Decomposition of $B_n\, J(A)$ and extension of A_n-maps to A_∞-maps.

M. Cuvilliez*

Abstract

In this paper, we generalize the result of James $\Sigma J(A) \simeq \bigvee_{i=1}^{\infty} (\Sigma \bigwedge^i A)$ about the suspension of $J(A)$, the free monoid generated A, to the n^{th} step $B_n J(A)$ of the construction of classifying space of $J(A)$: $B_n J(A) \simeq \Sigma A \vee \Sigma^n \bigwedge^{n+1} A \wedge \frac{S^0}{\bigwedge^n (S^0 - A)}$ where $\frac{S^0}{\bigwedge^n (S^0-A)} = S^0 \vee \bigvee_{j=1}^{\infty} \bigvee^{\frac{n\cdot(n+1)\ldots(n+j-1)}{j!}} (\bigwedge^j A)$ and apply it to the computation of the set $[\Omega S^p, \Omega X]_{A_n}$ of A_n-homotopy classes of A_n-maps from ΩS^p to ΩX. We prove also that each A_{n+1}-map from $J(A)$ in a loop space is A_n-homotopic to an A_∞-map.

For beginning, we introduce some notations. Let $f\colon A \longrightarrow X$ and $g\colon B \longrightarrow X$ two continuous maps. Denote by $A \underset{X}{\times} B$ the standard homotopy pullback [4] of f and g, by $A \underset{X}{\bowtie} B$ the standard homotopy pushout of the projections $A \underset{X}{\times} B \longrightarrow A$ and $A \underset{X}{\times} B \longrightarrow B$, and by $f \bowtie g\colon A \underset{X}{\bowtie} B \longrightarrow X$ the induced map. Let $A \wedge B$ be the cofiber of the inclusion $A \vee B \longrightarrow A \times B$. Let $\bigwedge^0 A = S^0$, $\bigwedge^{n+1} A = A \wedge \bigwedge^n A$, $\overset{1}{\underset{X}{\bowtie}} A = A$ and $\overset{n+1}{\underset{X}{\bowtie}} A = A \underset{X}{\bowtie} (\overset{n}{\underset{X}{\bowtie}} A)$. Denote by C_f the homotopy cofiber of f and by F the homotopy fiber of the inclusion $\alpha\colon X \longrightarrow C_f$. For an strictly associative H-space X, $B_n X$ denotes the n^{th} step of the Stasheff bar construction [5]. In the special case where $X = \Omega A$, $B_n \Omega A$ is constructed by induction as follows: $b_0\colon B_0 \Omega A = * \to A$ is the pointed map, F_n is the homotopy fiber of b_n and $b_{n+1}\colon B_{n+1}\Omega A = B_n \Omega A \cup CF_n \to A$ is the induced map sending the

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cone on the basepoint. Denote by $cat\,\alpha$ (resp. by $cat\,X$ and by $Cl\,X$) be the L.S. category of a map α (resp. the L.S. category and the cone-length of the space X)[3].

Theorem *Suppose that X is connected and that C_f is 1-connected. If $cat\,\alpha \leq n-1$, then $B_n\Omega C_f \simeq C_f \bigcup_X (A \underset{X}{\bowtie} (\overset{n}{\underset{X}{\bowtie}} F))$.*

Let $J(A) \simeq \Omega\Sigma A$ the free monoid generated by A. This Theorem generalizes the result of James: $\Sigma J(A) \simeq \bigvee_{i=1}^{\infty} (\Sigma \bigwedge^i A)$. In fact, remark that $\Sigma\Omega\Sigma A \simeq B_1(J(A)) \simeq \Sigma A \vee (A \underset{*}{\bowtie} \Omega\Sigma A)$. Since $A \underset{*}{\bowtie} M \simeq A \wedge \Sigma M$, we get $\Sigma\Omega\Sigma A \simeq \Sigma A \vee (A \wedge \Sigma\Omega\Sigma A) \simeq \bigvee_{i=1}^{\infty} (\Sigma \bigwedge^i A)$.

By analogy with the power series extension of $\frac{1}{(1-x)^n}$, we denote $S^0 \vee \bigvee_{j=1}(\bigvee^{\frac{n\cdot(n+1)\ldots(n+j-1)}{j!}}(\bigwedge^j A))$ by $\frac{S^0}{\bigwedge^n(S^0-A)}$.

Corollary 1 *We have: $B_nJ(A) \simeq \Sigma A \vee \Sigma^n \bigwedge^{n+1} A \wedge \frac{S^0}{\bigwedge^n(S^0-A)}$.*

In particular,

$$B_n\Omega S^{p+1} \simeq S^{p+1} \vee S^{p(n+1)+p} \vee \bigvee_{i=1}^{\infty} \Big(\bigvee^{\frac{n(n+1)\ldots(n+i-1)}{i!}} S^{p(n+i+1)+n}\Big).$$

Proof of the Corollary: By the Theorem, we have

$$\begin{aligned}
B_n\Omega\Sigma A &\simeq \Sigma A \vee (A \underset{*}{\bowtie} (\overset{n}{\underset{*}{\bowtie}} \Omega\Sigma A)) \\
&\simeq \Sigma A \vee (A \wedge \bigwedge^{n}(\Sigma\Omega\Sigma A)) \\
&\simeq \Sigma A \vee (A \wedge \bigwedge^{n}(\bigvee_{i=1}^{\infty} (\Sigma \bigwedge^{i} A))) \\
&\simeq \Sigma A \vee (\Sigma^n \bigwedge^{n+1} A \wedge \bigwedge^{n}(\bigvee_{i=0}^{\infty} (\bigwedge^{i} A)) \\
&\simeq \Sigma A \vee \Sigma^n \bigwedge^{n+1} A \wedge \frac{S^0}{\bigwedge^n(S^0-A)}
\end{aligned}$$

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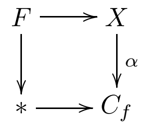
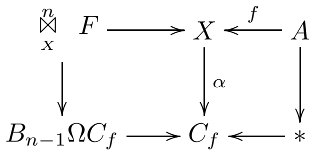
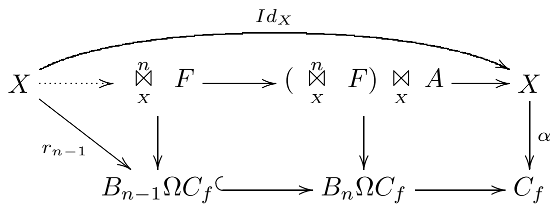
Proof of the Theorem: Apply $(n-1)$ times the Join Theorem [2] to the homotopy pullback [4]



Then the left square in the next diagram is a homotopy pullback.

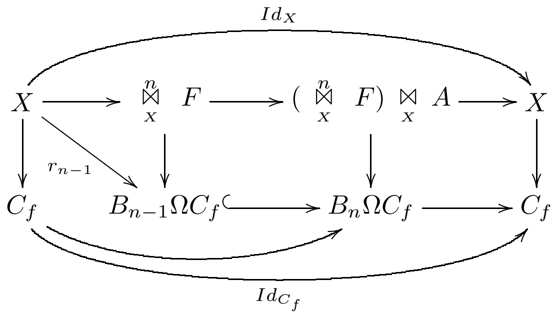


Apply the second join Theorem [2] to this diagram. Then the right square in the following diagram is a homotopy pushout.



Since $cat\, \alpha \leq n-1$, the map α factorizes up to homotopy through $B_{n-1}\Omega C_f$ [3] by r_{n-1}. Then, by universal property of homotopy pullback, we can construct a map $X \longrightarrow \overset{n}{\underset{X}{\bowtie}} F$ such that the previous diagram commutes up to homotopy.

Let F_{n-1} be the homotopy fiber of $b_{n-1} : B_{n-1}\Omega C_f \longrightarrow C_f$. Since the composed map $b_{n-1} \circ r_{n-1} \circ f : A \longrightarrow C_f$ is nullhomotopic, $r_{n-1} \circ f$ factorizes through F_{n-1}. Since $B_n\Omega C_f$ is the homotopy cofiber of the inclusion $F_{n-1} \longrightarrow B_{n-1}\Omega C_f$, Id_{C_f} factorizes through $B_n\Omega C_f$ and the following diagram is homotopy commutative.



The large rectangle and the right square are homotopy pushout, then, since C_f and $(\overset{n}{\underset{X}{\bowtie}} F) \underset{X}{\bowtie} A$ are 1-connected, the following square is a homotopy pushout [4, Lemma 41].

$$\begin{array}{ccc} X & \longrightarrow & (\overset{n}{\underset{X}{\bowtie}} F) \underset{X}{\bowtie} A \\ \downarrow & & \downarrow \\ C_f & \longrightarrow & B_n \Omega C_f \end{array}$$

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Let $g : W \longrightarrow Z$ be a map between two H-spaces. We said that g is an A_2-map if g preserves the H-spaces structures up to homotopy; we said that g is an A_1-map if g is just a continuous map. More generally, an A_n-map is a map which preserves H-spaces structure up to homotopy of degree n (see [5]). Denote by $[W, Z]_{A_n}$ the set of the A_n-homotopy classes of A_n-maps from W to Z. In [1], A. Deligiannis proves that $[\Omega X, \Omega Y]_{A_n}$ is isomorphic to $[B_n \Omega X, Y]$. By Corollary 1, we deduce that :

Corollary 2

$$[\Omega S^{p+1}, \Omega Y]_{A_n} \simeq \pi_{p+1}(Y) \oplus \pi_{p(n+1)+n}(Y) \oplus$$

$$\bigoplus_{i=1}^{\infty} \frac{n(n+1)\dots(n+i-1)}{i!} \pi_{p(n+i+1)+n}(Y)\,.$$

Proposition 1 *Let A be a space and C be a space with $Cat\, C \leq k$. Then an A_{n+1}-map from $J(A)$ in a loop space is A_n-homotopic to an A_∞-maps.*

More generally, an A_{n+k}-map from ΩC in a loop space is A_n-homotopic to an A_∞-map.

Proof : The morphism

$$\begin{array}{ccc} [\Omega\Sigma A, \Omega Z]_{n+1} & \xrightarrow{\quad g \quad} & [\Omega\Sigma A, \Omega Z]_n \\ \cong \uparrow & & \uparrow \cong \\ [\Sigma A, Z] \oplus [A \underset{*}{\bowtie} (\overset{n+1}{\underset{*}{\bowtie}} \Omega\Sigma A), Z] & \longrightarrow & [\Sigma A, Z] \oplus [A \underset{*}{\bowtie} (\overset{n}{\underset{*}{\bowtie}} \Omega\Sigma A), Z] \end{array}$$

is given by the inclusion $A \underset{*}{\bowtie} (\overset{n}{\underset{*}{\bowtie}} \Omega\Sigma A) \longrightarrow A \underset{*}{\bowtie} (\overset{n+1}{\underset{*}{\bowtie}} \Omega\Sigma A)$ which is nullhomotopic. Then if f is in $[\Omega\Sigma A, \Omega Z]_{n+1}$, $g(f)$ is in $[\Sigma A, Z] \cong [\Omega\Sigma A, \Omega Z]_{A_\infty}$.

More generally, if $Cat\, C \leq k$, there exists a cofibration $A \longrightarrow X \longrightarrow C$ with $cat\, C \leq k-1$ and the morphism $g : [\Omega C, \Omega Z]_{A_{n+k}} \longrightarrow [\Omega C, \Omega Z]_{A_n} \cong [C, Z] \bigoplus_{[X,Z]} [C, A \underset{X}{\bowtie} (\overset{n+k}{\underset{X}{\bowtie}} F)]$ is given by the map $(A \underset{X}{\bowtie} (\overset{n+k}{\underset{X}{\bowtie}} F)) \longrightarrow (A \underset{X}{\bowtie} (\overset{n}{\underset{X}{\bowtie}} F))$ which factorizes through X. Then $g([\Omega C, \Omega Z]_{A_{n+k}}) = [\Omega C, \Omega Z]_{A_\infty}$. ■

References

[1] A. Deligiannis, *A_n-spaces, Ganea spaces and adjoint functors*, Preprint UCL, Louvain-La-Neuve (2000).

[2] J. P. Doeraene, *L.S.-category in a model category*, J. of pure and applied algebra 84 (1993), pp. 215-261.

[3] I. M. James, *Handbook of Algebraic Topology*, Elsevier (1995).

[4] M. Mather, *Pull-backs in homotopy theory*, Can. J. Math., 28 (1976), pp. 225-263.

[5] J. Stasheff, *H-space from homotopy point of view*, Lectures Notes in Mathematics 161 (1970), Springer-Verlag.

M. Cuvilliez,
Centre de Recerca Matemàtica,
Institut d'Estudis Catalans,
Apartat 50,
E-08193 Bellaterra,
Spain

e-mail:mcuvilli@crm.es