Darboux theory of integrability for discrete dynamical systems

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Abstract

The classical Darboux theory of integrability deals with ordinary differential equations and, in certain cases, allows to construct a first integral for these differential equations from the knowledge of several of their invariant algebraic hypersurfaces. We extend this theory to discrete dynamical systems, giving a criterion of integrability and several examples of application.

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1 Introduction

Consider a discrete dynamical system (DDS for short)

$$x_{n+1} = F(x_n), \tag{1}$$

where $F : \mathcal{U} \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m$, being $\mathcal{U}$ an open subset. We will say that H is a *first integral* for (1) if H is a real valued function defined in $\mathcal{V}$, an open and dense subset of $\mathcal{U}$, satisfying

$$H(F(x)) = H(x), \text{ for all } x \in \mathcal{V}.$$

In this paper we will adapt to DDS the known as Darboux theory of integrability. This theory has been already developed for searching first integrals of ordinary differential equations (ODE), see for instance [5] for a survey or [2] for an application for $m = 3$. Although, as it becomes apparent from our results, this adaption can be done for any m, we will mainly fix our attention to $m = 2$.

Before describing our results we would like to comment that although for most DDS there is no way of finding an explicit first integral, the cases for which such first integral can be found have been extensively studied and they are relevant in the applications.

In the next section we will introduce the Darboux Method to give explicit first integrals of (1) for $m = 2$. Section 3 will be devoted to apply the method to several examples, most of them obtained from the literature. Among them we can quote: linear systems, the Lyness map, maps related to the Gauss Map, some rational maps. A nice property of this method is that it allows to present all the examples from a unified viewpoint. We end Section 3 with two representative examples for which such a first integral exists but it can not be found with the method that we present (which from now will be called Darboux Method), just to give the idea that as in the case of ODE this method is not universal, and with an application of the knowledge of the first integral to the determination of the limit of some DDS.

The last section will be devoted to relate the method for ODE with the one presented here.

2 Definitions and main results

The main idea of Darboux Method for ODE is the following (see Section 4 for a more detailed description): Consider an ODE

$$(\dot{x}, \dot{y}) = X(x, y), \tag{2}$$

and find for it as many invariant algebraic curves $f_i(x,y)=0$ as possible. Afterwards, from these invariants algebraic curves, try to construct a first integral of the form

$$H(x,y)=\prod_{i=1}^{s}|f_i(x,y)|^{\alpha_i}$$

for $\alpha_i\in\mathbb{R}$. It is also possible to extend this method by using complex invariant algebraic curves and other generalizations, see again [5], but for brevity here we restrict our explanation to the easiest case. Our approach is similar than the one explained above.

Definition 1. *Consider $R:\mathcal{V}\subset\mathbb{R}^2\longrightarrow\mathbb{R}$ being $\mathcal{V}$ an open subset, $\{(x,y)\in\mathbb{R}^2: R(x,y)=0\}\neq\emptyset$ and R not necessarily algebraic. We will say that $R(x,y)=0$ is an invariant curve for the DDS, $(x_{n+1},y_{n+1})=F(x_n,y_n)$ if*

$$R(F(x,y))=K(x,y)R(x,y),\quad \textit{for all}\quad (x,y)\in\mathcal{B},\qquad (3)$$

where $\mathcal{B}$ is an open and dense subset of $\mathcal{V}$, and $\{R(x,y)=0\}\cap\mathcal{B}=\emptyset$. The function K is called a cofactor of R with respect to F.

Remark 2. *When a curve R satisfies equation (3) but $\{(x,y)\in\mathbb{R}^2: R(x,y)=0\}=\emptyset$ we will say that R is an exponential factor of F with cofactor K. For instance any function of the form $R(x,y)=\exp\{\varphi(x,y)\}$ is an exponential factor with $K(x,y)=\exp\{\varphi(F(x,y))-\varphi(x,y)\}$. See [5] for a motivated introduction of the role of the exponential factors for ODE.*

Our main results is the following:

Theorem 3 (Darboux Method of integration for DDS). *Let*

$$(x_{n+1},y_{n+1})=F(x_n,y_n),\qquad (4)$$

be a discrete dynamic system defined in $\mathcal{U}\subset\mathbb{R}^2$. Let $R_i(x,y)$, $i=1,2,\cdots,s$ be either invariant curves or exponential factors of (4) with associated cofactor $K_i(x,y)$, $i=1,...,s$ respectively. Then if there exists $\alpha_1,\alpha_2,\cdots,\alpha_s\in\mathbb{R}$ such that

$$\prod_{i=1}^{s}|K_i(x,y)|^{\alpha_i}=1,\qquad (5)$$

in an open and dense subset of $\mathcal{U}$, then

$$H(x,y):=\prod_{i=1}^{s}|R_i(x,y)|^{\alpha_i},$$

is a first integral of (4).

Proof. Since we have that $R_i(F(x,y)) = K_i(x,y)R_i(x,y)$ for each $i = 1, \cdots, s$ in an open dense subset $\mathcal{B}_i$ of $\mathcal{U}$, we have that

$$H(F(x,y)) = \prod_{i=1}^{s} |R_i(F(x,y))|^{\alpha_i} = \prod_{i=1}^{s} |K_i(x,y)|^{\alpha_i} \prod_{i=1}^{s} |R_i(x,y)|^{\alpha_i}.$$

Using (5) we obtain $H(F(x,y)) = H(x,y)$ for any $(x,y) \in \cap_{i=1}^{s} \mathcal{B}_i$, which is an open and dense subset of $\mathcal{U}$, as we wanted to prove. ∎

Remark 4. *Note that if instead of obtaining a collection of α_i such that condition (5) holds we get $\prod_{i=1}^{s} |K_i(x,y)|^{\alpha_i} = a \neq 1$, for some constant a, we have obtained a kind of non–autonomous first integral because defining $H_n := H(x_n, y_n)$ we have that $H(F(x,y)) = aH(x,y)$, that is $H_{n+1} = aH_n$, or in other words $H_{n+1} = a^{n+1}H_0$.*

A main difference between invariants curves for ODE and DDS is the following: assume that $R(x,y) = T(x,y)S(x,y) = 0$ is an invariant solution for an ODE, then it is clear that both $S(x,y) = 0$ and $T(x,y) = 0$ are invariant solutions for the same ODE. On the other hand if $R(x,y) = T(x,y)S(x,y) = 0$ is an invariant curve for the DDS, $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$, being $\{S(x,y) = 0\}$ and $\{T(x,y) = 0\}$ both nonempty, maybe $S(x,y)$ and $T(x,y)$ are as well invariant for the DDS, but it also may occur that the DDS maps $\{S(x,y) = 0\}$ into $\{T(x,y) = 0\}$, and viceversa, or in other words $\{S(x,y) = 0\}$ and $\{T(x,y) = 0\}$ are invariants for $F^2 = F \circ F$.

Note also that the singularities of the map F that defines the DDS, the singularities of its iterates, as well as the preimages of all these singularities should also play an especial role in the dynamics generated by F.

The above comments lead to give the above following generalization of Theorem 3:

Remark 5 (Generalized Darboux Method of integration for DDS). *Let*

$$(x_{n+1}, y_{n+1}) = F(x_n, y_n), \tag{6}$$

be a DDS defined in $\mathcal{U} = \mathbb{R}^2 \setminus \cup_{k=1}^{l} \{G_k(x,y) = 0\}$ an open, dense subset of $\mathbb{R}^2$. Let $R_i(x,y)$, $i = 1, 2, \cdots, s$ be invariant curves for F^{k_i} for some $k_i \geq 1$, or exponential factors of F. Then a natural candidate to be a first integral for (6) is

$$H(x,y) = \prod_{i=1}^{s} |R_i(x,y)|^{\alpha_i} \prod_{j=0}^{m} \left(\prod_{k=1}^{l} |G_k \circ F^j(x,y)|^{\beta_{j,k}} \right), \tag{7}$$

where m is any given fixed natural number.

3 Examples

In this section we find a first integral of different planar DDS by applying to them the Darboux Method. We also give a four dimensional example. We end with two classical examples for which the method that we develop does not work and with an application of the knowledge of a first integral.

3.1 The Lyness Map

A first integral for the well-known Lyness map

$$F(x,y) = \left(y, \frac{a+y}{x}\right),$$

is already known (see [8]). Here we find it by using the Darboux Method.

By searching all the invariant lines and singularities for F, F^2 and F^3, we get that just the following linear invariants and singularities appear:

$x+y+1=0$;	Invariant under F^2.
$x+1=0$, $y+1=0$, and $x+y+a=0$;	Invariant under F^3.
$x=0$, $y=0$ and $y+a=0$;	Singularities of F,F^2, and F^3.

Therefore the Generalized Darboux Method, explained in Remark 5 provides a candidate to be a first integral of the form

$$H(x,y) = |x+y+1|^{\alpha_1}|x+1|^{\alpha_2}|y+1|^{\alpha_3}|x+y+a|^{\alpha_4}|x|^{\beta_1}|y|^{\beta_2}|y+a|^{\beta_3}.$$

By imposing that $H(F(x,y)) = H(x,y)$ we get the solutions

$$\beta_3 = 0, \alpha_1 = 0, \alpha_2 = \alpha_3 = \alpha_4 = -\beta_1 = -\beta_2.$$

Taking $\alpha_2 = 1$ we obtain that a first integral for the Lyness map is

$$H(x,y) = \frac{(x+1)(y+1)(x+y+a)}{xy}.$$

Just for information, we want to comment that there is a generalization of the Lyness map and of its first integral to higher dimensions, see [8].

3.2 Maps related with the Gauss Map

The two mappings studied in this section are also considered in [3]. Here we give a different deduction of their first integrals by using the Darboux Method.

Consider first for x ad y positive

$$F(x,y) = \left(\sqrt[p]{\frac{x(x^p - y^p)}{p(x-y)}}, \sqrt[p]{\frac{y(x^p - y^p)}{p(x-y)}} \right), \tag{8}$$

where $p \in \mathbb{N}, p \geq 2$. From direct inspection it is clear that $R_1(x,y) = y^p - x^p = 0$ and $R_2(x,y) = \ln(y/x) = 0$ are invariants under F. In fact

$$R_1(F(x,y)) = \frac{1}{p} R_1(x,y), \quad \text{and} \quad R_2(F(x,y)) = \frac{1}{p} R_2(x,y).$$

Therefore, since their cofactors coincide, by Theorem 3, we obtain that the function

$$H(x,y) = \frac{R_1(x,y)}{R_2(x,y)} = \frac{y^p - x^p}{\ln y - \ln x},$$

is a first integral of (8).

Note that for $p = 2$, (8) gives the map

$$F(x,y) = \left(\sqrt{x\frac{x+y}{2}}, \sqrt{y\frac{x+y}{2}} \right), \tag{9}$$

which is a kind of arithmetic-geometric mean, see Example A in Subsection 3.7.

As a second example consider, also for x and y positive and for instance $0 < x \leq y$.

$$F(x,y) = (F_1(x,y), F_2(x,y)) = \left(\frac{x+y}{2}, \sqrt{y\frac{x+y}{2}} \right).$$

This map was introduced by Gauss in 1800 in a letter to Pfaff and was also studied later by Borchardt.

Taking $R_1(x,y) = y^2 - x^2$ (as above when $p = 2$) we get a first invariant, because

$$R_1(F(x,y)) = \frac{1}{4} R_1(x,y).$$

A second invariant is more difficult to be found. Note that

$$\frac{F_1(x,y)}{F_2(x,y)} = \sqrt{\frac{x+y}{2y}} = \sqrt{\frac{1}{2}\frac{x}{y} + \frac{1}{2}},$$

and that the right hand side of the above formula is reminiscent of the right hand side of the well-known equality

$$\cos A = \sqrt{\frac{1}{2}\cos 2A + \frac{1}{2}}.$$

Hence it becomes natural to consider $\cos 2A = \frac{x}{y}$. Therefore we obtain $\cos A = \dfrac{F_1(x,y)}{F_2(x,y)}$, or equivalently,

$$\arccos\left(\frac{F_1(x,y)}{F_2(x,y)}\right) = \frac{1}{2}\arccos\left(\frac{x}{y}\right).$$

The above equality leads also to

$$R_2(F(x,y)) = \frac{1}{2}R_2(x,y),$$

being $R_2(x,y) = \arccos(\frac{x}{y})$ an invariant for (9). By using Theorem 3, we get that

$$H(x,y) = \frac{\arccos^2(x/y)}{y^2 - x^2},$$

is a first integral for (9).

Finally observe that in the study of both maps (8) and (9) we get constant cofactors. By using Remark 4, for instance in this second case, we get the following equalities associated to R_1 and R_2:

$$y_n^2 - x_n^2 = \left(\frac{1}{4}\right)^n (y_0^2 - x_0^2),$$

and

$$\arccos^2\left(\frac{x_n}{y_n}\right) = \left(\frac{1}{2}\right)^n \arccos^2\left(\frac{x_0}{y_0}\right).$$

Note that from the above two equalities it is easy to get (x_n, y_n) in terms of n, x_0 and y_0.

3.3 Linear Maps

Consider planar linear maps with real eigenvalues. In this section we study their integrability by the Darboux Method. By performing an affine change of variables we can assume that $F(x,y) = A\cdot(x,y)^t$, where A is a 2×2 matrix of one of the following forms:

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \text{with}\, a,b\in\mathbb{R}.$$

and

$$A_2 = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \quad \text{with}\, a\in\mathbb{R}.$$

Let us construct a first integral in each one of the cases:

(i) Case $A = A_1$; $F(x,y) = (ax, by)$. Taking the $R_1(x,y) = x$ and $R_2(x,y) = y$ we get $R_1(F(x,y)) = aR_1(x,y)$, $R_2(F(x,y)) = bR_2(x,y)$. In order to apply Theorem 3 we search for α_1, α_2 such that $|a|^{\alpha_1}|b|^{\alpha_2} = 1$. We just consider the cases $a\cdot b\neq 0$, $|a|\neq 1$, or $|b|\neq 1$, because otherwise the dynamical system is trivial. Taking for instance $\alpha_1 = \ln|b|$ and $\alpha_2 = -\ln|a|$ we obtain the first integral

$$H(x,y) = |x|^{\ln|b|}|y|^{-\ln|a|}.$$

For instance, consider the Fibonacci recurrence $f_{n+1} = f_n + f_{n-1}$. It is equivalent to the DDS generated by $F(x,y) = (y, x+y)$. The above reasoning lead us to the first integral

$$H(x,y) = |y-\alpha^+x|^{\ln|\alpha^+|}|y-\alpha^-x|^{\ln|\alpha^-|},$$

where $\alpha^\pm = \frac{1\pm\sqrt{5}}{2}$ are the roots of $x^2+x+1=0$.

(ii) Case $A = A_2$; $F(x,y) = (ax, x+ay)$. As above we just consider $a\neq 0$. As in the previous case $R_1(x,y) = x = 0$ is an invariant curve, and $R_1(F(x,y)) = aR_1(x,y)$.

On the other hand it is not difficult to verify that $R_2(x,y) = \exp(y/x)$ is an exponential factor for F. Observe that

$$R_2(F(x,y)) = \exp\left(\frac{x+ay}{ax}\right) = \exp\left(\frac{1}{a}\right)\exp\left(\frac{y}{x}\right).$$

Therefore, by using Theorem 3 again, we get that

$$H(x,y) = |x|\exp\left((-a\ln a)\frac{y}{x}\right),$$

is a first integral of the linear map in the case $A = A_2$.

3.4 A type of rational maps

In this section we obtain a first integral for the rational map studied by Sahadevan in [11]:

$$F(x,y) = \left(y, \frac{P(x,y)}{Q(x,y)}\right),$$

where

$$P(x,y) = \begin{vmatrix} I_0(y) & I_1(y) \\ I_3(y) & I_4(y) \end{vmatrix} - \begin{vmatrix} I_2(y) & I_0(y) \\ I_5(y) & I_3(y) \end{vmatrix},$$

and

$$Q(x,y) = \begin{vmatrix} I_2(y) & I_0(y) \\ I_5(y) & I_3(y) \end{vmatrix} - \begin{vmatrix} I_1(y) & I_2(y) \\ I_4(y) & I_5(y) \end{vmatrix},$$

being

$$\begin{array}{ll} I_0(y) = A_1 + A_2 y + A_3 y^2, & I_3(y) = B_1 + B_2 y + B_3 y^2, \\ I_1(y) = A_2 + A_5 y + A_6 y^2, & I_4(y) = B_2 + B_5 y + B_6 y^2, \\ I_2(y) = A_3 + A_6 y + A_9 y^2, & I_5(y) = B_3 + B_6 y + B_9 y^2. \end{array}$$

By using a computer algebra package we ge that $R_1(x,y) = I_0(x) + I_1(x)y + I_2(x)y^2$, and $R_2(x,y) = I_3(x) + I_4(x)y + I_5(x)y^2$, are invariants of the DDS. More concretely

$$R_1(F(x,y)) = \frac{R_8(y)}{Q^2(x,y)} R_1(x,y),$$

and

$$R_2(F(x,y)) = \frac{R_8(y)}{Q^2(x,y)} R_2(x,y),$$

being $R_8(y)$ a polynomial of degree 8 in the variable y with coefficients in the variables A_i and B_i. Since both invariants have the same cofactor, Theorem 3 implies that

$$H(x,y) = \frac{R_1(x,y)}{R_2(x,y)},$$

is a first integral of our rational map.

3.5 An example constructed from involutions

Consider the map

$$F(x,y) = \left(\frac{ay+b}{y-a}, \frac{ax+b}{x-a}\right), \tag{10}$$

where a and b are real numbers. By searching quadratic invariant curves with a computer algebra package we get that $R(x,y) = b + a(x-y) + xy = 0$ is invariant. In fact

$$R(F(x,y)) = \frac{b+a^2}{(x-a)(y-a)} R(x,y).$$

Furthermore the curves $x - a = 0$ and $y - a = 0$ are singularities of the map F. Hence, by using Remark 6 a good family of candidates to be a first integral for the DDS generated by F is

$$H_{\alpha,\beta,\gamma}(x,y) = |b + a(x-y) + xy|^{\alpha} |x-a|^{\beta} |y-a|^{\gamma}.$$

By imposing that $H(F(x,y)) = H(x,y)$, we get several solutions. In particular

$$H_{1,-1,0} = \frac{b + a(x-y) + xy}{x-a}, \qquad \text{and} \qquad H_{0,1,-1} = \frac{x-a}{y-a}$$

are independent first integrals of F.

Remark 6. *(a) Observe that the fact that the map (10) has two independent first integrals fully determine the dynamical system. This is not a possible situation in the case of planar ODE. If a planar system has two first integrals, then they are functionally dependent.*

(b) The map $\varphi(x) = \frac{ax+b}{x-a}$ is an involution ($\varphi(\varphi(x)) \equiv x$), and the structure of (10) is $F(x,y) = (\varphi(y), \varphi(x))$ which is also an involution. For these type of maps it is easy to check that $H_1(x,y) = y + \varphi(x)$ and $H_2(x,y) = x + \varphi(y)$ are always first integrals. In fact H_1 is the first integral $H_{1,-1,0}$ obtained by using the Darboux Method.

3.6 An example in $\mathbb{R}^4$: the cross-ratio as a first integral of Darboux type

Consider the homography $\varphi(x) := \frac{ax+b}{cx+d}$, and from it define the map $F : \mathcal{U} \subset \mathbb{R}^4 \to \mathbb{R}^4$, $F(x) = (\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4))$. If we take $R_{i,j}(x) := x_i - x_j$ it is satisfied that

$$R_{i,j}(F(x)) = \frac{\det A}{\varphi(x_i)\varphi(x_j)} R_{i,j}(x),$$

for all $i \neq j \in \{1,2,3,4\}$, where $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence the hypersurfaces $R_{i,j}(x) = 0$ are invariant for the DDS generated by F, and their cofactors are $\frac{\det A}{\varphi(x_i)\varphi(x_j)}$. If $\det A = 0$ then each $R_{i,j}(x)$ is a first integral for F. If $\det A \neq 0$ the Darboux Method given in Theorem 3 (extended to $\mathbb{R}^4$ in the natural way) gives that for $i, j, k, l \in \{1, 2, 3, 4\}$ being all different, the function

$$H(x) = \frac{R_{i,j}(x)R_{k,l}(x)}{R_{i,k}(x)R_{j,l}(x)}$$

is a first integral for the DDS generated dy F. Note that the equality $H(F(x)) = F(x)$ can be interpreted as the conservation of the cross-ratio for homographies.

3.7 Two examples of DDS for which the Darboux Method does not work.

The aim of this subsection is to clarify that although the Darboux Method introduced in this paper works for several DDS, it is not an universal method for searching first integrals.

Example A (The Gauss Map, [3, 7]). Consider x and y positive and

$$F(x, y) = \left(\frac{x+y}{2}, \sqrt{xy}\right). \tag{11}$$

The above map has the nice interpretation to combine the geometric and arithmetic means, and was studied by Gauss in 1799 and 1818, see [7]. He proved that

$$H(x, y) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{x^2 \cos^2\theta + y^2 \sin^2\theta}}$$

is a first integral of the dynamic system defined by (11). Note that the above expression is an elliptic integral. For a different proof of that of Gauss of the fact that $H(x, y)$ is a first integral of the Gauss map, and a study of similar related maps, see again [7].

Example B (The Cohen map, [10]). Consider the following map:

$$F(x, y) = \left(\sqrt{1+x^2} - y, x\right). \tag{12}$$

Although the numerical studies show exact integrability an explicit expression of its first integral is not known. In particular in [10], the authors prove that it does not exists any algebraic first integral of (12), that is, any map $H(x, y)$ such that $G(x, y, H(x, y)) = 0$ for some three variable polynomial G.

3.8 An application of the knowledge of a first integral

Assume that for the DDS generated by F it is proved that $\lim_{n\to\infty} F^n(x_0, y_0)$ exists. The knowledge of a first integral for the DDS allows to get this limit in terms of (x_0, y_0). We collect some examples of application of the above idea in next result.

Proposition 7. *Consider the four DDS given by*

$$F_1(x,y) = (y, \mu x + (1-\mu)y)\,, \qquad 0<\mu<1,\ (x_0,y_0)\in\mathbb{R}^2,$$

$$F_2(x,y) = \left(\sqrt[p]{\frac{x(x^p-y^p)}{p(x-y)}}, \sqrt[p]{\frac{y(x^p-y^p)}{p(x-y)}}\right), \qquad (x_0,y_0)\in(0,\infty)\times(0,\infty),\ p\in\mathbb{N}, p\geq 2,$$

$$F_3(x,y) = \left(\frac{x+y}{2}, \sqrt{y\frac{x+y}{2}}\right), \qquad 0<x_0\leq y_0,\ y_0\in(0,\infty),$$

$$F_4(x,y) = \left(\frac{x+y}{2}, \sqrt{xy}\right), \qquad (x_0,y_0)\in(0,\infty)\times(0,\infty),$$

then $\lim_{n\to\infty} F_j(x_0,y_0) = (\ell_j(x_0,y_0), \ell_j(x_0,y_0)),\ j=1,2,3,4,$ *where*

$$\ell_1(x_0,y_0) = \tfrac{\mu x_0+y_0}{\mu+1}, \qquad \ell_2(x_0,y_0) = \sqrt[p]{\tfrac{y_0^p-x_0^p}{p(\ln y_0-\ln x_0)}}\,,$$

$$\ell_3(x_0,y_0) = \sqrt{\tfrac{y_0^2-x_0^2}{\arccos^2(x_0/y_0)}}\,, \qquad \ell_4(x_0,y_0) = \tfrac{\pi}{2}\left(\int_0^{\pi/2}\tfrac{d\theta}{\sqrt{x_0^2\cos^2\theta+y_0^2\sin^2\theta}}\right)^{-1}.$$

Proof. The DDS generated by $F_j(x,y)$, $j=2,3,4$ are studied in Sections 3.2, 3.2 and 3.7 respectively, and a first integral $H_j(x,y)$ for each one of them is obtained. The DDS generated by $F_1(x,y)$ is linear. By using the results of Section 3.3 we get the first integral $H_1(x,y)=\mu x+y$. In all cases it is not difficult to prove that $\lim_{n\to\infty} F_j(x_0,y_0)$ exists and that it is of the form $(\ell_j(x_0,y_0), \ell_j(x_0,y_0))$. We will give the details for the case $j=2$. The other cases follow with a similar study. Firstly observe that all the points of the form (x_0,x_0) are fix points for F_2. Set $(x_n,y_n)=F_2(x_{n-1},y_{n-1})$. Assume for instance that the initial condition satisfies $x_0<y_0$. Therefore it is easy to check that $x_0<x_1<y_1<y_0$. Hence the sequence $\{x_n\}$ (resp. $\{y_n\}$) is monotone increasing (resp. decreasing) and bounded above (resp. bellow). So $\lim_{n\to\infty}(x_n,y_n)$ exists. By using the expression of F_2 it follows that the limit has to be of the form $(\ell_2(x_0,y_0), \ell_2(x_0,y_0))$, as we wanted to prove. In other words we have to solve the system

$$H_2(x,y) = \frac{y^p-x^p}{\ln y-\ln x} = \frac{y_0^p-x_0^p}{\ln y_0-\ln x_0}, \quad x=y.$$

By applying the L'Hôpital's rule we get the solution $x = y = \sqrt[p]{\frac{y_0^p - x_0^p}{p(\ln y_0 - \ln x_0)}} :=$ $\ell_2(x_0, y_0)$. ∎

Remark 8. *(i) In the case $j = 4$ of the above proposition the number $l_4(x_0, y_0)$ is called the arithmetic–geometric mean of x_0 and y_0 and has been already computed by Gauss.*

(ii) The case $j = 1$ of the above proposition corresponds also to the study of the second order linear difference equation $x_{n+1} = (1 - \mu)x_n + \mu x_{n-1}$. Note that the result proved implies that $\lim_{n\to\infty} x_n = (\mu x_0 + x_1)/(\mu + 1)$, and it is obtained without solving the difference equation.

(iii) Finally observe that all the results obtained in the above proposition correspond to the computation of some kind of generalized means of the numbers x_0 and y_0.

Remark 9. *The following problem was proposed by G. Ladas several years ago, see also [1]: Consider the second order difference equation $x_{n+1} = 1 + x_{n-1}/x_n$. It is known that every solution with initial condition x_1 and x_2 positive numbers tends to a period two solution located on the curve $\mathcal{D} := \{(x, y) : y = \frac{x}{x-1}\}$. Determine the two periodic solution in terms of the initial condition.*

The above problem can be reformulated in terms of $F^2 = F \circ F$, where $F(x, y) = (y, 1 + x/y)$: Consider the map $F^2(x, y) = (\frac{x+y}{y}, \frac{x+y+y^2}{x+y})$. As a consequence of the study of the difference equation we have that every solution with initial condition (x_1, y_1), both positive numbers, tends to a fix point of F^2, also located in $\mathcal{D}$. If we were able to find a first integral for F^2, arguing as in the proof of Proposition 7, the problem would be solved. A difficulty of this approach comes from the fact that the expression of first integrals is not always given by elementary functions, as for instance, in the case of the Gauss map.

4 Some relation between Darboux Method of integrability for ODE and for DDS

Let us recall the basic ideas of Darboux Theory of integrability for planar polynomial differential equations: Consider the ODE $(\dot{x}, \dot{y}) = X(x, y)$, being X a polynomial vector field of degree n. It is said that an algebraic curve $R(x, y) = 0$ is an *invariant algebraic curve* for it if $< \nabla R(x, y), X(x, y) >=$ $K(x, y)X(x, y)$, for some polynomial K, called *the cofactor.* Notice that $R(x, y)$ has degree at most $n - 1$. Here $<, >$ denotes the usual scalar product.

Exponential factors, complex curves, and other extensions are considered in the general theory, see again [5, 6], but we restrict our attention to the simplest situation. The Darboux Method is summarized in the following result:

Theorem 10 (Darboux Method of integrability for ODE). *Let*

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{13}$$

be a polynomial differential system of degree n. Then the following statements hold:

(i) If $R_i(x,y) = 0$ for $i = 1, \cdots, \frac{n(n+1)}{2} + 1$ are different invariant irreducible algebraic curves for (13), with associated cofactors $K_i(x,y)$ respectively, then the differential equation has a Darboux first integral of the form

$$H(x,y) = \prod_{i=1}^{\frac{n(n+1)}{2}+1} |R_i(x,y)|^{\alpha_i},$$

for some $\alpha_i \in \mathbb{R}$.

(ii) If (13) has s invariant algebraic curves $R_i(x,y) = 0$, $i = 1, \cdots, s$ such that there exist $\alpha_i \in \mathbb{R}$, $i = 1, \cdots, s$ satisfying $\sum_{i=1}^{s} \alpha_i K_i(x,y) = 0$, being $K_i(x,y)$ the cofactors of the curves $R_i(x,y)$, then the differential equation has the Darboux first integral

$$H(x,y) = \prod_{i=1}^{s} |R_i(x,y)|^{\alpha_i}.$$

Note that a modification of statement (ii) of the above result is the one that is applicable to DDS (see Theorem 3 and Remark 5). Also in the statement (ii), the the fact that both $R_i(x,y)$ and $K_i(x,y)$ are polynomials is not essential. On the other hand the proof of part (i) is strongly based on the fact that the cofactors are polynomials of degree at most $n-1$, and if the system has enough of them (that is $\frac{n(n+1)}{2} + 1$) they have to be linearly independent.

To fix the relation we are searching we proceed with a concrete example: Consider the differential system

$$\begin{cases} \dot{x} = -y + 4x^2y^2, \\ \dot{y} = x + 4xy^3. \end{cases} \tag{14}$$

(System H_4 of [4], p.45). This system of differential equations has an isochronous center at the origin (this follows because in polar coordinates (r, θ), it satisfies $\dot{\theta} \equiv 1$). We check that it has a Darboux first integral:

Set $X(x, y) = (-y + 4x^2y^2, x + 4xy^3)$. Consider $R_1(x, y) = x^2 + y^2$ and $R_2(x, y) = 1 + 4y^3$. Then

$$< \nabla R_1(x, y), X(x, y) >= 8xy^2 R_1(x, y) := K_1(x, y) R_1(x, y),$$

and

$$< \nabla R_2(x, y), X(x, y) >= 12xy^2 R_2(x, y) := K_2(x, y) R_2(x, y).$$

Therefore $3K_1(x, y) - 2K_2(x, y) = 0$ and by Theorem 10.(ii)

$$H(x, y) = \frac{(x^2 + y^2)^3}{(1 + 4y^3)^2}, \tag{15}$$

is a first integral of system (14).

Consider now the flow associated to system (14). It is given by

$$\varphi(t, (x_0, y_0)) = \left(\frac{x_0 \cos t - y_0 \sin t}{\sqrt[3]{1 + 4y_0^3 - 4(x_0 \sin t + y_0 \cos t)^3}}, \right.$$
$$\left. \frac{x_0 \sin t + y_0 \cos t}{\sqrt[3]{1 + 4y_0^3 - 4(x_0 \sin t + y_0 \cos t)^3}} \right).$$

Notice that in a neighborhood of the origin the above map is the identity for $t = 2\pi$ (that is the flow is isochronous). Consider the flow at the time $t = \pi$. Then define

$$F(x, y) := \varphi(\pi, (x, y)) = \left(\frac{-x}{\sqrt[3]{1 + 8y^3}}, \frac{-y}{\sqrt[3]{1 + 8y^3}} \right). \tag{16}$$

The above map should have the same first integral than (14), because (16) comes from the flow–map. Let us test that the Darboux Method introduced in this paper also works for (16) by using the same invariants and giving rise again to (15).

Take $R_1(x, y) = x^2 + y^2$ and $R_2(x, y) = 1 + 4y^3$. Then

$$R_1(F(x, y)) = \frac{1}{(1 + 8y^3)^{\frac{2}{3}}} R_1(x, y) := \tilde{K}_1(x, y) R_1(x, y), \text{ and}$$
$$R_2(F(x, y)) = \frac{1}{(1 + 8y^3)} R_2(x, y) := \tilde{K}_2(x, y) R_2(x, y).$$

Notice that

$$\frac{\tilde{K}_1^3(x,y)}{\tilde{K}_2^2(x,y)} = 1,$$

and therefore by Theorem 3, (15) is a first integral of (16).

Observe that in fact the map (16) has another independent first integral given by $H_2(x,y) = y/x$, and, as in Example 3.5, both first integrals fully determine the dynamical system generated by F.

Next result generalizes the relation between the integrability of ODE and DDS detected in the previous example:

Theorem 11 (Relation between the Darboux Method for ODE and DDS.). *Let*

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{17}$$

be a differential system, such that it has s invariant algebraic curves $R_i(x,y) = 0$, $x = 1, \cdots, s$, with cofactors $K_i(x,y)$, such that there exist $\alpha_i \in \mathbb{R}$, $i = 1, \cdots, s$ satisfying $\sum_{i=1}^{s} \alpha_i K_i(x,y) = 0$. Let $\varphi(t,(x_0,y_0))$ the solution of the Cauchy problem defined by system (17) plus the initial conditions $x(0) = x_0, y(0) = y_0$. Consider $F_t(x,y) := \varphi(t,(x_0,y_0))$ for some $t \in \mathbb{R}$ for which the flow φ is well defined. Note that $\varphi : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Therefore the following statements hold:

(i) The curves $R_i(x,y) = 0$ are also invariant curves for F_t.

(ii) The cofactors of each $R_i(x,y)$ are

$$K_i^t(x,y) = \exp\left[\int_0^t K_i(\varphi(s,(x,y)))ds\right].$$

(iii) $\prod_{i=1}^{s} |K_i^t(x,y)|^{\alpha_i} = 1$.

(iv) Both dynamical systems, the ODE, and the DDS generated by F_t have the same first integral

$$H(x,y) = \prod_{i=1}^{s} |R_i(x,y)|^{\alpha_i}.$$

Proof. It suffices to prove (i) and (ii). The other results follow straightforward from them. Let us prove both together. If $R(x,y) = 0$ is an invariant by (17) we have that $R(\varphi(t,(x,y))) = 0$ for all t in the interval of definition of the solution. Therefore we write

$$R(\varphi(t,(x,y))) = K(t,(x,y))R(x,y), \tag{18}$$

for some $K(t,(x,y))$. Let us compute this $K(t,(x,y))$. Making derivatives with respect to t we get

$$< \nabla R(\varphi(t,(x,y))), f(\varphi(t,(x,y))) >= \frac{\partial K(t,(x,y))}{\partial t} R(x,y).$$

Since $\{R(x,y)=0\}$, is invariant for (17) with cofactor $K(x,y)$ we obtain

$$K(\varphi(t,(x,y)))R(\varphi(t,(x,y))) = \frac{\partial K(t,(x,y))}{\partial t} R(x,y).$$

By using (18) we get

$$\frac{\partial K(t,(x,y))}{\partial t} = K(\varphi(t,(x,y)))K(t,(x,y)),$$

or equivalently (ii) as we wanted to prove. Observe that in statement (ii) we have denoted $K(t,(x,y))$ by $K^t(x,y)$. ∎

Corollary 12. *If an ODE can be integrated by the Darboux Method then any DDS defined by the flow of this ODE can be integrated by the Darboux Method as well.*

Another example is given by the following isochronous system studied by Loud in [9],

$$\begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x(1+y). \end{cases} \tag{19}$$

Its flow at time $\pi/2$ is

$$F(x,y) := \varphi\left(\frac{\pi}{2},(x,y)\right) = \left(\frac{-y}{1+y-x}, \frac{x}{1+y-x}\right). \tag{20}$$

It has the invariants $R_1(x,y) = x^2+y^2$ and $R_2(x,y) = 1+y$ and the first integral, for both (19) and (20) can be obtained from the Darboux Method giving the first integral:

$$H(x,y) = \frac{x^2+y^2}{(1+y)^2}.$$

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