Approximation to the $\mathcal{F}$-killing length of a space.

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Abstract

In this paper, we give estimates for the $\mathcal{F}$-killing length of a space in terms of the cone-length and of the weak category.

In [1], the authors introduced a new homotopy invariant for spaces:

Definition *Let $\mathcal{F}$ be a family of spaces. A $\mathcal{F}$-killing decomposition of X with length m is a sequence of cofibrations $L_i \longrightarrow X_i \longrightarrow X_{i+1}$, $0 \leq i < m$, such that $X_0 \simeq X$, $X_m \simeq *$, and each L_i is a wedge of spaces belonging to $\mathcal{F}$. The $\mathcal{F}$-killing length of X, $kl_{\mathcal{F}}(X)$, is defined as follows. If X is contractible, then $kl_{\mathcal{F}}(X) = 0$; otherwise $kl_{\mathcal{F}}(X)$ is the smallest integer m such that there exists an $\mathcal{F}$-killing decomposition of X with length m.*

In this paper, we give upper and lower bounds for this invariant in terms of two classical invariants: The cone-length and the weak L.S. category. Recall that the cone-length of a space X, $Cl\, X$, is the least integer n such that X have the homotopy type of a space X' which can be covered by $(n+1)$ *contractible open sets* (see [3] or [6] for instance) and that the L.S. category of a space X, denoted by $cat\, X$, is the least integer n such that X can be covered by $(n+1)$ *open sets which are contractible in* X (see [6] for instance). For example, $Cl\, X \leq 1$ if and only if X is a suspension and $cat\, X \leq 1$ if and only if X is a co-H-space.

Denote by $T_1(X^n)$ the set of n-tuples $(x_1, ..., x_n) \in X^n$ such that at least one x_i is the base point. The Ganea map $p_n : G_n(X) \to X$ is the

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homotopy pullback of the inclusion $T_1(X^{n+1}) \hookrightarrow X^{n+1}$ along the diagonal map $\Delta : X \to X^{n+1}$.

- The weak L.S. category of a space X, $wcat\, X$, is the least integer n such that the composition $X \to X^{n+1} \to X^{n+1}/T_1(X^{n+1})$ is nullhomotopic;
- the weak Ganea L.S. category of a space X, $wcat_G\, X$, is the least integer n such that the quotient map $X \to X/G_n(X)$ is nullhomotopic;
- the cup-length of X, $c_{\Bbbk}(X)$, is the greatest integer n such that there exists a sequence $(\alpha_j)_{0\leq j\leq n}$ in $H^+(X;\Bbbk)$ with $\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_n$ non trivial;
- the Toomer invariant, $e_{\Bbbk}(X)$, is the least integer n such that the Ganea map $p_n : G_n(X) \to X$ is injective in cohomology with coefficient in $\Bbbk$.

Clearly, we have $c_{\Bbbk}(X) \leq e_{\Bbbk}(X) \leq cat\ X \leq Cl\ X$ and $wcat\, X \leq wcat_G\, X \leq cat\, X$. In fact, the Toomer invariant and the weak category coincide in the rational case.

Proposition 1

(a) $e_{\Bbbk}(X) \leq wcat_G\, X$,

(b) $e_{\mathbb{Q}}(X) = wcat_G\, X_0$,

(c) $c_{\mathbb{Q}}(X) < e_{\mathbb{Q}}(X)$, *then* $wcat\, X_0 < e_{\mathbb{Q}}$.

Following Arkowitz and Strom, we want to approximate the $\mathcal{F}$-killing length of a space in terms of category-like homotopy invariants.

Let $\lceil a \rceil$ be the least integer greater than or equal to a. In [1], M. Arkowitz and J. Strom prove that $kl_{\Sigma}(X) \leq \lceil log_2(\frac{Dim\, X+1}{n}) \rceil$ when X is $(n-1)$-connected and they suggest that $kl_{\Sigma}(X) \leq \lceil log_2(cat\, X+1) \rceil$. In fact, we have:

Theorem 2 *If $\mathcal{F}$ contains all spaces L with $Cl\ L \leq p-1$, then we have $kl_{\mathcal{F}}(X) \leq \lceil log_p(Cl\, X+1) \rceil$. In particular, if Σ is the family of all suspensions, we have $kl_{\Sigma}(X) \leq \lceil log_2(Cl\, X+1) \rceil$.*

Remark. In the previous formula, we can not replace Cl by cat like suggested by M. Arkowitz and J. Strom. For instance, consider an co-H-space X which is not a suspension; Then $kl_{\Sigma}(X) > 1$, but $\lceil log_2(cat\, X+1) \rceil = 1$.

To get lower bounds for $kl_{\mathcal{F}}$ in terms of $wcat$, we need to evaluate $wcat\, X$ in a cofibration $A \xrightarrow{i} X \xrightarrow{f} B$. We obtain

Proposition 3 *Consider a cofibration* $A \xrightarrow{i} X \xrightarrow{f} B$*. Then*

$$wcat\, X \leq (wcat\, A + 1)(wcat\, B + 1) - 1\,.$$

In [2], M. Arkowitz and J. Strom prove $kl_\Sigma(X) \geq log_2(wcat\, X + 1)$. We extend their result to more general families $\mathcal{F}$.

Theorem 4 *If each* $F \in \mathcal{F}$ *verifies* $wcat\, F \leq p - 1$*, then* $kl_\mathcal{F}(X) \geq log_p(wcat\, X + 1)$.

Example 1. There is a cofibration (see section 2 of this paper)

$$\bigvee_{i=1}^{7} S^3 \longrightarrow X \longrightarrow \bigvee_{i=1}^{12} S^6 \vee \bigvee_{i=1}^{12} S^9 \vee S^{11} \vee S^{14}$$

with $wcat_G\, X = e_\mathbb{Q}(X) = 4$. That implies that

- $wcat\, X = 3$ by Proposition 3,
- Proposition 3 can not be extended to $e_\Bbbk X$ or $wcat_G$,
- $kl_\Sigma X = 2$,
- Theorem 4 can not be extended to $wcat_G$.

Example 2. Let X be a formal space. Since $c_\mathbb{Q}(X) = wcat\, X_0 = cat\, X_0 = Cl\, X_0$, we have $kl_\mathcal{F}(X) = \lceil log_p(cat\, X + 1) \rceil$

Exemple 3. In [6, Example 1 pp. 432], the authors give an example of a space X with $e(X) = 2$ and $cat\ X = \infty$. Moreover, they construct a cofibration $\bigvee_\alpha S_\mathbb{Q}^{n_\alpha} \longrightarrow X \longrightarrow \bigvee_\beta S_\mathbb{Q}^{n_\beta}$ which shows that $kl_\Sigma(X) = 2$. This example shows that Proposition 3 can not be extended to *cat* and that there is not hope to find lower bound of $kl_\Sigma(X)$ in terms of $cat\, X$.

1 Proofs

Proof of Proposition 1: (a). Consider the homotopy cofibration

$$G_n X \xrightarrow{p_n} X \xrightarrow{q_n} X/G_n X$$

and remark that $H^*(p_n)$ is injective if and only if $\tilde{H}^*(q_n) = 0$. Then $e_\Bbbk(X) \leq wcat_G\, X$.

(b). Let $(\Lambda V, d)$ be the minimal model of X and denote by $p'_n : E_n(X) \to X$ a fibration homotopy equivalent to the geometric realization of the canonical projection $(\Lambda V, d) \to (\Lambda V/\Lambda^{>n}V, d)$. By [4] or [5], there exists continuous map ϕ and ψ,

$$\begin{array}{ccc} G_nX & \overset{\psi}{\underset{\phi}{\rightleftarrows}} & E_n(X) \\ & {\scriptstyle p_n}\searrow \quad \swarrow{\scriptstyle p'_n} & \\ & X & \end{array}$$

with $p'_n \circ \psi = p_n$ and $p_n \circ \phi = p'_n$ Then $q_n : X \to X/G_n(X)$ is nullhomotopic if and only if $q'_n : X \to X/E_n(X)$ is nullhomotopic. A model of the cofibration

$$E_n(X) \longrightarrow X \longrightarrow X/E_n(X)$$

is given by

$$(\mathbb{Q} \oplus \Lambda^{>n}V, d) \longrightarrow (\Lambda V, d) \longrightarrow (\Lambda V/\Lambda^{>n}V, d))$$

Denote by $\theta : (\Lambda W, d) \longrightarrow (\mathbb{Q} \oplus \Lambda^{>n}V, d)$ the Sullivan minimal model of $(\mathbb{Q} \oplus \Lambda^{>n}V, d)$. Let $(\Lambda W, d) \longrightarrow (\Lambda W \otimes \Lambda \bar{W}, D)$ be the relative Sullivan model of the path fibration on X ([6]). We construct by induction on a basis of W a morphism θ' such that the following diagram commutes

$$\begin{array}{ccccc} (\mathbb{Q} \oplus \Lambda^{>n}V, d) & \xrightarrow{\mu} & (\Lambda V, d) & \longrightarrow & (\Lambda V/\Lambda^{>n}, d)) \\ \uparrow{\scriptstyle \theta} & & \uparrow{\scriptstyle \theta'} & & \\ (\Lambda W, d) & \longrightarrow & (\Lambda W \otimes \Lambda \bar{W}, D) & & \end{array} .$$

By commutativity, $\theta' = \theta$ on W. Suppose to have constructed θ' on $\bar{W}_{<n}$ and $x \in \bar{W}_n$. Since $Dx \in \Lambda^+ W \otimes \Lambda \bar{W}$, the cocycle $\theta'(Dx)$ belongs to $\Lambda^{>n}V$ and is therefore a coboundary, i.e., there exists $\alpha \in \Lambda V$ with $\theta'(Dx) = d\alpha$. We put $\theta'(x) = \alpha$.

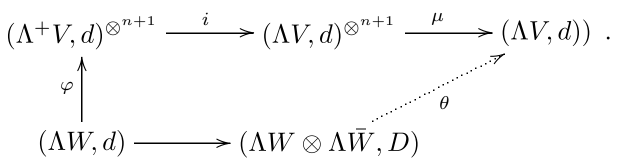
(c). We suppose now that $c_{\mathbb{Q}}(X) < e_{\mathbb{Q}}(X)$. Denote by $(\Lambda V, d)$ the Sullivan minimal model of X, then *wcat* X is the least integer n such that the composition

$$(\Lambda W, D) \xrightarrow{\varphi} (\Lambda^+ V, d)^{\otimes^{n+1}} \xrightarrow{i} (\Lambda V, d)^{\otimes^{n+1}} \xrightarrow{\mu} (\Lambda V, d)$$

is homotopically trivial. Here φ is the minimal model of $(\Lambda^+ V, d)^{\otimes^{n+1}}$, i is the canonical inclusion and μ is the multiplication.

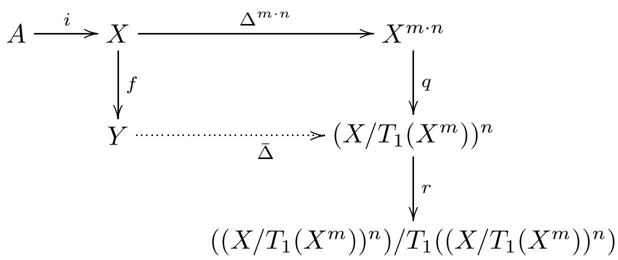
Let $w \in W$. Then either w is a cocycle, or else Dw is decomposable. In the second case, $\varphi Dw \in (\Lambda^{\geq 2}V, d)^{\otimes^{n+1}}$ and $\varphi Dw = D\gamma$ with $\gamma \in \Lambda^{\geq r_1}V \otimes \Lambda^{\geq r_2}V \otimes \ldots \otimes \Lambda^{\geq r_{n+1}}V$, and $r_i \geq 2$ for all i except maximum one. We can therefore suppose $\varphi(w) = \gamma$ so that $\mu \circ i \circ \varphi(w) \in \Lambda^{\geq 2n+1}V$.

Suppose $n+1 = e_{\mathbb{Q}}(X) = wcat\, X_0$. We show that we can extend the map $\mu \circ i \circ \varphi$ to the acyclic differential graded algebra $(\Lambda W \otimes \Lambda \bar{W}, D)$



If $w \in W$ is a cocycle, then $\mu \circ i \circ \varphi(w)$ is a sum of products of $n+1$ cocycles, and is a coboundary, because $c_{\mathbb{Q}}(X) < e_{\mathbb{Q}}(X)$. This defines $\theta(\bar{w})$. If $D(w) \neq 0$, then $D\bar{w} - w \in \Lambda^+ W \otimes \Lambda^+ \bar{W}$, and $\theta(D\bar{w}) \in \Lambda^{\geq n+2}V$. There exists therefore $\beta \in \Lambda V$ such that $\theta(D\bar{w}) = D\beta$, and we put $\theta(\bar{w}) = \beta$. ■

Proof of Proposition 3:
Suppose that $wcat\, A = n-1$ and $wcat\, B = m-1$ and consider the following diagram



Remark that $q \circ \Delta^{m\cdot n} \circ i$ factorize through $A \to A/T_1(A^n)$ and since $wcat\, A = n-1$, $q \circ \Delta^{m\cdot n} \circ i$ is nullhomotopic. Thus $q \circ \Delta^{m\cdot n}$ extends in a map $\bar{\Delta} : Y \to (X/T_1(X^m))^n$. Since $r \circ \bar{\Delta}$ factorize through $Y/T_1(Y^m)$, $r \circ \bar{\Delta}$ is nullhomotopic. Finally, remark that $((X/T_1(X^m))^n)/T_1((X/T_1(X^m))^n)$ is $X^{m\cdot n}/T_1(X^{m\cdot n})$ and that the quotient map $r \circ q$ is nullhomotopic. Therefore $wcat\, X \leq m \cdot n - 1$. ■

Proof of Theorem 2: Let X be a space with $Cl X \leq n$ and let $(Y_j)_{0\leq j\leq n}$ be a covering of X by contractible open sets. Let

$$L = (Y_0 \cup Y_1 \cup ... \cup Y_{p-1}) \vee (Y_p \cup ... \cup Y_{2p-1}) \vee ... \vee (Y_{kp} \cup ... \cup Y_n).$$

Note that $Cl\,L \leq p-1$. Let C be the homotopy cofiber of the canonical map $L \longrightarrow X$ and remark that C can be covered by the open contractible subsets $Cone(Y_0 \cup Y_1 \cup ... \cup Y_{p-1})$, $Cone(Y_p \cup ... \cup Y_{2p-1})$, ... , $Cone(Y_{kp} \cup ... \cup Y_n)$. Then $Cl\, C + 1 \leq \lceil \frac{Cl\, Y+1}{p} \rceil$. Let $m = \lceil log_p(Cl\, X + 1) \rceil$. Apply inductively m times the previous construction to X and remark that $\left\lceil \frac{\lceil \frac{a}{p} \rceil}{p} \right\rceil = \left\lceil \frac{a}{p^2} \right\rceil$. We obtain a sequence of cofibrations $L_i \longrightarrow X_i \longrightarrow X_{i+1}$, $L_i \in \mathcal{F}$, $X_0 = X$ such that

$$Cl\, X_m + 1 \leq \left\lceil \frac{Cl\, X_{m-1} + 1}{p^2} \right\rceil \leq \left\lceil \frac{Cl\, X_{m-2} + 1}{p^2} \right\rceil \leq ... \leq \left\lceil \frac{Cl\, X + 1}{p^m} \right\rceil = 1$$

Then $X_m \simeq *$ and $kl_{\mathcal{F}}(X) \leq m$.

■

Proof of Theorem 2:

Suppose $kl_{\mathcal{F}}(X) = n$ and consider a sequence of cofibrations $L_i \longrightarrow X_i \longrightarrow X_{i+1}$ with $X_0 \simeq X$ and $X_n \simeq *$. Proposition 3 gives the following sequence of inequalities

$$wcat\, X_0 + 1 \leq p \cdot (wcat X_1 + 1) \leq \ldots \leq p^n (wcat\, X_n + 1) = p^n.$$

Thus, $kl_{\mathcal{F}}(X) \geq log_p(wcat\, X + 1)$. ■

2 Description of Example 1

We construct a space X with $e_{\mathbb{Q}}(X) = 4$, $wcat\, X = 3$ and $kl_{\Sigma}(X) = 2$.

To define X, we first consider the Sullivan minimal model $(\Lambda Z, d)$ with

- $Z = (a, b, c_1, c_2, d_1, d_2, u, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$,
- a, b, c_1, c_2, d_1 and d_2 are in degree 3;
- the x_i and u are in degree 5,
- $du = c_1 b - d_1 d_2$, $dx_1 = ad_1$, $dx_2 = ad_2$, $dx_3 = bc_2$,
- $dx_4 = bd_1$, $dx_5 = bd_2$, $dx_6 = c_1 c_2$, $dx_7 = c_1 d_1$ et $dx_8 = c_1 d_2$.

This differential algebra is bigraded, i.e. $Z = Z_0 \oplus Z_1$ $dZ_0 = 0$, $dZ_1 \subset \Lambda Z_0$, with $Z_0 = (a, b, c_1, c_2, d_1, d_2)$ and $Z_0 = (u, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$.

Therefore $H^*(\Lambda Z, d)$ is also bigradued : $H^* = \oplus_{n\geq 0} H^*_n$. It is easy to verify that $H_0 = H^0_0 \oplus H^3_0 \oplus H^6_0$. Moreover the algebra H^* contains 34 linearly independant generators in bidegree $(1, 8)$ and at least a generator in bidegree $(1, 11)$ represented by $w = uac_2 + x_1 d_2 c_2 + abx_6$. We introduce new generators in degrees ≥ 2 in order to obtain on bigraded differential algebra $(\Lambda T, d)$ satisfying $H_{\geq 2}(\Lambda T, d)$, $dim(H_1(\Lambda T, d)) = 1$ and such that a generator of $H_1(\Lambda T, d)$ is represented by w. In particular, T^7_2 is a vector space of dimension 34 introduced to kill $H^8_1(\Lambda Z, d)$. We denote by S the geometric realization of $(\Lambda T, d)$. We get $\dim(H^3(S; \mathbb{Q})) = 6$, $\dim H^6(S; \mathbb{Q}) = 6$, $\dim H^{11}(S; \mathbb{Q}) = 1$ and $H^r(S; \mathbb{Q}) = 0$ for $r \neq 0, 3, 6, 11$.

The space X is the space $S^3 \times S$. Since $e_{\mathbb{Q}}(S)$ is the maximal n such that there is a non trivial class in $\Lambda^{\geq n} T$, we have $e_{\mathbb{Q}}(S) = 3$ and $e_{\mathbb{Q}}(X) = 4$. A minimal model for the space X is $(\Lambda T, d) \otimes \Lambda t, 0)$ with t in degree 3.

We now prove that the homotopy cofiber C of a map $\phi : \bigvee_{i=1}^{7} S^3 \longrightarrow X$ which induces an isomorphism on $H^3(-, \mathbb{Q})$, is rationally a wedge of spheres. This shows that $kl_\Sigma(X) = 2$ and that $wcat\, X \leq 3$. Since $c_{\mathbb{Q}}(X) = 3$, we have $wcat\, X = 3$.

A model for the cofiber C is given by the subalgebra (A, d) of $(\Lambda T, d) \otimes (\Lambda t, 0)$ where

$$A = \mathbb{Q} \oplus t.\Lambda T \oplus \Lambda^{\geq 2} T \oplus T_{\geq 1}$$

$$H^*(A) = \mathbb{Q} \oplus t.H^3_0 \oplus H^6_0 \oplus t.H^6_0 \oplus H^{11}_1 \oplus t.H^{14}_1$$

Thus the reduced homology is concentrated in degrees 6, 9, 11 and 14 and for degree reasons all products are zero.

Denote by $(\Lambda R, D)$ the Sullivan minimal model of the graded vector space $(H^*(A, d), 0)$. The differential algebra $(\Lambda R, D)$ is the Sullivan minimal model of $\bigvee_{i=1}^{12} S^6 \vee \bigvee_{i=1}^{12} S^9 \vee S^{11} \vee S^{14}$.

It is well know that the differential D in $(\Lambda R, D)$ satisfies [7] :

- $R = \bigoplus_{i\geq 0} R_i$
- $D(R_i) \subset (\Lambda^2 R)_{i-1}$
- $H^*_0(\Lambda R, D) = H^*(\Lambda R, D)$

Therefore $R_0 = R^{\geq 6}_0$, $R_1 = R^{\geq 11}_1$, $R_2 = R^{\geq 16}_2$... The choice of a basis of the cohomology of (A, d) defines a map $\phi : (\Lambda R_0, d) \longrightarrow (A, d)$ such that $\phi : R_0 \longrightarrow \tilde{H}^*(A, d)$ is an isomorphism. Since the products are zero in $H^*(A, d)$, ϕ can be extended to a map $\phi : (\Lambda R_{\leq 1}, d) \longrightarrow (A, d)$. Now, since $R_{\geq 2} = R^{\geq 16}_{\geq 2}$ and $H^{\geq 17}(A, d) = 0$, there is no obstruction to extend ϕ into a map $\phi : (\Lambda R, d) \longrightarrow (A, d)$. Since $H^*(\Lambda R, D) \cong \mathbb{Q} \oplus R$, $H^*(\phi)$ is an isomorphism.

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