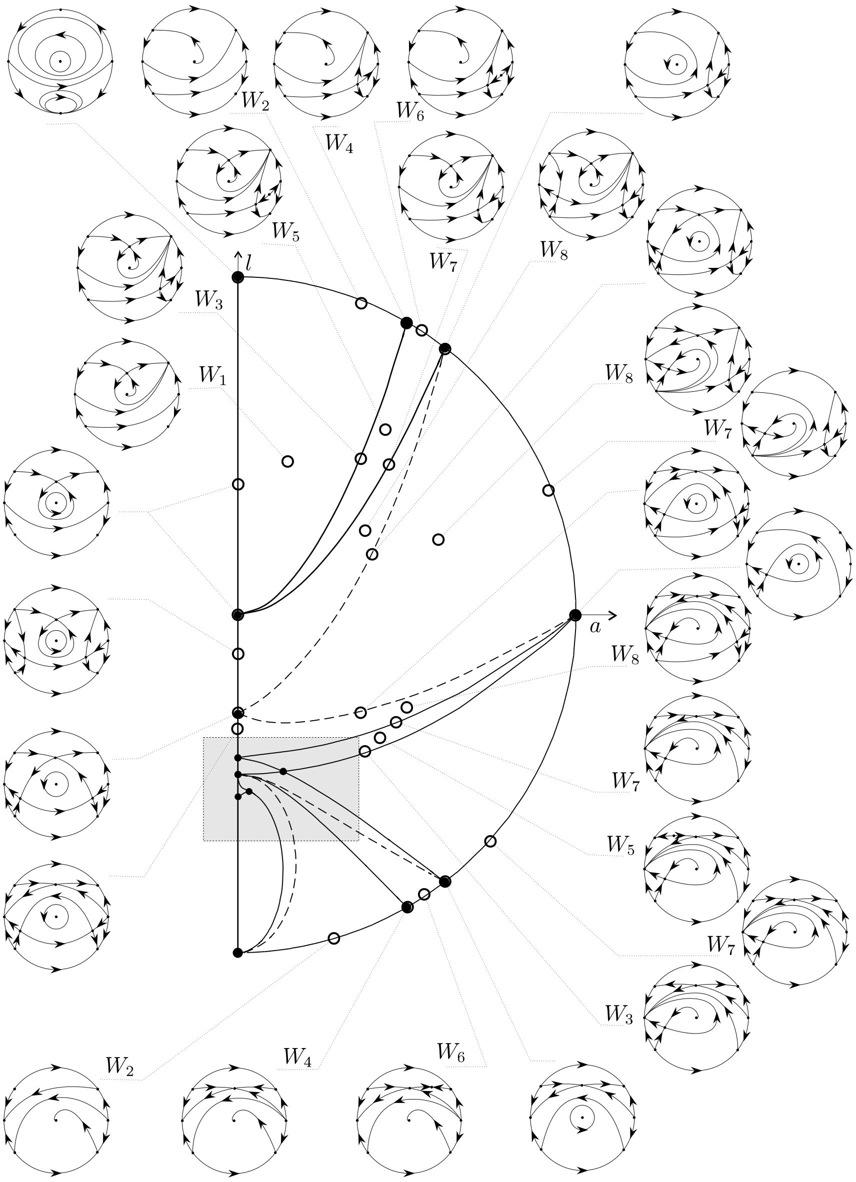

Figure 2: Phase portraits.

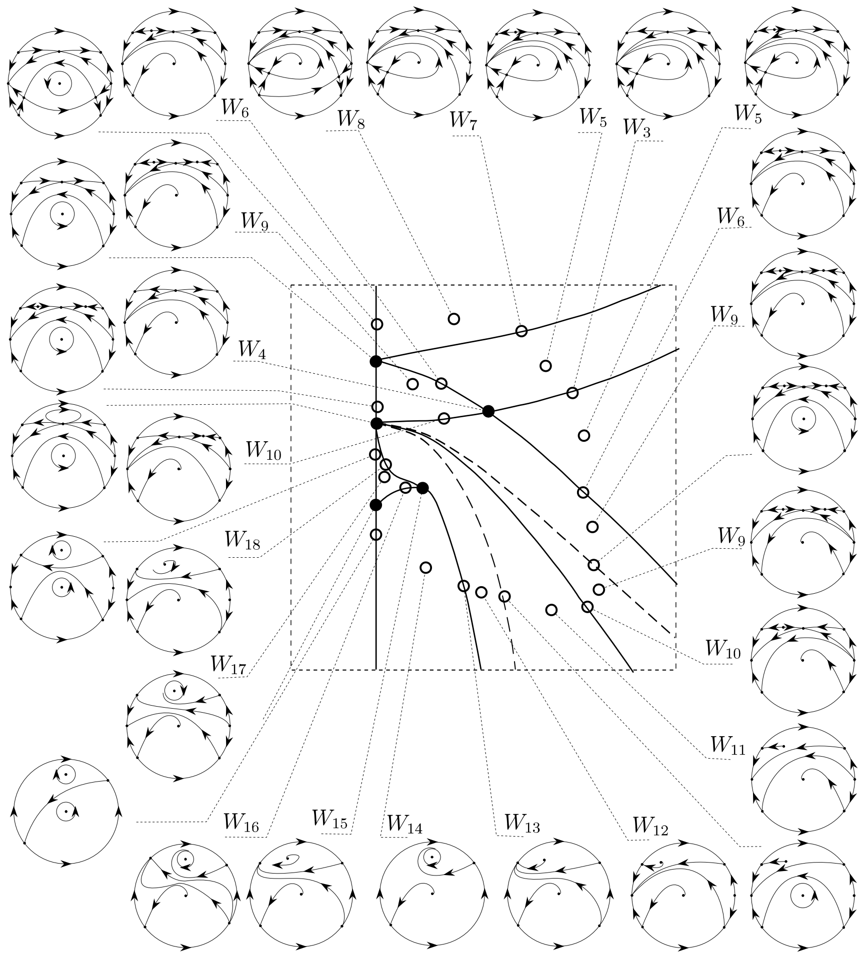

Figure 3: Phase portraits.

of third order. It is very likely that such arguments will be needed in future work on classifying quadratic systems.

First we describe how to obtain the phase portraits corresponding to values of parameters in the regions R_1 and its boundary, ∂R_1. Then, with the exception of the region R_8 and its partial boundary $\Delta = 0$, where more arguments will be needed, the phase portraits in all the remaining regions could be obtained in a similar way.

In what follows when we say for instance: the phase portrait W_1 on R_1, we mean the phase portrait W_1 which is obtained for values of the parameter λ in the region R_1.

The phase portrait W_1 on R_1 follows easily from the phase portrait corresponding to the $\partial R_1 \cap \{a = 0\}$. For obtaining it we use the behaviour of the vector field on the explicit expression of the straight invariant line, and on the straight line which connects two diametrically opposite infinite saddles.

The phase portrait W_2 on n_1 is obtained from the phase portrait W_1 on R_1 taking into account that on n_1 the saddle $[0 : 1 : n]$ of W_2 collides with an infinite node of W_2, creating a saddle–node at infinity.

The phase portrait W_3 on δ_1 follows from the phase portrait W_1 on R_1 knowing that on δ_1 there appears an additional singularity which is a finite saddle–node.

The phase portrait W_4 at $[1 : \sqrt{3} : 0]$ can be obtained easily from the phase portraits W_2 on n_1 and W_3 on δ_1. So we have described all the phase portraits on the region R_1 and its boundary points not on $a = 0$. Now we shall describe the phase portraits on the region R_8 and on $\Delta = 0$.

We start by using the curve $\mathcal{L}_2$ of centers. These centers are algebraically integrable systems which in [36] are of Class I. This curve enables us to find the phase portraits corresponding to any λ sufficiently close to $\mathcal{L}_2$. For such λ's we obtain two distinct phase portraits, one for λ such that $2a^2 + n(l + 2n) < 0$ and another one for λ such that $2a^2 + n(l + 2n) > 0$. These phase portraits are respectively W_{11} and W_{12}. For obtaining them we use the behaviour of the vector field on the algebraic curve which bounds the period annulus around the center of the phase portrait associated to $\mathcal{L}_2$ and look at how the local separatrices behave with respect to this curve in the perturbations.

In the subregion of R_8 where $2a^2 + n(l + 2n) > 0$ we have the curve $\Gamma = 25a^2 + 12(l + 2n)n = 0$ (or simply Γ) on which the singular point $(0, 1/n) = [0 : 1 : n]$ passes from being a node to a focus. This curve Γ has endpoints $[0 : -2 : 1]$ and $[0 : -1 : 0]$ and it is contained inside the part of R_8 which is limited by $\mathcal{L}_2$ and $\Delta = 0$; it does not intersect these last two curves except at its endpoints. We claim that on Γ and close to it there is no topological change in the phase portrait but only a C^∞ change when crossing Γ. Therefore the phase portrait in Γ and also close around it, is thus W_{12}.

Now we prove the claim. By Zhang Pingguang result [49], crossing Γ from a node to a focus there could appear at most one limit cycle surounding the focus, which must be semistable near the bifurcation curve due to the fact that the focus or node is unstable and there is a stable separatrix coming from infinity surounding it (see phase portrait W_{12}). But if for a system X in $QW3$ such a semistable limit cycle exists, then close to it there would be two limit cycles surrounding the focus, one stable and the other unstable, (because the $QW3$ systems form a rotated family with respect to the parameter a in an open neighborhood of Γ, see [48] for a definition and properties), and this is in contradiction with Zhang Pingguan result. So the claim is proved.

We now discuss the behaviour of the systems on $\Delta = 0$, or simply Δ. First we divide the curve Δ into three pieces. The first one, Δ_1, is the open arc of Δ having endpoints: $[0 : -1 : 0]$ and the point $\Delta_2 = [a : l : \sqrt{1 - a^2 - l^2}]$[2] of Δ where the coordinate l takes its maximum value. Finally, the third piece of Δ is the open arc Δ_3 having endpoints: Δ_2 and $[0 : -5 : 2]$.

In the region R_9 the phase portrait is W_{14} and in the region R_8 near Δ_1 is W_{12}. Then, moving from R_8 to R_9 on a small segment with $l =$ constant which intersects Δ_1, just over Δ_1 an infinite saddle and an infinite node of W_{12} collide, and by continuity we obtain on Δ_1 the phase portrait W_{13}. If we continue moving in the same direction we get the phase portrait W_{14} on R_9.

We consider now the open segment γ on $a = 0$ having endpoints $[0 : -5 : 2]$ and $[0 : -2 : 1]$. On γ we know the phase portrait because the system has two centers. So, studying the behaviour of the vector field on the branches of the hyperbola that limit the annular regions around the two centers, we obtain the phase portrait W_{17} on the points of R_8 which are sufficiently near to γ.

9.3 The saddle to saddle connection bifurcation curve $\mathcal{G}_3$

Now we consider a segment with l a constant larger than the coordinate l of the point Δ_2 in affine coordinates. This segment starts on points very near γ (see above) and ends at a point of Γ. At the endpoints of such a

[2] An easy computation shows that $\Delta_2 = [0.11275910243331191\ldots : -2.40320432 29540636\ldots : 1]$

segment we have the phase portraits W_{17} and W_{12}, respectively. Therefore, by continuity there must exist at least a point on this segment where we must have a connection between the separatrices of two infinite saddles, obtaining the phase portrait W_{18}. The uniqueness of such a point for every segment with l constant has been observed numerically. Again, by continuity, varying the constant for l of the above segment we get a continuous curve connecting the points $[0 : -2 : 1]$ and Δ_2, such that on it the phase portrait is W_{18}. We denote by $\mathcal{G}_3$ this bifurcation curve. Numerical computations shows that the curve $\mathcal{G}_3$ has the form presented in Figure 3.

Now we consider sufficiently small open segments $l =$ constant, having endpoints on both sides of the curve Δ_3. Thus at the endpoint inside R_9 the phase portrait is W_{14} and at the endpoint inside R_8 and very near Δ_3 the phase portrait is W_{17}. Then moving from W_{17} to W_{14} on this vertical segment we obtain that just over Δ_3 an infinite saddle and an infinite node of W_{17} collide on Δ_3 giving a saddle–node, and by continuity we obtain the phase portrait W_{16} on Δ_3. If we continue the motion the saddle–node at infinity disappears and we get the phase portrait W_{14}.

We note that the phase portraits W_{14}, W_{16} and W_{17} have a limit cycle, due to the stability of the strong focus and the behaviour of the separatrix whose ω–limit set tends to turn around this strong focus. The uniqueness of this limit cycle follows from previous arguments given at the beginning of this section.

Let $\mathcal{G}$ be the bifurcation curve which separates in D the phase portraits having limit cycles from those without limit cycles. Then $\mathcal{G} = \Delta_1 \cup \Delta_2 \cup \mathcal{G}_3$. We also denote $\mathcal{G}_1 = \Delta_1$ and $\mathcal{G}_2 = \Delta_2$.

From the above arguments, modulo uniqueness of the curve $\mathcal{G}_3$ which is observed numerically, we thus have obtained the following theorem.

Theorem 18. *The bifurcation diagram of the class $QW3$, viewed in the quarter of the disc D is as indicated in Figure 2 with its grey area enlarged in Figure 3.*

10 Theorems on the global geometrical classification of quadratic systems with third order foci

All the information about the dynamics of the systems is contained in the bifurcation diagram with the corresponding phase portraits for the various values of the parameters. This bifurcation diagram was done with respect to a specific normal form. Some phase portraits which appear in distinct regions of the bifurcation diagram turn out to be topologically equivalent. If the phase portraits are simple, it is easy to detect and prove their topological equivalence. However, when the number of separatrices is larger, the topological equivalence of phase portraits is not so easily detected. Thus in [5], it was claimed that we have 20 topologically distinct phase portraits. Actually, the number is 18. In identifying two of the phase portraits, we were helped by the constructed invariants. A similar coincidence occurs in [2]. Indeed, in Figure 23 (Puc. 23) of [2], the phase portraits 42 and 48 are identical.

We need to retain the essential information from the bifurcation diagram and classify the systems according to their intrinsic geometric properties. Our strategy is to first give an intrinsic stratification, grouping phase portraits into classes, according to their most basic intrinsic, geometrical properties. Secondly, more such properties will be inserted to distinguish the phase portraits in the same class, until we reach a topological classification.

A crude grouping of the systems into classes according to their properties, was given in terms of the values of $\mathbf{N_C}$ in Corollary 12 of the preceeding section. We shall now split the three classes there obtained, into finer ones by using the remaining three invariants: $\mathbf{N}_{\mathbf{R},f}$, $\deg(DI_f)$, $\mathbf{N}_{\mathbf{R},\infty}$, the four invariants being ordered according to their weight in the classification problem. More information is encoded in the divisors and zero-cycles which are in fact the principal geometric objects here. In some situations, knowing the first invariants determines the values of the remaining ones and even implies up to topological equivalence, a unique phase portrait as indicated in the next theorem. Using these four invariants we obtain a partition of the parameter space $\mathbf{RP}^2$, yielding for each class of systems with the exception of two cases a unique phase portrait, as indicated in the next result.

Theorem 19. *The following statements hold.*
(I) Consider the family $QW3$ of all quadratic systems with a weak focus of third order. The values of the affine invariant

$$\mathcal{J}(S) = (\mathbf{N_C}, \mathbf{N}_{\mathbf{R},f}, \deg(DI_f), \mathbf{N}_{\mathbf{R},\infty})\,(S)$$

given in the following diagram yield a partition of the family $QW3$[3] *as follows:*

[3] We point out that in the cases other than (i) and (ii), whenever just the first component (or the first two components) of $\mathcal{J}(S)$ suffice to yield a single phase portrait, we do not write the values of the remaining invariants.

$$
\mathbf{N}_{\mathbf{C}}(S) = \begin{cases}
7 \text{ and } \mathbf{N}_{\mathbf{R},f}(S) = \begin{cases}
2 \text{ and } \deg(DI_f(S)) = \begin{cases} 0 \ (W_1)\ , \\ 2 \text{ and } \mathbf{N}_{\mathbf{R},\infty}(S) = \begin{cases} 1 \ (W_{14}), \\ 3\ , \end{cases} \end{cases} \\
4 \text{ and } \deg(DI_f(S)) = \begin{cases} -2 \ (W_8)\ , \\ 0 \ (W_5)\ , \\ 2 \ (W_9)\ , \end{cases}
\end{cases} \\
6 \text{ and } \mathbf{N}_{\mathbf{R},f}(S) = \begin{cases}
1 \ (W_2)\ , \\
2\ , \\
3 \text{ and } \deg(DI_f(S)) = \begin{cases} -1 \ (W_3)\ , \\ 0 \ (W_7)\ , \\ 1 \ (W_6)\ , \\ 2 \ (W_{10})\ , \end{cases}
\end{cases} \\
5 \ (W_4)\ .
\end{cases}
$$

Furthermore for each value of $\mathcal{J}(S)$ in this diagram with the exception of two cases:

(i) $\mathcal{J}(S) = (7, 2, 2, 3)$ *which occurs in R_8, and*

(ii) $\mathbf{N}_{\mathbf{C}}(S) = 6$ *and* $\mathbf{N}_{\mathbf{R},f}(S) = 2$ *which occurs on Δ,*

there corresponds a single phase portrait; i.e. if S and S' are such that $\mathcal{J}(S) = \mathcal{J}(S')$ and this common value does not satisfy (i) or (ii) above, S and S' are topologically equivalent. This unique phase portrait has neither limit cycles, nor graphics (for the definition of a graphic see [14]), with the exception of the phase portrait W_{14} corresponding to $\mathcal{J}(S) = (7, 2, 2, 1)$ which has a unique limit cycle. The phase portrait W_{14} is the only one in the family $QW3$ having complex singularities at infinity (two) in addition to a real one.
(II) In the set of all quadratic systems we consider the topology of the coefficients. Consider the family formed by the systems in the class $QW3$ which have neither a limit cycle nor a graphic; i.e. whose phase portrait is different from W_i for $i = 14, 16, 17$, or $i = 13, 15, 18$, respectively. Then there exists a neighborhood of this family such that any quadratic system in this neighborhood has at most three limit cycles.

Proof: Statement (I) follows from careful analysis of the bifurcation diagram in Figures 2 and 3 of Section 9.

Now we prove Statement (II). If the perturbation is sufficiently small within the quadratic family, the only limit cycles which could be obtained are those produced by the weak focus and in view of Bautin's theorem in [6], they are at most three because a weak focus of third order cannot be surounded by a limit cycle. We note that if we want to control the limit cycles of a small perturbation of our $QW3$ systems we do not need to take care of the C^∞ bifurcation curve Γ of nodes into foci for the same arguments as those given after the definition of the curve G. ∎

We now consider the cases (i) and (ii) left out in Theorem 19. We first note that case (i) corresponds to the region denoted by R_8 and the case (ii) is the part of the boundary of R_8 denoted in Figure 1 by Δ (i.e. $\Delta = 0$). We thus need to consider the systems S whose normal form (2) corresponds to $\lambda \in R_8 \cup \Delta$ and spell out their geometrical features. We shall first define some new concepts.

Definition. *Let us consider the Poincaré compactification on the sphere. Let $\bar{H} = H \cup S^1$ where H is the upper hemisphere and S^1 is its equator. Let $D_{Sep,\infty} = \sum_{W \in S^1} s(W)W$, $D_{Sep,f} = \sum_{W \in H} s(W)W$, $D_{Sep} = \sum_{W \in \bar{H}} s(W)$ where $s(W)$ is the number of global nonequilibrium separatrices contained in S^1 (respectively in H (or in $\mathbf{R}^2$), respectively in $\bar{H}$), which start or end at W. Let $m_\infty = max_{W \in S^1}\{s(W)\}$ where $s(W)$ is defined as above.*

There is a great difference between the divisor $D_{Sep,\infty}$ on S^1, the zero-cycles $D_{Sep,f}$ on R^2, D_{Sep} on the semi-algebraic variety $\bar{H}$, and the previouly defined zero–cycles or divisors. $D_{Sep,\infty}$, $D_{Sep,f}$, D_{Sep} are the first global invariants which depend on global solutions of the phase portrait, the (global) separatrices, all the others depending only on local solutions or properties. m_∞ is a global integer-valued affine invariant.

For a good definition of a separatrix of a 2–dimensional flow see, for instance, Newman [28]. It is not difficult to prove that the only separatrices of polynomial vector fields on the Poincaré sphere are singular points, limit cycles and the boundary orbits of the hyperbolic sectors of singularities; for more details see [24].

We denote by p_i' the infinite singular point diametrically opposed to the infinite singular point p_i for the Poincaré compactification of a planar polynomial vector field.

We divide the open set R_8 as follows: R_8^u is the open subset of R_8 whose boundary is formed by the curves $\delta_2 \cup n_8 \cup \mathcal{L}_2$; R_8^d is the open subset of R_8 whose boudary is formed by the curves $\mathcal{L}_2 \cup \mathcal{G}$; and R_8^{dd} is the open subset defined by $R_8 \setminus (R_8^u \cup \text{cl}(R_8^d))$, where as usually $\text{cl}(A)$ denotes the closure of the set A.

A straightforward computation from the phase portraits of Figures 2 and 3 yields the following proposition:

Proposition 20. *For all values of the parameter $\lambda \in \mathbf{RP}^2$ the zero–cycle $D_{Sep}(\lambda)$ for a system (2) with $L_3 \neq 0$ is well defined and it is equal to*

$1q_1 + 4q_2$	$+0p_1 + 1p_2 + 3p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in R_1$,	W_1,
$1q_1 + 4q_2 + 2q_3 + 4q_4$	$+1p_1 + 1p_2 + 4p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in R_2 \cup R_4 \cup R_5$,	W_5,
$1q_1 + 4q_2 + 4q_3 + 4q_4$	$+1p_1 + 1p_2 + 2p_3 + 1p'_1 + 2p'_2 + 4p'_3$	if $\lambda \in R_3$,	W_8,
$1q_1 + 4q_2 + 2q_3 + 2q_4$	$+1p_1 + 1p_2 + 1p_3 + 1p'_1 + 2p'_2 + 1p'_3$	if $\lambda \in R_6 \cup R_7$,	W_9,
$1q_1 + 1q_2$	$+1p_1 + 1p_2 + 1p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in R_8^u$,	W_{11},
$1q_1 + 1q_2$	$+1p_1 + 0p_2 + 1p_3 + 1p'_1 + 2p'_2 + 1p'_3$	if $\lambda \in R_8^d$,	W_{12},
$1q_1 + 0q_2$	$+1p_1 + 1p'_1$	if $\lambda \in R_9$,	W_{14},
$1q_1$	$+0p_1 + 1p_2 + 1p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in n_1 \cup n_8$,	W_2,
$1q_1 + 2q_3 + 4q_4$	$+1p_1 + 1p_2 + 2p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in n_2 \cup n_7 \cup \overline{\Omega}_{31} \cup \overline{\Omega}_{32}$,	W_6,
$1q_1 + 4q_3 + 4q_4$	$+1p_1 + 1p_2 + 1p_3 + 1p'_1 + 1p'_2 + 4p'_3$	if $\lambda \in n_3 \cup n_5 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2$,	W_7,
$1q_1 + 4q_3 + 4q_3$	$+1p_1 + 1p_2 + 4p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in \delta_1 \cup ln_2$,	W_3,
$1q_1 + 4q_3 + 2q_3$	$+1p_1 + 2p_2 + 1p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in \delta_2 \cup ln_1$,	W_{10},
$1q_1 + 1q_2$	$+1p_1 + 1p_2 + 3p'_1 + 1p'_2$	if $\lambda \in \Delta_1 = \mathcal{G}_1$,	W_{13},
$1q_1 + 0q_2$	$+1p_1 + 1p_2 + 2p'_1 + 1p'_2$	if $\lambda = \Delta_2 = \mathcal{G}_2$,	W_{15},
$1q_1 + 0q_2$	$+2p_1 + 1p_2 + 2p'_1 + 1p'_2$	if $\lambda \in \Delta_3$,	W_{16},
$1q_1 + 0q_2$	$+1p_1 + 0p_2 + 1p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in \mathcal{G}_3$,	W_{18},
$1q_1 + 4q_3$	$+1p_1 + 1p_2 + 2p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in \mathrm{Sing(C)}$,	W_4,
$1q_1 + 0q_2$	$+1p_1 + 1p_2 + 1p_3 + 1p'_1 + 1p'_2 + 1p'_3$	if $\lambda \in R_8^{dd}$,	W_{17}.

When λ belongs to a union of several sets (for instance $\lambda \in R_2 \cup R_4 \cup R_5$), then the value of $D_{Sep}(\lambda)$ in Proposition 20 corresponds to λ in the first set (so in the example above $\lambda \in R_2$).

Remark. The zero–cycle D_{Sep} itself distinguishes among the 18 distinct topological phase portraits of quadratic systems having a weak focus of third order with the unique exception given by the phase portraits W_3 and W_7. These two phase portraits are different because W_3 has a saddle–node and at its corresponding place W_7 has a saddle. These phase portraits are well distinguished by the degree $\deg(DI_f)$ of the zero–cycle DI_f. More precisely $\deg(DI_f) = 0$ for W_3 and $\deg(DI_f) = -1$ forW_7.

Corollary 21. *For $\lambda \in \mathbf{RP}^2$ the degree of the zero–cycle $D_{Sep}(\lambda)$ for a system (2) with $L_3 \neq 0$ is equal to:*

12	if $\lambda \in R_1$,	W_1,	18	if $\lambda \in n_3 \cup n_5 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2$,	W_7,
20	if $\lambda \in R_2 \cup R_4 \cup R_5$,	W_5,	18	if $\lambda \in \delta_1 \cup ln_2$,	W_3,
24	if $\lambda \in R_3$,	W_8,	14	if $\lambda \in \delta_2 \cup ln_1$,	W_{10},
16	if $\lambda \in R_6 \cup R_7$,	W_9,	8	if $\lambda \in \Delta_1 = \mathcal{G}_1$,	W_{13},
8	if $\lambda \in R_8^u$,	W_{11},	6	if $\lambda = \Delta_2 = \mathcal{G}_2$,	W_{15},
8	if $\lambda \in R_8^d$,	W_{12},	7	if $\lambda \in \Delta_3$,	W_{16},
3	if $\lambda \in R_9$,	W_{14},	6	if $\lambda \in \mathcal{G}_3$,	W_{18},
6	if $\lambda \in n_1 \cup n_8$,	W_2,	12	if $\lambda \in \mathrm{Sing(C)}$,	W_4,
14	if $\lambda \in n_2 \cup n_7 \cup \overline{\Omega}_{31} \cup \overline{\Omega}_{32}$,	W_6,	7	if $\lambda \in R_8^{dd}$,	W_{17}.

Proof: This is an immediate computation from Proposition 20. ∎

We note that the degree of the zero–divisor $D_{Sep}(\lambda)$ can be even or odd. When it is even it means that every separatrix associated to a hyperbolic sector has been counted twice. This is due to the fact that those separatrices start and end at different singular points. When it is odd (only for the λ's belonging to R_9, Δ_3 and R_8^{dd}) it means that one of such separatrices ends or starts at a limit cycle, and consequently it is counted only once in the degree of $D_{Sep}(\lambda)$.

We now define the full integer–valued invariant which classifies topologically all systems in $QW3$:

$$\mathcal{I} = (\mathbf{N_C}, \mathbf{N_{R,\mathit{f}}}, \deg(DI_f), \mathbf{N_{R,\infty}}, m_\infty) \ .$$

As it follows from Proposition 20 and from Theorem 19, this invariant distinguishes the 18 distinct topological phase portraits of quadratic systems having a weak focus of third order. In particular, it distinguishes all the cases covered by (i) and (ii), left out in Theorem 19. In short, we have proved the following theorem.

Theorem 22. *Let S and S' be quadratic systems having a weak focus of third order. Then $\mathcal{I}(S) = \mathcal{I}(S')$ if and only if S and S' are topologically equivalent.*

We now sum up the global geometrical characteristics we have obtained for the class $QW3$.

Theorem 23. *The class $QW3$ is partitioned in the following three subclasses:*

(I) Systems without a limit cycle and without graphics. These systems yield a total of twelve phase portraits which are: W_i with $i = 1, \ldots, 12$. The systems with W_i, $i = 1, \ldots, 10$ are classified by $\mathcal{J}(S)$ (see Theorem 19). $\mathcal{J}(S) = (7, 2, 2, 3)$ for W_{11} and for W_{12} and these phase portraits are distinguished by m_∞: $m_\infty = 1$ for W_{11} and $m_\infty = 2$ for W_{12}.

(II) Systems with a limit cycle. These have no graphic and the limit cycle is unique. These systems yield a total of three phase portraits which are topologically classified by $\mathcal{J}(S)$. More precisely we have:

(II.1) $\mathcal{J}(S) = (7, 2, 2, 1)$, phase portrait W_{14};

(II.2) $\mathcal{J}(S) = (7, 2, 2, 3)$, phase portrait W_{17};

(II.3) $\mathcal{J}(S) = (6, 2, 2, 2)$, phase portrait W_{16}.

The third case occurs as a bifurcation from (II.2) to (II.1), when two of the three points at infinity collide. For each one of these cases we have a unique (up to topologically equivalence) phase portrait. (In Figure 3 the region where we have limit cycles is delimited by the curves $\mathcal{G}$ and $a = 0$: $\lambda \in R_9 \cup \Delta_3 \cup R_8^{dd}$).

(III) Systems with a graphic. These have no limit cycle and the graphic is unique, surrounding a strong focus. We have a total of three phase portraits in this class. These are: W_{18} *with* $\mathcal{J}(S) = (7, 2, 2, 3)$ *and* W_{13}*,* W_{15}*, both with* $\mathcal{J}(S) = (6, 2, 2, 2)$*.* W_{13} *is distinguished from* W_{15} *by* m_∞*. For* W_{13} $m_\infty = 3$*, for* W_{15} $m_\infty = 2$*.*

We note that if $\mathcal{J}(S) = (7, 2, 2, 3)$, S could be of any of the types (I), (II), (III).

It is worthwhile noting that inside the region of systems with limit cycles, on Δ_3, we have only one bifurcation curve of phase portraits, where we have two separatrices connecting two opposite points at infinity which are both saddle–nodes.

We remark that all graphics in quadratic systems are known, see [14], and their interior in the Poincaré disc are convex sets. This last claim can be proved easily following the proof that the interior of any limit cycle of a quadratic system is a convex set, see for instance [10].

Finally, in the next two propositions we point out relations among some integer–valued invariants attached to systems in $QW3$.

Proposition 24. *Consider a quadratic system* S *with a weak focus of third order.*

(a) If $\mathbf{N_C}(S) = 7$*, then* S *has either* 2 *or* 4 *finite real singularities.*

(b) Suppose $\mathbf{N_C}(S) = 6$*. Then either all singularities are real and we have* $N_{\mathbf{R},f}(S) = 3 = \mathbf{N}_{\mathbf{R},\infty}$*, or we have two singularities which are not real. In this case either* $N_{\mathbf{R},f}(S) = 2 = \mathbf{N}_{\mathbf{R},\infty}$ *or* $N_{\mathbf{R},f}(S) = 1$*,* $\deg(DI_f) = 1$ *and* $\mathbf{N}_{\mathbf{R},\infty} = 3$*.*

(c) If $\mathbf{N_C}(S) = 5$*, then* $N_{\mathbf{R},f}(S) = 2$*,* $\mathbf{N}_{\mathbf{R},\infty} = 3$ *and* $\deg(DI_f) = 1$*.*

Let S be a quadratic system with a weak focus of third order. We denote by $\mathbf{N}_c(S)$ the number of canonical regions of S, and by $\mathbf{N}_{fs}(S)$ the number of finite saddles or of finite saddle-nodes of S.

Sometimes we shall write $\mathbf{N}_c$ and $\mathbf{N}_{fs}$ instead of $\mathbf{N}_c(S)$ and $\mathbf{N}_{fs}(S)$, respectively.

Proposition 25. *If* S *is a quadratic system with a weak focus of third order, then the following statements hold.*

(a) With the exception of the bounded canonical region inside a limit cycle, all canonical regions of a quadratic system with a third order focus are unbounded.

(b) $2 \leq \mathbf{N}_c \leq 9$ *and* $N_c(S)$ *takes all the values from 1 to 9 except 6 and 8.*

(c) *If S has a limit cycle, then $\mathbf{N}_c$ is even. If $\mathbf{N}_c = 2$ we have a unique phase portrait W_{14}. $N_c = 4$ occurs only for W_{16} and W_{17}. $m_\infty = 2$ for both but this maximum is attained at two distinct points for W_{16}, the only phase portrait where this happens.*

(d) *If S has a graphic, then $\mathbf{N}_c = 3$.*

(e) *If S has neither limit cycles, nor graphics, then $\mathbf{N}_c$ takes all the values 3, 5, 7 or 9 and we have $\mathbf{N}_c = 3 + 2\mathbf{N}_{fs}$. More precisely we have:*

  (e3) *If $\mathbf{N}_c = 3$ then $\mathbf{N}_{\mathbf{R},f}(S)$ is either 1 or 2, and all the finite singularities have topological index 1. Hence $\mathbf{N}_{fs} = 0$. In this case there are 3 possible phase portraits W_2, W_{11}, W_{12}.*

  (e5) *If $\mathbf{N}_c = 5$ then S has either a unique finite saddle, or a unique finite saddle–node so $\mathbf{N}_{fs}(S) = 1$. In this case there are 5 possible phase portraits W_1, W_4, W_6, W_9 and W_{10}.*

  (e7) *$\mathbf{N}_c = 7$ if and only if $\mathbf{N}_{fs}(S) = 2$. In this case there are 3 possible phase portraits W_3, W_5 and W_7.*

  (e9) *$\mathbf{N}_c = 9$ if and only if $\mathbf{N}_{fs}(S) = 3$. In this case there is a unique phase portrait W_8.*

11 Comparison with other results obtained in the literature and a view towards future work

The quadratic systems with a third order focus were first studied in [3] (written in catalan) and then in [2] (written in russian). [5] is a new version, in english, of [3].

In these articles the authors constructed phase portraits and gave (two distinct!) bifurcation diagrams, for the class $QW3$. These studies were done with respect to specific charts and normal forms.

We were interested in doing a much more geometric study, using invariants built from pure geometric objects which have a clear dynamic meaning. Thus for example the degree of the divisor $D_{\lambda,\mathbf{C}}(P,Q;Z)$ tells us how many finite singularities split from the singularities at infinity of the system, in a quadratic perturbation of the system with parameters λ.

The fact that we work with invariants was instrumental in determining that the number of topologically distinct phase portraits of the class $QW3$ is 18 and not 20 as it was claimed in [5]. The topological coincidence of two of the phase portraits in [20] was observed only after we saw that these portraits were put into the same class by the first components of the integer–valued invariant $\mathcal{I}(S)$. The two phase portraits were denoted by W_6 and W_9 in the notation of [5]. This supports the view that the construction of invariants helps us identifying or distinguishing (depending on the case) phase portraits. The amount of detail on a picture is such that the eye alone cannot easily detect these differences. The multiplicity divisors, real and complex as well as the global topological index $\deg(DI_f)$, helped us see that two portraits assumed to be topologically distinct in [5] turned out to be equivalent. Two other phase portraits in the list in [5] turn out to be topologically equivalent: W_{13} and W_{14}. These are in fact distinct C^∞ phase portraits (we have a node and a focus in one and two foci in the other) but topologically equivalent.

For the proof of these results we first determined the bifurcation diagram, which was also obtained in [5]. Apart from what we mentioned above, our diagram is an improvement over the bifurcation diagram given in [5] because we constructed it in the adequate parameter space for this study: the real projective plane, allowing us to view it on just one picture: a disc with opposite points on its boundary identified.

In [2] Andronova also studied the class $QW3$ and constructed a bifurcation diagram for this class. This was done in $\mathbf{R}^3$ by taking sections of the space obtained by making l (denoted by k in [2]) constant for $l = 0$ or $l \neq 0$. However her arguments on pages 122 and 123 of [2] where she deals with the more delicate cases, encountered by us in the part of the half disc which we denoted by $\Delta \cup R_8$, are incomplete. She says: *If the separatrices coalesce, then there must exist still another bifurcation curve, at each point of which, a separatrix of a saddle (or of a saddle–node) goes from it to a saddle. If the coalescence of separatrices does not occur, then on the curve* $\Delta = 0$ *for* $|a| > -l/(12\sqrt{5})$ *there could exist only the decomposition* 36, ... *It is exactly in this last case that we get the bifurcation diagram in Figure 22*[4]. This paragraph contains an "if... if" situation and she chooses one of the two cases and gives the bifurcation diagram for this case. We see numerically that actually it is the other situation which occurs. Due to this, her description of the phase portraits on the bifurcation curve where we only have two points at infinity, (our curve Δ in Figure 1), is correct in only one part of this curve, as she says that the phase portraits on $\Delta = 0$ with the exception of only one point, are all topologically equivalent. Actually on the above mentioned curve we have three phase portraits. The phase portrait we denoted here by W_{13} is completely missing from Andronova's list. Although she notes the presence of a bifurcation curve of connections in the

[4]This is an almost word for word translations of the authors of Andronova's russian text

area which in our Figure 3 is “inside” $\mathcal{L}_2$ but not “inside” Δ, she does not observe that this curve may actually coincide with Δ at some of the points. We see numerically that this happens and so on the curve $\mathcal{G}$ we have three phase portraits instead of just two. The numerical computations performed in [5] which could be made as accurate as we wish were very helpful is seeing more clearly what actually occurs. Andronova did not mention that she relied on numerical computations and we believe she did not. This work shows clearly the importance of numerical work in such studies. Finally, we recall that her phase portraits 42 and 48 in her Figure 23 (Puc. 23) are actually identical.

We shall now indicate some directions for future work. The class QW3 was a case study for us. We chose this class for two reasons: firstly because these systems naturally seem to be the first next case in line after the center case, systems which where studied in [47], [36] and [30]. Secondly, this class plays a significant role in Hilbert’s 16–th problem. Indeed, all the quadratic systems we know which have most limit cycles (four) which can be shown to exist are obtained by perturbing systems in $QW3$. The first known examples of quadratic systems for which it was possible to show that they have at least four limit cycles are the example of Shi Songling [41] and that of Chen and Wang [8]. Both these examples are produced by perturbing systems S in the region of $QW3$ determined by $\mathcal{J}(S) = (7, 2, 2, 1)$. In the early 1980’s it became clear from the work [3] that there are two other distinct subsets of $QW3$ from where two new types of phase portraits with at least four limit cycles could be obtained by perturbations of systems in the class $QW3$. In this work we see these sets clearly on Figures 2 and 3. They are R_9, R_8^{dd} and Δ_3. The region where we have limit cycles is bounded by the curve $\mathcal{G}$ and by points on $a = 0$ which correspond to symmetric systems with two centers. On $a = 0$, on one side of the bifurcation point the phase portraits have an invariant hyperbola which at the bifurcation point $[0 : -5 : 2]$ becomes a double line. The bifurcation point corresponding to a system where the invariant hyperbola on one side becomes a double line.

Clearly other classes of low degree polynomial differential systems could be studied along the lines we developped in this article. On the other hand, to gain insight in the global geometry of quadratic differential systems, other global problems about this class need to be considered. An example is the problem of classifying the class of quadratic differential systems according to the topology of their phase curves in the neighborhood of the infinity. This problem was treated in [40].

The invariants we used in this work (based on divisors and zero–cycles) are very simple and have a clear dynamic meaning. In contrast to these simple invariants, the algebraic invariants and comitants used in the work of K.S. Sibirsky and his school (cf. [44]) are much more complicated. These invariants and comitants are tensorially defined using the multi–index notation and rather artificial looking polynomial expressions are defined from these invariants and comitants. Classifications are done using then these polynomial expressions, see for example [29]. Conditions on the parameters are then expressed in the form of equalities or inequalities involving these polynomial expressions. This method is technically very complex and the geometrical meaning is mostly missing. The power of the method resides in its computational aspect: no matter how the system may be presented, independent of particular chart and normal form, one can compute the corresponding invariants and comitants and see the class the system belongs to. Furthermore, the invariants and comitants can be programmed on a computer. We have therefore the motivation to relate in future work our geometrical invariants with the algebraic invariants and comitants for the study of the class $QW3$. Such a parallel was drawn in [40] for the problem of classifying quadratic vector fields according to their behaviour in the neighbourhood of infinity. In view of the computational power of the methode of algebraic invariants, for the algebraic part of the classification problems or for classifications of an algebraic nature, such connections need to be made for other classes of systems or for other problems on low degree polynomial vector fields.

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