Besov regularity of stochastic integrals with respect to the fractional Brownian motion with parameter $H > 1/2$

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Abstract

Let $\{B_t, t \in [0,1]\}$ be a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Using the techniques of the Malliavin calclulus we show that the trajectories of the indefinite divergence integral $\int_0^t u_s \delta B_s$ belong to the Besov space $\mathcal{B}_{p,q}^{\alpha}$ for all $q \geq 1$, $\frac{1}{p} < \alpha < H$, provided the integrand u belongs to the space $\mathbb{L}^{p,1}$. Moreover, if u is bounded and belongs to $\mathbb{L}^{\delta,2}$ for some even integer $p \geq 2$ and for some δlarge enough, then trajectories of the indefinite divergence integral $\int_0^t u_s \delta B_s$ belong to the Besov space $\mathcal{B}_{p,\infty}^{H}$.

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1 Introduction

Let $B = \{B_t, t \in [0,1]\}$ be a fractional Brownian motion with Hurst parameter H. If $H \neq \frac{1}{2}$, the process B is not a semimartingale and we cannot apply the stochastic calculus developed by Itô in order to define stochastic integrals with respect to B. Different approaches have been used in order to define stochastic integrals $\int_0^t u_s dB_s$. From $E|B_t - B_s|^2 = |t-s|^{2H}$ it follows that B has α-Hölder continuous paths for all $\alpha < H$. As a consequence, if the process u has β-Hölder continuous paths with $\alpha + \beta > 1$, then we can define the pathwise integral $\int_0^t u_s dB_s$ using Young's approach (see [24]).

One can weaken the regularity assumptions on the process u using Besov spaces. The trajectories of the fractional Brownian motion belong to the Besov space $\mathcal{B}^H_{p,\infty}$ if $\frac{1}{p} < H$. In [7] Ciesielski, Kerkyacharian and Roynette (see also [20]) have proved that if the process u has trajectories in the Besov space $\mathcal{B}^{1-H}_{p,1}$, where $\frac{1}{p} < H < 1 - \frac{1}{p}$, then the integral process $\int_0^t u_s dB_s$ has its paths in the Besov space $\mathcal{B}^H_{p,\infty}$.

A different approach to construct stochastic integrals with respect to the fractional Brownian is based on the Malliavin calculus. In fact, as in the case of the Brownian motion, the divergence operator with respect to B can be interpreted as a stochastic integral called the Skorohod integral [23]. This idea has been developed by Decreusefond and Üstünel [9], Carmona and Coutin [6], Alòs, Mazet and Nualart [2, 3], Duncan, Hu and Pasik-Duncan [10] and Hu and Øksendal [11]. The integral constructed by this method has zero mean, and can be obtained as the limit of Riemann sums defined using Wick products. The divergence integral, denoted by $\int_0^t u_s \delta B_s$, turns out to be equal to the symmetric integral in the sense of Russo and Vallois [21] plus an absolutely continuous term (see [4] when $H > \frac{1}{2}$ and [1] for the case $H < \frac{1}{2}$). Unlike the pathwise approach, in this method the symmetric (pathwise) integral exists under regularity assumptions of the process u in the sense of the Malliavin calculus, and no regularity on the paths of u is assumed.

In this framework, one can study the Besov regularity of the indefinite divergence integral $X_t := \int_0^t u_s \delta B_s$. In the case of the standard Brownian motion ($H = \frac{1}{2}$) Lorang [16] has proved that if $u \in L^{\delta,2}$ for some even integer $p \geq 2$ and $\delta > 2(p+1)$ and u is bounded then the indefinite Skorohod integral X belongs a.s. to the Besov space $\mathcal{B}^{1/2}_{p,\infty}$. This result generalizes the work by Roynette (see [20]) in the adapted case. A generalization of this result has been obtained by Berkaoui and Ouknine in [5] where it is required that $u \in L^{p+1,2}$ for some even integer $p \geq 4$.

The aim of this paper is to apply the methodology introduced by Lorang in [16] to the case of the fBm with Hurst parameter $H > \frac{1}{2}$. In order to carry out this program we make use of the stochastic calculus for the divergence integral with respect to the fBm with parameter $H > \frac{1}{2}$ developed in [4]. We show in Theorem 4 that the indefinite divergence integral $\int_0^t u_s \delta B_s$ belong to the Besov space $\mathcal{B}^\alpha_{p,q}$ for all $q \geq 1$, $\frac{1}{p} < \alpha < H$, if the integrand u belongs to the space $\mathbb{L}^{p,1}$. In Theorem 5 we prove that $\int_0^t u_s \delta B_s$ belongs to the Besov space $\mathcal{B}^H_{p,\infty}$ if the process u is bounded and belongs to the space $\mathbb{L}^{\delta,2}$ for some even integer $p \geq 4$ and $\delta > \frac{2(p-1)}{1-H} \vee \frac{2}{2H-1}$. The Besov regularity of the indefinite divergence integral with respect to the fBm with parameter $H < \frac{1}{2}$ is studied in the paper [14].

The paper is organized as follows. In Section 2 we recall some definitions on the Besov norms and spaces. Section 3 is devoted to present the main facts of the Malliavin calculus and stochastic integrals with respect to the fractional Brownian motion. Finally Section 4 contains our main results and their proofs.

2 Preliminaries on Besov spaces

The modulus of continuity of a function $f\colon [0,1] \to \mathbb{R}$ in the L^p norm, where $1 \leq p < \infty$, is defined as

$$\omega_p(f,t) = \sup_{|h| \leq t} \left(\int_{I_h} |f(s+h) - f(s)|^p \, ds \right)^{\frac{1}{p}},$$

where $I_h = \{s \in [0,1] : s+h \in [0,1]\}$. The Besov spaces $\mathcal{B}^\alpha_{p,q}$, where $\alpha \in [0,1]$ and $1 \leq p, q < \infty$ are Banach spaces of functions on $[0,1]$ equipped with the norm

$$\|f\|_{\alpha,p,q} = \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}} + \left(\int_0^1 \left(\frac{\omega_p(f,t)}{t^\alpha} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

If $q = \infty$, then the Besov space $\mathcal{B}^\alpha_{p,\infty}$ is the set of functions f in $L^p([0,1])$ such that

$$\sup_{0<t<1} \frac{\omega_p(f,t)}{t^\alpha} < \infty.$$

Let $\{\varphi_n, n \geq 1\}$ the Schauder basis on $[0,1]$ defined by $\varphi_1(s) = s$, and

$$\varphi_n(s) = 2^{-\left(\frac{j}{2}+1\right)} \varphi(2^{j+1}s - 2k + 1),$$

for $n = 2^j + k$, $j \in \mathbb{N}$, and $k = 1, \ldots, 2^j$, with $\varphi(u) = \max(1 - |u|, 0)$. Let $\{\chi_n, n \geq 1\}$ the orthonormal Haar basis in $L^2([0,1])$, with $\chi_1 = 1$, and

$\chi_{jk} = \chi_n$ for $n = 2^j + k$. That is

$$\chi_n = 2^{j/2}\mathbf{1}_{\left[\frac{k-1}{2^j}, \frac{2k-1}{2^{j+1}}\right)} - 2^{j/2}\mathbf{1}_{\left[\frac{2k-1}{2^{j+1}}, \frac{k}{2^j}\right)}.$$

We know that for any continuous function f on $[0,1]$ vanishing at zero we have

$$f(s) = \sum_{n=1}^{\infty} f_n \varphi_n(s)$$

where $f_1 = f(1)$ and

$$f_n = f_{jk} = 22^{\frac{j}{2}} \left[f\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2}\left(\left(\frac{2k}{2^{j+1}}\right) + \left(\frac{2k-2}{2^{j+1}}\right)\right)\right]$$

for $n = 2^j + k$.

In [7] Ciesielski, Kerkyacharian and Roynette have proved that for $\alpha > \frac{1}{p}$ and $q < \infty$ we have

$$\|f\|_{\alpha,p,q} \backsim \max\left(|f_1|, \left\{ \sum_{j\geq 0} 2^{-jq\left(\frac{1}{2}-\alpha+\frac{1}{p}\right)} \left(\sum_{k=1}^{2^j} |f_{jk}|^p \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \right),$$

and for $q = \infty$, the l^q norm is replaced by the l^∞ norm.

3 Stochastic integral with respect to the fractional Brownian motion

Fix $H \in (1/2, 1)$. Let $B = \{B_t, t \in [0,1]\}$ be a fractional Brownian motion with parameter H. That is, B is a zero mean Gaussian process with the covariance

$$R_H(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1}$$

We assume that B is defined in a complete probability space $(\Omega, \mathcal{F}, P)$.

We denote by $\mathcal{E} \subset \mathcal{H}$ the set of step functions on $[0,1]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with B. We will denote this isometry by $\varphi \longrightarrow B(\varphi)$.

It is easy to see that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr, \tag{2}$$

where $\alpha_H = H(2H-1)$.

The Hilbert space $\mathcal{H}$ coincides with the space of distributions f such that $s^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}}(s^{H-\frac{1}{2}} f)$ is a square integrable function, where $I_{0+}^{H-\frac{1}{2}}$ is the left-sided fractional integral of order $H-\frac{1}{2}$ (see [22]). The continuous embedding $L^{\frac{1}{H}}([0,1]) \subset \mathcal{H}$ has been proved in [17].

The process $B = \{B_t, t \in [0,T]\}$ is Gaussian and, hence, we can develop a stochastic calculus of variations (or Malliavin calculus) with respect to it (see [4]). Let us recall the basic notions of this calculus.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{3}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$. The derivative operator D of a smooth and cylindrical random variable F of the form (3) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i.$$

The derivative operator D is then a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. For any integer $k \geq 1$ we denote by D^k the iteration of the derivative operator. For any $p \geq 1$ the Sobolev space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = \mathbb{E}|F|^p + \mathbb{E} \sum_{j=1}^k \left\| D^j F \right\|_{\mathcal{H}^{\otimes j}}^p.$$

In a similar way, given a Hilbert space V we denote by $\mathbb{D}^{k,p}(V)$ the coresponding Sobolev space of V-valued random variables.

The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E} \langle DF, u \rangle_{\mathcal{H}}.$$

We denote by Dom δ the domain of the divergence in L^2, which contains the space $\mathbb{D}^{1,2}(\mathcal{H})$.

The divergence operator satisfies the following property:

$$\mathbb{E}\delta(u)\delta(v) = \mathbb{E}\left\langle u, v\right\rangle_{\mathcal{H}} + \mathbb{E}\left\langle Du, (Dv)^{*}\right\rangle_{\mathcal{H}\otimes\mathcal{H}}.$$

where $(Dv)^{*}$ denotes the adjoint of Dv in $\mathcal{H}\otimes\mathcal{H}$. This allows us to deduce the estimate

$$\begin{aligned} |\mathbb{E}\delta(u)\delta(v)| \leq & \alpha_H \mathbb{E}\int_0^1\int_0^1 |r-s|^{2H-2}|u_r v_s| drds \\ & + \alpha_H^2 \int_{[0,1]^4} |D_r u_\theta D_s u_t|\, |r-t|^{2H-2}|\theta-s|^{2H-2} drd\theta dsdt. \end{aligned} \tag{4}$$

We have the continuous embedding $L^{\frac{1}{H}}([0,1]^2) \subset \mathcal{H}\otimes\mathcal{H}$. For all $p>1$ we denote by $\mathbb{L}_H^{p,1}$ the set of stochastic processes $u = \{u_t, t\in[0,1]\}$ such that

$$\|u\|_{\mathbb{L}_H^{p,1}}^p := \mathbb{E}\,\|u\|_{L^{1/H}([0,1])}^p + \mathbb{E}\,\|Du\|_{L^{1/H}([0,1]^2)}^p < \infty.$$

Notice as a consequence of Meyer inequalities (see for example [18]) for all $p>1$ the space $\mathbb{L}_H^{p,1}$ is included in the domain of the divergence in L^p and for all $u\in\mathbb{L}_H^{p,1}$ we have (see [4])

$$\mathbb{E}\left|\delta\left(u\right)\right|^p \leq C_{p,H}\left(\mathbb{E}\,\|u\|_{L^{1/H}([0,1])}^p + \mathbb{E}\,\|Du\|_{L^{1/H}([0,1]^2)}^p\right). \tag{5}$$

The symmetric integral introduced by Russo and Vallois in [21] of a stochastic process $u=\{u_t, t\in[0,1]\}$ with integrable trajectories is defined as the limit in probability as ε tend sto zero of

$$(2\varepsilon)^{-1}\int_0^1 u_s(B_{s+\varepsilon} - B_{s-\varepsilon})ds,$$

and it is denoted by $\int_0^1 u_t dB_t$. By convention we will assume that all processes and functions vanish outside the interval $[0,1]$. The following proposition (see [4]) shows that the divergence operator can also be interpreted as a stochastic integral. We recall that the in the case of the classical Brownian motion (i.e. if $H=\frac{1}{2}$) then the divergence operator coincides with an extension of the Itô integral for anticipating processes introduced by Skorohod (see [23]).

Proposition 1 *Let $u\in\mathbb{L}_H^{2,1}$, and assume that a.s.*

$$\int_0^1\int_0^1 |D_s u_t|\,|t-s|^{2H-2}\,dsdt < \infty. \tag{6}$$

Then the symmetric integral of u in any inteval $[0,t]\subset[0,1]$ exists and we have

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + \alpha_H \int_0^t \int_0^1 D_r u_s \left|r-s\right|^{2H-2} drds, \tag{7}$$

where $\int_0^t u_s \delta B_s = \delta(1_{[0,t]}u)$ denotes the indefinite divergence integral.

For any $p>1$ we define the seminorm

$$\|u\|_{\mathbb{L}^{p,1}}^p = \mathbb{E}\int_0^1 |u_s|^p ds + \mathbb{E}\int_0^1\int_0^1 |D_r u_s|^p dsdr,$$

and we define the space $\mathbb{L}^{p,1}$ as the set of processes $u=\{u_t, t\in[0,1]\}$ such that $\|u\|_{\mathbb{L}^{p,1}}<\infty$. Notice that if $pH\geq 1$

$$\|u\|_{\mathbb{L}_H^{p,1}} \leq \|u\|_{\mathbb{L}^{p,1}}$$

and $\mathbb{L}^{p,1}\subset\mathbb{L}_H^{p,1}$. If u belongs to $\mathbb{L}^{p,1}$ and $pH>1$ then the indefinite divergence integral $\int_0^t u_s\delta B_s$ possesses a continuous version (see [4]). We also denote by $\mathbb{L}^{p,2}$ the space of processes such that $\|u\|_{\mathbb{L}^{p,2}}<\infty$, where

$$\begin{aligned}\|u\|_{\mathbb{L}^{p,2}}^p =& \mathbb{E}\int_0^1 |u_s|^p ds + \mathbb{E}\int_0^1\int_0^1 |D_r u_s|^p dsdr \\ &+ \mathbb{E}\int_{[0,1]^3} |D_\theta D_r u_s|^p dsdrd\theta.\end{aligned}$$

The following Itô's formula has also been proved in [4]:

Theorem 2 *Let F be a function of class $C^2(\mathbb{R})$. Assume that $u=\{u_t, t\in[0,1]\}$ is a process in the space $\mathbb{L}^{2,2}$. Assume that $\|u\|_2$ belongs to $\mathcal{H}$. Then for each $t\in[0,T]$ the following formula holds*

$$\begin{aligned}F(X_t) =& F(0) + \int_0^t F'(X_s) u_s \delta B_s \\ &+ \alpha_H \int_0^t F''(X_s)\, u_s \left(\int_0^1 |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right) d\sigma\right) ds \\ &+ \alpha_H \int_0^t F''(X_s) u_s \left(\int_0^s u_\theta \,(s-\theta)^{2H-2}\, d\theta\right) ds.\end{aligned}$$

The above Itô's formula holds assuming only that the process u belongs locally to the space $\mathbb{D}^{2,2}(|\mathcal{H}|)$ introduced in [4] which contains the space $\mathbb{L}^{2,2}$.

4 Besov regularity of the divergence integral

Fix $pH \geq 1$, and consider a process $u \in \mathbb{L}^{p,1}$. Define $X_t = \int_0^t u_s \delta B_s$, and for each $j \geq 1$, and $1 \leq k \leq 2^j$, set

$$X_{jk}(t) = \int_0^t \chi_{jk}(s) u_s \delta B_s.$$

We denote by $\mathbf{1}_{jk}$ the indicator function of the set $\left[\frac{k-1}{2^j}, \frac{k}{2^j}\right)$. We will make use of the following technical lemma.

Lemma 3 *For all $p > \frac{1}{H}$ and $\delta \geq 1$ we have*

$$\begin{aligned}\mathbb{E}\left|X_{jk}(t)\right|^p \leq & C_{p,H} 2^{jp\left(\frac{1}{2}-H+\frac{H}{\delta}\right)} \left\{ \mathbb{E}\left(\int_0^1 \mathbf{1}_{jk}(s) |u_s|^{\frac{\delta}{H}} ds \right)^{\frac{pH}{\delta}} \right. \\ & \left. + \mathbb{E}\left(\int_0^1 \int_0^1 \mathbf{1}_{jk}(s) |D_r u_s|^{\frac{\delta}{H}} ds dr \right)^{\frac{pH}{\delta}} \right\},\end{aligned}$$

where $C_{p,H}$ is the constant appearing in (5).

Proof. Using the estimate (5) for the moment of order p of the divergence operator we can write

$$\begin{aligned}\mathbb{E}\left|X_{jk}(t)\right|^p = & \mathbb{E}\left| \int_0^t \chi_{jk}(s) u_s \delta B_s \right|^p \\ & \leq C_{p,H} \left\{ \mathbb{E}\left(\int_0^1 |\chi_{jk}(s)|^{\frac{1}{H}} |u_s|^{\frac{1}{H}} ds \right)^{pH} \right. \\ & \left. + \mathbb{E}\left(\int_0^1 \int_0^1 |\chi_{jk}(s)|^{\frac{1}{H}} |D_r u_s|^{\frac{1}{H}} ds dr \right)^{pH} \right\}.\end{aligned}$$

Assume $\delta > 1$. Applying Hölder inequality with $\frac{1}{\delta} + \frac{1}{\delta'} = 1$ yields

$$\begin{aligned}\mathbb{E}\left|X_{jk}(t)\right|^p \leq & C_{p,H} \left(\int_0^1 |\chi_{jk}(s)|^{\frac{\delta'}{H}} ds \right)^{\frac{pH}{\delta'}} \left\{ \mathbb{E}\left(\int_0^1 \mathbf{1}_{jk}(s) |u_s|^{\frac{\delta}{H}} ds \right)^{\frac{pH}{\delta}} \right. \\ & \left. + \mathbb{E}\left(\int_0^1 \int_0^1 \mathbf{1}_{jk}(s) |D_r u_s|^{\frac{\delta}{H}} ds dr \right)^{\frac{pH}{\delta}} \right\}.\end{aligned}$$

Finally, the desired result follows from the equality

$$\left(\int_0^1 |\chi_{jk}(s)|^{\frac{\delta'}{H}}\, ds\right)^{\frac{pH}{\delta'}} = 2^{jp\left(\frac{1}{2}-H+\frac{H}{\delta}\right)}.$$

The case $\delta = 1$ is obvious because $|\chi_{jk}(s)| \le 2^{j/2}$. ■

As a consequence of Lemma 3 we get the following inequality:

$$\mathbb{E}\,|X_{jk}(t)|^p \le C_{p,H} 2^{jp\left(\frac{1}{2}-H+\frac{H}{\delta}\right)} \|u\|^p_{\mathbb{L}^{\delta/H,1}}\,. \tag{8}$$

On the other hand, taking $\delta = pH$, we obtain

$$\sum_{k=1}^{2^j} \mathbb{E}\,|X_{jk}(t)|^p \le C_{p,H} 2^{jp\left(\frac{1}{2}-H+\frac{1}{p}\right)} \|u\|^p_{\mathbb{L}^{p,1}}\,. \tag{9}$$

Theorem 4 *Let $0 < \alpha < H$, $\frac{1}{\alpha} < p < \infty$, and $u \in \mathbb{L}^{p,1}$. Then the trajectories of the indefinite divergence integral $\int_0^t u_s dB_s$ belong almost surely to the Besov space $\mathcal{B}^{\alpha}_{p,q}$ for all $1 \le q \le \infty$.*

Proof. Fix $0 < \beta < H - \alpha$. By Thebychev inequality and using (9) we obtain

$$\mathbb{P}\left(2^{-jp\left(\frac{1}{2}-\alpha+\frac{1}{p}\right)} \sum_{k=1}^{2^j} |X_{jk}(1)|^p > 2^{-jp\beta}\right)$$
$$\le 2^{-jp\left(-\beta+\frac{1}{2}-\alpha+\frac{1}{p}\right)} \mathbb{E} \sum_{k=1}^{2^j} |X_{jk}(t)|^p \le c_p 2^{jp(H-\alpha-\beta)} \|u\|^p_{\mathbb{L}^{p,1}}\,.$$

Then the desired result follow by Borel-Cantelli lemma. ■

The following is the main result of this paper.

Theorem 5 *Let $p \ge 2$ be an even integer and let $\delta > \frac{2(p-1)}{1-H} \vee \frac{2}{2H-1}$. If the process u is bounded and belongs to the space $\mathbb{L}^{\delta,2}$, then the trajectories of the indefinite divergence integral $\int_0^t u_s dB_s$ belong almost surely to the Besov space $\mathcal{B}^{H}_{p,\infty}$.*

Proof. In order to show that the paths of the indefinite integral $\int_0^t u_s dB_s$ belong almost surely to the Besov space $\mathcal{B}^{H}_{p,\infty}$ we need to show that

$$\sup_j 2^{-jp\left(\frac{1}{2}-H+\frac{1}{p}\right)} \sum_{k=1}^{2^j} X^p_{jk}(1) < \infty \tag{10}$$

almost surely. Using the Itô's formula for the divergence integral (see (2)) we can write

$$
\begin{aligned}
X_{jk}^{p}(t) =& p\int_0^t X_{jk}^{p-1}(s)\chi_{jk}(s)u_s\delta B_s \\
&+ \alpha_{p,H}\int_0^t X_{jk}^{p-2}(s)\chi_{jk}(s)u_s \\
&\times \left(\int_0^1 |s-\sigma|^{2H-2}\left(\int_0^s \chi_{jk}(\theta)D_\sigma u_\theta \delta B_\theta\right)d\sigma\right)ds \\
&+ \alpha_{p,H}\int_0^t X_{jk}^{p-2}(s)\chi_{jk}(s)u_s\left(\int_0^s \chi_{jk}(\theta)u_\theta|s-\theta|^{2H-2}d\theta\right)ds,
\end{aligned}
\tag{11}
$$

where $\alpha_{p,H} = p(p-1)H(2H-1)$. The proof of the estimate (10) will be done in several steps:

Step 1. Set

$$
\gamma_{jk}^{(p)}(t) = \int_0^t X_{jk}^{p-1}(s)\chi_{jk}(s)u_s\delta B_s.
$$

We are going to show that

$$
\sup_j 2^{-jp\left(\frac{1}{2}-H+\frac{1}{p}\right)}\left|\sum_{k=1}^{2^j}\gamma_{jk}^{(p)}(1)\right| < \infty \tag{12}
$$

almost surely. The estimate (12) is a consequence of Borel-Cantelli lemma and the inequality

$$
\mathbb{E}\left(\sum_{k=1}^{2^j}\gamma_{jk}^{(p)}(1)\right)^2 \le C2^{j\theta}, \tag{13}
$$

where $\theta < 2 + p(1-2H)$.

Along the proof C will denote a constant depending on p, H and $\|u\|_\infty$ and that may vary from one formula to another one.

In order to show (13) we need to evaluate terms of the form $\mathbb{E}\left(\gamma_{jk}^{(p)}(1)\gamma_{jk}^{(p)}(1)\right)$, where $1 \le k,l \le 2^j$. Applying the estimate (4) for the expectation of two divergences we obtain

$$
\begin{aligned}
\left|\mathbb{E}\left(\gamma_{jk}^{(p)}(1)\gamma_{jk}^{(p)}(1)\right)\right| \le& C\mathbb{E}\int_0^1\int_0^1 |\chi_{jk}(s)\chi_{jl}(t)|\left|X_{jk}^{p-1}(s)X_{jl}^{p-1}(t)\right| \\
&\times |s-t|^{2H-2}dsdt
\end{aligned}
$$

$$
\begin{aligned}
&+ C\mathbb{E}\int_{[0,1]^4}\left|D_r\left[X_{jk}^{p-1}(\theta)u_\theta\right]D_s\left[X_{jl}^{p-1}(t)u_t\right]\right| \\
&\times|\chi_{jk}(\theta)\chi_{jl}(t)|\,|r-t|^{2H-2}|\theta-s|^{2H-2}drd\theta dsdt \\
&:= B_{jkl}^{(1)}+B_{jkl}^{(2)}.
\end{aligned}
$$

For the first term we have

$$
B_{jkl}^{(1)} \le C2^j\int_0^1\int_0^1 \mathbf{1}_{jk}(s)\mathbf{1}_{jl}(t)\left(\mathbb{E}X_{jk}^{2(p-1)}(s)\mathbb{E}X_{jl}^{2(p-1)}(t)\right)^{\frac12}|s-t|^{2H-2}dsdt. \tag{14}
$$

From (8) with the exponent $2(p-1)>\frac{1}{H}$ we deduce the estimate

$$
\mathbb{E}\,X_{jk}^{2(p-1)}(s)\le C2^{j(p-1)\left(1-2H+\frac{2H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)}, \tag{15}
$$

for any $\delta\ge1$. Substituting (15) into (14) yields

$$
B_{jkl}^{(1)}\le C2^{j\left((p-1)(1-2H+\frac{2H}{\delta})+1\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)}\int_0^1\int_0^1\mathbf{1}_{jk}(s)\mathbf{1}_{jl}(t)|s-t|^{2H-2}dsdt.
$$

Hence,

$$
\sum_{k,l=1}^{2^j}B_{jkl}^{(1)}\le C2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)}c_H,
$$

where

$$
c_H=\int_0^1\int_0^1|s-t|^{2H-2}dsdt.
$$

This implies the desired inequality provided $2H+\frac{2(p-1)H}{\delta}<2$, which means

$$
\delta>\frac{H}{1-H}(p-1). \tag{16}
$$

In order to estimate the term $B_{jkl}^{(2)}$ let us compute

$$
\begin{aligned}
D_r\left(X_{jk}^{p-1}(s)u_s\right)=&(p-1)X_{jk}^{p-2}(s)u_s\int_0^s\chi_{jk}(\theta)D_ru_\theta\delta B_\theta \\
&+(p-1)X_{jk}^{p-2}(s)u_s\chi_{jk}(r)u_r\mathbf{1}_{[0,s]}(r) \\
&+X_{jk}^{p-1}(s)D_ru_s.
\end{aligned} \tag{17}
$$

We are going to use this decomposition in order to deduce an estimate for

$$
\mathbb{E}\left(D_r\left(X_{jk}^{p-1}(s)u_s\right)\right)^2.
$$

We have, by Hölder's inequality with $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$

$$\mathbb{E}\left(X_{jk}^{2(p-2)}(s)\left|\int_0^s \chi_{jk}(\theta)D_r u_\theta \delta B_\theta\right|^2\right)$$
$$\leq \left(\mathbb{E}\left|X_{jk}(s)\right|^{2(p-2)\lambda}\right)^{\frac{1}{\lambda}}\left(\mathbb{E}\left|\int_0^s \chi_{jk}(\theta)D_r u_\theta \delta B_\theta\right|^{2\lambda'}\right)^{\frac{1}{\lambda'}}. \qquad (18)$$

From (8) with the exponent $2(p-2)\lambda > \frac{1}{H}$ we deduce the estimate

$$\left(\mathbb{E}\left|X_{jk}(s)\right|^{2(p-2)\lambda}\right)^{\frac{1}{\lambda}} \leq C 2^{j(p-2)\left(1-2H+\frac{2H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}, \qquad (19)$$

for all $\delta \geq 1$. Applying the estimate (8) to the process $\{D_r u_s, s\in[0,1]\}$ with the exponent $2\lambda' > \frac{1}{H}$ yields

$$\left(\mathbb{E}\left|\int_0^s \chi_{jk}(\theta)D_r u_\theta \delta B_\theta\right|^{2\lambda'}\right)^{\frac{1}{\lambda'}} \leq C 2^{j\left(1-2H+\frac{2H}{\delta}\right)}\|D_r u\|_{\mathbb{L}^{\delta/H,1}}^2, \qquad (20)$$

for all $\delta \geq 1$. Substituting (19) and (20) into (18) we obtain

$$\mathbb{E}\left(X_{jk}^{2(p-2)}(s)\left|\int_0^s \chi_{jk}(\theta)D_r u_\theta \delta B_\theta\right|^2\right)$$
$$\leq C 2^{j\left((p-1)(1-2H)+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}\|D_r u\|_{\mathbb{L}^{\delta/H,1}}^2. \qquad (21)$$

For the second term in the expression of $D_r\left(X_{jk}^{p-1}(s)u_s\right)$ we have using (8) with the exponent $2(p-2) > \frac{1}{H}$,

$$\left|\chi_{jk}(r)\right|^2 \mathbb{E}X_{jk}^{2(p-2)}(s) \leq C 2^{j\left((p-2)\left(1-2H+\frac{2H}{\delta}\right)+1\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}\mathbf{1}_{jk}(r). \qquad (22)$$

Finally, by Hölder's inequality with $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$

$$\mathbb{E}\left(\left|X_{jk}(s)\right|^{2(p-1)}\left|D_r u_s\right|^2\right) \leq \left(\mathbb{E}\left|X_{jk}(s)\right|^{2(p-1)\lambda}\right)^{\frac{1}{\lambda}}\left(\mathbb{E}|D_r u_s|^{2\lambda'}\right)^{\frac{1}{\lambda'}},$$

and by (8) with the exponent $2(p-1)\lambda > \frac{1}{H}$, we obtain

$$\mathbb{E}\left(\left|X_{jk}(s)\right|^{2(p-1)}\left|D_r u_s\right|^2\right)$$
$$\leq C 2^{j(p-1)\left(1-2H+\frac{2H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)}\left(\mathbb{E}\left|D_r u_s\right|^{2\lambda'}\right)^{\frac{1}{\lambda'}}. \qquad (23)$$

Using the inequalities (21), (22) and (23) into (17) we deduce

$$\begin{aligned}
&\mathbb{E}\left(D_r\left(X_{jk}^{p-1}(s)u_s\right)\right)^2 \\
&\qquad \le C2^{jp(1-2H)}\left\{2^{j\left(2H-1+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}\|D_r u\|_{\mathbb{L}^{\delta/H,1}}^2\right. \\
&\qquad + C2^{j\left(4H-1+\frac{2(p-2)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}\mathbf{1}_{jk}(r) \\
&\qquad \left.+C2^{j\left(2H-1+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)}\left(\mathbb{E}\left|D_r u_s\right|^{2\lambda'}\right)^{\frac{1}{\lambda'}}\right\}. \qquad (24)
\end{aligned}$$

We are going to use this estimate in order to treat the second term given by:

$$\begin{aligned}
B_{jkl}^{(2)} \le& C2^j\int_{[0,1]^4}\left(\mathbb{E}\left(D_r\left[X_{jk}^{p-1}(\theta)u_\theta\right]\right)^2\mathbb{E}\left(D_s\left[X_{jl}^{p-1}(t)u_t\right]\right)^2\right)^{\frac{1}{2}} \\
&\times\mathbf{1}_{jk}(\theta)\mathbf{1}_{jl}(t)|r-t|^{2H-2}|\theta-s|^{2H-2}drd\theta dsdt. \qquad (25)
\end{aligned}$$

Applying the inequality (24) we can write

$$\left(\mathbb{E}\left(D_r\left[X_{jk}^{p-1}(\theta)u_\theta\right]\right)^2\mathbb{E}\left(D_s\left[X_{jl}^{p-1}(t)u_t\right]\right)^2\right)^{\frac{1}{2}} \le C2^{jp(1-2H)}\sum_{i=1}^{6}C_{jkl}^{(i)},$$

where

$$\begin{aligned}
C_{jkl}^{(1)} =& 2^{j\left(2H-1+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}\|D_r u\|_{\mathbb{L}^{\delta/H,1}}\|D_s u\|_{\mathbb{L}^{\delta/H,1}}, \\
C_{jkl}^{(2)} =& 2^{j\left(4H-1+\frac{2(p-2)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)}\mathbf{1}_{jk}(r)\mathbf{1}_{jl}(s), \\
C_{jkl}^{(3)} =& 2^{j\left(2H-1+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)}\left(\mathbb{E}\left|D_r u_\theta\right|^{2\lambda'}\mathbb{E}\left|D_s u_t\right|^{2\lambda'}\right)^{\frac{1}{2\lambda'}}, \\
C_{jkl}^{(4)} =& 2^{j\left(3H-1+\frac{(2p-3)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-2)} \\
&\times\left[\|D_r u\|_{\mathbb{L}^{\delta/H,1}}\mathbf{1}_{jk}(s)+\|D_s u\|_{\mathbb{L}^{\delta/H,1}}\mathbf{1}_{jl}(r)\right], \\
C_{jkl}^{(5)} =& 2^{j\left(2H-1+\frac{2(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-3} \\
&\times\left[\|D_r u\|_{\mathbb{L}^{\delta/H,1}}\left(\mathbb{E}\left|D_s u_t\right|^{2\lambda'}\right)^{\frac{1}{2\lambda'}}+\|D_s u\|_{\mathbb{L}^{\delta/H,1}}\left(\mathbb{E}\left|D_r u_\theta\right|^{2\lambda'}\right)^{\frac{1}{2\lambda'}}\right]
\end{aligned}$$

and

$$C^{(6)}_{jkl} = 2^{j\left(3H-1+\frac{(2p-3)H}{\delta}\right)} \|u\|^{2p-3}_{\mathbb{L}^{\delta/H,1}} \times \left[\mathbf{1}_{jk}(r) \left(\mathbb{E}\left|D_s u_t\right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} + \mathbf{1}_{jl}(s) \left(\mathbb{E}\left|D_r u_\theta\right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} \right].$$

Replacing these six terms into (25) yields

$$\sum_{k,l=1}^{2^j} B^{(2)}_{jkl} \le C 2^{j(p(1-2H)+1)} \sum_{i=1}^{6} \sum_{k,l=1}^{2^j} \int_{[0,1]^4} C^{(i)}_{jkl} \times \mathbf{1}_{jk}(\theta) \mathbf{1}_{jl}(t) |r-t|^{2H-2} |\theta - s|^{2H-2} dr d\theta ds dt := \sum_{i=1}^{6} d_i.$$

We have

$$d_1 \le C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|^{2(p-1)}_{\mathbb{L}^{\delta/H,2}} \left(\sup_r \int_0^1 |r-t|^{2H-2} dt \right)^2,$$

$$d_2 \le C 2^{j\left(p(1-2H)+4H+\frac{2(p-2)H}{\delta}\right)} \|u\|^{2(p-2)}_{\mathbb{L}^{\delta/H,1}} \sum_{k,l=1}^{2^j} \int_{[0,1]^4} \mathbf{1}_{jk}(r) \mathbf{1}_{jl}(s) \times \mathbf{1}_{jk}(\theta) \mathbf{1}_{jl}(t) |r-t|^{2H-2} |\theta - s|^{2H-2} dr d\theta ds dt.$$

We claim that for all k, l

$$\int_0^1 \int_0^1 \mathbf{1}_{jk}(r) \mathbf{1}_{jl}(t) |r-t|^{2H-2} dr dt \le C 2^{-2Hj}. \qquad (26)$$

Indeed, the maximum value of this integral corresponds to the case $k = l$ where it is equal to $\frac{1}{H(2H-1)} 2^{-2Hj}$. As a consequence we deduce the inequality

$$d_2 \le C 2^{j\left(p(1-2H)+2H+\frac{2(p-2)H}{\delta}\right)} \|u\|^{2(p-2)}_{\mathbb{L}^{\delta/H,1}}.$$

We have

$$d_3 \le C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|^{2(p-1)}_{\mathbb{L}^{\delta/H,1}} \int_{[0,1]^4} \left(\mathbb{E}\left|D_r u_\theta\right|^{2\lambda'} \mathbb{E}\left|D_s u_t\right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} \times |r-t|^{2H-2} |\theta - s|^{2H-2} dr d\theta ds dt.$$

Applying Hölder's inequality with $\lambda > \frac{1}{2H-1}$ we obtain

$$d_3 \leq C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2(p-1)} \left(\int_{[0,1]^2} \left(\mathbb{E} \left| D_s u_t \right|^{2\lambda'} \right)^{\frac{\lambda}{2\lambda'}} dsdt \right)^{\frac{2}{\lambda}},$$

and taking $\lambda' = \frac{\lambda}{2} = \frac{\delta}{2H}$ and assuming $\delta > \frac{2}{2H-1}$ we get

$$d_3 \leq C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p} .$$

We have

$$\begin{aligned} d_4 \leq & C 2^{j\left(p(1-2H)+3H+\frac{(2p-3)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-4} \\ & \times \sum_{k,l=1}^{2^j} \int_{[0,1]^4} \left[\|D_r u\|_{\mathbb{L}^{\delta/H,1}} \mathbf{1}_{jk}(s) + \|D_s u\|_{\mathbb{L}^{\delta/H,1}} \mathbf{1}_{jl}(r) \right] \\ [2mm] & \times \mathbf{1}_{jk}(\theta) \mathbf{1}_{jl}(t) |r-t|^{2H-2} |\theta - s|^{2H-2} drd\theta dsdt. \end{aligned} \tag{27}$$

Using the estimate $\sup_{\theta,k} \int_0^1 \mathbf{1}_{jk}(s) |\theta - s|^{2H-2} ds \leq C 2^{j(1-2H)}$ we can write

$$\begin{aligned} \sum_{k,l=1}^{2^j} \int_{[0,1]^4} & \|D_r u\|_{\mathbb{L}^{\delta/H,1}} \mathbf{1}_{jk}(\theta) \mathbf{1}_{jl}(t) \mathbf{1}_{jk}(s) |r-t|^{2H-2} |\theta - s|^{2H-2} drd\theta dsdt \\ & \leq C 2^{j(1-2H)} \int_0^1 \|D_r u\|_{\mathbb{L}^{\delta/H,1}} dr. \end{aligned} \tag{28}$$

Substituting (28) into (27) yields

$$d_4 \leq C 2^{j\left(p(1-2H)+H+1+\frac{(2p-3)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,2}}^{2p-3} .$$

For the fifth term we can write the estimate

$$\begin{aligned} d_5 \leq \ & C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-3} \\ & \times \int_{[0,1]^4} \left[\|D_r u\|_{\mathbb{L}^{\delta/H,1}} \left(\mathbb{E} \left| D_s u_t \right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} + \|D_s u\|_{\mathbb{L}^{\delta/H,1}} \left(\mathbb{E} \left| D_r u_\theta \right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} \right] \\ & \times |r-t|^{2H-2} |\theta - s|^{2H-2} drd\theta dsdt . \end{aligned}$$

Hence, using the continuous embedding of $\mathcal{H}$ into $L^{\frac{1}{H}}([0,1])$ we obtain taking $\lambda' = \frac{\delta}{2H}$

$$
\begin{aligned}
d_5 \leq & C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-3} \\
& \times \left(\int_0^1 \left(\int_0^1 \left(\mathbb{E}\left|D_s u_t\right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} ds \right)^{\frac{1}{H}} dt \right)^H \left(\int_0^1 \|D_r u\|_{\mathbb{L}^{\delta/H,1}}^{\frac{1}{H}} dr \right)^H \\
& \leq C 2^{j\left(p(1-2H)+2H+\frac{2(p-1)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,2}}^{2p-1}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
d_6 \leq & C 2^{j\left(p(1-2H)+3H+\frac{(2p-3)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-3} \\
& \times \sum_{k,l=1}^{2^j} \int_{[0,1]^4} \left[\mathbf{1}_{jk}(r) \left(\mathbb{E}\left|D_s u_t\right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} + \mathbf{1}_{jl}(s) \left(\mathbb{E}\left|D_r u_\theta\right|^{2\lambda'} \right)^{\frac{1}{2\lambda'}} \right] \\
& \times \mathbf{1}_{jk}(\theta) \mathbf{1}_{jl}(t) |r-t|^{2H-2} |\theta - s|^{2H-2} dr d\theta ds dt.
\end{aligned}
$$

Using the estimate $\sup_{t,k} \int_0^1 \mathbf{1}_{jk}(r)|r-t|^{2H-2} dr \leq C 2^{j(1-2H)}$ we can write

$$
\begin{aligned}
d_6 \leq & C 2^{j\left(p(1-2H)+H+1+\frac{(2p-3)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-3} \\
& \times \int_0^1 \int_0^1 \left(\mathbb{E}\left|D_s u_t\right|^{2\lambda'} \right)^{\frac{\lambda}{2\lambda'}} ds dt.
\end{aligned}
$$

Finally choosing $\lambda' = \frac{\delta}{2H}$ we get

$$
d_6 \leq C 2^{j\left(p(1-2H)+H+1+\frac{(2p-3)H}{\delta}\right)} \|u\|_{\mathbb{L}^{\delta/H,1}}^{2p-2}.
$$

As a consequence we deduce

$$
\sum_{k,l=1}^{2^j} B_{jkl}^{(2)} \leq C 2^{j\left(p(1-2H)+H+1+\frac{2(p-1)H}{\delta}\right)} \left(\|u\|_{\mathbb{L}^{\delta/H,2}}^{2p} \vee 1 \right).
$$

This inequality provides the desired estimation if

$$
H + 1 + \frac{2(p-1)H}{\delta} < 2
$$

which means $\frac{\delta}{H} > \frac{2(p-1)}{1-H} \vee \frac{2}{2H-1}$.

Step 2. Define

$$
\psi_{jk}^{(p)}(t) = \int_0^t X_{jk}^{p-2}(s) \chi_{jk}(s) u_s \left(\int_0^1 |s-\sigma|^{2H-2} \left(\int_0^s \chi_{jk}(\theta) D_\sigma u_\theta \delta B_\theta \right) d\sigma \right) ds.
$$

We want to show that

$$\sup_j 2^{-jp\left(\frac{1}{2}-H+\frac{1}{p}\right)} \sum_{k=1}^{2^j} \left|\psi_{jk}^{(p)}(1)\right| < \infty$$

almost surely, and by Borel-Cantelli lemma it suffices to check that

$$\mathbb{E} \sum_{k=1}^{2^j} \left|\psi_{jk}^{(p)}(1)\right| \le C 2^{j\theta},$$

where $\theta < 1 + p(\frac{1}{2} - H)$. We have

$$\begin{aligned} \mathbb{E} \sum_{k=1}^{2^j} \left|\psi_{jk}^{(p)}(1)\right| \le & C 2^{\frac{j}{2}} \sum_{k=1}^{2^j} \int_0^1 \int_0^1 \mathbf{1}_{jk}(s) \left|s-\sigma\right|^{2H-2} \\ & \times \mathbb{E}\left[X_{jk}^{p-2}(s) \left|\int_0^s \chi_{jk}(\theta) D_\sigma u_\theta \delta B_\theta\right|\right] d\sigma ds. \end{aligned} \tag{29}$$

By Hölder's inequality with $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$ we can write

$$\begin{aligned} & \mathbb{E}\left[X_{jk}^{p-2}(s) \left|\int_0^s \chi_{jk}(\theta) D_\sigma u_\theta \delta B_\theta\right|\right] \\ & \le \left(\mathbb{E}\left|X_{jk}(s)\right|^{(p-2)\lambda}\right)^{\frac{1}{\lambda}} \left(\mathbb{E}\left|\int_0^s \chi_{jk}(\theta) D_\sigma u_\theta \delta B_\theta\right|^{\lambda'}\right)^{\frac{1}{\lambda'}}. \end{aligned} \tag{30}$$

Applying the estimate (8) to the processes u and $\{D_r u_s, s \in [0,1]\}$ yields

$$\left(\mathbb{E}\left|X_{jk}(s)\right|^{(p-2)\lambda}\right)^{\frac{1}{\lambda}} \le C 2^{j(p-2)\left(\frac{1}{2}-H+\frac{H}{\delta}\right)} \left\|u\right\|_{\mathbb{L}^{\delta/H,1}} \tag{31}$$

and

$$\left(\mathbb{E}\left|\int_0^s \chi_{jk}(\theta) D_\sigma u_\theta \delta B_\theta\right|^{\lambda'}\right)^{\frac{1}{\lambda'}} \le C 2^{j\left(\frac{1}{2}-H+\frac{H}{\delta}\right)} \left\|D_\sigma u\right\|_{\mathbb{L}^{\delta/H,1}} \tag{32}$$

and substituting (31) and (32) into (30) we obtain

$$\begin{aligned} & \mathbb{E}\left[X_{jk}^{p-2}(s) \left|\int_0^s \chi_{jk}(\theta) D_\sigma u_\theta \delta B_\theta\right|\right] \\ & \le C 2^{j\left((p-1)\left(\frac{1}{2}-H\right)+\frac{(p-1)H}{\delta}\right)} \left\|u\right\|_{\mathbb{L}^{\delta_2/H,1}}^{p-2} \left\|D_\sigma u\right\|_{\mathbb{L}^{\delta_1/H,1}}. \end{aligned} \tag{33}$$

Finally, replacing (33) into (29) we obtain

$$\mathbb{E}\sum_{k=1}^{2^j}\left|\psi_{jk}^{(p)}(1)\right| \leq C2^{j\left(p\left(\frac{1}{2}-H\right)+H+\frac{(p-1)H}{\delta}\right)}\|u\|_{\mathbb{L}^{\delta_{2/H,1}}}^{p-1}.$$

Hence, we need

$$H+\frac{(p-1)H}{\delta}<1,$$

that means, $\delta > \frac{(p-1)H}{1-H}$.

Step 3. In order to complete the proof of the theorem it remains to show that

$$\sup_j 2^{-jp\left(\frac{1}{2}-H+\frac{1}{p}\right)}\sum_{k=1}^{2^j}\left|\eta_{jk}^{(p)}(1)\right|<\infty, \tag{34}$$

where

$$\eta_{jk}^{(p)}(t)=\int_0^t X_{jk}^{p-2}(s)\chi_{jk}(s)u_s\left(\int_0^s \chi_{jk}(\theta)u_\theta|s-\theta|^{2H-2}d\theta\right)ds.$$

We have

$$\begin{aligned}\left|\eta_{jk}^{(p)}(t)\right| \leq& C\int_0^t\int_0^s X_{jk}^{p-2}(s)\left|\chi_{jk}(s)\chi_{jk}(\theta)\right|\,|s-\theta|^{2H-2}d\theta ds\\ =&C\int_0^t X_{jk}^{p-2}(s)\rho_{jk}(s)ds,\end{aligned}$$

where

$$\rho_{jk}(s)=2^j\int_0^s \mathbf{1}_{jk}(s)\mathbf{1}_{jk}(\theta)\ |s-\theta|^{2H-2}d\theta\,.$$

Notice that the term $\rho_{jk}(s)$ is deterministic and

$$\int_0^1\rho_{jk}(s)ds=\frac{1}{2\alpha_H}2^{j(1-2H)}. \tag{35}$$

In order to show (34) we have to show the estimate

$$\sup_j 2^{-jq\left(\frac{1}{2}-H+\frac{1}{q}\right)}\sum_{k=1}^{2^j}\int_0^1 X_{jk}^{q-2}(s)\rho_{jk}(s)ds<\infty, \tag{36}$$

almost surely, for $q = p$. We are going to show the estimate (36) for $q = 2, 4, \ldots, p$ by a recurrence argument. For $q = 2$ it is immediate using (35). Suppose it holds for q and let us show that

$$\sup_j 2^{-j\left((q+2)\left(\frac{1}{2}-H\right)+1\right)} \sum_{k=1}^{2^j} \int_0^1 X_{jk}^q(s)\rho_{jk}(s)ds < \infty \tag{37}$$

almost surely. By Itô's formula (11) we can write

$$X_{jk}^q(s) = p\gamma_{jk}^{(q)}(s) + \alpha_{p,H}\psi_{jk}^{(q)}(s) + \alpha_{p,H}\eta_{jk}^{(q)}(s). \tag{38}$$

By the same arguments as in the proof of Step 2 we can show that

$$\sup_{j,s} 2^{-j\left(q\left(\frac{1}{2}-H\right)+1\right)} \sum_{k=1}^{2^j} \left|\psi_{jk}^{(q)}(s)\right| < \infty. \tag{39}$$

Moreover,

$$\sum_{k=1}^{2^j} \int_0^1 \left|\eta_{jk}^{(q)}(s)\right| \rho_{jk}(s)ds \leq C2^{j(1-2H)} \sum_{k=1}^{2^j} \int_0^1 X_{jk}^{q-2}(\theta)\rho_{jk}(\theta)d\theta$$

and applying the induction hypothesis we obtain

$$\sup_j 2^{-j\left((q+2)\left(\frac{1}{2}-H\right)+1\right)} \sum_{k=1}^{2^j} \int_0^1 \left|\eta_{jk}^{(q)}(s)\right| \rho_{jk}(s)ds < \infty. \tag{40}$$

In order to handle the term $\gamma_{jk}^{(q)}(s)$ in (38) we use the following integration by parts formula

$$\begin{aligned}\gamma_{jk}^{(q)}(1) \int_0^1 \rho_{jk}(s)ds &= \int_0^1 \gamma_{jk}^{(q)}(s)\rho_{jk}(s)ds \\ &\quad + \int_0^1 X_{jk}^{q-1}(s) \left(\int_0^s \rho_{jk}(\theta)d\theta\right) \chi_{jk}(s)u_s\delta B_s.\end{aligned}$$

From Step 1 we konw that

$$\sup_j 2^{-j\left(q\left(\frac{1}{2}-H\right)+1\right)} \left|\sum_{k=1}^{2^j} \gamma_{jk}^{(q)}(1)\right| < \infty. \tag{41}$$

On the other hand, $\int_0^s \rho_{jk}(\theta)d\theta$ is bounded by a constant. Hence, the proof of Step 1 also provides the estimate

$$\sup_j 2^{-j\left(q\left(\frac{1}{2}-H\right)+3-2H\right)} \left| \sum_{k=1}^{2^j} \int_0^1 X_{jk}^{q-1}(s) \left(\int_0^s \rho_{jk}(\theta)d\theta \right) \chi_{jk}(s) u_s \delta B_s \right| < \infty. \tag{42}$$

Finally, from (39), (40), (41) and (42) we deduce (37). ■

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