HOMOTOPY FINITE GROUP THEORY

CARLOS BROTO, RAN LEVI, AND BOB OLIVER

The main goal of this paper is to identify and study a certain class of spaces which in many ways behave like p-completed classifying spaces of finite groups. These spaces occur as the "classifying spaces" of certain algebraic objects, which we call "homotopy finite groups".

Homotopy finite group theory is aimed to be for finite group theory, what the theory of p-compact groups, due to Dwyer-Wilkerson [DW3] and studied by many others, is for compact Lie groups. It is thus likely to be of independent interest in algebraic topology. We also hope that the ideas presented here can eventually be related to group representation theory and in particular be applied to construct classifying spaces for blocks in p-adic group rings of finite groups.

This paper has as starting point in part the axiomatic treatment by Lluis Puig [Pu] of systems of fusion among subgroups of a given p-group, as well as the earlier paper [BLO] by the same authors on p-completed classifying spaces of finite groups.

A *saturated Frobenius system* $\mathcal{F}$ over a p-group S consists of a set of monomorphisms $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for each pair of subgroups $P,Q \le S$, which form a category under composition, include all monomorphisms induced by conjugation in S, and satisfy certain other axioms formulated by Puig (see Definition 1) which are clearly satisfied by the fusion homomorphisms in a finite group. We refer to [Pu] for more details of Puig's work on Frobenius systems; the definitions and results given here in Section 1 and Appendix A are only a very brief account of those results of Puig needed in this paper.

If $\mathcal{F}$ is a saturated Frobenius system over S, then two subgroups $P,P' \le S$ are called *$\mathcal{F}$-conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P,P') \ne \emptyset$, and P is called *$\mathcal{F}$-centric* if $C_S(P') \le P'$ for all P' which is $\mathcal{F}$-conjugate to P. Let $\mathcal{F}^c$ be the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S. An *associated $\mathcal{L}$-system* to $\mathcal{F}$ is a category $\mathcal{L}$, together with a functor $\mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}^c$ which is bijective on objects and surjective on morphisms, and which satisfies other axioms listed below in Definition 1. In particular, for each object P, the kernel of the induced map $\mathrm{Aut}_{\mathcal{L}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P)$ is isomorphic to $Z(P)$, and $\mathrm{Aut}_{\mathcal{L}}(P)$ contains a distinguished subgroup isomorphic to P.

The motivating examples for these definitions come from finite groups. If G is a finite group and p is a prime, then $\mathcal{F} = \mathcal{F}_S(G)$ is the Frobenius system over $S \in \mathrm{Syl}_p(G)$ such that for each $P,Q \le S$, $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the set of homomorphisms induced by conjugation in G (and inclusion). The $\mathcal{F}$-centric subgroups of S are the p-centric subgroups: those $P \le S$ such that $C_G(P) \cong Z(P) \times C'_G(P)$ for some $C'_G(P)$ of order prime to p. In [BLO], we defined a category $\mathcal{L}^c_p(G)$ whose objects are the p-centric subgroups, and where $\mathrm{Mor}_{\mathcal{L}^c_p(G)}(P,Q) = N_G(P,Q)/C'_G(P)$. Here, $N_G(P,Q)$ is the set of elements of G which conjugate P into Q. The category $\mathcal{L}^c_p(G)$, together with its projection to $\mathcal{F}_p(G)$ which sends an element $g \in N_G(P,Q)$ to conjugation by g, is the example which motivated our definition of an associated $\mathcal{L}$-system.

We define a *homotopy finite group* to be a triple $(S,\mathcal{F},\mathcal{L})$, where $\mathcal{L}$ is an associated $\mathcal{L}$-system to a saturated Frobenius system $\mathcal{F}$ over S. The *classifying space* of such a triple is the space $|\mathcal{L}|_p^\wedge$, where for any small category $\mathcal{C}$, the space $|\mathcal{C}|$ denotes the geometric realization of the nerve of $\mathcal{C}$. This is partly motivated by the result that $|\mathcal{L}_p^c(G)|_p^\wedge \simeq BG_p^\wedge$ for any finite G [BLO, Proposition 1.1]. But additional motivation comes from Proposition 2 below, which says that if $\mathcal{L}$ is an associated $\mathcal{L}$-system to $\mathcal{F}$, then $|\mathcal{L}| \simeq \underrightarrow{\mathrm{hocolim}}_{\mathcal{O}^c(\mathcal{F})}(\widetilde{B})$, where $\mathcal{O}^c(\mathcal{F})$ is a certain quotient "orbit" category of $\mathcal{F}^c$, and $\widetilde{B}$ is a lifting of the homotopy functor which sends P to BP. The classifying space of each homotopy finite group thus comes equipped with a homotopy decomposition as the homotopy colimit of a finite diagram of classifying spaces of p-groups.

We now state our main results. Our first result is that a homotopy finite group is determined by its classifying space up to isomorphism. What is meant by an isomorphism of homotopy finite groups will be explained later.

Theorem A. *A homotopy finite group $(S, \mathcal{F}, \mathcal{L})$ is determined by the homotopy type of $|\mathcal{L}|_p^\wedge$ (Theorem 5). In particular, if $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are two homotopy finite groups and $|\mathcal{L}|_p^\wedge \simeq |\mathcal{L}'|_p^\wedge$, then $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are isomorphic.*

The next theorem gives an explicit description of the mapping space from the classifying space of a finite p-group into the classifying space of a homotopy finite group. It is stated precisely as Theorem 4 and Corollary 4.

Theorem B. *For any homotopy finite group $(S, \mathcal{F}, \mathcal{L})$, and any p-group Q,*

$$[BQ, |\mathcal{L}|_p^\wedge] \cong \mathrm{Rep}(Q, \mathcal{L}) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(Q, S)/(\mathcal{F}\text{-conjugacy}).$$

Furthermore, each component of the mapping space has the homotopy type of the classifying space of a homotopy finite group which can be thought of as the "centralizer" of the image of the corresponding homomorphism $Q \longrightarrow S$.

Next we study the cohomology of homotopy finite groups. As one may expect we have the following theorem, which appears as Theorem 6.

Theorem C. *For any homotopy finite group $(S, \mathcal{F}, \mathcal{L})$, $H^*(|\mathcal{L}|_p^\wedge; F_p)$ is noetherian.*

[**Add something (remark only?) on $H^*(|L|)$ and invariants in $H^*(S)$.**]

The next result describes the space of self equivalences of the classifying space of a homotopy finite group. It is a generalization of [BLO, Theorem C]. For a small category $\mathcal{C}$, let $\mathcal{A}ut(\mathcal{C})$ denote the groupoid whose objects are self equivalences of $\mathcal{C}$, and whose morphisms are natural isomorphisms of functors. Let $\mathcal{L}$ be an $\mathcal{L}$-system associated to a saturated Frobenius system $\mathcal{F}$. Self equivalences of $\mathcal{L}$ which are structure preserving, in a sense to be made precise in section 7 below, are said to be isotypical. We let $\mathcal{A}ut_{\text{typ}}(\mathcal{L})$ denote the subgroupoid of $\mathcal{A}ut(\mathcal{L})$ whose objects are the isotypical self equivalences of $\mathcal{L}$. For a space X, let $Aut(X)$ denote the topological monoid of all self homotopy equivalences of X. The following theorem is restated below as Theorem 7.

Theorem D. *Fix a homotopy finite group $(S, \mathcal{F}, \mathcal{L})$. Then $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$ and $|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|$ are equivalent as topological monoids in the sense that their classifying spaces are homotopy equivalent. In particular, they have isomorphic groups of components and each component is aspherical.*

The statement of Theorem 7 also includes a description of the homotopy groups of $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$.

So far we have not mentioned the question of existence and uniqueness of $\mathcal{L}$-systems associated to a saturated Frobenius system. Of course, as pointed out above, any finite group G gives rise to an associated homotopy finite group. However there are Frobenius systems which do not occur as the Frobenius system of any finite group. Thus a tool for deciding existence and uniqueness would be useful. The following theorem, restated later as Propositions 3 and 2, settles this problem for p-groups of small rank. [**Say more about the obstruction theory**].

Theorem E. *Fix a saturated Frobenius system $\mathcal{F}$ over a p-group S. If $\mathrm{rk}_p(S) < p^3$, then there always exists an associated $\mathcal{L}$-system, and if $\mathrm{rk}_p(S) < p^2$, then the associated $\mathcal{L}$-system is unique.*

Finally we present a construction of a certain family of homotopy finite groups whose underlying Frobenius systems were shown by Solomon [Sol] not to correspond to any finite group. These homotopy finite groups, constructed in Section 9, and are based on Frobenius systems $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ over a Sylow 2-subgroup of $\mathrm{Spin}(7,q)$, which are generated by the fusion in $\mathrm{Spin}(7,q)$ together with the relation that all elements of order two are $\mathcal{F}$-conjugate. A precise restatement of the following is as Theorem 9.

Theorem F. *Let q be an odd prime power, and fix $S \in \mathrm{Syl}_2(\mathrm{Spin}(7,q))$. Then there is a saturated Frobenius system $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ and a unique associated $\mathcal{L}$-system $\mathcal{L} = \mathcal{L}_{\mathrm{Sol}}(q)$. In particular the classifying space of the respective homotopy finite group $(S, \mathcal{F}, \mathcal{L})$ is not homotopy equivalent to $BG^\wedge_2$ for any finite group G.*

Similar (presumably equivalent) spaces were constructed by Benson [Be, §8] using very different methods. Using the same ideas, the first author and J. Møller constructed further examples of homotopy finite groups, whose classifying spaces are not equivalent to p-completed classifying spaces of finite groups [BrM].

[**LIST RELATED KNOWN RESULTS: Linckelmann-Webb. Ask Markus's permission to mention their results (and which ones??).**]

The basic definitions of saturated Frobenius systems and their associated $\mathcal{L}$-systems are included in Section 1. The obstruction theory for the existence and uniqueness of associated $\mathcal{L}$-systems is handled in Section 2, and some results involving the obstruction groups are shown in Section 3. Maps from the classifying space of a p-group to the classifying space of a homotopy finite group are studied in Section 4, and the characterization of classifying spaces of homotopy finite groups in Section 5. The cohomology rings of classifying spaces of homotopy finite groups are dealt with in Section 6, and their spaces of self equivalences in Section 7. "Exotic" examples of homotopy finite groups are constructed in Sections 8 and 9. Finally, some other material needed for reference (on Frobenius systems, liftings of homotopy diagrams, and the groups $\mathrm{Spin}(n,q)$) is included in the three sections of the appendix.

We would like to thank Dave Benson and Jesper Grodal for their many suggestions in the course of this work. In particular, Dave had earlier written and distributed notes which contained some of the ideas of our $\mathcal{L}$-systems. We would also like to thank Lluis Puig for giving us a copy of his unpublished notes on saturated Frobenius systems. Also, Markus Linckelmann, Haynes Miller, and Bill Dwyer have all shown interest and made helpful comments in the course of this work.

We would especially like to thank the universities of Aberdeen and Paris-Nord, the UAB in Barcelona, and the Max-Planck Institut in Bonn for their hospitality in helping the three authors get together in various combinations; and also the homotopy theory network under the European Union [**be more specific??**] for helping to finance these visits.

1. Frobenius systems and $\mathcal{L}$-systems

We begin with the precise definitions of saturated Frobenius systems and their associated $\mathcal{L}$-systems. Additional results about Frobenius systems due to Puig [Pu] are in Appendix A.

Given two finite groups P, Q, let $\mathrm{Hom}(P,Q)$ denote the set of group homomorphisms from P to Q, and let $\mathrm{Inj}(P,Q)$ denote the set of monomorphisms. If P and Q are subgroups of a larger group G, $\mathrm{Hom}_G(P,Q) \subseteq \mathrm{Inj}(P,Q)$ denotes the subset of homomorphisms induced by conjugation by elements of G.

Definition 1.1. *A* Frobenius system $\mathcal{F}$ *over a finite* p*-group* S *is a category whose objects are the subgroups of* S*, and whose morphism sets* $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ *satisfy the following conditions:*

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ *for all* $P, Q \leq S$.

(b) *Every morphism in* $\mathcal{F}$ *factors as an isomorphism in* $\mathcal{F}$ *followed by an inclusion.*

[Remark that we write $\mathrm{Hom}_{\mathcal{F}} = \mathrm{Mor}_{\mathcal{F}}$ **to emphasize that morphisms in the category are all homomorphisms!]**

Note that what we call a Frobenius system here is what Puig calls a divisible Frobenius system.

If $\mathcal{F}$ is a Frobenius system over S and $P, Q \leq S$, then we write $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ for the set of isomorphisms in $\mathcal{F}$. Thus $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_{\mathcal{F}}(P,Q)$ if $|P| = |Q|$, and $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \emptyset$ otherwise. Also, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$ and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. Two subgroups $P, P' \leq S$ are called *$\mathcal{F}$-conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P,P') \neq \emptyset$. We also write c_g (for $g \in S$) to denote any homomorphism $(x \mapsto gxg^{-1})$ induced by conjugation by g.

The Frobenius systems we consider here will all satisfy the following additional condition.

Definition 1.2. *Let* $\mathcal{F}$ *be a Frobenius system over a* p*-group* S*. A subgroup* $P \leq S$ *is* saturated in $\mathcal{F}$ *if* $|C_S(P)| \geq |C_S(P')|$ *for all* $P' \leq S$ *which is* $\mathcal{F}$*-conjugate to* P*. And* $\mathcal{F}$ *is a* saturated Frobenius system *if the following two conditions hold:*

(I) *Each* $\mathcal{F}$*-conjugacy class of subgroups of* S *contains at least one subgroup* P *which is saturated, and which satisfies* $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

(II) *If* $P \leq S$ *and* $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ *are such that* φP *is saturated, and if we set*
$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ *such that* $\bar{\varphi}|_P = \varphi$.

The above definition is slightly different from the definition of a saturated Frobenius system as formulated by Lluis Puig [Pu, §1], but is equivalent to his definition (see the remarks after Proposition A). Note, in condition (II) above, that N_φ is defined to be the largest subgroup of $N_S(P)$ to which φ could possibly be extended, and that $N_\varphi \geq C_S(P){\cdot}P$.

One motivating example for this definition is the Frobenius system of a finite group G. For any $S \in \mathrm{Syl}_p(G)$, we let $\mathcal{F}_S(G)$ be the Frobenius system S defined by setting $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for all $P, Q \leq S$.

In general, for any P and any $g \in P$, $c_g \in \mathrm{Aut}(P)$ denotes the inner automorphism $c_g(x) = gxg^{-1}$.

Proposition 1.3. *Let G be a finite group, and let S be a Sylow p-subgroup of G. Then the Frobenius system $\mathcal{F}_S(G)$ over S is saturated, and a subgroup $P \le S$ is saturated in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$.*

Proof. For each $H \le G$ and each $g \in G$, we write ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for short. Fix some $P \le S$, and choose $g \in G$ such that S^g contains a Sylow p-subgroup of $N_G(P)$. Then ${}^gP \le S$ and $N_{S^g}(P) \in \mathrm{Syl}_p(N_G(P))$, and so $N_S({}^gP) \in \mathrm{Syl}_p(N_G({}^gP))$. In particular,

$$\mathrm{Aut}_S({}^gP) \in \mathrm{Syl}_p(\mathrm{Aut}_G({}^gP)) \qquad \text{and} \qquad C_S({}^gP) \in \mathrm{Syl}_p(C_G({}^gP)).$$

This also shows that $|C_S({}^gP)| \ge |C_S(P')|$ for all $P' \le S$ which is G-conjugate to P and ${}^gP$, and thus that ${}^gP$ is saturated in $\mathcal{F}_S(G)$. This proves (I) in Definition 1, and also shows that each $P' \le S$ which is G-conjugate to P is saturated if and only if $C_S(P') \in \mathrm{Syl}_p(C_G(P'))$.

To see condition (II), let $P \le S$ and $g \in G$ be such that ${}^gP \le S$ and is saturated in $\mathcal{F}_S(G)$. Thus $C_S({}^gP) \in \mathrm{Syl}_p(C_G({}^gP))$, and so there is $h \in C_G({}^gP)$ such that ${}^h({}^gC_S(P)) \le C_S({}^gP)$. Then $c_{hg} = c_g$ as elements of $\mathrm{Hom}_G(P,S)$. And if we set

$$N = \{x \in N_S(P) \,|\, c_{hg} \circ c_x \circ c_{hg}^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$

then ${}^{(hg)}N \le S$, and so $c_{hg} \in \mathrm{Hom}_{\mathcal{F}_S(G)}(N,S)$ extends $c_g \in \mathrm{Hom}_{\mathcal{F}_S(G)}(P,S)$. □

Puig's original motivation for defining Frobenius systems came from block theory. Let k be an algebraically closed field of characteristic $p \ne 0$. A block in a group ring $k[G]$ is an indecomposable 2-sided ideal which is a direct summand. Puig described in [Pu] how to construct, for any block b in $k[G]$, a saturated Frobenius category over the defect group b based on the Brauer pairs associated to b (the "b-subgroups"). See, for example, [Alp, ??], for definitions of defect groups and Brauer pairs of blocks.

In order to help motivate the next constructions, we recall some definitions from [BLO]. If G is a finite group and p is a prime, then a p-subgroup $P \le G$ is *p-centric* if $C_G(P) = Z(P) \times C'_G(P)$ where $C'_G(P)$ has order prime to p. For any $P, Q \le G$, let $N_G(P,Q)$ denote the *transporter*: the set of all $g \in G$ such that $gPg^{-1} \le Q$. For any $S \in \mathrm{Syl}_p(G)$, $\mathcal{L}^c_S(G)$ denotes the category whose objects are the p-centric subgroups of S, and where $\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q) = N_G(P,Q)/C'_G(P)$. By comparison, $\mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P)$. Hence there is a functor from $\mathcal{L}^c_S(G)$ to $\mathcal{F}_S(G)$ which is the inclusion on objects, and which sends the morphism corresponding to $g \in N_G(P,Q)$ to $c_g \in \mathrm{Hom}_G(P,Q)$.

Definition 1.4. *Let $\mathcal{F}$ be any Frobenius system over a p-group S. A subgroup $P \le S$ is* $\mathcal{F}$-centric *if P and all of its $\mathcal{F}$-conjugates contain their S-centralizers. Let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.*

We are now ready to define "$\mathcal{L}$-systems" associated to a saturated Frobenius system.

Definition 1.5. *Let $\mathcal{F}$ be a Frobenius system over the p-group S. An* $\mathcal{L}$-system *associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor*

$$\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c,$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \le S$, which satisfy the following conditions.

(A) *π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \le \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection*

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\;\cong\;} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) *For each $\mathcal{F}$-centric subgroup $P \le S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.*

(C) *For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:*

$$\begin{array}{ccc} P & \xrightarrow{\;f\;} & Q \\ \Big\downarrow{\scriptstyle \delta_P(g)} & & \Big\downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{\;f\;} & Q. \end{array}$$

One easily checks that for any G and any $S \in \mathrm{Syl}_p(G)$, $\mathcal{L}_S^c(G)$ is an associated $\mathcal{L}$-system to the Frobenius category $\mathcal{F}_S(G)$. Condition (C) is motivated in part because it always holds in $\mathcal{L}_S^c(G)$ for any G. Note that conditions (A) and (B) imply that P and Q each acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$, and (C) describes how the action of Q determines the action of P. For example, condition (C) will be important in Proposition 2 below, where we show that the nerve of any $\mathcal{L}$-system is equivalent to the homotopy colimit of a certain functor.

Throughout the rest of this paper, whenever we refer to conditions (A), (B), or (C), it will mean the conditions in the above Definition 1.

Definition 1.6. *A* homotopy finite group *is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated Frobenius system over the p-group S and $\mathcal{L}$ is an associated $\mathcal{L}$-system to $\mathcal{F}$. The* classifying space *of the homotopy finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|_p^\wedge$.*

The following notation will be used when working with homotopy finite groups. For any group G, let $\mathcal{B}(G)$ denote the category with one object o_G, and one morphism denoted $\check{g}$ for each $g \in G$.

Notation 1.7. *Let $(S, \mathcal{F}, \mathcal{L})$ be a homotopy finite group, where $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ denotes the projection functor. For each $\mathcal{F}$-centric subgroup $P \le S$, and each $g \in P$, we write*

$$\widehat{g} = \delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P),$$

and let

$$\theta_P\colon \mathcal{B}(P) \longrightarrow \mathcal{L}$$

denote the functor which sends the unique object $o_P \in \mathrm{Ob}(\mathcal{B}(P))$ to P and which sends a morphism $\check{g}$ (for $g \in P$) to $\widehat{g} = \delta_P(g)$. And if f is any morphism in $\mathcal{L}$, we let $[f] = \pi(f)$ denote its image in $\mathcal{F}$.

The following lemma lists some easy properties of associated $\mathcal{L}$-systems to saturated Frobenius systems.

Lemma 1.8. *Fix a homotopy finite group $(S, \mathcal{F}, \mathcal{L})$, and let $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ be the projection. Fix $\mathcal{F}$-centric subgroups P, Q, R in S. Then the following hold.*

(a) *Fix any sequence $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ of morphisms in $\mathcal{F}^c$, and let $\widetilde{\varphi} \in \pi^{-1}(\varphi)$, $\widetilde{\psi} \in \pi^{-1}(\psi)$, and $\widetilde{\psi\varphi} \in \pi^{-1}(\psi\varphi)$ be arbitrary liftings. Then there are unique morphisms $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and $\beta \in \mathrm{Mor}_{\mathcal{L}}(Q,R)$ such that*

$$\widetilde{\psi} \circ \alpha = \widetilde{\psi\varphi} = \beta \circ \widetilde{\varphi}, \tag{1}$$

and $[\alpha] = \varphi$ *and* $[\beta] = \psi$*.*

(b) *If* $\varphi, \varphi' \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ *are such that* $\pi_{P,Q}(\varphi)$ *and* $\pi_{P,Q}(\varphi')$ *are conjugate modulo* $\mathrm{Inn}(Q)$*, then there is a unique element* $g \in Q$ *such that* $\varphi' = \widehat{g} \circ \varphi$ *in* $\mathrm{Mor}_{\mathcal{L}}(P, Q)$*.*

Proof. Point (b) follows immediately from (a), so we need only prove (a). By (A), there is a unique element $g \in Z(P)$ such that $\widetilde{\psi\varphi} = \widetilde{\psi} \circ \widetilde{\varphi} \circ \widehat{g}$. Hence Equation (1) holds if we set $\alpha = \widetilde{\varphi} \circ \widehat{g}$ and $\beta = \widetilde{\psi} \circ \widehat{\varphi(g)}$ (the second equality follows from (C)). Conversely, if α and β are any morphisms in $\mathcal{L}$ which satisfy Equation (1), then $[\alpha] = \varphi$ and (since P is $\mathcal{F}$-centric) $[\beta] = \psi$, and hence the choice is unique by (A) again. □

Lemma 1 implies in particular that all morphisms in $\mathcal{L}$ are monomorphisms in the categorical sense.

The next proposition describes how an associated $\mathcal{L}$-system $\mathcal{L}$ over a p-group S contains the category with the same objects and whose morphisms are the sets $N_S(P, Q)$.

Proposition 1.9. *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a homotopy finite group, and let* $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ *be the canonical projection. For each* $P \le S$*, fix a choice of "inclusion" morphism* $\iota_P \in \mathrm{Mor}_{\mathcal{L}}(P, S)$ *such that* $[\iota_P] = \mathrm{incl} \in \mathrm{Hom}(P, S)$*. Then there are unique injections*

$$\delta_{P,Q}\colon N_S(P, Q) \longrightarrow \mathrm{Mor}_{\mathcal{L}}(P, Q),$$

defined for all $\mathcal{F}$*-centric subgroups* $P, Q \le S$*, which have the following properties.*

(a) *For all* $\mathcal{F}$*-centric* $P, Q \le S$ *and all* $g \in N_S(P, Q)$*,* $[\delta_{P,Q}(g)] = c_g \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$*.*

(b) *For all* $\mathcal{F}$*-centric* $P \le S$*,* $\delta_{P,S}(1) = \iota_P$*, and* $\delta_{P,P}(g) = \delta_P(g)$ *for* $g \in P$*.*

(c) *For all* $\mathcal{F}$*-centric* $P, Q, R \le S$ *and all* $g \in N_S(P, Q)$ *and* $h \in N_S(Q, R)$*,* $\delta_{Q,R}(h) \circ \delta_{P,Q}(g) = \delta_{P,R}(hg)$*.*

Proof. For each $\mathcal{F}$-centric P and Q and each $g \in N_S(P, Q)$, there is by Lemma 1(a) a unique morphism $\delta_{P,Q}(g) \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ such that $[\delta_{P,Q}(g)] = c_g$, and such that the following square commutes:

$$\begin{array}{ccc} P & \xrightarrow{\iota_P} & S \\ {\scriptstyle \delta_{P,Q}(g)}\downarrow & & \downarrow{\scriptstyle \delta_S(g)} \\ Q & \xrightarrow{\iota_Q} & S. \end{array}$$

Conditions (b) and (c) above also follow from the uniqueness property in Lemma 1(a). The injectivity of $\delta_{P,Q}$ follows from condition (A), since $[\delta_{P,Q}(g)] = [\delta_{P,Q}(h)]$ in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ if and only if $h^{-1}g \in C_S(P) = Z(P)$. □

We finish the section with the following proposition, which shows that the classifying space of any homotopy finite group is p-complete, and also provides some control over its fundamental group.

Proposition 1.10. *Let* $(S, \mathcal{F}, \mathcal{L})$ *be any homotopy finite group at the prime* p*. Then* $|\mathcal{L}|$ *is* p*-good. Also, the composite*

$$S \xrightarrow{\pi_1(|\theta_S|)} \pi_1(|\mathcal{L}|) \longrightarrow \pi_1(|\mathcal{L}|^\wedge_p),$$

induced by the inclusion $\mathcal{B}(S) \xrightarrow{\theta_S} \mathcal{L}$*, is surjective.*

Proof. For each $\mathcal{F}$-centric subgroup $P \le S$, fix a morphism $\iota_P \in \mathrm{Mor}_{\mathcal{L}}(P,S)$ which lifts the inclusion (and set $\iota_S = \mathrm{Id}_S$). By Lemma 1, for each $P \le Q \le S$, there is a unique morphism $\iota_P^Q \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $\iota_Q \circ \iota_P^Q = \iota_P$.

Regard the vertex S as the basepoint of $|\mathcal{L}|$. Define

$$\omega\colon \mathrm{Mor}(\mathcal{L}) \longrightarrow \pi_1(|\mathcal{L}|)$$

by sending each $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to the loop formed by the edges ι_P, φ, and ι_Q (in that order). Clearly, $\omega(\psi \circ \varphi) = \omega(\psi)\cdot\omega(\varphi)$ whenever ψ and φ are composable, and $\omega(\iota_P^Q) = \omega(\iota_P) = 1$ for all $P \le Q \le S$. And $\pi_1(|\mathcal{L}|)$ is generated by $\mathrm{Im}(\omega)$ since any loop in $|\mathcal{L}|$ can be split up as a composite of loops of the above form.

By Alperin's fusion theorem for saturated Frobenius systems (Theorem A), each morphism in $\mathcal{F}$, and hence each morphism in $\mathcal{L}$, is (up to inclusions) a composite of automorphisms in $\mathcal{L}$. Thus $\pi_1(|\mathcal{L}|)$ is generated by the subgroups $\omega(\mathrm{Aut}_{\mathcal{L}}(P))$ for all $\mathcal{F}$-centric $P \le S$.

Choose a set $S = P_0, P_1, \dots, P_k$ of N-saturated $\mathcal{F}$-conjugacy class representatives for the $\mathcal{F}$-centric subgroups of S. In particular, $N_S(P_i) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}}(P_i))$ for each i. Then $\pi_1(|\mathcal{L}|)$ is generated by the subgroups $\omega(\mathrm{Aut}_{\mathcal{L}}(P_i))$ for $0 \le i \le k$ together with images under ω of isomorphisms which send each $\mathcal{F}$-centric subgroup of S to one of the P_i; and each such isomorphism is equivalent modulo inclusions to a composite of automorphisms of larger subgroups (by the fusion theorem again). So by downwards induction on the orders of the subgroups, we see that $\pi_1(|\mathcal{L}|)$ is generated by the subgroups $\omega(\mathrm{Aut}_{\mathcal{L}}(P_i))$.

Let $K \lhd \pi_1(|\mathcal{L}|)$ be the subgroup generated by all elements of finite order prime to p. For each i, $\mathrm{Aut}_{\mathcal{L}}(P_i)$ is generated by its Sylow subgroup $N_S(P_i)$ together with elements of order prime to p. Hence $\pi_1(|\mathcal{L}|)$ is generated by K together with the subgroups $\omega(N_S(P_i))$; and $\omega(N_S(P_i)) \le \omega(S)$ for each i. This shows that ω sends S surjectively onto $\pi_1(|\mathcal{L}|)/K$, and in particular that this quotient group is a finite p-group.

Set $\pi = \pi_1(|\mathcal{L}|)/K$ for short. Since K is generated by elements of order prime to p, the same is true of its abelianization, and hence $H_1(K;\mathbb{F}_p) = 0$. Thus, K is p-perfect. Let X be the cover of $|\mathcal{L}|$ with fundamental group K. Then X is p-good and $X_p^\wedge$ is simply connected since $\pi_1(X)$ is p-perfect [BK, VII.3.2]. Also, $H_i(X;\mathbb{F}_p)$ is finite for all i since $|\mathcal{L}|$ and hence X has finite skeleta. Hence $X_p^\wedge \longrightarrow |\mathcal{L}|_p^\wedge \longrightarrow B\pi$ is a fibration sequence and $|\mathcal{L}|_p^\wedge$ is p-complete by [BK, II.5.2(iv)]. So $|\mathcal{L}|$ is p-good, and $\pi_1(|\mathcal{L}|_p^\wedge) \cong \pi$ is a quotient group of S. (Alternatively, this follows from a "mod-p plus construction" on $|\mathcal{L}|$: there is a space Y and an $\mathbb{F}_p$-homology equivalence $|\mathcal{L}| \longrightarrow Y$ such that $\pi_1(Y) \cong \pi_1(|\mathcal{L}|)/K = \pi$, and $|\mathcal{L}|$ is p-good since Y is.) □

2. Obstruction theory for associated $\mathcal{L}$-systems

We now consider the obstructions to the existence and uniqueness of $\mathcal{L}$-systems associated to a given Frobenius system. We first define the orbit category of a Frobenius system.

Definition 2.1. *The ($\mathcal{F}$-centric)* orbit category *of a Frobenius system $\mathcal{F}$ over a p-group S is the category $\mathcal{O}^c(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S, and whose morphisms are defined by*

$$\mathrm{Mor}_{\mathcal{O}^c(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

If $\mathcal{L}$ is an associated $\mathcal{L}$-system to $\mathcal{F}$, then we let $\widetilde{\pi}$ denote the composite functor

$$\widetilde{\pi}\colon \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}^c \longrightarrow\!\!\!\!\rightarrow \mathcal{O}^c(\mathcal{F}).$$

And if f is any morphism in $\mathcal{L}$, we let $[\![f]\!] = \widetilde{\pi}f$ denote it images in $\mathcal{O}^c(\mathcal{F})$.

[Emphasize the difference between the orbit category of a Frobenius system and the orbit category of a group!! Use $\mathrm{Rep}_{\mathcal{F}}(P,Q)$ or $\mathrm{Mor}_{\mathcal{O}\mathcal{F}}(P,Q)$ to denote morphisms in the orbit category??]

For any Frobenius system $\mathcal{F}$,

$$\mathcal{Z} = \mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}^c(\mathcal{F}) \longrightarrow \mathsf{Ab}$$

denotes the functor defined by $\mathcal{Z}_{\mathcal{F}}(P) = Z(P)$.

Proposition 2.2. *Fix a saturated Frobenius system $\mathcal{F}$ over the p-group S. Then there is an element $\eta(\mathcal{F}) \in \varprojlim^3_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z})$ such that $\mathcal{F}$ has an associated $\mathcal{L}$-system if and only if $\eta(\mathcal{F}) = 0$. And if there are any associated $\mathcal{L}$-systems, then the group $\varprojlim^2_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z})$ acts freely and transitively on the set of all isomorphism classes of $\mathcal{L}$-systems associated to $\mathcal{F}$.*

Proof. The obstruction to the existence of an associated $\mathcal{L}$-system will be handled in Step 1, and the action of $\varprojlim^2(\mathcal{Z})$ in Step 2. Let $C^*(\mathcal{O}^c(\mathcal{F});\mathcal{Z})$ denote the *normalized* chain complex for $\mathcal{Z}$:

$$C^n(\mathcal{O}^c(\mathcal{F});\mathcal{Z}) = \prod_{P_0\to\cdots\to P_n} \mathcal{Z}(P_0),$$

where the product is taken over all composable sequences of nonidentity morphisms. For simplicity, we regard cochains as functions defined on all sequences of morphisms, which send a sequence to $1 \in \mathcal{Z}(P_0)$ if any of the morphisms is an identity. Then

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^i(\mathcal{Z}) \cong H^i\bigl(C^*(\mathcal{O}^c(\mathcal{F});\mathcal{Z}),\delta\bigr),$$

where δ is the obvious coboundary map, by the same argument as that given for the unnormalized chain complex in [GZ, Appendix II, Proposition 3.3] or [Ol, Lemma 2].

<u>Step 1</u>: Fix a section $\sigma\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \longrightarrow \mathrm{Mor}(\mathcal{F}^c)$ which sends identity maps to identity maps, and write $\widetilde{\varphi} = \sigma(\varphi)$ for short. For each pair of $\mathcal{F}$-centric subgroups $P, Q \le S$, set

$$X(P,Q) = Q \times \mathrm{Mor}_{\mathcal{O}}(P,Q)$$

and define

$$X(P,Q) \xrightarrow{\ \pi_\sigma^{P,Q}\ } \mathrm{Hom}_{\mathcal{F}}(P,Q)$$

by setting

$$\pi_\sigma^{P,Q}(g,\varphi) = c_g \circ \widetilde{\varphi}.$$

For each composable pair of morphisms $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ in the orbit category, choose some $t(\varphi,\psi) \in R$ such that

$$\widetilde{\psi} \circ \widetilde{\varphi} = c_{t(\varphi,\psi)} \circ \widetilde{\psi\varphi}, \tag{1}$$

and such that

$$t(\varphi,\psi) = 1 \quad \text{if } \varphi = \mathrm{Id}_Q \text{ or } \psi = \mathrm{Id}_Q. \tag{2}$$

Define maps

$$X(Q,R) \times X(P,Q) \xrightarrow{\ *\ } X(P,R)$$

by setting

$$(h,\psi)*(g,\varphi)=\bigl(h\cdot\widetilde{\psi}(g)\cdot t(\varphi,\psi),\psi\varphi\bigr). \tag{3}$$

Definition of $u(\varphi,\psi,\chi)$: By definition, if $P,Q\le S$, and $\pi_\sigma^{P,Q}(g,\varphi)=\pi_\sigma^{P,Q}(g',\varphi)$ for some $g,g'\in Q$ and $\varphi\in\mathrm{Hom}(P,Q)$, then $c_g=c_{g'}\in\mathrm{Aut}(Q)$, so $g^{-1}g'\in Z(Q)$ since P is $\mathcal{F}$-centric, and $(g,\varphi)=(g',\varphi)*(u,\mathrm{Id}_P)$ for some $u\in Z(P)$. Also, by construction, the following square commutes for each triple of objects P,Q,R:

$$\begin{array}{ccc} X(Q,R)\times X(P,Q) & \xrightarrow{\quad *\quad} & X(P,R)\\ {\scriptstyle \pi_\sigma^{Q,R}\times\pi_\sigma^{P,Q}}\Big\downarrow & & {\scriptstyle \pi_\sigma^{P,R}}\Big\downarrow\\ \mathrm{Hom}_{\mathcal{F}}(Q,R)\times\mathrm{Hom}_{\mathcal{F}}(P,Q) & \xrightarrow{\text{composition}} & \mathrm{Hom}_{\mathcal{F}}(P,R). \end{array}$$

Hence for each triple of composable maps

$$P\xrightarrow{\ \varphi\ }P'\xrightarrow{\ \psi\ }Q\xrightarrow{\ \chi\ }R$$

in the orbit category, there is a unique element $u(\varphi,\psi,\chi)=u_{\sigma,t}(\varphi,\psi,\chi)\in Z(P)$ such that

$$\bigl((1,\chi)*(1,\psi)\bigr)*(1,\varphi)=\bigl[(1,\chi)*\bigl((1,\psi)*(1,\varphi)\bigr)\bigr]*(u(\varphi,\psi,\chi),\mathrm{Id}_P). \tag{4}$$

We regard $u\in C^3(\mathcal{O}^c(\mathcal{F});\mathcal{Z})$ as a normalized 3-cochain. Upon substituting formula (3) into (4), we get the following formula for $u(\varphi,\psi,\chi)$:

$$\widetilde{\chi\psi\varphi}(u(\varphi,\psi,\chi))=t(\psi\varphi,\chi)^{-1}\cdot\widetilde{\chi}(t(\varphi,\psi))^{-1}\cdot t(\psi,\chi)\cdot t(\varphi,\chi\psi). \tag{5}$$

And after combining this with (3) again, we get that for each $g\in P'$, $h\in Q$, and $k\in R$,

$$\bigl((k,\chi)*(h,\psi)\bigr)*(g,\varphi)=\bigl[(k,\chi)*\bigl((h,\psi)*(g,\varphi)\bigr)\bigr]*(u(\varphi,\psi,\chi),\mathrm{Id}_P). \tag{6}$$

Proof that u is a 3-cocycle: Fix a sequence of morphisms

$$P\xrightarrow{\ \varphi\ }P'\xrightarrow{\ \psi\ }Q\xrightarrow{\ \chi\ }Q'\xrightarrow{\ \omega\ }R$$

in $\mathcal{O}^c(\mathcal{F})$. Then

$$\delta u(\varphi,\psi,\chi,\omega)=u(\varphi,\psi,\omega\chi)^{-1}\cdot u(\psi\varphi,\chi,\omega)^{-1}\cdot u(\varphi,\psi,\chi)\cdot u(\varphi,\chi\psi,\omega)\cdot\widetilde{\varphi}^{-1}(u(\psi,\chi,\omega)), \tag{7}$$

(each term lies in the abelian group $\Phi(Z(P))\le R$), and we must show that this vanishes.

Set $\Phi=\sigma(\omega\chi\psi\varphi)\in\mathrm{Hom}(P,R)$ for short. Then by (1),

$$\begin{aligned}\Phi(u(\varphi,\psi,\chi))&=t(\chi\psi\varphi,\omega)^{-1}\cdot\bigl(\widetilde{\omega}\circ\widetilde{\chi\psi\varphi}(u(\varphi,\psi,\chi))\bigr)\cdot t(\chi\psi\varphi,\omega)\\ \Phi\circ\widetilde{\varphi}^{-1}(u(\psi,\chi,\omega))&=t(\varphi,\omega\chi\psi)^{-1}\cdot\bigl(\widetilde{\omega\chi\psi}(u(\psi,\chi,\omega))\bigr)\cdot t(\varphi,\omega\chi\psi).\end{aligned}$$

Together with (5), this gives the formulas

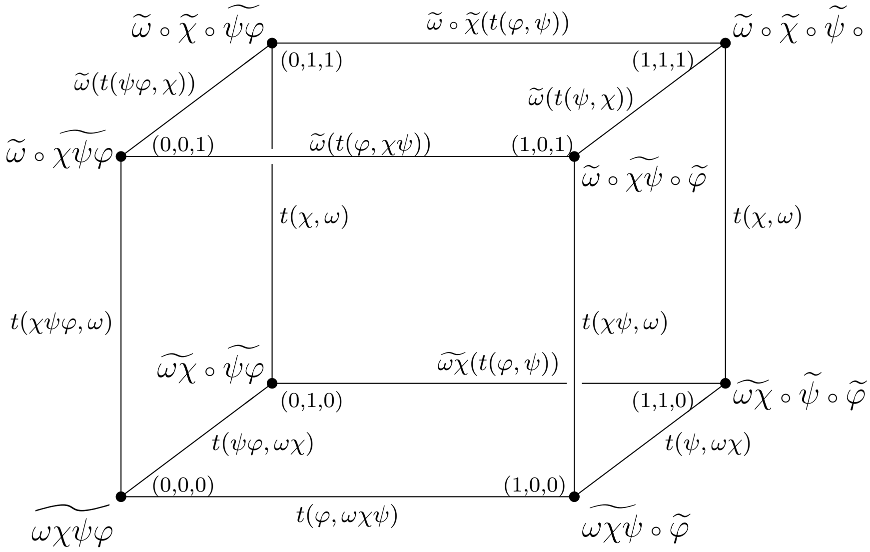
$$\begin{aligned}\Phi(u(\varphi,\psi,\chi))&=t(\chi\psi\varphi,\omega)^{-1}\cdot\widetilde{\omega}(t(\psi\varphi,\chi))^{-1}\cdot\bigl(t(\chi,\omega)\cdot\widetilde{\omega\chi}(t(\varphi,\psi))^{-1}\cdot t(\chi,\omega)^{-1}\bigr)\\ &\qquad\cdot\widetilde{\omega}(t(\psi,\chi))\cdot\widetilde{\omega}(t(\varphi,\chi\psi))\cdot t(\chi\psi\varphi,\omega)\\ \Phi(u(\varphi,\psi,\omega\chi))&=t(\psi\varphi,\omega\chi)^{-1}\cdot\widetilde{\omega\chi}(t(\varphi,\psi))^{-1}\cdot t(\psi,\omega\chi)\cdot t(\varphi,\omega\chi\psi)\\ \Phi(u(\varphi,\chi\psi,\omega))&=t(\chi\psi\varphi,\omega)^{-1}\cdot\widetilde{\omega}(t(\varphi,\chi\psi))^{-1}\cdot t(\chi\psi,\omega)\cdot t(\varphi,\omega\chi\psi)\\ \Phi(u(\psi\varphi,\chi,\omega))&=t(\chi\psi\varphi,\omega)^{-1}\cdot\widetilde{\omega}(t(\psi\varphi,\chi))^{-1}\cdot t(\chi,\omega)\cdot t(\psi\varphi,\omega\chi)\\ \Phi(\widetilde{\varphi}^{-1}(u(\psi,\chi,\omega)))&=t(\varphi,\omega\chi\psi)^{-1}\cdot t(\chi\psi,\omega)^{-1}\cdot\widetilde{\omega}(t(\psi,\chi))^{-1}\cdot t(\chi,\omega)\cdot t(\psi,\omega\chi)\cdot t(\varphi,\omega\chi\psi).\end{aligned} \tag{8}$$

Upon substituting these into (7), we get that

$$\Phi(\delta u(\varphi, \psi, \chi, \omega)) = 1,$$

and hence that $\delta u(\varphi, \psi, \chi, \omega) = 1$.

To see this more geometrically, consider the following cube, where each vertex is labelled by a homomorphism $P \longrightarrow R$ in the conjugacy class $\omega\chi\psi\varphi \in \mathrm{Rep}(P, R)$, and where each edge is labelled with an element of R:



The vertices of the cube are given the coordinatewise partial ordering, and we regard each edge as being oriented from the smaller to the larger vertex. Whenever an edge in the cube is labelled by $g \in R$ and its endpoints by $f_0, f_1 \in \mathrm{Hom}(P, R)$ (in that order), then $c_g \circ f_0 = f_1$. In particular, the product of the successive edges of any loop in the diagram (multiplied from right to left, and where an element is inverted if the orientation is reversed) lies in $f(Z(P))$ if $f \in \mathrm{Hom}(P, R)$ is the label of the basepoint of the loop.

The "back face" $(*, 1, *)$ represents an identity in R (by (1)). Each of the other five faces, when regarded as a loop based at $(0, 0, 0)$, represents one of the terms in $\Phi(\delta u(\varphi, \psi, \chi, \omega))$. For example, the two faces $(*, *, 1)$ and $(1, *, *)$ represent the first and last formulas in (8), with extra terms coming from the edge which connects these faces to the vertex $(0, 0, 0)$. And the other three formulas in (8) correspond to the three faces which contain $(0, 0, 0)$. Using this picture, we see directly that the product in (7) (rather, its image under Φ) vanishes, and hence that $\delta u = 1$.

Independence of the choice of $t(\varphi, \psi)$: Let $t'(\varphi, \psi)$, for each composable pair of morphisms in $\mathcal{O}^c(\mathcal{F})$, be another collection of elements which satisfy (1). Let $u' = u_{\sigma,t'} \in Z^3(\mathcal{O}^c(\mathcal{F}); \mathcal{Z})$ be the 3-cochain defined using (3) and (4) (after replacing t by t'). By the previous argument, u' is a 3-cocycle, and (5) now takes the form

$$t'(\psi, \chi) \cdot t'(\varphi, \chi\psi) = \widetilde{\chi}(t'(\varphi, \psi)) \cdot t'(\psi\varphi, \chi) \cdot \widetilde{\chi\psi\varphi}(u'(\varphi, \psi, \chi)). \tag{9}$$

For each composable sequence $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$, conjugation by $t'(\varphi, \psi)$ and by $t(\varphi, \psi)$ define the same automorphism of R, and hence there is a unique element $c(\varphi, \psi)$ such that

$$t'(\varphi, \psi) = t(\varphi, \psi) \cdot \widetilde{\psi\varphi}(c(\varphi, \psi)). \tag{10}$$

Then $c \in C^2(\mathcal{O}^c(\mathcal{F});\mathcal{Z})$ is a (normalized) 2-cochain. Upon substituting (9) into (8), and using the relations

$$\widetilde{\chi\psi} = c_{t(\varphi,\chi\psi)} \circ \widetilde{\chi\psi\varphi} \circ \widetilde{\varphi}^{-1} \qquad \text{and} \qquad \widetilde{\chi} \circ \widetilde{\psi\varphi} = c_{t(\chi,\psi\varphi)} \circ \widetilde{\chi\psi\varphi},$$

we get the relation

$$\begin{aligned} t(\psi,\chi)\cdot t(\varphi,\chi\psi)\cdot\widetilde{\chi\psi\varphi}(\widetilde{\varphi}^{-1}c(\psi,\chi))\cdot\widetilde{\chi\psi\varphi}(c(\varphi,\chi\psi)) \\ = \widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi)\cdot\widetilde{\chi\psi\varphi}(c(\varphi,\psi))\cdot\widetilde{\chi\psi\varphi}(c(\psi\varphi,\chi))\cdot\widetilde{\chi\psi\varphi}(u'(\varphi,\psi,\chi)). \end{aligned}$$

The four factors in this equation which are not in the image of $\widetilde{\chi\psi\varphi}$ can be replaced by $\widetilde{\chi\psi\varphi}(u(\varphi,\psi,\chi))$ using (5), and we thus get that

$$u(\varphi,\psi,\chi)\cdot\widetilde{\varphi}^{-1}(c(\psi,\chi))\cdot c(\varphi,\chi\psi) = c(\varphi,\psi)\cdot c(\psi\varphi,\chi)\cdot u'(\varphi,\psi,\chi).$$

Since all terms in this equation lie in $Z(P)$, this shows that

$$u^{-1}\cdot u' = \delta c. \tag{11}$$

A different choice of t thus results in changing u by a coboundary, and does not change the class $[u] \in \varprojlim^3(\mathcal{Z})$.

Independence of the choice of σ: This follows upon observing that under a different choice of section σ', the resulting sets and maps $X'(P,Q) \xrightarrow{\pi_{\sigma'}^{P,Q}} \mathrm{Hom}_{\mathcal{F}}(P,Q)$ can be identified with $X(P,Q)$ and $\pi_\sigma^{P,Q}$ in an obvious way. This induces elements $t'(\varphi,\psi)$ such that the X' and the X have the same composition under these identifications. But by (6), this shows that $u_{\sigma',t'} = u_{\sigma,t}$, and thus that $[u]$ is not changed by this different choice of section.

Existence of an $\mathcal{L}$-system if $[u]=0$: Formula (11) also shows that if $u = u_{\sigma,t}$ is a coboundary, then we can choose t' such that $u_{\sigma,t'} = 0$, and hence get a category $\mathcal{L}$ with $\mathrm{Mor}_{\mathcal{L}}(P,Q) = X(P,Q)$ and with composition defined using (3) but using $t'(\varphi,\psi)$ instead of $t(\varphi,\psi)$. In this case, we set $\widehat{g} = (g,\mathrm{Id})$ and $\pi(g,\varphi) = c_g \circ \widetilde{\varphi}$ for $g \in P$ and $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$. Conditions (A–C) are easily checked to hold. For example, for any $(a,\varphi) \in X(P,Q)$ and any $g \in P$,

$$(x,\varphi) * (g,\mathrm{Id}_P) = (x\cdot\widetilde{\varphi}(g),\varphi) = (x\widetilde{\varphi}(g)x^{-1},\mathrm{Id}_Q) * (x,\varphi) = (c_x \circ \widetilde{\varphi}(g),\mathrm{Id}_Q) * (x,\varphi)$$

by (2) and (3), and this implies (C). So $\mathcal{L}$ is an $\mathcal{L}$-system associated to $\mathcal{F}$.

Vanishing of $[u]$ if there is an $\mathcal{L}$-system: Let $\mathcal{L}$ be any $\mathcal{L}$-system associated to $\mathcal{F}$, and fix a section σ as above. This can be lifted to a section $\mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \longrightarrow \mathrm{Mor}(\mathcal{L})$, which in turn defines bijections $X(P,Q) \cong \mathrm{Mor}_{\mathcal{L}}(P,Q)$ in the obvious way. Since (C) holds, composition in $\mathcal{L}$ must correspond to multiplication of the $X(P,Q)$ (as defined by (3)) for some choice of elements $t(\varphi,\psi)$; and thus $u_{\sigma,t} = 0$ in this case. This shows that $[u] = 0$ whenever there exist associated $\mathcal{L}$-systems.

<u>Step 2</u>: Assume that $\mathcal{L}_1 \xrightarrow{\pi_1} \mathcal{F}^c$ and $\mathcal{L}_2 \xrightarrow{\pi_2} \mathcal{F}^c$ are two $\mathcal{L}$-systems associated to $\mathcal{F}$. Let $\mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \xrightarrow{\sigma} \mathrm{Mor}(\mathcal{F}^c)$ be as above, and fix sections

$$\mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \xrightarrow{\widetilde{\sigma}_1} \mathrm{Mor}(\mathcal{L}_1) \qquad \text{and} \qquad \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \xrightarrow{\widetilde{\sigma}_2} \mathrm{Mor}(\mathcal{L}_2)$$

which send identity morphisms to identity morphisms and such that $\pi_i \circ \widetilde{\sigma}_i = \sigma$ for $i = 1,2$. For each $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ in $\mathcal{O}^c(\mathcal{F})$, let $t_i(\varphi,\psi) \in R$ be the element (unique by Lemma 1(b)) such that

$$\widetilde{\sigma}_i(\psi) \circ \widetilde{\sigma}_i(\varphi) = \widehat{t_i(\varphi,\psi)} \circ \widetilde{\sigma}_i(\psi\varphi) \tag{12}$$

in $\mathcal{L}_i$ ($i = 1, 2$). These satisfy (1) and (2) above, as well as (3) when we identify $(g, \varphi) = \widehat{g} \circ \widetilde{\sigma}_i(\varphi)$. There is thus an element $c(\varphi, \psi) \in Z(P)$ such that

$$t_2(\varphi, \psi) = t_1(\varphi, \psi) \cdot \widetilde{\psi\varphi}(c(\varphi, \psi)). \tag{13}$$

By (11) ($u_1 = u_2 = 1$ in this case since $\mathcal{L}_1$ and $\mathcal{L}_2$ are actual categories), c is a (normalized) 2-cocycle.

Now assume that $\widetilde{\sigma}'_i$ ($i = 1$ or 2) is another section (over the fixed section σ), and define elements $t'_i(\varphi, \psi)$ using (12). By condition (A), there is a unique 1-cochain $w \in C^1(\mathcal{O}^c(\mathcal{F}); \mathcal{Z})$ such that for each morphism φ in $\mathcal{O}^c(\mathcal{F})$, $\widetilde{\sigma}'_i(\varphi) = \widetilde{\sigma}'_i(\varphi) \cdot w(\varphi)$. Upon substituting this into the definition of t'_i, we get that

$$\widetilde{\sigma}_i(\psi) \circ \widehat{w(\psi)} \circ \widetilde{\sigma}_i(\varphi) \circ \widehat{w(\varphi)} = \widehat{t'_i(\varphi, \psi)} \circ \widetilde{\sigma}_i(\psi\varphi) \circ \widehat{w(\psi\varphi)} \in \mathrm{Mor}_{\mathcal{L}_i}(P, R)$$

for each $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ in $\mathcal{O}^c(\mathcal{F})$; and hence (using condition (C)) that

$$\delta_R\big(\widetilde{\psi}(w(\psi)) \cdot \widetilde{\psi}\widetilde{\varphi}(w(\varphi))\big) \circ \widetilde{\sigma}_i(\psi) \circ \widetilde{\sigma}_i(\varphi) = \delta_R\big(t'_i(\varphi, \psi) \cdot \widetilde{\psi\varphi}(w(\psi\varphi))\big) \circ \widetilde{\sigma}_i(\psi\varphi).$$

After substituting (12) into this we get

$$\widetilde{\psi}(w(\psi)) \cdot \widetilde{\psi}\widetilde{\varphi}(w(\varphi)) \cdot t_i(\varphi, \psi) = t'_i(\varphi, \psi) \cdot \widetilde{\psi\varphi}(w(\psi\varphi)) \in R.$$

¿From this, together with the relation $c_{t_i(\varphi,\psi)} \circ \widetilde{\psi\varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$, it follows that

$$\begin{aligned} t_i(\varphi, \psi)^{-1} \cdot t'_i(\varphi, \psi) &= c^{-1}_{t_i(\varphi,\psi)}\big(\widetilde{\psi}(w(\psi)) \cdot \widetilde{\psi}\widetilde{\varphi}(w(\varphi))\big) \cdot \widetilde{\psi\varphi}(w(\psi\varphi)^{-1}) \\ &= \widetilde{\psi\varphi}\big(\widetilde{\varphi}^{-1}(w(\psi)) \cdot w(\varphi) \cdot w(\psi\varphi)^{-1}\big) = \widetilde{\psi\varphi}(\delta w(\varphi, \psi)). \end{aligned}$$

In other words, a change in $\widetilde{\sigma}_i$ corresponds to changing c by a coboundary, and hence the class $[c] \in \varprojlim^2(\mathcal{Z})$ is uniquely defined (depending only on $\mathcal{L}_1$ and $\mathcal{L}_2$). Also, $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic as categories over $\mathcal{O}^c(\mathcal{F})$ (i.e., there is a functor $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ which is bijective on objects and morphisms and commutes with the π_i and the $P \longrightarrow \mathrm{Aut}_{\mathcal{L}_i}(P)$) if and only if $t_1 = t_2$ for some choice of these sections, if and only if $[c] = 0$. Finally, for fixed $\mathcal{L}_1$ and $\widetilde{\sigma}_1$, any 2-cocycle c can be realized by some appropriate choice of $\mathcal{L}_2$ and $\widetilde{\sigma}_2$: first define t_2 using (13), and then define $\mathcal{L}_2$ using (3). This finishes the proof that $\varprojlim^2(\mathcal{Z})$ acts freely and transitively on the set of isomorphism classes of $\mathcal{L}$-systems associated to $\mathcal{F}$. □

The obstruction groups of Dwyer and Kan [DK2] to the existence and uniqueness of liftings of the homotopy functor $P \mapsto BP$ are exactly the same as the obstructions of Proposition 2 to the existence and uniqueness of associated $\mathcal{L}$-systems. So it is not surprising that there should be a correspondence between the two. This connection will be described in more detail in the next two propositions, where Top denotes the category of spaces.

Proposition 2.3. *A saturated Frobenius system $\mathcal{F}$ has an associated $\mathcal{L}$-system if and only if the homotopy functor $P \mapsto BP$ on $\mathcal{O}^c(\mathcal{F})$ lifts to* Top.

Proof. As in the proof of Proposition 2, we fix a section $\sigma\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \longrightarrow \mathrm{Mor}(\mathcal{F})$ which sends identity morphisms to identity morphisms, write $\widetilde{\varphi} = \sigma(\varphi)$ for short, and choose elements $t(\varphi, \psi) \in R$, for each $P \xrightarrow{\varphi} Q \xrightarrow{\psi} R$ in $\mathcal{O}^c(\mathcal{F})$, which satisfy the conditions

$$\widetilde{\psi} \circ \widetilde{\varphi} = c_{t(\varphi,\psi)} \circ \widetilde{\psi\varphi} \qquad \text{and} \qquad t(\varphi, \mathrm{Id}) = 1 = t(\mathrm{Id}, \psi).$$

Let [1] denote the category with two objects $0,1$ and one nonidentity morphism $0 \to 1$. For each such φ, ψ, let

$$\mathcal{T} = \mathcal{T}(\varphi,\psi)\colon \mathcal{B}(P) \times [1] \longrightarrow \mathcal{B}(R)$$

be the functor which sends both objects to o_R, and where

$$\mathcal{T}(-,\mathrm{Id}_0) = \mathcal{B}(\widetilde{\psi\varphi}), \quad \mathcal{T}(-,\mathrm{Id}_1) = \mathcal{B}(\widetilde{\psi} \circ \widetilde{\varphi}), \quad \text{and} \quad \mathcal{T}(\mathrm{Id}, 0 \to 1) = t(\varphi,\psi).$$

Let

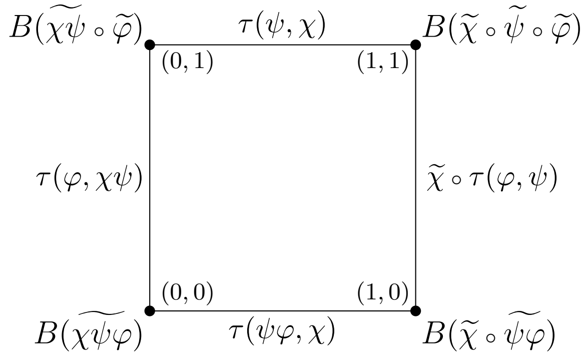
$$\tau(\varphi,\psi)\colon BP \times I \longrightarrow BR$$

be the homotopy from $B(\widetilde{\psi\varphi})$ to $B(\widetilde{\psi} \circ \widetilde{\varphi})$ induced by $\mathcal{T}$. These choices thus allow us to construct explicitly the 2-skeleton of a homotopy colimit of the functor $P \mapsto BP$.

Since we are working only with $\mathcal{F}$-centric subgroups, for each morphism $P \xrightarrow{\varphi} Q$ in $\mathcal{O}^c(\mathcal{F})$, the mapping space $\mathrm{Map}(BP,BQ)_{B\widetilde{\varphi}}$ has the homotopy type of $BZ(P)$. So by [DK2], the only obstruction to constructing a homotopy colimit, and thus to lifting $(P \mapsto BP)$ to the category of spaces, is an element $[v] \in \varprojlim^3(\mathcal{Z})$. By Proposition B, for each sequence of morphisms

$$P \xrightarrow{\ \varphi\ } P' \xrightarrow{\ \psi\ } Q \xrightarrow{\ \chi\ } R,$$

$v(\varphi,\psi,\chi) \in \pi_1(\mathrm{Map}(BP,BR)_{B\chi\psi\varphi}) \cong Z(P)$ is the loop described by the following diagram:



By formula (3) in the proof of Proposition 2, this obstruction is equal to

$$\widetilde{\chi\psi\varphi}(u(\varphi,\psi,\chi)) \in \widetilde{\chi\psi\varphi}(Z(P)) \cong \pi_1\big(\mathrm{Map}(BP,BR)_{B\widetilde{\chi\psi\varphi}}\big);$$

and hence the obstructions to the existence of an associated $\mathcal{L}$-system and to that of a homotopy lifting are equal. □

It remains to look at the homotopy type of the nerve of an associated $\mathcal{L}$-system.

Proposition 2.4. *Fix a saturated Frobenius system $\mathcal{F}$ and an associated $\mathcal{L}$-system $\mathcal{L}$, and let $\widetilde{\pi}\colon \mathcal{L} \longrightarrow \mathcal{O}^c(\mathcal{F})$ be the projection functor. Let*

$$\widetilde{B}\colon \mathcal{O}^c(\mathcal{F}) \longrightarrow \mathsf{Top}$$

be the left homotopy Kan extension of the constant functor $\mathcal{L} \xrightarrow{\ *\ } \mathsf{Top}$. Then $\widetilde{B}$ is a homotopy lifting of the homotopy functor $P \mapsto BP$, and

$$|\mathcal{L}| \simeq \underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}). \tag{1}$$

Proof. Point (1) follows from [HV, Theorem 5.5], .

Recall that we write $\mathrm{Rep}_{\mathcal{F}}(P,Q)$ to denote morphisms in $\mathcal{O}^c(\mathcal{F})$. By definition, for each $\mathcal{F}$-centric subgroup $P \le S$, $\widetilde{B}(P)$ is the nerve (homotopy colimit of the point functor) of the overcategory $\widetilde{\pi}{\downarrow}P$, whose objects are pairs (Q,α) for $\alpha \in \mathrm{Rep}_{\mathcal{F}}(Q,P)$, and where

$$\mathrm{Mor}_{\widetilde{\pi}\downarrow P}\big((Q,\alpha),(R,\beta)\big) = \big\{\varphi \in \mathrm{Mor}_{\mathcal{L}}(Q,R) \,\big|\, \alpha = \beta \circ \widetilde{\pi}_{Q,R}(\varphi)\big\}.$$

Since $|\mathcal{L}| \cong \underrightarrow{\mathrm{hocolim}}_{\mathcal{L}}(*)$, (1) holds by [HV, Theorem 5.5] (and the basic idea is due to Segal [Se, Proposition B.1]). It remains only to show that $\widetilde{B}$ is a lifting of the homotopy functor $P \mapsto BP$.

Let $\mathcal{B}'(P) \subseteq \widetilde{\pi}{\downarrow}P$ be the subcategory with one object (P,Id) and with morphisms $\{\widehat{g} \,|\, g \in P\}$. In particular, $|\mathcal{B}'(P)| \simeq BP$. We claim that $|\mathcal{B}'(P)|$ is a deformation retract of $|\widetilde{\pi}{\downarrow}P|$. To see this, we must define a functor $\Psi\colon \widetilde{\pi}{\downarrow}P \longrightarrow \mathcal{B}'(P)$ such that $\Psi|_{\mathcal{B}'(P)} = \mathrm{Id}$, together with a natural transformation $f\colon \mathrm{Id} \longrightarrow \mathrm{incl}\circ\Psi$ of functors from $\widetilde{\pi}{\downarrow}P$ to itself. Fix a section $\widetilde{\sigma}\colon \mathrm{Mor}(\mathcal{O}^c(\mathcal{F})) \longrightarrow \mathrm{Mor}(\mathcal{L})$ of $\widetilde{\pi}$ which sends identity morphisms to identity morphisms. To define Ψ, send each object to the unique object (P,Id) of $\mathcal{B}'(P)$, and send $\varphi \in \mathrm{Mor}_{\widetilde{\pi}\downarrow P}((Q,\alpha),(R,\beta))$ to the unique map $\widehat{g} = \Psi(\varphi)$ (for $g \in P$, see Lemma 1(b)) such that the following square commutes:

$$\begin{array}{ccc} Q & \xrightarrow{\ \varphi\ } & R \\ {\scriptstyle \widetilde{\sigma}(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\sigma}(\beta)} \\ P & \xrightarrow{\ \widehat{g}\ } & P. \end{array}$$

Finally, define $f\colon \mathrm{Id} \longrightarrow \mathrm{incl}\circ\Psi$ by sending each object (Q,α) to the morphism $\widetilde{\sigma}(\alpha) \in \mathrm{Mor}_{\mathcal{L}}(Q,P)$. This is clearly a natural transformation of functors, and thus

$$|\widetilde{\pi}{\downarrow}P| \simeq |\mathcal{B}'(P)| \simeq BP.$$

To finish the proof that $\widetilde{B}$ is a lifting of the homotopy functor $P \mapsto BP$, we must show, for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, that the following square commutes up to natural equivalence:

$$\begin{array}{ccc} \mathcal{B}'(P) & \xrightarrow{\ \mathrm{incl}\ } & \widetilde{\pi}{\downarrow}P \\ \big\downarrow{\scriptstyle B\varphi} & & \big\downarrow{\scriptstyle \bar{\varphi}\circ -} \\ \mathcal{B}'(Q) & \xrightarrow{\ \mathrm{incl}\ } & \widetilde{\pi}{\downarrow}Q. \end{array}$$

Here, $\bar{\varphi} \in \mathrm{Rep}_{\mathcal{F}}(P,Q)$ denotes the class of φ. This means constructing a natural transformation $F_1 \xrightarrow{\ \Phi\ } F_2$ of functors $\mathcal{B}'(P) \longrightarrow \widetilde{\pi}{\downarrow}Q$, where $F_1 = (\bar{\varphi}\circ -)\circ\mathrm{incl}$ and $F_2 = \mathrm{incl}\circ B\varphi$ are given by the formulas

$$F_1(P,\mathrm{Id}) = (P,\bar{\varphi}), \quad F_1(\widehat{g}) = \widehat{g}, \qquad \text{and} \qquad F_2(P,\mathrm{Id}) = (Q,\mathrm{Id}), \quad F_2(\widehat{g}) = \widehat{\varphi g}.$$

Let $\widetilde{\varphi} \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ be any lifting of φ. Then by condition (C), Φ can be defined by sending the object (P,Id) to the morphism $\widetilde{\varphi} \in \mathrm{Mor}_{\widetilde{\pi}\downarrow P}\big((P,\bar{\varphi}),(Q,\mathrm{Id})\big)$. □

3. Higher limits over the orbit category of a Frobenius system

We now show that higher limits of functors over an $\mathcal{F}$-centric orbit category $\mathcal{O}^c(\mathcal{F})$ of a saturated Frobenius system $\mathcal{F}$ can be computed using the same techniques as those already used to compute higher limits over orbit categories of finite groups. The main tools for doing this are certain graded groups $\Lambda^*(\Gamma; M)$, defined [JMO] for any finite group Γ and any $\mathbb{Z}_{(p)}[\Gamma]$-module M by setting

$$\Lambda^*(\Gamma; M) = \varprojlim_{\mathcal{O}_p(\Gamma)}{}^*(F_M),$$

where $F_M\colon \mathcal{O}_p(\Gamma) \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ is the functor which sends $P \le \Gamma$ to M^P.

Lemma 3.1. *Let $\mathcal{F}$ be a saturated Frobenius system over S. Let*

$$\Phi : \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be any functor which vanishes except on the isomorphism class of some fixed $\mathcal{F}$-centric subgroup $Q \le S$. Then

$$\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^*(\Phi) \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(Q); \Phi(Q)).$$

[I think the following argument holds for any functor to Ab.]

Proof. Since the result is independent of the choice of Q in its $\mathcal{F}$-conjugacy class, we can assume that Q is N-saturated in $\mathcal{F}$. Set $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$ for short.

Since $\mathcal{O}^c(\mathcal{F})$ is a full terminal subcategory of $\mathcal{O}_p(\mathcal{F})$ (i.e., any subgroup of S which contains an $\mathcal{F}$-centric subgroup is also $\mathcal{F}$-centric), it suffices to prove the result for $\varprojlim^*(\Phi)$. We first fix, for each $P \le S$, an (arbitrary) total ordering on the set $\mathrm{Rep}_{\mathcal{F}}(Q, P)$.

Let $\mathcal{O}_p(\mathcal{F})_{\amalg}$ be the category of formal finite "sums" of objects in $\mathcal{O}_p(\mathcal{F})$ (and where a morphism sends each summand in the source object to exactly one summand in the target). Let $\mathfrak{Set}_p(\Gamma) \cong \mathcal{O}_p(\Gamma)_{\amalg}$ be the category whose objects are finite Γ-sets whose isotropy subgroups are p-groups, together with a chosen basepoint in each orbit whose isotropy subgroup lies in $\mathrm{Out}_S(Q)$. Morphisms in $\mathfrak{Set}_p(\Gamma)$ are Γ-maps (independent of the basepoints).

Define functors

$$\mathfrak{Set}_p(\Gamma) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{O}_p(\mathcal{F})_{\amalg}$$

as follows. For each $P \le S$, set $\beta(P) = \mathrm{Rep}_{\mathcal{F}}(Q, P)$, with the action of $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$ induced by conjugation (and with the obvious definition on morphisms). The basepoint of each Γ-orbit is chosen to be the least element, under the ordering chosen above, whose isotropy subgroup is contained in $\mathrm{Out}_S(Q)$. Note that there always is such an element, since the isotropy subgroup of any $\varphi \in \mathrm{Rep}_{\mathcal{F}}(Q, P)$ is a p-group, and hence conjugate to a subgroup of $\mathrm{Out}_S(Q) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(Q))$.

To define α, for each Γ-orbit X, with basepoint $x \in X$ such that $\Gamma_x \le \mathrm{Out}_S(Q)$, set $\alpha(X, x) = N_S^{\Gamma_x}(Q) \le S$. For any Γ-map $X \xrightarrow{\sigma} Y$ between orbits, with basepoints $x \in X$ and $y \in Y$, fix $\gamma \in \Gamma$ such that $\sigma(x) = \gamma y$, and let $\alpha(\sigma) \in \mathrm{Rep}_{\mathcal{F}}(\alpha(X, x), \alpha(Y, y))$ be the unique morphism which makes the following square commute:

$$\begin{array}{ccc} Q & \xrightarrow{\gamma^{-1}} & Q \\ {\scriptstyle\mathrm{incl}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{incl}} \\ \alpha(X,x) & \xrightarrow{\alpha(\sigma)} & \alpha(Y,y) \end{array}$$

The existence of this map follows from condition (II) in Definition 1, and its uniqueness follows from Proposition A. (Note that $Q \lhd \alpha(X, x)$; and that Γ_x is the stabilizer of the inclusion $Q \longrightarrow \alpha(X, x)$ and similarly for Γ_y.)

[Find a better way to do this, without basepoints??]

We next claim that α is a left adjoint to β. To see this, fix a Γ-orbit X with basepoint $x\in X$, and a subgroup $P\le S$. Define maps

$$\mathrm{Map}_\Gamma(X,\mathrm{Rep}_{\mathcal F}(Q,P)) \underset{\nu}{\overset{\mu}{\rightleftarrows}} \mathrm{Rep}_{\mathcal F}(\alpha(X,x),P)$$

as follows. If a Γ-map $X\xrightarrow{f}\mathrm{Rep}_{\mathcal F}(Q,P)$ sends x to φ, then let $\alpha(X,x)\xrightarrow{\mu(f)}P$ be the unique map such that $\mu(f)|_Q=\varphi$ (by condition (II) and Proposition A again). And for any $\alpha(X,x)\xrightarrow{\psi}P$, let $X\xrightarrow{\nu(\psi)}\mathrm{Rep}_{\mathcal F}(Q,P)$ be the Γ-map which sends x to $\psi|_Q$. These maps are easily seen to be well defined, and inverses to each other.

For $\mathcal C=\mathcal O_p(\Gamma)$ or $\mathcal O_p(\mathcal F)$, we let $\mathcal C$-$\mathtt{mod}$ be the category of functors $\mathcal C^{\mathrm{op}}\longrightarrow\mathtt{Ab}$. Since this is equivalent to the category of functors $(\mathcal C_{\amalg})^{\mathrm{op}}\longrightarrow\mathtt{Ab}$ which send disjoint unions to direct sums, composition with α and β induce functors

$$\mathcal O_p(\Gamma)\text{-}\mathtt{mod} \underset{\beta^*}{\overset{\alpha^*}{\leftrightarrows}} \mathcal O_p(\mathcal F)\text{-}\mathtt{mod}.$$

Then α^* is a left adjoint to β^*, since α is a left adjoint to β. Also, α^* and β^* both preserve exact sequences, and hence β^* sends injectives to injectives.

Now let $\underline{\mathbb Z}$ denote the constant functor on $\mathcal O_p(\mathcal F)^{\mathrm{op}}$ which sends each object to $\mathbb Z$ and each morphism to the identity. Then $\alpha^*\underline{\mathbb Z}$ is the constant functor on $\mathcal O_p(\Gamma)$, since α sends objects of $\mathcal O_p(\Gamma)$ to objects of $\mathcal O_p(\mathcal F)$ (not to formal sums of objects). If $D:\mathcal O_p(\mathcal F)^{\mathrm{op}}\longrightarrow\mathtt{Ab}$ is any functor, then

$$\varprojlim_{\mathcal O_p(\mathcal F)}(D)\cong\mathrm{Hom}_{\mathcal O_p(\mathcal F)\text{-}\mathtt{mod}}(\underline{\mathbb Z},D);$$

and similarly for functors on $\mathcal O_p(\Gamma)$.

Let $\bar\Phi:\mathcal O_p(\Gamma)\longrightarrow\mathtt{Ab}$ be the functor which sends $1\le\Gamma$ to $\Phi(Q)$ (with the given action of Γ) and all other subgroups to 0. Then β^* sends an injective resolution I_* of $\bar\Phi$ to an injective resolution β^*I_* of $\beta^*(\bar\Phi)$. It follows that

$$\begin{aligned}\Lambda^*(\Gamma;\Phi(Q)) \overset{\text{def}}{=} \varprojlim_{\mathcal O_p(\Gamma)}{}^*(\bar\Phi) &\cong H^*\big(\mathrm{Mor}_{\mathcal O_p(\Gamma)\text{-}\mathtt{mod}}(\alpha^*\underline{\mathbb Z},I_*)\big)\\ &\cong H^*\big(\mathrm{Mor}_{\mathcal O_p(\mathcal F)\text{-}\mathtt{mod}}(\underline{\mathbb Z},\beta^*I_*)\big)\cong\varprojlim_{\mathcal O_p(\mathcal F)}{}^*(\beta^*\bar\Phi).\end{aligned}$$

By definition, $\bar\Phi=\alpha^*\Phi$, and it remains only to show that $\beta^*(\alpha^*\Phi)\cong\Phi$. For each $P\le S$, fix Γ-orbit representatives $\varphi_i\in\mathrm{Rep}_{\mathcal F}(Q,P)$ $(1\le i\le m)$ such that $K_i\overset{\text{def}}{=}\Gamma_{\varphi_i}\le\mathrm{Out}_S(Q)$ for each i, and set $P_i=N_S^{K_i}(Q)$. Then $K_i\cong P_i/Q$ and

$$\alpha(\beta(P))=\alpha(\mathrm{Rep}_{\mathcal F}(Q,P))=\coprod_{i=1}^m P_i,$$

and so

$$\beta^*(\alpha^*\Phi)(P)\cong\bigoplus_{i=1}^m\Phi(P_i).$$

If Q is not isomorphic to P, then for any $\varphi\in\mathrm{Rep}_{\mathcal F}(Q,P)$, $1\ne\varphi^{-1}(\mathrm{Out}_P(\varphi Q))\varphi\le\Gamma_\varphi$. So the action of Γ is not free on any orbit of $\mathrm{Rep}_{\mathcal F}(Q,P)$, hence $P_i\gneqq Q$ for each i, $\Phi(P_i)=0$ by the assumption on Φ, and thus $\beta^*(\alpha^*\Phi)(P)=0$. Finally, if $Q\cong P$ in

$\mathcal{F}$, then $\mathrm{Rep}_{\mathcal{F}}(Q,P)$ consists of one free orbit of Γ, so $m=1$, $P_1=Q$, and we are done. □

We say that a category $\mathcal{C}$ has "bounded limits at p" if there is an integer d such that for any functor $\Phi : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$, $\varprojlim^i(\Phi)=0$ for $i>d$.

Corollary 3.2. *Let $\mathcal{F}$ be any saturated Frobenius system. Then the $\mathcal{F}$-centric orbit category $\mathcal{O}^c(\mathcal{F})$ has bounded limits over p, in the sense that there is an integer $k>0$ such that $\varprojlim^i(F)=0$ for all functors*

$$F\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \xrightarrow{\qquad} \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

and all $i>k$.

Proof. This follows from Lemma 3, together with the result the result (cf. [JMO2, Proposition 4.11]) that for any finite group Γ, there is some k such that $\Lambda^i(\Gamma;M)=0$ for all $\mathbb{Z}_{(p)}[\Gamma]$-modules M and all $i>k$. □

Another consequence of Lemma 3 is:

Proposition 3.3. *Let $\mathcal{F}$ be any saturated Frobenius system over a p-group S. If $\mathrm{rk}_p(S)<p^3$, then there exists an associated $\mathcal{L}$-system for S. And if $\mathrm{rk}_p(S)<p^2$, then there exists a unique associated $\mathcal{L}$-system for S.*

[Thm or Prop??]

Proof. By [BLO, Proposition 5.8], for any finite group Γ and any finite $\mathbb{Z}_{(p)}[\Gamma]$-module M, $\Lambda^i(\Gamma,M)=0$ if $\mathrm{rk}(M)<p^i$. So if $\mathrm{rk}_p(S)<p^3$, then $\Lambda^3(N(P)/P;Z(P))=0$ for all $\mathcal{F}$-centric $P\le S$, and hence $\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^3(\mathcal{Z}_{\mathcal{F}})=0$ by Lemma 3. By the same argument, $\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^2(\mathcal{Z}_{\mathcal{F}})=0$ if $\mathrm{rk}_p(S)<p^2$. The result now follows from Proposition 2. □

4. Spaces of maps

We next describe the mapping spaces $\mathrm{Map}(BQ,|\mathcal{L}|^\wedge_p)$, where $|\mathcal{L}|^\wedge_p$ is the classifying space of a homotopy finite group $(S,\mathcal{F},\mathcal{L})$. We will see in Theorem 4 that the description of these spaces is nearly identical to that of the mapping spaces $\mathrm{Map}(BQ,BG^\wedge_p)$ for a finite group G. Throughout this section, $H^*(-)$ denotes cohomology with coefficients in $\mathbb{F}_p$.

We first show that under certain conditions, mapping spaces $\mathrm{Map}(BQ,-)$ "commute" with homotopy colimits. A similar result was shown by the first author and Nitu Kitchloo in [BrK, Theorem 6.11].

Lemma 4.1. *Fix a prime p, and let V be a group of order p. Let $\mathcal{C}$ be a finite category with bounded limits at p, and let*

$$F:\mathcal{C}\xrightarrow{\qquad} V\text{-}\mathsf{Spaces}$$

be a functor such that for each c in $\mathcal{C}$, $F(c)$ and $F(c)^{hV}$ are both p-complete and have finite mod p cohomology in each degree. Then the natural map

$$\big[\underrightarrow{\mathrm{hocolim}}_{\mathcal{C}}\big(F(-)^{hV}\big)\big]^\wedge_p \xrightarrow{\qquad} \big[\big(\underrightarrow{\mathrm{hocolim}}_{\mathcal{C}}(F)\big)^\wedge_p\big]^{hV}$$

is a homotopy equivalence.

Proof. To simplify the notation, set

$$X = \underset{\mathcal{C}}{\underrightarrow{\text{hocolim}}}(F) \qquad \text{and} \qquad Z = \underset{\mathcal{C}}{\underrightarrow{\text{hocolim}}}\big(F(-)^{hV}\big).$$

Write $X = \bigcup_{i=0}^{\infty} X_i$ and $Z = \bigcup_{i=0}^{\infty} Z_i$, where X_i and Z_i are the "skeleta" of the homotopy colimits.

We first recall the notation of Lannes [La, §4]. For any M in H^*V-$\mathcal{U}$, i.e., any unstable module over the Steenrod algebra with compatible H^*V-module structure, $T_V(M)$ is a $T_V(H^*V)$-module, and $T_V(H^*V) \cong \prod_{\mathrm{Hom}(V,V)} H^*V$. Lannes defines

$$\mathrm{Fix}(M) = \mathbb{F}_p((1)) \otimes_{T_V(H^*V)} T_V(M),$$

where $\mathbb{F}_p((1))$ is the factor $\mathbb{F}_p$ corresponding to $\mathrm{Id} \in \mathrm{Hom}(V,V)$ (regarded as a quotient algebra). For any V-space Y, Lannes defines $H^{V*}(Y) = \mathrm{Fix}(H_V^*(Y))$, where $H_V^*(Y) = H^*(EV \times_V Y)$. The goal of [La, §4] is to find conditions under which $H^{V*}(Y) \cong H^*((Y_p^\wedge)^{hV})$. By [La, Theorem 4.6.1.1], Fix is an exact functor.

Consider the following exact couples:

$$\begin{array}{ccccccc} \bigoplus H_V^*(X_i) & \longrightarrow & \bigoplus H_V^*(X_i) & \qquad & \bigoplus H^*(Z_i) & \longrightarrow & \bigoplus H^*(Z_i) \\ & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & \bigoplus H_V^*(X_{i+1}, X_i) & & & & \bigoplus H^*(Z_{i+1}, Z_i) & , \end{array}$$

and let $E_r^{pq}(X_{hV})$ and $E_r^{pq}(Z)$ denote the induced spectral sequences. Then $E_r^{p*}(X_{hV})$ is a spectral sequence of modules in H^*V-$\mathcal{U}$. The equivariant maps

$$EV \times Z_i \longrightarrow X_i$$

(where V acts trivially on Z_i) induce via adjointness a homomorphism of spectral sequences

$$T_V(E_r^{p*}(X_{hV})) \longrightarrow E_r^{p*}(Z),$$

and hence after tensoring over $T_V(H^*V)$ with $\mathbb{F}_p((1))$ a homomorphism of spectral sequences

$$\Phi\colon \mathrm{Fix}(E_r^{p*}(X_{hV})) \longrightarrow E_r^{p*}(Z).$$

We first consider the case $r = 2$. For each object c in $\mathcal{C}$, the spaces $F(c)$ and $F(c)^{hV}$ are p-complete by assumption. Hence $H^*(F(c)^{hV}) \cong \mathrm{Fix}(H_V^*(F(c)))$ by [La, Theorem 4.9.1] (applied with $Z = F(c)^{hV}$ and $X = F(c)$). Thus

$$E_2^{p*}(X_{hV}) \cong \underset{\mathcal{C}}{\underleftarrow{\lim}}{}^p(H_V^*(F(-))) \qquad \text{and} \qquad E_2^{p*}(Z) \cong \underset{\mathcal{C}}{\underleftarrow{\lim}}{}^p(\mathrm{Fix}(H_V^*(F(-)))),$$

and $\Phi\colon \mathrm{Fix}(E_2^{p*}(X_{hV})) \longrightarrow E_2^{p*}(Z)$ is the natural map. Since Fix is exact and commutes with finite products, and since $\mathcal{C}$ is a finite category,

$$\underset{\mathcal{C}}{\underleftarrow{\lim}}{}^p(\mathrm{Fix}(H_V^*(F(-)))) \cong \mathrm{Fix}\big(\underset{\mathcal{C}}{\underleftarrow{\lim}}{}^p(H_V^*(F(-)))\big)$$

for each p. It follows that Φ is an isomorphism when $r = 2$.

Thus Φ is also an isomorphism when $r = \infty$. Since $\mathcal{C}$ has bounded limits at p by assumption, there are only a finite number of nonzero columns in each spectral sequence, and so the resulting filtrations of $H^*(Z)$ and $\mathrm{Fix}(H_V^*(X))$ are both finite. Hence (using the exactness of Fix again) Φ induces an isomorphism

$$\mathrm{Fix}(H_V^*(X)) \longrightarrow H^*(Z).$$

By [La, Theorem 4.9.1] again, this implies that

$$Z_p^\wedge \longrightarrow (X_p^\wedge)^{hV}$$

is a homotopy equivalence, which is what we wanted to show. □

This will now be applied to describe maps into a homotopy colimit in certain cases.

Proposition 4.2. *Fix a prime p and a p-group Q. Let $\mathcal{C}$ be a finite category with bounded limits at p, and let*

$$F : \mathcal{C} \longrightarrow \mathsf{Top}$$

be a functor such that for each c in $\mathcal{C}$ and each $Q_0 \le Q$, $\mathrm{Map}(BQ_0, F(c))$ is p-complete and has finite mod p cohomology in each degree. Then the natural map

$$\big[\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}}\big(\mathrm{Map}(BQ,F)\big)\big]_p^\wedge \longrightarrow \mathrm{Map}\big(BQ, \big(\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}} F\big)_p^\wedge\big)$$

is a homotopy equivalence. Here, $\mathrm{Map}(BQ,F)$ denotes the functor which sends c to $\mathrm{Map}(BQ, F(c))$.

Proof. We prove this by induction on $|Q|$; the result is clear when $|Q|=1$. So assume $Q \neq 1$, let $Q_0 \lhd Q$ be a normal subgroup of index p, and set $V = Q/Q_0$.

By the induction hypothesis, the map

$$\big[\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}}\big(\mathrm{Map}(EQ/Q_0,F)\big)\big]_p^\wedge \xrightarrow{\ \simeq\ } \mathrm{Map}\big(EQ/Q_0, \big(\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}} F\big)_p^\wedge\big)$$

is a homotopy equivalence. It is also equivariant with respect to the V-actions induced by the action of V on EQ/Q_0, and hence induces a homotopy equivalence

$$\begin{aligned}\Big(\big[\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}}\big(\mathrm{Map}(EQ/Q_0,F)\big)\big]_p^\wedge\Big)^{hV} \xrightarrow{\ \simeq\ } &\Big(\mathrm{Map}\big(EQ/Q_0, \big(\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}} F\big)_p^\wedge\big)\Big)^{hV} \\ &\simeq \mathrm{Map}\big(BQ, \big(\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}} F\big)_p^\wedge\big).\end{aligned} \tag{1}$$

Furthermore, by assumption, the mapping spaces

$$\mathrm{Map}(BQ_0, F(c)) \qquad \text{and} \qquad \big(\mathrm{Map}(BQ_0,F(c))\big)^{hV} \simeq \mathrm{Map}(BQ,F(c))$$

are p-complete for each $c \in \mathrm{Ob}(\mathcal{C})$, and Proposition 4 applies to show that

$$\big[\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}}\big(\mathrm{Map}(BQ,F)\big)\big]_p^\wedge \xrightarrow{\ \simeq\ } \Big(\big[\operatorname*{hocolim}_{\overrightarrow{\mathcal{C}}}\big(\mathrm{Map}(EQ/Q_0,F)\big)\big]_p^\wedge\Big)^{hV} \tag{2}$$

is a homotopy equivalence. The proposition now follows from (1) and (2). □

We want to use Proposition 4 to describe spaces of maps to classifying spaces of homotopy finite groups. We first prove a more general result. Note that the definition of an associated $\mathcal{L}$-system makes sense, for any full subcategory of a Frobenius system, whether or not it is saturated, and whether or not it contains all of the centric subgroups.

Proposition 4.3. *Fix a Frobenius system $\mathcal{F}$ over S, and let $\mathcal{F}' \subseteq \mathcal{F}^c$ be any full subcategory. Let $\mathcal{L}$ be an associated $\mathcal{L}$-system to $\mathcal{F}'$; i.e., a category together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}'$ which satisfies conditions (A), (B), and (C). Fix a finite p-group Q, and let $\mathcal{L}_Q$ be the category whose objects are the pairs (P,α) for $P \in \mathrm{Ob}(\mathcal{L})$ and $\alpha \in \mathrm{Hom}(Q,P)$, and where*

$$\mathrm{Mor}_{\mathcal{L}_Q}\big((P,\alpha),(P',\alpha')\big) = \big\{\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,P') \,\big|\, \alpha' = \pi(\varphi)\circ\alpha \in \mathrm{Hom}(Q,P')\big\}.$$

Let $\Phi\colon \mathcal{L}_Q \times \mathcal{B}(Q) \longrightarrow \mathcal{L}$ *be the functor defined by setting*

$$\Phi\big((P,\alpha),o_Q\big) = P \qquad \text{and} \qquad \Phi\Big((P,\alpha) \xrightarrow{\ \varphi\ } (P',\alpha')\,,\,\check{x}\Big) = \varphi\circ\widehat{\alpha(x)}$$

Then the map

$$|\Phi|'\colon |\mathcal{L}_Q|^\wedge_p \longrightarrow \operatorname{Map}(BQ,|\mathcal{L}|^\wedge_p)$$

adjoint to $|\Phi|$ *is a homotopy equivalence.*

Proof. Note first that $\Phi(\varphi,\check{x}) = \varphi\circ\widehat{\alpha(x)} = \widehat{\alpha'(x)}\circ\varphi$ by condition (C). Let $\mathcal{O}'\subseteq\mathcal{O}^c(\mathcal{F})$ be the full subcategory with $\operatorname{Ob}(\mathcal{O}') = \operatorname{Ob}(\mathcal{F}') = \operatorname{Ob}(\mathcal{L})$, and let $\widetilde{\pi}\colon \mathcal{L}\longrightarrow\mathcal{O}'$ be the projection functor. As usual, we write $[\![\alpha]\!] = \widetilde{\pi}(\alpha)$ for a morphism α in $\mathcal{L}$. Let $\widetilde{\pi}_Q\colon \mathcal{L}_Q\longrightarrow\mathcal{O}'$ be the functor $\widetilde{\pi}_Q(P,\alpha) = P$ and $\widetilde{\pi}_Q(\varphi) = \widetilde{\pi}(\varphi)$. Let

$$\widetilde{B}_Q, \widetilde{B}\colon \mathcal{O}' \longrightarrow \mathsf{Top}$$

be the left Kan extensions over $\widetilde{\pi}_Q$ and $\widetilde{\pi}$, respectively, of the constant functors $*$. Then

$$|\mathcal{L}| \simeq \underset{\mathcal{O}'}{\underrightarrow{\operatorname{hocolim}}}(\widetilde{B}) \qquad \text{and} \qquad |\mathcal{L}_Q| \simeq \underset{\mathcal{O}'}{\underrightarrow{\operatorname{hocolim}}}(\widetilde{B}_Q)$$

(cf. [HV, Theorem 5.5]).

Consider the commutative triangle

$$\begin{array}{ccc} \mathcal{L}_Q\times\mathcal{B}(Q) & \xrightarrow{\quad\Phi\quad} & \mathcal{L} \\ & {\scriptstyle \widetilde{\pi}_Q\circ\mathrm{pr}_1}\searrow \qquad \swarrow{\scriptstyle\widetilde{\pi}} & \\ & \mathcal{O}'\ . & \end{array}$$

The left Kan extension over $\widetilde{\pi}_Q\circ\mathrm{pr}_1$ of the constant functor $*$ is the functor $\widetilde{B}_Q(-)\times BQ$, and so the triangle induces a natural transformation of functors

$$\Phi'\colon \widetilde{B}_Q(-)\times BQ \longrightarrow \widetilde{B}.$$

The adjoint map $\widetilde{\Phi}\colon \widetilde{B}_Q \longrightarrow \operatorname{Map}(BQ,\widetilde{B})$ to Φ' is also a natural transformation of functors from $\mathcal{O}'$ to Top, and induces a commutative diagram

$$\begin{array}{ccccc} \underset{\mathcal{O}'}{\underrightarrow{\operatorname{hocolim}}}(\widetilde{B}_Q)^\wedge_p & \xrightarrow{\operatorname{hocolim}(\widetilde{\Phi})} & \underset{\mathcal{O}'}{\underrightarrow{\operatorname{hocolim}}}\operatorname{Map}(BQ,\widetilde{B})^\wedge_p & \xrightarrow[\simeq]{\ \omega\ } & \operatorname{Map}\big(BQ,\underset{\mathcal{O}'}{\underrightarrow{\operatorname{hocolim}}}(\widetilde{B})^\wedge_p\big) \\ {\scriptstyle\simeq}\big\downarrow & & & & \big\downarrow{\scriptstyle\simeq} \\ |\mathcal{L}_Q|^\wedge_p & & \xrightarrow{\qquad\qquad |\Phi|' \qquad\qquad} & & \operatorname{Map}(BQ,|\mathcal{L}|^\wedge_p)\ . \end{array}$$

For each $P\le S$ and $Q_0\le Q$, each component of $\operatorname{Map}(BQ_0,BP)$ is of the form $BC_P(\rho(Q_0))$ for some $\rho\in\operatorname{Hom}(Q_0,P)$. So all such mapping spaces are p-complete and have finite mod p cohomology in each degree, and hence ω is a homotopy equivalence by Proposition 4. It thus remains only to show that $\widetilde{\Phi}(P)$ is a homotopy equivalence for each P.

For each P in $\mathcal{O}'$, $\widetilde{B}(P)$ is the nerve of the overcategory $\widetilde{\pi}{\downarrow}P$, whose objects are the pairs (R,χ) for $R\in\operatorname{Ob}(\mathcal{L}) = \operatorname{Ob}(\mathcal{O}')$ and $\chi\in\operatorname{Rep}_{\mathcal{F}}(R,P)$, and where

$$\operatorname{Mor}_{\widetilde{\pi}\downarrow P}\big((R,\chi),(R',\chi')\big) = \big\{\varphi\in\operatorname{Mor}_{\mathcal{L}}(R,R') \,|\, \chi = \chi'\circ[\![\varphi]\!]\big\}.$$

Let $\mathcal{B}'(P)$ be the full subcategory of $\widetilde{\pi}{\downarrow}P$ with the unique object (P,Id), and with morphisms the group of all $\widehat{g}$ for $g\in P$.

Similarly, $\widetilde{B}_Q(P)$ is the nerve of the category $\widetilde{\pi}_Q{\downarrow}P$, whose objects are the triples (R,α,χ) for $R\in\mathrm{Ob}(\mathcal{L})=\mathrm{Ob}(\mathcal{O}')$, $\alpha\in\mathrm{Hom}(Q,R)$, and $\chi\in\mathrm{Rep}_{\mathcal{F}}(R,P)$; and where

$$\mathrm{Mor}_{\widetilde{\pi}_Q\downarrow P}\big((R,\alpha,\chi),(R',\alpha',\chi')\big)=\big\{\varphi\in\mathrm{Mor}_{\mathcal{L}}(R,R')\,|\,\alpha'=[\varphi]\circ\alpha,\ \chi=\chi'\circ[\![\varphi]\!]\big\}.$$

Let $\mathcal{B}'_Q(P)$ be the full subcategory of $\widetilde{\pi}_Q{\downarrow}P$ with objects the triples (P,α,Id) for $\alpha\in\mathrm{Hom}(Q,P)$.

Fix a section $\widetilde{\sigma}\colon\mathrm{Mor}(\mathcal{O}')\longrightarrow\mathrm{Mor}(\mathcal{L})$ which sends identity morphisms to identity morphisms. Retractions

$$\widetilde{\pi}{\downarrow}P\xrightarrow{\ \Psi\ }\mathcal{B}'(P)\qquad\text{and}\qquad\widetilde{\pi}_Q{\downarrow}P\xrightarrow{\ \Psi_Q\ }\mathcal{B}'_Q(P)$$

are defined by setting

$$\Psi(R,\chi)=(P,\mathrm{Id})\qquad\text{and}\qquad\Psi_Q(R,\alpha,\chi)=(P,\pi\widetilde{\sigma}(\chi)\circ\alpha,\mathrm{Id});$$

and by sending φ in $\mathrm{Mor}_{\widetilde{\pi}\downarrow P}((R,\chi),(R',\chi'))$ or $\mathrm{Mor}_{\widetilde{\pi}_Q\downarrow P}((R,\alpha,\chi),(R',\alpha',\chi'))$ to the unique morphism $\widehat{g}\in\mathrm{Aut}_{\mathcal{L}}(P)$ such that $\widetilde{\sigma}(\chi')\circ\varphi=\widehat{g}\circ\widetilde{\sigma}(\chi)$ in $\mathrm{Mor}_{\mathcal{L}}(R,P)$ (Lemma 1(b)). There are natural transformations

$$\mathrm{Id}_{\widetilde{\pi}\downarrow P}\longrightarrow\mathrm{incl}\circ\Psi\qquad\text{and}\qquad\mathrm{Id}_{\widetilde{\pi}_Q\downarrow P}\longrightarrow\mathrm{incl}\circ\Psi_Q$$

of functors which send an object (R,χ) to $\chi\in\mathrm{Mor}_{\widetilde{\pi}\downarrow P}((R,\chi),(P,\mathrm{Id}))$ and similarly for an object (R,α,χ). This shows that $|\mathcal{B}'(P)|\subseteq|\widetilde{\pi}{\downarrow}P|$ and $|\mathcal{B}'_Q(P)|\subseteq|\widetilde{\pi}_Q{\downarrow}P|$ are deformation retracts.

It remains to show for each P that $\widetilde{\Phi}(P)$ restricts to a homotopy equivalence

$$\widetilde{\Phi}_0(P)\colon|\mathcal{B}'_Q(P)|\longrightarrow\mathrm{Map}(BQ,|\mathcal{B}'(P)|)\tag{1}$$

Two objects (P,α,Id) and (P,α',Id) in $\mathcal{B}'_Q(P)$ are isomorphic if and only if α and α' are conjugate in P, and the automorphism group of (P,α,Id) is isomorphic to $C_P(\alpha Q)$. This shows that

$$\widetilde{B}_Q(P)\simeq\coprod_{\alpha\in\mathrm{Rep}(Q,P)}BC_P(\alpha Q).$$

Since $|\mathcal{B}'(P)|\cong BP$, and since $\widetilde{\Phi}'$ is induced by the homomorphisms $(\mathrm{incl}\cdot\alpha)$ from $C_P(\alpha Q)\times Q$ to P, it follows that (1) is an equivalence. □

We will apply Proposition 4 to describe maps $BQ\longrightarrow|\mathcal{L}|^\wedge_p$ when $(S,\mathcal{F},\mathcal{L})$ is a homotopy finite group. For a finite group G, the components of $\mathrm{Map}(BQ,BG^\wedge_p)$ are described via centralizers of images of homomorphisms from Q to G. By analogy, the components of $\mathrm{Map}(BQ,|\mathcal{L}|^\wedge_p)$ will be described here via associated $\mathcal{L}$-systems to centralizers in Frobenius systems.

Recall that for a saturated Frobenius system $\mathcal{F}$ over S, a subgroup $Q\le S$ is saturated in $\mathcal{F}$ if $|C_S(Q)|$ is maximal among the $|C_S(Q')|$ for Q' $\mathcal{F}$-conjugate to Q (Definition 1). For any such Q, $C_{\mathcal{F}}(Q)$ is the Frobenius system over $C_S(Q)$ defined by setting

$$\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P,P')=\big\{\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,P')\,\big|\,\exists\bar{\varphi}\in\mathrm{Hom}_{\mathcal{F}}(PQ,P'Q),\ \bar{\varphi}|_P=\varphi,\ \bar{\varphi}_Q=\mathrm{Id}_Q\big\}$$

for all $P,P'\le C_S(Q)$ (see Definition A). We next construct an $\mathcal{L}$-system associated to $C_{\mathcal{F}}(Q)$.

Definition 4.4. *Fix a homotopy finite group $(S,\mathcal{F},\mathcal{L})$, and a subgroup $Q\le S$ which is saturated in $\mathcal{F}$. Define $C_{\mathcal{L}}(Q)$ to be the category whose objects are the $C_{\mathcal{F}}(Q)$-centric subgroups $P\le C_S(Q)$, and where* $\mathrm{Mor}_{C_{\mathcal{L}}(Q)}(P,P')$ *is the set of those morphisms $\varphi\in\mathrm{Mor}_{\mathcal{L}}(PQ,P'Q)$ whose underlying homomorphisms are the identity on Q and send P into P'.*

We will need the following properties of these categories. Recall that if $\mathcal{F}$ is a Frobenius system over S, then a subgroup $P \le S$ is $\mathcal{F}$-centric if $C_S(P') = Z(P')$ (equivalently $C_S(P') \le P'$) for all P' which is $\mathcal{F}$-conjugate to P.

Proposition 4.5. *Fix a saturated Frobenius system $\mathcal{F}$ over a p-group S, and a subgroup $Q \le S$ which is saturated in $\mathcal{F}$. Then the following hold:*

(a) *A subgroup $P \le C_S(Q)$ is $C_{\mathcal{F}}(Q)$-centric if and only if $P \ge Z(Q)$ and PQ is $\mathcal{F}$-centric; and if this holds then $Z(P) = Z(PQ)$.*

(b) *$C_{\mathcal{F}}(Q)$ is a saturated Frobenius system over $C_S(Q)$.*

(c) *If $\mathcal{L}$ is an $\mathcal{L}$-system associated to $\mathcal{F}$ (i.e., if $(S, \mathcal{F}, \mathcal{L})$ is a homotopy finite group), then $C_{\mathcal{L}}(Q)$ is an associated $\mathcal{L}$-system to $C_{\mathcal{F}}(Q)$.*

Proof. We first check point (a). Fix $P \le C_S(Q)$. If PQ is $\mathcal{F}$-centric and $P \ge Z(Q)$, then

$$C_{C_S(Q)}(P) = C_S(P) \cap C_S(Q) = C_S(PQ) = Z(PQ) = Z(P)\cdot Z(Q) = Z(P):$$

the last two steps since $[P, Q] = 1$ and $P \ge Z(Q)$. The same computation applies to any P' which is $C_{\mathcal{F}}(Q)$-conjugate to P, and so P is $C_{\mathcal{F}}(Q)$-centric in this case. Conversely, if P is $C_{\mathcal{F}}(Q)$-centric, then clearly $P \ge Z(Q)$. To see that PQ is $\mathcal{F}$-centric, fix any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$; we must show that $C_S(\varphi(PQ)) \le \varphi(PQ)$. Since Q is saturated in $\mathcal{F}$, there is a homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(\varphi(Q))\cdot\varphi(Q), C_S(Q)\cdot Q)$ such that $\psi|_{\varphi(Q)} = (\varphi|_Q)^{-1}$. Set $\varphi' = \psi \circ \varphi$; thus $\varphi'|_Q = \mathrm{Id}_Q$ and hence $\varphi'|_P \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, C_S(Q))$. Then

$$C_S(\varphi'(PQ)) = C_S(\varphi'(P)\cdot Q) = C_{C_S(Q)}(\varphi'(P)) \le \varphi'(P) \le \varphi'(PQ)$$

since P is $C_{\mathcal{F}}(Q)$-centric, so $C_S(\varphi(PQ)) \le \varphi(PQ)$ since ψ sends $C_S(\varphi(PQ))$ injectively into $C_S(\varphi'(PQ))$; and thus PQ is $\mathcal{F}$-centric.

Point (b) is a special case of Proposition A.

When showing that $C_{\mathcal{L}}(Q)$ is an associated $\mathcal{L}$-system to $C_{\mathcal{F}}(Q)$, note first that the category is well defined by (a): PQ is $\mathcal{F}$-centric whenever P is $C_{\mathcal{F}}(Q)$-centric. Conditions (B) and (C) are immediate. And condition (A) — the requirement that $Z(P)$ act freely on $\mathrm{Aut}_{C_{\mathcal{L}}(Q)}(P)$ with orbit set $\mathrm{Aut}_{C_{\mathcal{F}}(Q)}(P)$ — follows since $Z(P) = Z(PQ)$ by (a). □

In the above situation, where $\mathcal{L}$ is an associated $\mathcal{L}$-system to $\mathcal{F}$ and Q is saturated in $\mathcal{F}$, we define a functor

$$\Gamma = \Gamma_{\mathcal{L},Q} \colon C_{\mathcal{L}}(Q) \times \mathcal{B}(Q) \longrightarrow \mathcal{L}$$

by setting $\Gamma(P, o_Q) = PQ$ for each $C_{\mathcal{L}}(Q)$-centric $P \le C_S(Q)$, and

$$\Gamma_{(P,o_Q),(P',o_Q)}(\varphi, \breve{g}) = \varphi \circ \widehat{g} = \widehat{g} \circ \varphi.$$

The last equality follows from condition (C), since the underlying homomorphism of $\varphi \in \mathrm{Mor}_{\mathcal{L}}(PQ, P'Q)$ is the identity on Q.

These "centralizer $\mathcal{L}$-systems" can now be used to describe components of certain mapping spaces. This is analogous to the description of components of mapping spaces $\mathrm{Map}(BQ, BG)$ in terms of centralizers of images of homomorphisms $Q \longrightarrow G$.

Theorem 4.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a homotopy finite group, and let $f \colon BS \longrightarrow |\mathcal{L}|^\wedge_p$ be the natural inclusion followed by completion. Then the following hold for any p-group Q.*

(a) *Each map* $BQ \longrightarrow |\mathcal{L}|^\wedge_p$ *is homotopic to* $f \circ B\rho$ *for some* $\rho \in \operatorname{Hom}(Q,S)$.

(b) *Given any two homomorphisms* $\rho, \rho' \in \operatorname{Hom}(Q,S)$, $f \circ B\rho \simeq f \circ B\rho'$ *as maps from* BQ *to* $|\mathcal{L}|^\wedge_p$ *if and only if there is some* $\chi \in \operatorname{Hom}_{\mathcal{F}}(\rho Q, \rho' Q)$ *such that* $\rho' = \chi \circ \rho$.

(c) *For each* $\rho \in \operatorname{Hom}(Q,S)$ *such that* $\rho(Q)$ *is saturated in* $\mathcal{F}$, $\Gamma_{\mathcal{L},Q}$ *induces a homotopy equivalence*

$$|C_{\mathcal{L}}(\rho Q)|^\wedge_p \xrightarrow{\ \simeq\ } \operatorname{Map}(BQ, |\mathcal{L}|^\wedge_p)_{f\circ B\rho}.$$

In particular, $\operatorname{Map}(BQ, |\mathcal{L}|^\wedge_p)$ *is p-complete.*

Proof. By Proposition 4, $\operatorname{Map}(BQ,|\mathcal{L}|^\wedge_p) \simeq |\mathcal{L}_Q|^\wedge_p$, where $\mathcal{L}_Q$ is the category whose objects are the pairs (P,α) for $P \le S$ $\mathcal{F}$-centric and $\alpha \in \operatorname{Hom}(Q,P)$, and where a morphism from (P,α) to (P',α') is a morphism $\varphi \in \operatorname{Mor}_{\mathcal{L}}(P,P')$ such that $\alpha' = [\varphi]\circ\alpha$. Points (a) and (b) follow immediately.

Fix some $\rho \in \operatorname{Hom}(Q,S)$ such that ρQ is saturated in $\mathcal{F}$. The component in $|\mathcal{L}_Q|$ of ρ is the nerve of the full subcategory $\mathcal{L}_{Q,\rho}$ with objects those pairs (P,α) such that $\rho = \varphi\circ\alpha$ for some $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)$. Let $\mathcal{L}'_{Q,\rho} \subseteq \mathcal{L}_{Q,\rho}$ be the full subcategory whose objects are the (P,α) such that $P \le C_S(\alpha Q)\cdot\alpha Q$. There is a retraction Ψ of $\mathcal{L}_{Q,\rho}$ onto $\mathcal{L}'_{Q,\rho}$ which sends each P to $C_P(\alpha Q)\cdot\alpha Q$, and a natural transformation of functors $\Psi \longrightarrow \operatorname{Id}_{\mathcal{L}_{Q,\rho}}$ defined by inclusion. Thus, $|\mathcal{L}'_{Q,\rho}|$ is a deformation retract of $|\mathcal{L}_{Q,\rho}|$. Consider the functor

$$\Phi\colon C_{\mathcal{L}}(\rho Q) \longrightarrow \mathcal{L}'_{Q,\rho},$$

defined by setting $\Phi(P) = (P\cdot\rho Q, \rho)$. This is injective on objects, and induces bijections on morphism sets by definition of $C_{\mathcal{L}}(\rho Q)$. Also, since ρQ is saturated, each object in $\mathcal{L}'_{Q,\rho}$ is isomorphic to one of the form $(P,\rho) = \Phi(C_P(Q))$ for some $\mathcal{F}$-centric P such that $\rho Q \le P \le C_S(\rho Q)\cdot\rho Q$. Thus Φ is an equivalence of categories, and hence

$$|\mathcal{L}_{Q,\rho}| \simeq |\mathcal{L}'_{Q,\rho}| \simeq |C_{\mathcal{L}}(\rho Q)|. \qquad \square$$

If $(S,\mathcal{F},\mathcal{L})$ is a homotopy finite group, then for any finite p-group Q we define

$$\operatorname{Rep}(Q,\mathcal{L}) = \operatorname{Hom}(Q,S)/{\sim},$$

where $\sim$ is the equivalence relation defined by setting $\rho \sim \rho'$ if there is some $\chi \in \operatorname{Hom}_{\mathcal{F}}(\rho Q, \rho' Q)$ such that $\rho' = \chi\circ\rho$. Theorem 4(a,b) can now be restated as follows.

Corollary 4.7. *Fix a homotopy finite group* $(S,\mathcal{F},\mathcal{L})$, *and let* $f\colon BS \longrightarrow |\mathcal{L}|^\wedge_p$ *be the natural inclusion followed by completion. Then the map*

$$\operatorname{Rep}(Q,\mathcal{L}) \xrightarrow{\ \cong\ } [BQ, |\mathcal{L}|^\wedge_p],$$

defined by sending the class of $\rho\colon Q \longrightarrow S$ *to* $f\circ B\rho$, *is a bijection.* $\square$

We finish the section by describing a second decomposition of $|\mathcal{L}|$ as a homotopy colimit, analogous to the centralizer decomposition of BG of Jackowski and McClure [JM], and to the centralizer decomposition of certain algebras of Dwyer and Wilkerson [DW1].

Theorem 4.8. *Fix a homotopy finite group* $(S,\mathcal{F},\mathcal{L})$. *Let* $\mathcal{F}^e$ *be the full subcategory of* $\mathcal{F}$ *whose objects are the nontrivial saturated elementary abelian p-subgroups of* S. *Then the natural map*

$$\underset{E\in\mathcal{F}^e}{\underrightarrow{\operatorname{hocolim}}}\ \operatorname{Map}(BE, |\mathcal{L}|^\wedge_p)_{\mathrm{incl}} \longrightarrow |\mathcal{L}|^\wedge_p,$$

induced by evaluation at basepoints, is a mod p homology equivalence.

Proof. Let $\widehat{\mathcal{L}}$ denote the category whose objects are the pairs (P,E) for $\mathcal{F}$-centric subgroups $P \le S$ and elementary abelian subgroups $E \le Z(P)$, and where a morphism from (P,E) to (P',E') is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{L}}(P,P')$ such that $[\varphi](E) \ge E'$. For each $P \le S$, let $\mathcal{E}(P) \le Z(P)$ denote the subgroup of elements of order p in the center. There are obvious functors

$$\mathcal{L} \underset{T}{\overset{S}{\rightleftarrows}} \widehat{\mathcal{L}}$$

defined by setting $S(P) = (P, \mathcal{E}(P))$ and $T(P,E) = P$, and a morphism of functors $S \circ T \longrightarrow \mathrm{Id}_{\widehat{\mathcal{L}}}$. This shows that $|\mathcal{L}| \simeq |\widehat{\mathcal{L}}|$.

Let $\tau\colon \widehat{\mathcal{L}} \longrightarrow (\mathcal{F}^e)^{\mathrm{op}}$ be the functor which sends an object (P,E) to E. Then by [HV, Theorem 5.5],

$$|\mathcal{L}| \simeq |\widehat{\mathcal{L}}| = \underset{\widehat{\mathcal{L}}}{\underrightarrow{\mathrm{hocolim}}}(*) \simeq \underset{E\in\mathcal{F}^e}{\underrightarrow{\mathrm{hocolim}}}\, |\tau{\downarrow}E|,$$

since $E \mapsto |\tau{\downarrow}E|$ is the left Kan extension of the trivial functor over τ. It remains to identify this (up to p-completion) with the functor which sends E to $\mathrm{Map}(BE, |\mathcal{L}|^\wedge_p)_{\mathrm{incl}}$.

By definition, $\tau{\downarrow}E$ is the overcategory whose objects are the triples (P,E',α) for (P,E') in $\widehat{\mathcal{L}}$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}^e}(E,E')$. This contains as homotopy retract the full subcategory whose objects are the triples (P,E',α) for which $E' = \alpha(E)$. This can be identified with the category $\mathcal{C}$ of pairs (P,α) for $\alpha \in \mathrm{Hom}_{\mathcal{F}}(E, Z(P))$, where a morphism from (P,α) to (P',α') is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{L}}(P,P')$ such that $[\varphi]\circ\alpha = \alpha'$. This in turn is a homotopy retract of the component $(\mathcal{L}_E)_{\mathrm{incl}}$ of the category $\mathcal{L}_E$ of Proposition 4 corresponding to the inclusion $E \hookrightarrow S$, and so

$$|\tau{\downarrow}E|^\wedge_p \simeq |\mathcal{C}|^\wedge_p \simeq |(\mathcal{L}_E)_{\mathrm{incl}}|^\wedge_p \simeq \mathrm{Map}(BE, |\mathcal{L}|^\wedge_p)_{\mathrm{incl}}$$

by Proposition 4. □

5. A topological characterization

In Definition 1, we defined the classifying space of a homotopy finite group $(S,\mathcal{F},\mathcal{L})$ to be the space $|\mathcal{L}|^\wedge_p$. In this section, we show that the triple $(S,\mathcal{F},\mathcal{L})$ is in fact determined up to isomorphism by the homotopy type of $|\mathcal{L}|^\wedge_p$ (Theorem 5). Afterwards, we prove a more intrinsic characterization of these spaces, by showing that a p-complete space X is the classifying space of some homotopy finite group if and only if it satisfies certain conditions listed in Theorem 5.

Definition 5.1. *A space X is called* quasi-finite *at a prime p if $\mathrm{Map}_*(B\mathbb{Z}/p, X)$ is contractible for all choices of basepoint in X. A map $f\colon X \longrightarrow Y$ is called a* homotopy monomorphism *at p if its homotopy fiber F (over any connected component of Y) is quasi-finite at p.*

By [La, Theorem 0.5], a p-complete space X is quasi-finite at p if and only if $T_{\mathbb{Z}/p}(H^*X) \cong H^*X$; and by [LS, Proposition 6.3.2] this is the case if and only if $H^*(X;\mathbb{F}_p)$ is locally finite as a module over the Steenrod algebra. This is what provided the original motivation for the above definition.

We will show in Corollary 6 below that a map $X \xrightarrow{f} X'$ between classifying spaces of homotopy finite groups is a homotopy monomorphism if and only if $H^*(X;\mathbb{F}_p)$ is a finitely generated $H^*(X';\mathbb{F}_p)$-module via f.

Lemma 5.2. *The composite of two homotopy monomorphisms is a homotopy monomorphism.*

Proof. Let $X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z$ be homotopy monomorphisms. We can assume, without changing the homotopy types, that ψ and φ are fibrations. Fix basepoints $y \in Y$ and $z = \varphi(y) \in Z$, and set $F = \psi^{-1}(y)$, $E = (\varphi\psi)^{-1}(z)$, and $B = \varphi^{-1}(z)$. Then F and B are quasi-finite by assumption, $F \xrightarrow{i} E \xrightarrow{f} B$ is a fibration sequence, and we must show that the fiber E of $\varphi \circ \psi$ is quasi-finite.

Fix a basepoint $e_0 \in E$, set $b_0 = f(e_0)$, and assume that $F = f^{-1}(b_0)$. Then

$$\mathrm{Map}_*(B\mathbb{Z}/p, F) \xrightarrow{i\circ -} \mathrm{Map}_*(B\mathbb{Z}/p, E) \xrightarrow{f\circ -} \mathrm{Map}_*(B\mathbb{Z}/p, B)$$

(spaces of pointed maps with respect to these basepoints) is a fibration sequence, and $\mathrm{Map}_*(B\mathbb{Z}/p, E)$ is contractible since the other two spaces are. □

Definition 5.3. *A* p-subgroup *of a space X is a pair (P, f) where P is a p-group and $f\colon BP \longrightarrow X$ is a homotopy monomorphism at p. A p-subgroup (P, f) of X is* centric *if f is a centric map; i.e., if the induced map*

$$\mathrm{Map}(BP, BP)_{\mathrm{Id}} \xrightarrow{f\circ -} \mathrm{Map}(BP, X)_f$$

is a homotopy equivalence. A Sylow p-subgroup *of X is a p-subgroup (S, f) such that for any finite p-group P, each map $BP \longrightarrow X$ factors up to homotopy through $BS \xrightarrow{f} X$.*

One can also define a *p-centric subgroup* of X to be a p-subgroup P such that the map from $\mathrm{Map}(BP, BP)_{\mathrm{Id}}$ to $\mathrm{Map}(BP, X)_f$ is a mod p homology equivalence. The next proposition still holds if stated for p-centric subgroups rather than for centric subgroups. But since we do not forsee working in any situations where the two concepts differ, results are stated here for centric subgroups only.

Proposition 5.4. *Fix a p-complete space X with Sylow p-subgroup (S, f). Then the following hold.*

(a) *For any finite p-group P and any $\rho \in \mathrm{Hom}(P, S)$, $f \circ B\rho\colon BP \longrightarrow X$ is a homotopy monomorphism if and only if ρ is injective.*

(b) *For any subgroup $P \le S$ such that the map $BP \xrightarrow{f|_{BP}} X$ is centric, P is centric in S.*

(c) *For any $P \le S$ such that the map $f|_{BP}$ is centric, $f|_{BQ}$ is centric for all Q such that $P \le Q \le S$.*

(d) *If P is a finite p-group, and if $\rho, \rho' \in \mathrm{Hom}(P, S)$ are homomorphisms such that $f \circ B\rho \simeq f \circ B\rho'$, then $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho')$. If, furthermore, X has the property that $\mathrm{Map}(B\mathbb{Z}/p, X)_{\mathrm{const}} \simeq X$, and if we set $K = \mathrm{Ker}(\rho)$ and let $\bar\rho$ and $\bar\rho'$ denote the induced monomorphisms from P/K to S, then $f \circ B\bar\rho \simeq f \circ B\bar\rho'$.*

Proof. **(a)** Let F be the homotopy fiber of $f \circ B\rho$. If ρ is injective, then $f \circ B\rho$ is the composite of two homotopy monomorphisms, so F is the total space of a fibration whose base and fiber are both quasi-finite at p, and is thus itself quasi-finite at p. So $f \circ B\rho$ is a homotopy monomorphism in this case.

Conversely, if ρ is not injective, let $\sigma \in \mathrm{Hom}(\mathbb{Z}/p, P)$ be an injection into $\mathrm{Ker}(\rho)$. Then $B\mathbb{Z}/p \xrightarrow{B\sigma} BP$ is null homotopic in X, and hence lifts to a map $B\mathbb{Z}/p \longrightarrow F$

which is not null homotopic. So F is not quasi-finite at p, and $f\circ B\rho$ is not a homotopy monomorphism.

(b) Assume otherwise: let $P\le S$ be such that $C_S(P)\gneqq Z(P)$ and $f|_{BP}$ is a centric map. Fix $Q\ge P$ such that $Q\le C_S(P){\cdot}P$ and $[Q{:}P]=p$. Then there is some $\psi\in\operatorname{Hom}(C_Q(P),Z(P))$ such that the following square commutes up to homotopy:

$$\begin{array}{ccc} BC_Q(P) & \overset{B\psi}{\dashrightarrow} & \operatorname{Map}(BP,BP)_{\mathrm{Id}}\simeq BZ(P) \\ \simeq\downarrow & & \simeq\downarrow f|_{BP}\circ- \\ \operatorname{Map}(BP,BQ)_{\mathrm{incl}} & \xrightarrow{f|_{BQ}\circ-} & \operatorname{Map}(BP,X)_f\ . \end{array}$$

Upon restricting the mapping spaces to their evaluation at the basepoint of BP, this shows that

$$f|_{BZ(P)}\circ B\psi\simeq f|_{BC_Q(P)}\colon BC_Q(P)\longrightarrow X.$$

And this is a contradiction, since $f|_{BC_Q(P)}$ is a homotopy monomorphism by (a), while $f|_{BZ(P)}\circ B\psi$ is not a homotopy monomorphism since ψ is not injective ($C_Q(P)\gneqq Z(P)$).

(c) Since there is always a subnormal series of subgroups joining P and Q, it suffices to prove this result when $P\lhd Q$. Consider the following maps between mapping spaces:

$$\operatorname{Map}(BP,BP)_{\mathrm{Id}}\xrightarrow{\mathrm{incl}\circ-}\operatorname{Map}(BP,BQ)_{\mathrm{incl}}\xrightarrow{f|_{BQ}\circ-}\operatorname{Map}(BP,X)_{f|_{BP}}.$$

The composite is a homotopy equivalence by assumption ($f|_{BP}$ is centric), and the first map is a homotopy equivalence since $C_S(P)=C_Q(P)=Z(P)$. Thus, the second map is a homotopy equivalence. So if we replace BP by EQ/P and take homotopy fixed point sets of the Q/P-action on the last two mapping spaces induced by the action on EQ/P, we get homotopy equivalences

$$\operatorname{Map}(BQ,BQ)_{\mathrm{Id}}\simeq\big(\operatorname{Map}(EQ/P,BQ)_{\mathrm{incl}}\big)^{hQ/P}\xrightarrow{(f|_{BQ}\circ-)^{hQ/P}}$$
$$\big(\operatorname{Map}(EQ/P,X)_{f|_{BP}}\big)^{hQ/P}\simeq\operatorname{Map}(BQ,X)_{f|_{BQ}};$$

and thus $f|_{BQ}$ is centric.

(d) Now assume that $\rho,\rho'\in\operatorname{Hom}(P,S)$ are such that $f\circ B\rho\simeq f\circ B\rho'$, and set $K=\operatorname{Ker}(\rho)$ and $K'=\operatorname{Ker}(\rho')$. Then $f\circ B(\rho'|_K)\simeq *$, so $BK\xrightarrow{B(\rho'|_K)}BS$ lifts to the homotopy fiber of $BS\xrightarrow{f}X$, and hence must be null homotopic since the homotopy fiber is quasi-finite. Thus, $K\le\operatorname{Ker}(\rho')=K'$, and $K'\le K$ by symmetry.

Assume in addition that $\operatorname{Map}(B\mathbb{Z}/p,X)_{\mathrm{const}}\simeq X$. We apply the following result of Zabrodsky (cf. [Dw, Proposition 3.5]): if $E\xrightarrow{\psi}B$ is a fibration with fiber F and connected base space B, and if Y is any space such that $\operatorname{Map}(F,Y)_{\mathrm{const}}\simeq Y$, then the induced map

$$\operatorname{Map}(B,Y)\ \underset{\simeq}{\xrightarrow{-\circ\psi}}\ \operatorname{Map}(E,Y)_{[F]}\tag{1}$$

is an equivalence, where $\operatorname{Map}(E,Y)_{[F]}$ denotes the subspace of maps whose restriction to F is nullhomotopic. Thus $\operatorname{Map}(BQ,X)_{\mathrm{const}}\simeq X$ for all p-groups Q: this follows from (1) and induction on $|Q|$ since it holds by assumption when $|Q|=p$. In particular, $\operatorname{Map}(BK,X)_{\mathrm{const}}\simeq X$, and so (1) applies again to show that

$$\operatorname{Map}(B(P/K),X)\simeq\operatorname{Map}(BP,X)_{[BK]}.$$

And thus $f\circ B\bar\rho\simeq f\circ B\bar\rho'$, where $\bar\rho,\bar\rho'\in\operatorname{Hom}(P/K,S)$ are induced by ρ,ρ'. □

We now look at the case where X is the classifying space of a homotopy finite group.

Proposition 5.5. *Let $(S,\mathcal{F},\mathcal{L})$ be a homotopy finite group, and let $f\colon BS \longrightarrow |\mathcal{L}|^\wedge_p$ be the natural inclusion followed by completion. Then (S,f) is a Sylow p-subgroup of X. Furthermore, for any subgroup $P \le S$, $(P, f|_{BP})$ is a centric p-subgroup ($f|_{BP}$ is a centric map) if and only if P is $\mathcal{F}$-centric.*

Proof. We first show that f is a homotopy monomorphism. By adjointness,

$$\mathrm{Map}_*(B\mathbb{Z}/p, \Omega(|\mathcal{L}|^\wedge_p)) \simeq \Omega(\mathrm{Map}_*(B\mathbb{Z}/p, |\mathcal{L}|^\wedge_p)) \simeq *,$$

where the basepoints of the mapping space and loop space are constant maps, and where the last step holds since $\mathrm{Map}(B\mathbb{Z}/p, |\mathcal{L}|^\wedge_p)_{\mathrm{triv}} \simeq |\mathcal{L}|^\wedge_p$ by Theorem 4. Thus, the component of the constant loops in $\Omega(|\mathcal{L}|^\wedge_p)$ is quasi-finite at p, and hence all components are quasi-finite since they all have the same homotopy type.

Now let F be the homotopy fiber of f, and consider the fibration sequence

$$\Omega(|\mathcal{L}|^\wedge_p) \longrightarrow F \longrightarrow BS.$$

Any map $B\mathbb{Z}/p \longrightarrow F$ must be nullhomotopic in BS, since otherwise $B\mathbb{Z}/p \longrightarrow BS$ is induced by a monomorphism $\mathbb{Z}/p \longrightarrow S$ and hence is nontrivial in $|\mathcal{L}|^\wedge_p$ by Theorem 4(b). But $\mathrm{Map}_*(B\mathbb{Z}/p, BS)_{\mathrm{triv}}$ is contractible, the space of pointed maps to any component of the fiber is contractible, and thus each component of $\mathrm{Map}_*(B\mathbb{Z}/p, F)$ is contractible. So F is quasi-finite at p, and f is a homotopy monomorphism. By Theorem 4(a), all maps $BP \longrightarrow |\mathcal{L}|^\wedge_p$ (for any p-group P) factor through f, and this finishes the proof that (S,f) is a Sylow p-subgroup.

It remains to show, for any $P \le S$, that $f|_{BP}$ is centric if and only if P is $\mathcal{F}$-centric. The "only if" part was shown in Proposition 5(c). Using Theorem 4(b), it remains only to show, for each $\mathcal{F}$-centric subgroup $P \le S$ which is saturated in $\mathcal{F}$, that $f|_{BP}$ is centric. But for such P, $C_{\mathcal{F}}(P)$ is the Frobenius system over $Z(P)$ containing inclusion morphisms only,

$$\mathrm{Map}(BP, |\mathcal{L}|^\wedge_p)_{f|_{BP}} \simeq |C_{\mathcal{L}}(P)|^\wedge_p \cong BZ(P)$$

by Theorem 4(c), and thus $f|_{BP}$ is centric. □

The following proposition shows that Sylow subgroups behave as expected with respect to maps between classifying spaces of homotopy finite groups.

Proposition 5.6. *Let $X = |\mathcal{L}|^\wedge_p$ and $X' = |\mathcal{L}'|^\wedge_p$ be classifying spaces of homotopy finite groups $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$. Let $BS \xrightarrow{f} X$ and $BS' \xrightarrow{f'} X'$ denote the inclusions. Then for any map $\varphi\colon X \longrightarrow X'$, there is a homomorphism $\rho\colon S \longrightarrow S'$, unique up to $\mathcal{F}'$-conjugacy, such that the following square commutes up to homotopy:*

$$\begin{array}{ccc} BS & \xrightarrow{\;f\;} & X \\ \big\downarrow{\scriptstyle B\rho} & & \big\downarrow{\scriptstyle \varphi} \\ BS' & \xrightarrow{\;f'\;} & X' \end{array}.$$

Furthermore, φ is a homotopy monomorphism if and only if ρ is a monomorphism of groups.

Proof. The composite $\varphi \circ f$ factors through BS' by Theorem 4(a), and the resulting homomorphism ρ is unique up to $\mathcal{F}'$-conjugacy by Theorem 4(b). We know that f and

f' are homotopy monomorphisms (Proposition 5). So if φ is a homotopy monomorphism, then so is $\varphi\circ f\simeq f'\circ B\rho$ (Lemma 5), and hence ρ is injective by Proposition 5(a).

Assume now that ρ is injective; we must show that φ is a homotopy monomorphism. Assume for simplicity that φ is a fibration, fix base points $x_0\in X$ and $x_0'=\varphi(x_0)\in X'$, set $F=\varphi^{-1}(x_0')$, and let $F\xrightarrow{\;\iota\;}X$ be the inclusion. For any map $\alpha\colon B\mathbb{Z}/p\longrightarrow F$, there is a homomorphism $\sigma\in\mathrm{Hom}(\mathbb{Z}/p,S)$ such that $\iota\circ\alpha=f\circ B\sigma$; then $f'\circ B(\rho\circ\sigma)\simeq\varphi\circ\iota\circ\alpha\simeq *$, and so $\rho\circ\sigma$ is the trivial homomorphism by Theorem 4. Since ρ is injective, this implies that σ is the trivial homomorphism, and thus that $\iota\circ\alpha\simeq *$. Hence $\mathrm{Map}_*(B\mathbb{Z}/p,F)$ is the fiber of the fibration

$$\mathrm{Map}_*(B\mathbb{Z}/p,X)_{\mathrm{const}}\xrightarrow{\;\varphi\circ-\;}\mathrm{Map}_*(B\mathbb{Z}/p,X')_{\mathrm{const}}.$$

If Y is the classifying space of any homotopy finite group, then $\mathrm{Map}(B\mathbb{Z}/p,Y)_{\mathrm{const}}\simeq Y$ by Theorem 4(c), and hence $\mathrm{Map}_*(B\mathbb{Z}/p,Y)_{\mathrm{const}}\simeq *$. Thus $\mathrm{Map}_*(B\mathbb{Z}/p,F)\simeq *$, and φ is a homotopy monomorphism. □

For any space X, we define categories $\mathcal{F}_p(X)$ and $\mathcal{L}_p^c(X)$ as follows. The objects in $\mathcal{F}_p(X)$ are the p-subgroups (P,α) of X, while the objects in $\mathcal{L}_p^c(X)$ are the p-centric subgroups. Morphisms are defined by setting

$$\mathrm{Mor}_{\mathcal{F}_p(X)}\big((P,\alpha),(Q,\beta)\big)=\{\varphi\in\mathrm{Hom}(P,Q)\,|\,\alpha\simeq\beta\circ B\varphi\}$$

and

$$\begin{aligned}&\mathrm{Mor}_{\mathcal{L}_p^c(X)}\big((P,\alpha),(Q,\beta)\big)\\&\qquad=\big\{(\varphi,[H])\,\big|\,\varphi\in\mathrm{Hom}(P,Q),\ [H]\in\mathrm{Mor}_{\pi(\mathrm{Map}(BP,X))}(\alpha,\beta\circ B\varphi)\big\}.\end{aligned}$$

Here, $\pi(\mathrm{Map}(BP,X))$ denotes the fundamental groupoid of the mapping space; thus H is a homotopy $BP\times I\longrightarrow X$ and $[H]$ is its homotopy class as a path relative to its endpoints. In both of these categories, the sets of objects (and of morphisms) are given the discrete topology.

Let $(S,\mathcal{F},\mathcal{L})$ be a homotopy finite group, and let $\pi\colon\mathcal{L}\longrightarrow\mathcal{F}^c$ be the projection functor. For each $P\le S$, let

$$\theta_P\colon\mathcal{B}(P)\longrightarrow\mathcal{L}$$

be the functor which sends o_P to P and sends a morphism $\check{g}$ (for $g\in P$) to $\widehat{g}\in\mathrm{Aut}_{\mathcal{L}}(P)$. For each $\varphi\in\mathrm{Hom}_{\mathcal{L}}(P,Q)$, let

$$\eta_\varphi\colon\theta_P\longrightarrow\theta_Q\circ\pi_{P,Q}(\varphi)$$

be the natural transformation of functors $\mathcal{B}(P)\longrightarrow\mathcal{L}$ which sends the object o_P to the morphism φ. Now define functors

$$\xi_{\mathcal{F}}\colon\mathcal{F}\longrightarrow\mathcal{F}_p(|\mathcal{L}|_p^\wedge)\qquad\text{and}\qquad\xi_{\mathcal{L}}\colon\mathcal{L}\longrightarrow\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$$

as follows. On objects, for all $P\le S$,

$$\xi_{\mathcal{F}}(P)=\xi_{\mathcal{L}}(P)=(P,|\theta_P|_p^\wedge).$$

And for each $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$,

$$\xi_{\mathcal{F}}(\varphi)=\varphi;$$

while for each morphism $\varphi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$,

$$\xi_{\mathcal{L}}(\varphi)=\big(\pi_{P,Q}(\varphi),[|\eta_\varphi|]\big),$$

where $|\eta_\varphi|$ is regarded as a homotopy $BP\times I\longrightarrow|\mathcal{L}|_p^\wedge$.

Define an isomorphism $(S,\mathcal{F},\mathcal{L}) \longrightarrow (S',\mathcal{F}',\mathcal{L}')$ of homotopy finite groups to consist of a triple $(\alpha,\alpha_{\mathcal{F}},\alpha_{\mathcal{L}})$, where

$$S \xrightarrow{\ \alpha\ } S', \qquad \mathcal{F} \xrightarrow{\ \alpha_{\mathcal{F}}\ } \mathcal{F}', \qquad \text{and} \qquad \mathcal{L} \xrightarrow{\ \alpha_{\mathcal{L}}\ } \mathcal{L}'$$

are isomorphisms of groups and categories such that $\alpha_{\mathcal{F}}(P)=\alpha_{\mathcal{L}}(P)=\alpha(P)$ for all $P\le S$, and such that they commute in the obvious way with the projections $\mathcal{L}\longrightarrow\mathcal{F}$ and the structure maps $P\longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$. The following theorem describes how a homotopy finite group $(S,\mathcal{F},\mathcal{L})$ is determined up to isomorphism by the homotopy type of $|\mathcal{L}|_p^\wedge$.

Theorem 5.7. *The following hold for any homotopy finite group $(S,\mathcal{F},\mathcal{L})$.*

(a) *The functor $\xi_{\mathcal{F}}\colon \mathcal{F}\longrightarrow \mathcal{F}_p(|\mathcal{L}|_p^\wedge)$ is an equivalence of categories.*

(b) *The functor $\xi_{\mathcal{L}}\colon \mathcal{L}\longrightarrow \mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$ is an equivalence of categories.*

Proof. We keep the above notation. Set $f=|\theta_S|_p^\wedge$. For all $P\le S$, let $i_P\in\mathrm{Inj}(P,S)$ be the inclusion, and note that $f\circ Bi_P\simeq|\theta_P|_p^\wedge$. So by Theorem 4 and Proposition 5, $\xi_{\mathcal{F}}$ and $\xi_{\mathcal{L}}$ both induce bijections on the sets of isomorphism classes of objects.

By Corollary 4, for all $P,Q\le S$,

$$\begin{aligned}\mathrm{Mor}_{\mathcal{F}_p(|\mathcal{L}|_p^\wedge)}\big((P,\theta_P),(Q,\theta_Q)\big) &\cong \big\{\varphi\in\mathrm{Inj}(P,Q)\,\big|\,|\theta_P|_p^\wedge\simeq|\theta_Q|_p^\wedge\circ B\varphi\big\}\\ &=\big\{\varphi\in\mathrm{Inj}(P,Q)\,\big|\,\exists\psi\in\mathrm{Hom}_{\mathcal{F}}(P,S),\ \psi=i_Q\circ\varphi\big\}=\mathrm{Hom}_{\mathcal{F}}(P,Q).\end{aligned}$$

So $\xi_{\mathcal{F}}$ induces bijections on morphism sets, and is thus an equivalence of categories.

It remains to show that $\xi_{\mathcal{L}}$ induces bijections on morphism sets. Fix a pair of $\mathcal{F}$-centric subgroups $P,Q\le S$, and consider the following commutative diagram:

$$\begin{array}{ccc}\mathrm{Mor}_{\mathcal{L}}(P,Q) & \xrightarrow{(\xi_{\mathcal{L}})_{P,Q}} & \mathrm{Mor}_{\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)}\big((P,\theta_P),(Q,\theta_Q)\big)\\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'}\\ \mathrm{Hom}_{\mathcal{F}}(P,Q) & \xrightarrow[\cong]{(\xi_{\mathcal{F}})_{P,Q}} & \mathrm{Mor}_{\mathcal{F}_p(|\mathcal{L}|_p^\wedge)}\big((P,\theta_P),(Q,\theta_Q)\big).\end{array}$$

Here, π' is the "forgetful" functor which is the identity on objects and sends a morphism $(\alpha,[H])$ to α. Since P is $\mathcal{F}$-centric, $\pi_1\big(\mathrm{Map}(BP,|\mathcal{L}|_p^\wedge)_{|\theta_P|}\big)\cong Z(P)$, and hence π' is the orbit map of a free action of $Z(P)$ on $\mathrm{Mor}_{\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)}\big((P,\theta_P),(Q,\theta_Q)\big)$ (by definition of $\mathcal{L}_p^c(-)$). By condition (A), π is the orbit map of a free $Z(P)$-action on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$. Also, $(\xi_{\mathcal{L}})_{P,Q}$ is easily checked to be equivariant, and hence is a bijection since the orbit map $(\xi_{\mathcal{F}})_{P,Q}$ is a bijection. □

Note in particular that by Theorem 5, the space $|\mathcal{L}|_p^\wedge$ determines the Frobenius system $\mathcal{F}$ and its associated $\mathcal{L}$-system $\mathcal{L}$. This will be made more precise in the next theorem.

If (S,f) is a Sylow p-subgroup of X, then $\mathcal{F}_{(S,f)}(X)$ denotes the Frobenius system over S defined by setting

$$\mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(P,Q)=\big\{\varphi\in\mathrm{Hom}(P,Q)\,\big|\,f|_{BP}\simeq f|_{BQ}\circ B\varphi\big\}$$

for each $P,Q\le S$.

We are now ready to prove the following characterization of classifying spaces of homotopy finite groups.

Theorem 5.8. *A p-complete space X is the classifying space of some homotopy finite group if and only if the following conditions hold:*

(a) *X has a centric Sylow p-subgroup $BS \xrightarrow{f} X$,*

(b) *the Frobenius system $\mathcal{F}_{(S,f)}(X)$ is saturated,*

(c) *there is a homotopy equivalence $X \simeq |\mathcal{L}_p^c(X)|_p^\wedge$, and*

(d) *$f|_{BP}$ is a centric map for each $\mathcal{F}_{(S,f)}(X)$-centric subgroup $P \leq S$.*

When these hold, $\mathcal{L}_p^c(X)$ is equivalent to an associated $\mathcal{L}$-system to $\mathcal{F}_{(S,f)}(X)$.

Proof. Assume first that $X = |\mathcal{L}|_p^\wedge$, where $\mathcal{L}$ is an associated $\mathcal{L}$-system to a Frobenius system $\mathcal{F}$ over S. Then conditions (a) and (d) hold by Proposition 5, condition (c) follows from Theorem 5(b), while (b) holds since $\mathcal{F}_S(X) \cong \mathcal{F}$ by Theorem 5(a).

Now assume that X is a p-complete space which satisfies conditions (a–d). Fix a Sylow subgroup (S, f) of X, set $\mathcal{F} = \mathcal{F}_{(S,f)}(X)$, and let $\mathcal{L}$ be the category whose objects are the $\mathcal{F}$-centric subgroups of S and where

$$\mathrm{Mor}_{\mathcal{L}}(P, Q) = \mathrm{Mor}_{\mathcal{L}_p^c(X)}\big((P, f|_{BP}), (Q, f|_{BQ})\big).$$

We will show that $(S, \mathcal{F}, \mathcal{L})$ is a homotopy finite group, and that $X \simeq |\mathcal{L}|_p^\wedge$.

By condition (d), for each $P \in \mathrm{Ob}(\mathcal{L})$, $(P, f|_{BP})$ is an object in $\mathcal{L}_p^c(X)$. Conversely, by Proposition 5(a,c), each object in $\mathcal{L}_p^c(X)$ is isomorphic to one of the form $(P, f|_{BP})$ for some $\mathcal{F}$-centric subgroup $P \leq S$. So if we identify $\mathcal{L}$ with the full subcategory of $\mathcal{L}_p^c(X)$ with objects the $(P, f|_{BP})$, then the inclusion $\mathcal{L} \subseteq \mathcal{L}_p^c(X)$ is an equivalence of categories, and by (c):

$$X \simeq |\mathcal{L}_p^c(X)|_p^\wedge \simeq |\mathcal{L}|_p^\wedge.$$

It remains to show that $\mathcal{L}$ is an associated $\mathcal{L}$-system to $\mathcal{F}$. Define $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ by sending $g \in P$ to the pair $(c_g, [H_g])$, where H_g is the homotopy

$$H_g \colon BP \times I \xrightarrow{|\eta_g|} BP \subseteq BS \xrightarrow{f} X$$

induced by the natural transformation of functors $\mathrm{Id} \xrightarrow{\eta_g} c_g$ which sends the object o_P in $\mathcal{B}(P)$ to the morphism $\widehat{g} \in \mathrm{Aut}_{\mathcal{L}}(P)$. Condition (B) is clear; and (A) holds by (d) and since $\mathrm{Map}(BP, X)_{f|_{BP}} \simeq BZ(P)$ for each P. Condition (C) means showing, for each $(\varphi, [H]) \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, that the following square commutes:

$$\begin{array}{ccc} P & \xrightarrow{(\varphi,[H])} & Q \\ \downarrow{\scriptstyle (c_g,[H_g])} & & \downarrow{\scriptstyle (c_{\varphi(g)},[H_{\varphi(g)}])} \\ P & \xrightarrow{(\varphi,[H])} & Q. \end{array}$$

Here $[H]$ is the homotopy class of a path H in $\mathrm{Map}(BP, X)$. Clearly, $\varphi \circ c_g = c_{\varphi(g)} \circ \varphi$. It remains to check that the following two paths in $\mathrm{Map}(BP, X)$ are homotopic:

$$i_P \xrightarrow{H} i_Q \circ B\varphi \xrightarrow{H_{\varphi(g)} \circ B\varphi} i_Q \circ B(\varphi \circ c_g) \quad \text{and} \quad i_P \xrightarrow{H_g} i_P \circ Bc_g \xrightarrow{H \circ Bc_g} i_Q \circ B(\varphi \circ c_g).$$

The map

$$F \colon BP \times I \times I \longrightarrow X \qquad \text{defined by} \qquad F(x, s, t) = H\big(|\eta_g|(x, t), s\big)$$

is such a homotopy, since

$$\begin{aligned} F(x,s,0) &= H(x,s), \\ F(x,1,t) &= f\circ B\varphi\circ|\eta_g|(x,t) = H_{\varphi(g)}(B\varphi(x),t), \\ F(x,0,t) &= f\circ|\eta_g|(x,t) = H_g(x,t), \qquad \text{and} \\ F(x,s,1) &= H(Bc_g(x),s). \end{aligned}$$ □

The above proof, together with Proposition 4, also shows that X satisfies conditions (a–c) if and only if $X \simeq |\mathcal{L}_0|^\wedge_p$ for some associated $\mathcal{L}$-system $\mathcal{L}_0$ to a full subcategory $\mathcal{F}_0 \subseteq \mathcal{F}^c$, where $\mathcal{F}$ is a saturated Frobenius system, and where $\mathcal{F}_0$ is such that any overgroup of a subgroup in $\mathcal{F}_0$ is also in $\mathcal{F}_0$.

6. The cohomology ring of a classifying space

Throughout this section, we fix a homotopy finite group $(S,\mathcal{F},\mathcal{L})$. All cohomology is taken with coefficients in $\mathbb{F}_p$. Since $|\mathcal{L}|$ is p-good by Proposition 1, $H^*(|\mathcal{L}|^\wedge_p) \cong H^*(|\mathcal{L}|)$.

We first recall some definitions and theorems due to Henn, Lannes, and Schwartz [HLS]. Their idea is to compare unstable algebras over the Steenrod algebra with profinite right $\mathrm{End}(V)$-sets for certain elementary abelian p-groups V.

Fix some $d \geq 1$, and set $V_d = (\mathbb{Z}/p)^d$. Let $\mathcal{A}$ be the mod p Steenrod algebra, and let $\mathcal{K}_d$ be the category of unstable $\mathcal{A}$-algebras of transcendence degree at most d. Let $\mathcal{PS}-\mathrm{End}(V_d)$ be the category of profinite sets with a continuous right action of $\mathrm{End}(V_d)$. Define functors

$$b_d\colon (\mathcal{PS}-\mathrm{End}(V_d))^{\mathrm{op}} \longrightarrow \mathcal{K} \qquad \text{and} \qquad s_d\colon \mathcal{K}_d \longrightarrow (\mathcal{PS}-\mathrm{End}(V_d))^{\mathrm{op}}$$

by setting

$$b_d(S) = \mathrm{Hom}_{\mathcal{PS}-\mathrm{End}(V_d)}(S, H^*(V_d)) \qquad \text{and} \qquad s_d(K) = \mathrm{Hom}_{\mathcal{K}}(K, H^*(V_d)).$$

There are obvious natural transformations of functors $\mathrm{Id} \longrightarrow b_d\circ s_d$ and $\mathrm{Id} \longrightarrow s_d\circ b_d$, analogous to the inclusion of a vector space in its double dual. In fact, it is not hard to see that b_d is a right adjoint to s_d.

Theorem 6.1 ([HLS])**.** *Fix some $d \geq 1$.*

(a) *The natural transformation $S \longrightarrow (s_d\circ b_d)(S)$ is an isomorphism for all profinite* $\mathrm{End}(V_d)$*-sets S, and the natural transformation $K \longrightarrow (b_d\circ s_d)(K)$ is an F-isomorphism for all unstable $\mathcal{A}$-algebras K of transcendence degree d. Thus s_d and b_d define an equivalence of categories*

$$\mathcal{K}_d/\mathcal{N}il_d \longrightarrow (\mathcal{PS}-\mathrm{End}(V_d))^{\mathrm{op}}.$$

(b) *If K is a noetherian unstable $\mathcal{A}$-algebra, then $s_d(K)$ is a noetherian* $\mathrm{End}(V_d)$*-set.*

(c) *If S is a noetherian* $\mathrm{End}(V_d)$*-set, then $b_d(S)$ is a noetherian algebra.*

Proof. Point (a) is shown in [HLS, Theorem II.2.4], and points (b) and (c) in [HLS, Theorem II.7.1]. □

Recall that for any finite p-group Q, we define

$$\mathrm{Rep}(Q,\mathcal{L}) = \mathrm{Hom}(Q,S)/\sim,$$

where $\rho \sim \rho'$ if there is some $\chi \in \mathrm{Iso}_{\mathcal{F}}(\rho Q, \rho' Q)$ such that $\rho' = \chi\circ\rho$.

Proposition 6.2 ([HLS]). *The following hold for any elementary abelian p-group V.*

(a) *For each right* $\operatorname{End}(V)$*-set S and each $s \in S$, there is a unique subgroup* $\operatorname{Ker}(s) \le V$ *such that for all $\alpha \in \operatorname{End}(V)$, $s \in S\alpha$ if and only if* $\operatorname{Ker}(s) \ge \operatorname{Ker}(\alpha)$.

(b) *For each p-group P, and each $\rho \in \operatorname{Rep}(V,P)$, $\operatorname{Ker}(\rho)$ is the same whether one regards ρ as a homomorphism or as an element of the* $\operatorname{End}(V)$*-set* $\operatorname{Rep}(V,P)$.

(c) *For each $\rho \in \operatorname{Rep}(V,\mathcal{L})$, $\operatorname{Ker}(\rho)$ is the same whether one regards ρ as a homomorphism or as an element of the* $\operatorname{End}(V)$*-set* $\operatorname{Rep}(V,\mathcal{L})$.

Proof. To prove (a), note first that if $\alpha,\alpha' \in \operatorname{End}(V)$ are such that $\operatorname{Ker}(\alpha) \le \operatorname{Ker}(\alpha')$, then $\alpha' = \beta\alpha$ for some $\beta \in \operatorname{End}(V)$, and hence $S\alpha' \subseteq S\alpha$. So the only problem is to show that if $s \in S\alpha \cap S\beta$, then there is some $\gamma \in \operatorname{End}(V)$ such that $s \in S\gamma$ and $\operatorname{Ker}(\gamma) = \operatorname{Ker}(\alpha)\cdot\operatorname{Ker}(\beta)$. And upon replacing α and β by appropriate idempotents with the same kernel, we can take $\gamma = \alpha\beta$.

Points (b) and (c) are clear. For example, for given elements $\rho \in \operatorname{Rep}(V,P)$ and $\alpha \in \operatorname{End}(V)$, there is $\rho' \in \operatorname{Rep}(V,P)$ such that $\rho = \rho'\alpha$ if and only if $\operatorname{Ker}(\alpha)$ is contained in the kernel of ρ as a homomorphism (rather, an equivalence class of homomorphisms). □

A map $\varphi\colon S \longrightarrow S'$ of $\operatorname{End}(V)$-sets is said to *preserve kernels* if $\operatorname{Ker}(s) = \operatorname{Ker}(\varphi(s))$ for each $s \in S$.

Theorem 6.3 ([HLS]). *The following hold for any morphism $\varphi\colon K \longrightarrow L$ of unstable $\mathcal{A}$-algebras.*

(a) *If L is a finitely generated K-module via φ, then $s_d(\varphi)$ preserves kernels for all d.*

(b) *If K and L are noetherian, L has transcendence degree d, and $s_d(\varphi)$ preserves kernels, then L is a finitely generated K-module via φ.*

Proof. See [HLS, Proposition II.7.8]. □

For any elementary abelian p-group V, a right $\operatorname{End}(V)$-set S is called *noetherian* if $|S| < \infty$ and $\operatorname{Ker}(s\alpha) = \alpha^{-1}(\operatorname{Ker}(s))$ for all $s \in S$ and $\alpha \in \operatorname{End}(V)$.

Lemma 6.4. *For any elementary abelian p-group V, $\operatorname{Rep}(V,\mathcal{L})$ is noetherian as an* $\operatorname{End}(V)$*-set.*

Proof. Clearly, $\operatorname{Rep}(V,\mathcal{L})$ is a finite set. For all $\rho \in \operatorname{Rep}(V,\mathcal{L})$ and $\alpha \in \operatorname{End}(V)$, $\alpha^{-1}(\operatorname{Ker}(\rho)) = \operatorname{Ker}(\rho\alpha)$ by Proposition 6(c), since this relation is clear for kernels of homomorphisms. □

Lemma 6.5. *The cohomology algebra $H^*(|\mathcal{L}|)$ is the limit of a spectral sequence of $H^*(|\mathcal{L}|)$-modules, where each column in the E_1-term is a finite sum of copies of $H^*(BP)$ for subgroups $P \le S$, and where the E_2-term has only a finite number of nonzero columns. Furthermore, if $d = \operatorname{rk}_p(S)$, then $H^*(|\mathcal{L}|)$ has transcendence degree at most d.*

Proof. By Proposition 2, $|\mathcal{L}| \simeq \underrightarrow{\operatorname{hocolim}}(\widetilde{B})$, for some functor $\widetilde{B}\colon \mathcal{F}^c \longrightarrow \mathsf{Top}$ which sends each P to a space with the homotopy type of BP. Hence, in the spectral sequence for the cohomology of the homotopy colimit, each column in the E_1-term is a sum of rings $H^*(BP)$ for subgroups $P \le S$. Also, $E_2^{p*} \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^p(H^*(B(-)))$, and hence there are only finitely many nonzero columns by Corollary 3.

In particular, each column in the E_1-term has polynomial growth of degree at most $d-1$ by [Qu, 2.5 & 7.7] (applied with $X = \text{pt}$). So the same holds for each column in the E_∞-term, and hence for $H^*(|\mathcal{L}|)$ itself since there are only finitely many nonzero columns. It follows that $H^*(|\mathcal{L}|)$ has transcendence degree at most d. □

As an easy consequence of Lemma 6, we have:

Corollary 6.6. *For any homotopy finite group $(S, \mathcal{F}, \mathcal{L})$, the induced homomorphism*

$$R_{\mathcal{L}} \colon H^*(|\mathcal{L}|; \mathbb{F}_p) \longrightarrow \varprojlim_{P \in \mathcal{O}^c(\mathcal{F})} H^*(BP; \mathbb{F}_p)$$

is an F-isomorphism, in the sense that its kernel is nilpotent, and some power of each element in the target group lies in the image.

Let $\mathcal{F}^e$ denote the full subcategory of $\mathcal{F}$ whose objects are the nontrivial elementary abelian p-subgroups of S.

Proposition 6.7. *The natural map*

$$q_{\mathcal{L}} \colon H^*(|\mathcal{L}|) \longrightarrow \varprojlim_{E \in \mathcal{F}^e} H^*(BE)$$

is an F-isomorphism, and $\varprojlim_{E \in \mathcal{F}^e} H^*(BE)$ is a noetherian ring.

Proof. Let d be the p-rank of S; we have just seen that $H^*(|\mathcal{L}|)$ has transcendence degree at most d. Fix an elementary abelian subgroup V of rank d. Since $|\mathcal{L}|_p^\wedge$ is p-complete by Proposition 1,

$$\mathrm{Hom}_{\mathcal{K}}(H^*(|\mathcal{L}|), H^*(BV)) \cong [BV, |\mathcal{L}|_p^\wedge]$$

by [La, Theorem 0.5]. By Corollary 4,

$$[BV, |\mathcal{L}|_p^\wedge] \cong \mathrm{Rep}(V, \mathcal{L}) \cong \varinjlim_{P \in \mathcal{F}} \mathrm{Hom}(V, P) \cong \varinjlim_{E \in \mathcal{F}^e} \mathrm{Hom}(V, E),$$

and

$$\begin{aligned} \mathrm{Map}_{\mathrm{End}(V)}(\mathrm{Rep}(V, \mathcal{L}), H^*(BV)) &= \mathrm{Map}_{\mathrm{End}(V)}(\varinjlim_{E \in \mathcal{F}^e} \mathrm{Hom}(V, E), H^*(BV)) \\ &\cong \varprojlim_{E \in \mathcal{F}^e} \mathrm{Map}_{\mathrm{End}(V)}(\mathrm{Hom}(V, E), H^*(BV)) \cong \varprojlim_{E \in \mathcal{F}^e} H^*(BE). \end{aligned}$$

Thus, in terms of the functors defined above,

$$s_d(H^*(|\mathcal{L}|)) \cong \mathrm{Rep}(V, \mathcal{L}) \qquad \text{and} \qquad b_d(\mathrm{Rep}(V, \mathcal{L})) \cong \varprojlim_{E \in \mathcal{F}^e} H^*(BE).$$

By Theorem 6(a), $K \longrightarrow b_d \circ s_d(K)$ is an F-isomorphism for any unstable algebra over the Steenrod algebra of transcendence degree at most d. In particular, this applies when $K = H^*(|\mathcal{L}|)$, and thus $q_{\mathcal{L}}$ is an F-equivalence. The last statement follows from Lemma 6, together with Theorem 6(c). □

Lemma 6.8. *There is a subalgebra K of $H^*(|\mathcal{L}|)$ which is noetherian, and such that for each object P of $\mathcal{L}$, the composite*

$$K \longrightarrow H^*(|\mathcal{L}|) \xrightarrow{f|_{BP}} H^*(BP)$$

makes $H^*(BP)$ into a finitely generated K-module.

[Replace K by A??]

Proof. Set $d = \mathrm{rk}_p(S)$, and fix an elementary abelian p-group V of rank d. By Lemma 6, $\mathrm{Rep}(V, \mathcal{L})$ is noetherian as an $\mathrm{End}(V)$-set. Furthermore, given an object P of $\mathcal{L}$, the inclusion $P \leq S$ induces a map of $\mathrm{End}(V)$-sets $\mathrm{Rep}(V, P) \longrightarrow \mathrm{Rep}(V, \mathcal{L})$. And this map preserves kernels, since any $\rho\colon V \longrightarrow P$ has the same kernel whether considered as a homomorphism to P or one to S.

Fix a finite set of algebra generators $x_1, \ldots, x_n$ of $\varprojlim_{E \in \mathcal{F}^e} H^*(BE)$, which is noetherian by Proposition 6. Since $q_{\mathcal{L}}$ is an F-isomorphism, there are elements $y_1, \ldots, y_n \in H^*(|\mathcal{L}|)$ such that $q_{\mathcal{L}}(y_i) = (x_i)^{p^k}$ for some fixed $k > 0$. Since $\mathrm{Ker}(q_{\mathcal{L}})$ is nilpotent, we can find $\ell \geq 0$ such that the subalgebra of $H^*(|\mathcal{L}|)$ generated by $y_1^{p^\ell}, \ldots, y_n^{p^\ell}$, is closed under the action of the Steenrod operations. Let K be the subalgebra of $H^*(|\mathcal{L}|)$ generated by these elements. It is noetherian by definition, and the inclusion $K \longrightarrow H^*(|\mathcal{L}|)$ is an F-isomorphism. In particular, we have isomorphisms of $\mathrm{End}(V)$-sets

$$\mathrm{Hom}_{\mathcal{K}}(K, H^*(BV)) \cong \mathrm{Hom}_{\mathcal{K}}(H^*(|\mathcal{L}|), H^*(BV)) \cong \mathrm{Rep}(V, \mathcal{L}).$$

The morphism of $\mathrm{End}(V)$-sets induced by the composite

$$K \longrightarrow H^*(|\mathcal{L}|) \xrightarrow{|i_P|^*} H^*(BP),$$

is thus isomorphic to the map $\mathrm{Rep}(V, P) \longrightarrow \mathrm{Rep}(V, \mathcal{L})$, which has been seen to preserve kernels. Since K and $H^*(BP)$ are both noetherian, $H^*(BP)$ is finitely generated as a K-module by Theorem 6(b). □

Theorem 6.9. *The cohomology algebra $H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p)$ of the classifying space of any homotopy finite group $(S, \mathcal{F}, \mathcal{L})$ is noetherian.*

Proof. Let $K \subseteq H^*(|\mathcal{L}|)$ be the noetherian subalgebra of Lemma 6. By Lemma 6, there is a spectral sequence of $H^*(|\mathcal{L}|)$-modules such that each column E_1^{p*} is finitely generated as a K-module, and such that E_2 has only finitely many columns. It follows that E_∞, and hence $H^*(|\mathcal{L}|)$, are finitely generated as K-modules. And thus $H^*(|\mathcal{L}|)$ is a noetherian algebra. □

One consequence of Theorem 6 is another criterion for a map between classifying spaces of homotopy finite groups to be a homotopy monomorphism.

Corollary 6.10. *Let X and X' be classifying spaces of homotopy finite groups. Then a map $\varphi\colon X \longrightarrow X'$ is a homotopy monomorphism if and only if $H^*(X)$ is finitely generated as a module over $H^*(X')$ via φ.*

Proof. Assume that $X = |\mathcal{L}|_p^\wedge$ and $X' = |\mathcal{L}'|_p^\wedge$, where $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are homotopy finite groups. Let $BS \xrightarrow{f} X$ and $BS' \xrightarrow{f'} X'$ denote the inclusions. According to Proposition 5, there is a homomorphism $\rho\colon S \longrightarrow S'$, unique up to $\mathcal{F}'$-conjugacy, such that the following square commutes up to homotopy:

$$\begin{array}{ccc} BS & \xrightarrow{\ f\ } & X \\ \big\downarrow{\scriptstyle B\rho} & & \big\downarrow{\scriptstyle \varphi} \\ BS' & \xrightarrow{\ f'\ } & X' \end{array} \qquad (1)$$

and φ is a homotopy monomorphism if and only if ρ is a monomorphism of groups.

Let V any elementary abelian p-group. Then φ and ρ induce a commutative diagram of $\mathrm{End}(V)$-sets

$$\begin{array}{ccccc} \mathrm{Rep}(V,\mathcal{L}) & \xrightarrow{\cong} & [BV,X] & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{K}}(H^*(X),H^*(BV)) \\ \downarrow{\scriptstyle\rho_\sharp} & & \downarrow{\scriptstyle\varphi_\sharp} & & \downarrow{\scriptstyle(\varphi^*)^\sharp} \\ \mathrm{Rep}(V,\mathcal{L}') & \xrightarrow{\cong} & [BV,X'] & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{K}}(H^*(X'),H^*(BV)) \end{array}$$

where $\rho_\sharp$ maps the class of $V \xrightarrow{\sigma} S$ in $\mathrm{Rep}(V,\mathcal{L})$ to the class of the composite $\rho\circ\sigma$ in $\mathrm{Rep}(V,\mathcal{L}')$, and is well defined by Theorem 5.

Now, if $\rho\colon S \longrightarrow S'$ is a monomorphism, then $\rho_\sharp$ clearly preserves kernels (as a map of $\mathrm{End}(V)$-sets), and hence $(\varphi^*)^\sharp$ also does so. Since $H^*(X)$ and $H^*(X')$ are both noetherian by Theorem 6, $H^*(X)$ is finitely generated as a $H^*(X')$-module via φ^* by Theorem 6(b).

Conversely, assume that $H^*(X)$ is finitely generated as a $H^*(X')$-module via φ^*. By Lemma 6, $H^*(BS)$ is finitely generated as a $H^*(X)$-module via f^*. Thus $H^*(BS)$ is also finitely generated as a $H^*(X')$-module, and hence as a $H^*(BS')$-module via the commutative square (1). Therefore $\rho\colon S \longrightarrow S'$ must be a monomorphism (cf. [BLO, Lemma 2.3]). □

One obvious question to ask is whether or not the cohomology of the classifying space of a homotopy finite group $(S,\mathcal{F},\mathcal{L})$ is always isomorphic to the "ring of invariants" in $H^*(BS;\mathbb{F}_p)$ determined by the Frobenius system $\mathcal{F}$, in the same way that this holds for the cohomology of a finite group. In the next theorem, we show that this is true with certain additional hypotheses on $\mathcal{F}$.

We need to work with (S,S)-bisets: sets which have left and right actions of S which commute with each other. We first establish some notation. If $P \le S$ and $\varphi \in \mathrm{Hom}(P,S)$, then $S \times_{(P,\varphi)} S$ denotes the biset

$$S \times_{(P,\varphi)} S = (S\times S)/{\sim}, \qquad \text{where} \qquad (x,gy) \sim (x\varphi(g),y) \quad \text{for } x,y\in S,\ g\in P.$$

Then $S \times_{(P,\varphi)} S$ is free as a left S-set (and also as a right S-set if φ is injective), and every (S,S)-biset with free left action is a disjoint union of bisets of this form. For each finite (S,S)-biset B whose left S-action is free, define an endomorphism $[B]$ of $H^*(BS)$ as follows. For each $P \le S$ and each $\varphi \in \mathrm{Hom}(P,S)$, set

$$[S \times_{(P,\varphi)} S] = \Big(H^*(BS) \xrightarrow{\varphi^*} H^*(BP) \xrightarrow{\mathrm{trf}_P} H^*(BS)\Big),$$

where trf_P denotes the transfer map. And if $B = \coprod_{i=1}^k B_i$ is a disjoint union of bisets B_i of this form, then $[B] = \sum_{i=1}^k [B_i]$.

If B is an (S,S)-biset, then for $P \le S$ and $\varphi \in \mathrm{Inj}(P,S)$, we let $B|_{(P,S)}$ denote the restriction of B to a (P,S)-biset, and let $B|_{(\varphi,S)}$ denote the (P,S)-biset where the left P-action is induced by φ.

Theorem 6.11. *Fix a homotopy finite group $(S,\mathcal{F},\mathcal{L})$. Assume that there is an (S,S)-biset Ω with the following properties:*

(a) *Each indecomposable component of Ω is of the form $S \times_{(P,\varphi)} S$ for some $P \le S$ and some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$.*

(b) *For each* $P \le S$ *and each* $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, $\Omega|_{(P,S)}$ *and* $\Omega|_{(\varphi,S)}$ *are isomorphic as* (P,S)*-bisets.*

(c) $|\Omega|/|S|$ *is prime to* p.

Then the natural homomorphism

$$R_{\mathcal{L}}\colon H^*(|\mathcal{L}|^\wedge_p;\mathbb{F}_p) \xrightarrow{\ \cong\ } \varprojlim_{P\in\mathcal{O}^c(\mathcal{F})} H^*(BP;\mathbb{F}_p)$$

is an isomorphism.

Note in particular, when $\mathcal{F}$ is the Frobenius system of a finite group G, that $\Omega = G$ (considered as an (S,S)-biset) satisfies the above hypotheses.

Proof. All cohomology is assumed to be with coefficients in $\mathbb{F}_p$. Set

$$R = \varprojlim_{P\in\mathcal{O}^c(\mathcal{F})} H^*(BP)$$

for short, regarded as a subring of $H^*(BS)$.

We want to apply [DW1, Theorem 1.2], to the inclusion of algebras $R \subseteq H^*(BS)$. The algebra $H^*(BS)$ has "nontrivial center" in the sense of [DW1, §4] since S has nontrivial center. Also, by Theorem 6, the rings R and $H^*(BS)$ are both noetherian, and hence finitely generated as algebras. And by Lemma 6 (applied with $P = S$), $H^*(BS)$ is finitely generated as an R-module.

It remains to check that condition (2) in [DW1, Theorem 1.2] holds, by showing that $[\Omega] \in \mathrm{End}(H^*(BS))$ defines a splitting to R with the right properties. For any $r \in R$ and any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, $\varphi^*(r) = \mathrm{Res}_P(r)$ (by definition of R as an inverse limit), and in particular

$$[S\times_{(P,\varphi)} S](r) = \mathrm{trf}_P(\varphi^*(r)) = \mathrm{trf}_P\circ\mathrm{Res}_P(r) = [S{:}P]\cdot r = \tfrac{|S\times_{(P,\varphi)}S|}{|S|}\cdot r.$$

Since Ω is a disjoint union of such bisets by (a), this shows that $[\Omega](r) = (|\Omega|/|S|)\cdot r$. And since $|\Omega|/|S|$ is invertible in $\mathbb{F}_p$ by (c), $[\Omega]|_R$ is multiplication by an element of $\mathbb{F}_p^*$. Furthermore, for all $r \in R$ and all $x \in H^*(BS)$,

$$[S\times_{(P,\varphi)} S](rx) = \mathrm{trf}_P\big(\mathrm{Res}_P(r)\cdot\varphi^*(x)\big) = r\cdot\mathrm{trf}_P(\varphi^*(x)) = r\cdot[S\times_{(P,\varphi)} S](x),$$

and hence $[\Omega]$ is R-linear. Finally, $\mathrm{Im}([\Omega]) \subseteq R$ by (b), and this shows that $[\Omega]$ is an R-linear splitting of the inclusion $R \subseteq H^*(BS)$. And $[\Omega]$ is a morphism of modules over the Steenrod algebra since the φ^* and the transfer homomorphisms are all such morphisms.

We have now shown that the inclusion $R \subseteq H^*(BS)$ satisfies the hypotheses of [DW1, Theorem 1.2]. Hence

$$R \xrightarrow{\ \cong\ } \varprojlim_{\mathbf{A}(R)}{}^0(\alpha_R) = \varprojlim_{\mathbf{A}(R)} \alpha_R \tag{1}$$

is an isomorphism, and

$$\varprojlim_{\mathbf{A}(R)}{}^i(\alpha_R) = 0 \tag{2}$$

for all $i > 0$. It remains to explain what this means, and to show that this implies that $R \cong H^*(|\mathcal{L}|^\wedge_p)$.

Let $\mathcal{K}$ be the category of unstable algebras over the mod p Steenrod algebra. For any K in $\mathcal{K}$, $\mathbf{A}(K)$ denotes the category whose objects are the pairs (V,f), where

$V \neq 0$ is an elementary abelian p-group and $f \in \mathrm{Mor}_{\mathcal{K}}(K, H^*(BV))$ makes $H^*(BV)$ into a finitely generated K-module. A morphism in $\mathbf{A}(K)$ from (V, f) to (V', f') is a monomorphism $V \xrightarrow{\varphi} V'$ such that $\varphi^* f' = f$. The functor $\alpha_K \colon \mathbf{A}(K) \longrightarrow \mathcal{K}$ is defined by setting $\alpha_K(V, f) = T_V(K; f)$, where T_V is Lannes's T-functor and $T_V(K; f)$ is the component in $T_V(K)$ of $f \in T_V^0(K) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*(BV))$.

By Corollary 6, the natural map $H^*(|\mathcal{L}|) \xrightarrow{R_{\mathcal{L}}} R$ is an F-isomorphism. Hence, by [**?????**], $R_{\mathcal{L}}$ induces an equivalence of categories $\mathbf{A}(R) \simeq \mathbf{A}(H^*(|\mathcal{L}|))$, and an natural isomorphism of functors $\alpha_R \cong \alpha_{H^*(|\mathcal{L}|)}$ over these categories. Furthermore, by Theorem 4, there is a homotopy equivalence

$$|\mathcal{L}| \simeq \underset{E \in \mathcal{F}^e}{\underrightarrow{\mathrm{hocolim}}} \mathrm{Map}(BE, |\mathcal{L}|^\wedge_p)_{\mathrm{incl}}.$$

Since $\mathrm{Map}(BE, |\mathcal{L}|^\wedge_p)$ is p-complete by Theorem 4,

$$H^*(\mathrm{Map}(BE, |\mathcal{L}|^\wedge_p)_{\mathrm{incl}}) \cong T_E(H^*(|\mathcal{L}|); \mathrm{incl})$$

by [La, Theorem 3.3.2]. So there is a spectral sequence

$$E_2^{i*} = \underset{E \in \mathcal{F}^e}{\underleftarrow{\lim}}{}^i (T_E(H^*(|\mathcal{L}|); \mathrm{incl})). \tag{3}$$

For any space X, let $\mathbf{A}(X)$ denote the category whose objects are the pairs (E, f) where $E \neq 1$ is elementary abelian and $f \colon BE \longrightarrow X$ is a homotopy monomorphism, and where a morphism from (E, f) to (E', f') is a monomorphism $E \xrightarrow{\varphi} E'$ such that $f' \circ B\varphi \simeq f$. Then

$$\mathcal{F}^e \simeq \mathbf{A}(|\mathcal{L}|^\wedge_p) \simeq \mathbf{A}(H^*(|\mathcal{L}|)) \simeq \mathbf{A}(R):$$

the first equivalence by Corollaries 4 and 6, the second by [La, ??], and the third since R and $H^*(|\mathcal{L}|)$ are F-isomorphic. Points (1) and (2) now apply to show that the spectral sequence of (3) collapses, and that

$$H^*(|\mathcal{L}|) \cong \underset{E \in \mathcal{F}^e}{\underleftarrow{\lim}}\, T_E(H^*(|\mathcal{L}|); \mathrm{incl}) \cong \underset{\mathbf{A}(R)}{\underleftarrow{\lim}}\, (\alpha_R) \cong R. \qquad \square$$

The above theorem was motivated by recent, still unpublished work of Markus Linckelmann and Peter Webb. They have shown that in many cases, the Frobenius system of a block has a biset which satisfies hypotheses (a–c) in Theorem 6. Thus, whenever the Frobenius system of a block has an associated $\mathcal{L}$-system, then the cohomology of the nerve of that $\mathcal{L}$-system is the cohomology of the block as defined by Linckelmann [????].

7. Spaces of self equivalences

We first recall some definitions from [BLO]. For any space X, $\mathrm{Aut}(X)$ denotes the monoid of self homotopy equivalences of X, $\mathrm{Out}(X) = \pi_0(\mathrm{Aut}(X))$ is the group of homotopy classes of equivalences, and $\mathcal{A}ut(X)$ is the fundamental groupoid of $\mathrm{Aut}(X)$. And for any discrete category $\mathcal{C}$, $\mathcal{A}ut(\mathcal{C})$ is the category whose objects are the self equivalences of $\mathcal{C}$ and whose morphisms are the natural isomorphisms between self equivalences, and $\mathrm{Out}(\mathcal{C}) = \pi_0(|\mathcal{A}ut(\mathcal{C})|)$ is the group of isomorphism classes of self equivalences. Both $\mathcal{A}ut(X)$ and $\mathcal{A}ut(\mathcal{C})$ are discrete categories. They are also strict monoidal categories, in the sense that composition defines a strictly associative functor

$$\mathcal{A}ut(-) \times \mathcal{A}ut(-) \longrightarrow \mathcal{A}ut(-)$$

with strict identity. The nerve of each $\mathcal{A}ut(X)$ or $\mathcal{A}ut(\mathcal{C})$ is thus a simplicial monoid, and $|\mathcal{A}ut(-)|$ is a topological monoid.

Recall that part of the structure of an associated $\mathcal{L}$-system $\mathcal{L}$ is a homomorphism $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each P in $\mathcal{L}$. We write $P_\delta = \mathrm{Im}(\delta_P)$, which we think of as a "distinguished subgroup" of $\mathrm{Aut}_{\mathcal{L}}(P)$ which can be identified with P. For the purposes of this paper, an equivalence of categories $\mathcal{L} \xrightarrow{\psi} \mathcal{L}$ will be called *isotypical* if for each P, $\psi_{P,P}$ sends the subgroup $P_\delta \le \mathrm{Aut}_{\mathcal{L}}(P)$ to the subgroup $\psi P_\delta \le \mathrm{Aut}_{\mathcal{L}}(\psi P)$. (This will be seen in Lemma 7 to be equivalent to the definition in [BLO].) Clearly, any equivalence which is naturally isomorphic to an isotypical equivalence is itself isotypical, and any inverse to an isotypical equivalence (inverse up to natural isomorphism of functors) is also isotypical. Thus, the subcategory $\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})$ is a union of connected components of $\mathcal{A}ut(\mathcal{L})$, and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is a subgroup of $\mathrm{Out}(\mathcal{L})$.

[Better notation for distinguished subgroup??]

The main result of this section is the following theorem:

Theorem 7.1. *Fix a homotopy finite group $(S, \mathcal{F}, \mathcal{L})$. Then $\mathrm{Aut}(|\mathcal{L}|^\wedge_p)$ and $|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|$ are equivalent as topological monoids in the sense that their classifying spaces are homotopy equivalent. In particular,*

$$\mathrm{Out}(|\mathcal{L}|^\wedge_p) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \qquad \text{and} \qquad \pi_i(\mathrm{Aut}(|\mathcal{L}|^\wedge_p)) \cong \begin{cases} \varprojlim_{\mathcal{O}^c(\mathcal{F})}^0(\mathcal{Z}) & \text{if } i = 1 \\ 0 & \text{if } i \ge 2. \end{cases}$$

Throughout the rest of the section, we fix a homotopy finite group $(S, \mathcal{F}, \mathcal{L})$, and let $\mathcal{L} \xrightarrow{\pi} \mathcal{F}^c$ denote the canonical projection. For any morphism α in $\mathcal{L}$, we set $[\alpha] = \pi(\alpha)$ for short.

Lemma 7.2. *Let $F\colon \mathcal{L} \longrightarrow \mathsf{Gr}$ denote the forgetful functor. Then for any equivalence $\mathcal{L} \xrightarrow{\psi} \mathcal{L}$, ψ is isotypical if and only if there is a natural isomorphism $F \xrightarrow[\cong]{\Psi} F \circ \psi$ of functors $\mathcal{L} \longrightarrow \mathsf{Gr}$. Also, if ψ is isotypical, and if $\psi_P\colon P \longrightarrow \psi P$ denotes the restriction of $\psi_{P,P}$ under the identifications $P = P_\delta$ and $\psi P = \psi P_\delta$, then $(P \mapsto \psi_P)$ is a natural isomorphism of functors $F \xrightarrow{\cong} F \circ \psi$.*

Proof. To simplify notation, we write $P' = \psi P$ for any P in $\mathcal{L}$. Assume first that there is a natural isomorphism $\Psi\colon F \xrightarrow{\cong} F \circ \psi$ of functors. Fix P, let $g \in P$ be any element, and set $\alpha = \psi_{P,P}(\widehat{g}) \in \mathrm{Aut}_{\mathcal{L}}(P')$. Then Ψ sends P to an isomorphism $\Psi_P \in \mathrm{Iso}(P, P')$ of groups, and

$$c_{\Psi_P(g)} \circ \Psi_P = \Psi_P \circ c_g = [\alpha] \circ \Psi_P :$$

the first equality holds when Ψ_P is replaced by any homomorphism $P \longrightarrow P'$, and the second holds by the naturality of Ψ with respect to $(P \xrightarrow{\widehat{g}} P)$. Thus $[\alpha] = c_{\Psi_P(g)}$, so $\alpha \overset{\text{def}}{=} \psi_{P,P}(g) = \widehat{x}$ for some $x \in P'$ such that $x^{-1}\Psi_P(g) \in Z(P')$. In particular, $\psi_{P,P}(\widehat{g}) \in P'_\delta$, and thus $\psi_{P,P}(P_\delta) \le P'_\delta$. And equality now holds since the distinguished subgroups are abstractly isomorphic (and $\psi_{P,P}$ is an isomorphism).

Now assume that $\psi_{P,P}(P_\delta) = P'_\delta$ for each P, and let $\psi_P\colon P \xrightarrow{\cong} P'$ be the restriction of $\psi_{P,P}$ under the identifications $P \cong P_\delta$ and $P' \cong P'_\delta$. We must show that $(P \mapsto \psi_P)$ is natural as an isomorphism of functors $F \longrightarrow F \circ \psi$; i.e., that

$$\psi_Q \circ [\alpha] = [\beta] \circ \psi_P \in \mathrm{Hom}(P, Q') \tag{1}$$

for any morphism $P \xrightarrow{\alpha} Q$ in $\mathcal{L}$, where $\beta = \psi_{P,Q}(\alpha)$. Fix $g \in P$ and set $h = [\alpha](g)$; then ψ sends $\alpha \circ \widehat{g} = \widehat{h} \circ \alpha$ in $\mathcal{L}$ to

$$\beta \circ \widehat{\psi_P(g)} = \widehat{\psi_Q(h)} \circ \beta \in \mathrm{Mor}_{\mathcal{L}}(P', Q').$$

Upon comparing this with condition (C) (and the uniqueness property shown in Lemma 1), we see that

$$\psi_Q([\alpha](g)) = \psi_Q(h) = [\beta](\psi_P(g))$$

for all $g \in P$, and this proves (1). □

From now on, for any isotypical equivalence $\mathcal{L} \xrightarrow[\simeq]{\psi} \mathcal{L}$ and any $\mathcal{F}$-centric $P \leq S$, we let $\psi_P \colon P \xrightarrow{\cong} \psi P$ denote the isomorphism obtained by restricting $\psi_{P,P}$.

We next define functors

$$\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\quad R \quad} \mathcal{A}ut(|\mathcal{L}|_p^\wedge) \xrightarrow{\quad L \quad} \mathcal{A}ut(\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)) \xrightarrow{\quad c_\xi \quad} \mathcal{A}ut(\mathcal{L})$$

whose composite will later be seen to be homotopic to the inclusion. The functor R is easily defined: it sends an object $\mathcal{L} \xrightarrow[\simeq]{\psi} \mathcal{L}$ — to the homotopy equivalence $|\mathcal{L}|_p^\wedge \xrightarrow{|\psi|_p^\wedge} |\mathcal{L}|_p^\wedge$, and sends a natural isomorphism of functors to its realization as a homotopy between the induced maps.

On objects, L sends a self homotopy equivalence $|\mathcal{L}|_p^\wedge \xrightarrow{f} |\mathcal{L}|_p^\wedge$ to the functor $\mathcal{L}_p^c(f)$ induced by composition with f. If $F : |\mathcal{L}|_p^\wedge \times I \longrightarrow |\mathcal{L}|_p^\wedge$ is a homotopy, representing a morphism in $\mathcal{A}ut(|\mathcal{L}|_p^\wedge)$ from f to f', then $L(F)$ is defined to be the natural isomorphism of functors which sends an object (P, α) to the morphism $(\mathrm{Id}_P, [F \circ (\alpha \times I)])$. (Note that this only depends on the homotopy class of F, as a path in $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$ from f to f'.) One easily checks that L preserves compositions of homotopies and of homotopy equivalences, and is thus a a well defined functor of monoidal categories.

Since $\xi_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$ is an inclusion and an equivalence of categories (Theorem 5), it has a left inverse ξ^*, defined by sending any object (P, α) in $\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$ not in the image of $\xi_{\mathcal{L}}$ to some $Q \leq S$ such that $(Q, i_Q) = \xi_{\mathcal{L}}(Q)$ is isomorphic to (P, α) in $\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$. Define

$$\mathcal{A}ut(\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)) \xrightarrow[\simeq]{\quad c_\xi \quad} \mathcal{A}ut(\mathcal{L})$$

by setting $c_\xi(\psi) = \xi^* \circ \psi \circ \xi_{\mathcal{L}}$ for any self equivalence ψ of $\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$, and similarly for morphisms.

Lemma 7.3. *The composite of the functors*

$$\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\quad R \quad} \mathcal{A}ut(|\mathcal{L}|_p^\wedge) \xrightarrow{\quad L \quad} \mathcal{A}ut(\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)) \xrightarrow[\simeq]{\quad c_\xi \quad} \mathcal{A}ut(\mathcal{L})$$

induces the inclusion $\pi_0(|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|) \longrightarrow \pi_0(|\mathcal{A}ut(\mathcal{L})|)$, and the identity on

$$\pi_1(|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|, \mathrm{Id}) = \pi_1(|\mathcal{A}ut(\mathcal{L})|, \mathrm{Id}) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^0(\mathcal{Z}). \tag{1}$$

Proof. **Step 1:** Fix an isotypical equivalence $\psi : \mathcal{L} \longrightarrow \mathcal{L}$, and consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{L} & \xrightarrow{\quad \xi_{\mathcal{L}} \quad} & \mathcal{L}_p^c(|\mathcal{L}|_p^\wedge) & & \\
\Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \mathcal{L}_p^c(|\psi|_p^\wedge) \,=\, L\circ R(\psi)} & & \\
\mathcal{L} & \xrightarrow{\quad \xi_{\mathcal{L}} \quad} & \mathcal{L}_p^c(|\mathcal{L}|_p^\wedge) & \xrightarrow{\quad \xi^* \quad} & \mathcal{L}.
\end{array}$$

Here, ξ^* is the left inverse of $\xi_{\mathcal{L}}$ used to define c_ξ. In particular, $c_\xi \circ L \circ R(\psi) = \xi^* \circ \mathcal{L}_p^c(|\xi|_p^\wedge) \circ \xi_{\mathcal{L}}$, and proving the first part of the proposition means showing that the square commutes up to a natural isomorphism of functors.

To simplify the notation, we regard $\xi_{\mathcal{L}}$ as a functor to $\mathcal{L}_p^c(|\mathcal{L}|)$ (the functor from $\mathcal{L}_p^c(|\mathcal{L}|)$ to $\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$ induced by inclusion is clearly natural with respect to ψ). Recall that $\xi_{\mathcal{L}}(P) = (P, |\theta_P|)$ for all $P \le S$, and that $\xi_{\mathcal{L}}(\alpha) = ([\alpha], |\eta_\alpha|)$ for each morphism $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$. Here, $\theta_P \xrightarrow{\eta_\alpha} \theta_Q \circ \mathcal{B}([\alpha])$ is the natural isomorphism of functors $\mathcal{B}(P) \longrightarrow \mathcal{L}$ which sends the object o_P to the morphism α.

We write for short $P' = \psi(P)$ for any object P in $\mathcal{L}$, and $\alpha' = \psi(\alpha)$ for any morphism α. Define

$$W(\psi) : \mathcal{L}_p^c(|\psi|) \circ \xi_{\mathcal{L}} \xrightarrow{\qquad} \xi_{\mathcal{L}} \circ \psi,$$

by sending an object P in $\mathcal{L}$ to the morphism

$$(\psi_P, C_P) \in \mathrm{Mor}_{\mathcal{L}_p^c(|\mathcal{L}|)}\big((P, |\psi| \circ |\theta_P|), (P', |\theta_{P'}|)\big).$$

Here C_P denotes the constant homotopy. To see that $W(\psi)$ is a natural isomorphism of functors, note first that the objects are correct: $\mathcal{L}_p^c(|\psi|)(\xi_{\mathcal{L}}(P)) = (P, |\psi| \circ \theta_P)$ and $\xi_{\mathcal{L}}(\psi P) = (P', \theta_{P'})$ by definition. Also, (ψ_P, C_P) is a morphism between these objects, since

$$|\psi| \circ |\theta_P| = |\psi \circ \theta_P| = |\theta_{P'} \circ \mathcal{B}(\psi_P)| = |\theta_{P'}| \circ B\psi_P$$

by definition of ψ_P. To show that $W(\psi)$ is natural, we must check, for each morphism $P \xrightarrow{\alpha} Q$ in $\mathcal{L}$, that the following square commutes:

$$\begin{array}{ccc}
(P, |\psi \circ \theta_P|) & \xrightarrow{(\psi_P, C_P)} & (P', |\theta_{P'}|) \\
\Big\downarrow{\scriptstyle ([\alpha], |\psi \circ \eta_\alpha|)} & & \Big\downarrow{\scriptstyle ([\alpha'], |\eta_{\alpha'}|)} \\
(Q, |\psi \circ \theta_Q|) & \xrightarrow{(\psi_Q, C_Q)} & (Q', |\theta_{Q'}|)
\end{array}$$

in $\mathcal{L}_p^c(|\mathcal{L}|)$. By Lemma 7, $(P \mapsto \psi_P)$ is a natural isomorphism of functors $F \xrightarrow{\cong} F \circ \psi$ (where $F \colon \mathcal{L} \longrightarrow \mathsf{Gr}$ is the forgetful functor), and thus $[\alpha'] \circ \psi_P = \psi_Q \circ [\alpha]$. So it remains to show that $|\eta_{\alpha'}| \circ (B\psi_P \times I) = |\psi \circ \eta_\alpha|$, and this follows since both are induced by the natural isomorphism $(o_P \mapsto \alpha')$ of functors $\mathcal{B}(P) \longrightarrow \mathcal{L}$.

Step 2: We next prove the isomorphism in (1). Since $\mathcal{A}ut(\mathcal{L})$ is a groupoid,

$$\pi_1(|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|, \mathrm{Id}) = \pi_1(|\mathcal{A}ut(\mathcal{L})|, \mathrm{Id}) \cong \mathrm{Aut}_{\mathcal{A}ut(\mathcal{L})}(\mathrm{Id}).$$

A natural isomorphism of functors $\mathrm{Id} \xrightarrow{\Psi} \mathrm{Id}$ is defined by morphisms $\Psi_P \in \mathrm{Aut}_{\mathcal{L}}(P)$, for all $\mathcal{F}$-centric $P \le S$, such that $\alpha \circ \Psi_P = \Phi_Q \circ \alpha$ for each morphism $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$. In particular, for each P, Ψ_P lies in the center of $\mathrm{Aut}_{\mathcal{L}}(P)$, so $\pi_{P,P}(\Psi_P) = \mathrm{Id}_P \in \mathrm{Aut}(P)$ by condition (C) (and Lemma 1(a)), and thus $\Psi_P = \widehat{g_P}$ for some $g_P \in Z(P)$. And the naturality of Ψ now shows that the elements g_P combine to define

$$\big(g_P\big)_{P \in \mathrm{Ob}(\mathcal{L})} \in \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^0(\mathcal{Z}).$$

The converse is clear — any such collection of elements $g_P \in Z(P)$ defines a natural isomorphism of functors — and this proves (1).

Step 3: It remains to show that $c_\xi \circ L \circ R$ induces the identity on $\pi_1(|\mathcal{A}ut(\mathcal{L})|)$. Fix an element in this group, represented by a natural isomorphism $\mathrm{Id} \xrightarrow{\Psi} \mathrm{Id}$ of functors, and write $\Psi_P = \widehat{g_P}$ where $g_P \in Z(P)$ for each P. Let $[1]$ denote the category with two objects $0,1$ and one nonidentity morphism $0 \xrightarrow{\iota} 1$. Then $R(\Psi)$ is the homotopy on $|\mathcal{L}|_p^\wedge$ induced by the functor

$$\Psi_* \colon \mathcal{L} \times [1] \longrightarrow \mathcal{L},$$

defined by setting $\Psi(P,t) = P$ $(t = 0,1)$, $\Psi(\alpha, \mathrm{Id}_t) = \alpha$, and $\Psi(\mathrm{Id}_P, \iota) = \Psi_P$. Hence $L(R(\Psi))$, as a natural isomorphism of functors from $\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$ to itself, sends each object $(P, |\theta_P|_p^\wedge)$ to the morphism

$$(\mathrm{Id}_P, |\Psi_* \circ (\theta_P \times \mathrm{Id}_{[1]})|_p^\wedge) = (\mathrm{Id}_P, |\eta_{g_P}|_p^\wedge) = \xi_{\mathcal{L}}(\widehat{g_P}) = \xi_{\mathcal{L}}(\Psi_P).$$

Since $\xi^* \circ \xi_{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}}$, this shows that $c_\xi \circ L \circ R(\Psi) = \Psi$. □

It remains to show that $|L|$ induces a monomorphism on homotopy groups.

Lemma 7.4. *The map*

$$|L| : |\mathcal{A}ut(|\mathcal{L}|_p^\wedge)| \longrightarrow |\mathcal{A}ut(\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge))|$$

induces monomorphisms on π_0 and on π_1. Also, $\pi_n(\mathrm{Aut}(|\mathcal{L}|_p^\wedge)) = 0$ for all $n > 1$.

Proof. The proof is based on the homotopy decomposition

$$\mathrm{pr} : \underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \xrightarrow{\ \simeq\ } |\mathcal{L}|$$

of Proposition 2, where $\widetilde{B} \colon \mathcal{O}^c(\mathcal{F}) \longrightarrow \mathsf{Top}$ is a lifting of the homotopy functor $P \mapsto BP$. In the following constructions, we regard $\underrightarrow{\mathrm{hocolim}}(\widetilde{B})$ as the union of skeleta:

$$\underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}{}^{(n)}(\widetilde{B}) = \Big(\coprod_{i=0}^{n} \coprod_{P_0 \to \cdots \to P_n} \widetilde{B}(P_0) \times D^i\Big)\Big/\!\sim$$

as described in Appendix B.

To simplify the notation, we write $i_P = |\theta_P|_p^\wedge \colon BP \longrightarrow |\mathcal{L}|_p^\wedge$ for each subgroup $P \leq S$. The obstructions to extending a map

$$\big(\underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \times S^{k-1}\big) \cup \big(\underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}{}^{(n-1)}(\widetilde{B}) \times D^k\big) \longrightarrow |\mathcal{L}|_p^\wedge$$

to $\big(\underrightarrow{\mathrm{hocolim}}^{(n)}(\widetilde{B}) \times D^k\big)$ lie in the groups

$$\pi_{n+k}\big(\mathrm{Map}(BP, |\mathcal{L}|_p^\wedge)_{Bi_P}\big) \cong \begin{cases} Z(P) & \text{if } n+k = 1 \\ 0 & \text{if } n+k > 1 \end{cases} \tag{1}$$

(see Theorem 4) for $\mathcal{F}$-centric subgroups $P \leq S$.

We prove the injectivity of π_0 in Step 1. The injectivity of π_1, together with the vanishing of higher homotopy groups, is shown in Step 2.

Step 1: Fix a homotopy equivalence $\varphi : |\mathcal{L}|_p^\wedge \longrightarrow |\mathcal{L}|_p^\wedge$ such that $[\varphi] \in \mathrm{Ker}(\pi_0(L))$, and let $\Psi : \mathcal{L}_p^c(\varphi) \longrightarrow \mathrm{Id}$ be a natural isomorphism. For each $\mathcal{F}$-centric $P \leq S$, write $\Psi(P, i_P) = (\sigma_P, \omega_P)$: an isomorphism from $(P, \varphi \circ i_P)$ to (P, i_P). Thus, σ_P is an automorphism of P, and ω_P is a homotopy (path) from $\varphi \circ i_P$ to $i_P \circ B\sigma_P$.

We first show, for each P, that $\sigma_P = \mathrm{Id}_P$. Set $\sigma = \sigma_P$ and $\omega = \omega_P$ for short. Fix $g \in P$, and consider the following two squares of morphisms in $\mathcal{L}_p^c(|\mathcal{L}|_p^\wedge)$:

$$\begin{array}{ccc} (P, \varphi\circ i_P) & \xrightarrow{(\sigma,\omega)} & (P, i_P) \\ \Big\downarrow {\scriptstyle (c_g, \varphi\circ i_P\circ\eta_g^P)} & & \Big\downarrow {\scriptstyle (c_g, i_P\circ\eta_g^P)} \\ (P, \varphi\circ i_P) & \xrightarrow{(\sigma,\omega)} & (P, i_P) \end{array} \quad \text{and} \quad \begin{array}{ccc} (P, \varphi\circ i_P) & \xrightarrow{(\sigma,\omega)} & (P, i_P) \\ \Big\downarrow {\scriptstyle (c_g, \varphi\circ i_P\circ\eta_g^P)} & & \Big\downarrow {\scriptstyle (c_{\sigma(g)}, i_P\circ\eta_{\sigma(g)}^P)} \\ (P, \varphi\circ i_P) & \xrightarrow{(\sigma,\omega)} & (P, i_P). \end{array} \tag{2}$$

Here, η_g^P denotes the homotopy $BP \times I \longrightarrow BP$ from Id to Bc_g induced by the homomorphism of functors $\mathcal{B}(P) \longrightarrow \mathcal{B}(P)$ which sends o_P to $\breve{g}$. The first square commutes by the naturality of Ψ with respect to $(P, i_P) \xrightarrow{(c_g, i_P\circ\eta_g^P)} (P, i_P)$. The second commutes since $\sigma \circ c_g = c_{\sigma(g)} \circ \sigma$, and since the square

$$\begin{array}{ccc} \varphi\circ i_P & \xrightarrow{\quad\omega\quad} & i_P \circ B\sigma \\ \Big\downarrow {\scriptstyle \varphi\circ i_P\circ\eta_g^P} & & \Big\downarrow {\scriptstyle i_P\circ\eta_{\sigma(g)}^P\circ B\sigma \,=\, i_P\circ B\sigma\circ\eta_g^P} \\ \varphi\circ i_P\circ Bc_g & \xrightarrow{\omega\circ Bc_g} & i_P\circ B\sigma\circ Bc_g = i_P\circ Bc_{\sigma(g)}\circ B\sigma \end{array}$$

of paths in $\mathrm{Map}(BP, |\mathcal{L}|_p^\wedge)$ commutes via the homotopy

$$H : I \times I \longrightarrow \mathrm{Map}(BP, |\mathcal{L}|_p^\wedge) \qquad \text{defined by} \qquad H(s,t) = \omega(s) \circ \eta_g^P.$$

Since all of the maps in (2) are isomorphisms, it now follows that $c_g = c_{\sigma(g)}$ (so $g^{-1}\sigma(g) \in Z(P)$), and that the loop $i_P \circ \eta_{g^{-1}\sigma(g)}^P$ in $\mathrm{Map}(BP, |\mathcal{L}|_p^\wedge)_{i_P}$ is null homotopic. Hence $g = \sigma(g)$ by (1), and $\sigma = \sigma_P = \mathrm{Id}$.

Now consider the composite

$$\underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \xrightarrow{\quad\mathrm{pr}\quad} |\mathcal{L}|_p^\wedge \xrightarrow{\quad\varphi\quad} |\mathcal{L}|_p^\wedge.$$

Since $\underrightarrow{\mathrm{hocolim}}^{(0)}(\widetilde{B})$ is the disjoint union of the $\widetilde{B}(P) \simeq BP$, and $\mathrm{pr}|_{BP} = i_P$, the $\omega_P\colon \varphi\circ i_P \xrightarrow{\simeq} i_P$ define a homotopy on $\underrightarrow{\mathrm{hocolim}}^{(0)}(\widetilde{B})$ between $\varphi\circ\mathrm{pr}$ and pr. The naturality of the ω_P implies that this can be extended to a homotopy on the 1-skeleton $\underrightarrow{\mathrm{hocolim}}^{(1)}(\widetilde{B})$, and hence by (1) to all of $\underrightarrow{\mathrm{hocolim}}(\widetilde{B})$. And since pr is an $\mathbb{F}_p$-homology equivalence, this shows that $\varphi \simeq \mathrm{Id}$.

Step 2: An element $[\Phi] \in \mathrm{Ker}(\pi_1(|L|))$ (at any basepoint) is a map

$$\Phi : S^1 \times |\mathcal{L}|_p^\wedge \longrightarrow |\mathcal{L}|_p^\wedge$$

such that for all $\mathcal{F}$-centric $P \le S$, $\Phi|_{S^1\times BP}$ extends to a map on $D^2 \times BP$. The composite

$$S^1 \times \underset{\mathcal{O}^c(\mathcal{F})}{\underrightarrow{\mathrm{hocolim}}}(\widetilde{B}) \xrightarrow{S^1\times\mathrm{pr}} S^1 \times |\mathcal{L}|_p^\wedge \xrightarrow{\quad\Phi\quad} |\mathcal{L}|_p^\wedge$$

can thus be extended to $D^2 \times \underrightarrow{\mathrm{hocolim}}^{(0)}(\widetilde{B})$, and hence to all of $D^2 \times \underrightarrow{\mathrm{hocolim}}(\widetilde{B})$ by (1). Since pr is an $\mathbb{F}_p$-homology equivalence, it now follows that $[\Phi] = 1$, and thus that $\pi_1(|T|)$ is injective.

The proof that $\pi_n(\mathrm{Aut}(|\mathcal{L}|_p^\wedge)) = 0$ for all $n > 1$ also follows easily from the homotopy colimit decomposition of $|\mathcal{L}|_p^\wedge$, together with (1). (See also [BL, Proposition 3.6]). □

We must show that $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$ and $|\mathcal{A}ut(\mathcal{L})|$ are homotopy equivalent, and moreover equivalent as monoids. There is no obvious way to construct a map between these two spaces which is both a homotopy equivalence and a morphism of monoids, so instead we connect them with a series of maps.

Let $S.\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$ denote the singular simplicial set of $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$; an n-simplex is thus a homotopy equivalence $\Delta^n \times |\mathcal{L}|_p^\wedge \longrightarrow |\mathcal{L}|_p^\wedge$. Let

$$\mathrm{Aut}(|\mathcal{L}|_p^\wedge) \xleftarrow{\ \mathrm{ev}\ } |S.\mathrm{Aut}(|\mathcal{L}|_p^\wedge)| \xrightarrow{\ \sigma\ } |\mathcal{A}ut(|\mathcal{L}|_p^\wedge)|$$

denote the obvious maps: the first is the evaluation map $|S.X| \longrightarrow X$ defined for any space X; and the second is the map $|S.X| \longrightarrow |\pi(X)|$ which sends each simplex to its homotopy class. Both are morphisms of monoids.

Proof of Theorem 7. The following maps are all morphisms of topological monoids:

$$\mathrm{Aut}(|\mathcal{L}|_p^\wedge) \xleftarrow{\ \mathrm{ev}\ } |S.\mathrm{Aut}(|\mathcal{L}|_p^\wedge)| \xrightarrow{\ \sigma\ } |\mathcal{A}ut(|\mathcal{L}|_p^\wedge)| \xleftarrow{\ |R|\ } |\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|.$$

The first is a (weak) homotopy equivalence by definition, and the second is a homotopy equivalence since $\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$ is aspherical (Lemma 7 or [BL]). Finally, $|R|$ induces isomorphisms on all homotopy groups by Lemmas 7 and 7. The classifying spaces $B\,\mathrm{Aut}(|\mathcal{L}|_p^\wedge)$ and $B|\mathcal{A}ut_{\mathrm{typ}}(\mathcal{L})|$ are thus homotopy equivalent since a morphism of monoids which is a homotopy equivalence induces a homotopy equivalence between the classifying spaces (cf. [GJ, Proposition IV.1.7]).

In particular, this shows that $\mathrm{Out}(|\mathcal{L}|_p^\wedge) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$. The isomorphism

$$\pi_1(\mathrm{Aut}(|\mathcal{L}|_p^\wedge)) \cong \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^0(\mathcal{Z})$$

was shown in Lemma 7. □

This description of $\mathrm{Aut}(BG_p^\wedge)$ will hopefully be useful when constructing fibrations with fiber the classifying space of a homotopy finite group.

8. Constructing saturated Frobenius systems

We first prove a very general condition for a Frobenius system to be saturated. Recall (Definition A) that for any Frobenius system $\mathcal{F}$ over a p-group S, and any $P \le S$, the "centralizer" Frobenius system $C_\mathcal{F}(P)$ over $C_S(P)$ is defined by setting

$$\mathrm{Hom}_{C_\mathcal{F}(P)}(Q,Q') = \bigl\{(\varphi|_Q) \bigm| \varphi \in \mathrm{Hom}_\mathcal{F}(PQ,PQ'),\ \varphi(Q) \le Q',\ \varphi|_P = \mathrm{Id}_P\bigr\}$$

for all $Q, Q' \le C_S(P)$. For any $g \in S$, we write $C_\mathcal{F}(g) = C_\mathcal{F}(\langle g\rangle)$.

Proposition 8.1. *Let $\mathcal{F}$ be any Frobenius system over a p-group S. Then $\mathcal{F}$ is saturated if and only if there is a set $\mathfrak{X}$ of elements of of order p in S such that the following conditions hold:*

(a) *Each $x \in S$ of order p is $\mathcal{F}$-conjugate to some element of $\mathfrak{X}$.*

(b) *If x and y are $\mathcal{F}$-conjugate and $y \in \mathfrak{X}$, then there is some $\psi \in \mathrm{Hom}_\mathcal{F}(C_S(x), C_S(y))$ such that $\psi(x) = y$.*

(c) *For each $x \in \mathfrak{X}$, $C_\mathcal{F}(x)$ is a saturated Frobenius category.*

Proof. Assume first that $\mathcal{F}$ is saturated, and let $\mathfrak{X}$ be the set of all $x \in S$ of order p such that $\langle x \rangle$ is saturated. Then conditions (a) and (b) hold by definition, and (c) holds by Proposition A.

Assume conversely that $\mathfrak{X}$ is chosen such that conditions (a–c) hold for $\mathcal{F}$. Define

$$\mathcal{U} = \{(P,x) \mid P \leq S,\ |x| = p,\ x \in Z(P)^T \text{ some } \mathrm{Aut}_S(P) \leq T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))\},$$

and let $\mathcal{U}_0 \subseteq \mathcal{U}$ be the set of pairs (P,x) such that $x \in \mathfrak{X}$. For any $P \leq S$, there is some x such that $(P,x) \in \mathcal{U}$ (since any action of a p-group on $Z(P)$ has nontrivial fixed set); but x need not be unique.

By definition, for any $(P,x) \in \mathcal{U}$, $\mathrm{Aut}_{C_S(x)}(P) = \mathrm{Aut}_S(P)$, and $\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)$ contains a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. In particular,

$$\forall (P,x) \in \mathcal{U}: \quad \mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P)) \iff \mathrm{Aut}_{C_S(x)}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)). \quad (1)$$

We next check that

$$(P,x) \in \mathcal{U}_0,\ P \text{ saturated in } C_{\mathcal{F}}(x) \implies P \text{ saturated in } \mathcal{F}. \quad (2)$$

Assume otherwise: let $P' \leq S$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P',P)$ be such that $|C_S(P')| > |C_S(P)|$. Set $x' = \varphi^{-1}(x) \leq Z(P')$. By (b), there exists $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(x))$ such that $\psi(x') = x$. Set $P'' = \psi(P')$. Then P'' is $C_{\mathcal{F}}(x)$-conjugate to P, and ψ sends $C_S(P')$ injectively into $C_S(P'')$. Thus $|C_S(P)| < |C_S(P'')|$, which contradicts the original assumption that P is saturated in $C_{\mathcal{F}}(x)$.

We are now ready to prove condition (I) in Definition 1: to show that each $P \leq S$ is $\mathcal{F}$-conjugate to some P' which is saturated in $\mathcal{F}$, and such that $\mathrm{Aut}_S(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$. We can assume that P is chosen such that $|N_S(P)|$ is maximal among all subgroups in its $\mathcal{F}$-conjugacy class. Choose $x \in Z(P)$ such that $(P,x) \in \mathcal{U}$. By (a) and (b), there is an element $y \in \mathfrak{X}$ and a homomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$. Set $P' = \psi P$. In particular, $N_S(P) \leq C_S(x)$ by definition of $\mathcal{U}$, and $\psi(N_S(P)) = N_S(P')$ by the maximality assumption. So if we set $x' = \psi(x)$, then $(P',x') \in \mathcal{U}_0$. Since $C_{\mathcal{F}}(x')$ is a saturated Frobenius system, the maximality of $|N_S(P')| = |N_{C_S(x')}(P')|$ together with Proposition A(a) imply that P' is N-saturated in $C_{\mathcal{F}}(x')$; i.e., that P' is saturated in $C_{\mathcal{F}}(x')$ and $\mathrm{Aut}_{C_S(x')}(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(x')}(P'))$. So by (1) and (2), P' is saturated in $\mathcal{F}$ and $\mathrm{Aut}_S(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$.

It remains to prove (II). Fix $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $P' \overset{\text{def}}{=} \varphi P$ is saturated in $\mathcal{F}$. Choose some $x' \in (Z(P'))^{N_S(P)}$ of order p, set $x = \varphi^{-1}(x')$, and let

$$N = \big\{ g \in N_S(P) \,\big|\, \varphi c_g \varphi^{-1} \in N_S(P') \big\}.$$

By construction, $x \in Z(N)$. Fix $y \in \mathfrak{X}$ which is $\mathcal{F}$-conjugate to x and x', and choose $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ and $\psi' \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(y))$. Set $Q = \psi(P)$ and $Q' = \psi'(P')$, and let $\tau = \psi'\varphi\psi^{-1} \in \mathrm{Iso}_{\mathcal{F}}(Q,Q')$. By construction, $\tau(y) = y$, and thus $\tau \in \mathrm{Iso}_{C_{\mathcal{F}}(y)}(Q,Q')$. Since Q' is saturated in $C_{\mathcal{F}}(y)$ ($|C_{C_S(y)}(Q')| = |C_S(Q')| \geq |C_S(P')|$ and P' is saturated in $\mathcal{F}$), condition (II) applied to the saturated Frobenius system $C_{\mathcal{F}}(y)$ shows that τ extends to some $\bar{\tau} \in \mathrm{Hom}_{C_{\mathcal{F}}(y)}(\psi N, C_S(y))$. Since P' is saturated, $\psi(C_S(P')) = C_S(Q')$. Hence by definition of N, $\bar{\tau}(\psi N) \leq \psi'(N_S(P'))$, and hence $\bar{\tau} \circ \psi$ lifts to a map $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N,S)$. □

The following is a topological version of the last proposition. Recall that for any space X and any p-subgroup (S,f), we denote by $\mathcal{F}_{(S,f)}(X)$ the Frobenius system over S defined by setting

$$\mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(P,Q) = \big\{ \varphi \in \mathrm{Hom}(P,Q) \,\big|\, f|_{BP} \simeq f|_{BQ} \circ B\varphi \big\}$$

for each $P, Q \le S$.

Proposition 8.2. *Let X be a space, let (S, f) be a Sylow p-subgroup of X, and let $\mathfrak{X}$ be a set of elements of order p in S. Assume these satisfy the following conditions.*

(i) $\mathrm{Map}(B\mathbb{Z}/p, X)_{const} \simeq X$.

(ii) *For all $x \in \mathfrak{X}$, $BC_S(x) \longrightarrow \mathrm{Map}(B\mathbb{Z}/p, X)_x$ is a Sylow for $\mathrm{Map}(B\mathbb{Z}/p, X)_x$.*

(iii) *For all $x \in \mathfrak{X}$, $\mathcal{F}_{(C_S(x))}(\mathrm{Map}(B\mathbb{Z}/p, X)_x)$ is saturated.*

(iv) *For all $x \in S$ of order p, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(\langle x\rangle, S)$ such that $\varphi(x) \in \mathfrak{X}$.*

Then $\mathcal{F}_{(S,f)}(X)$ is saturated.

Proof. Write $\mathcal{F} = \mathcal{F}_{(S,f)}(X)$ for short. Clearly, $\mathcal{F}$ is a Frobenius system over S. Condition (a) of Proposition 8 holds by (iv); and it remains to show that conditions (b) and (c) hold.

For each $x \in S$ of order p, let $i_x \in \mathrm{Hom}(\mathbb{Z}/p, S)$ denote the homomorphism $(1 \mapsto x)$, and let $\mathrm{Map}(B\mathbb{Z}/p, X)_x$ denote the component of $f \circ Bi_x$.

(b) Fix $x, y \in S$ of order p such that $y \in \mathfrak{X}$, and such that there is $\psi_0 \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle)$ with $\psi_0(x) = y$. We must show that ψ_0 extends to some $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$.

Since x and y are $\mathcal{F}$-conjugate,

$$[f \circ Bi_x] = [f \circ Bi_y] \in [B\mathbb{Z}/p, X],$$

and thus $\mathrm{Map}(B\mathbb{Z}/p, X)_x = \mathrm{Map}(B\mathbb{Z}/p, X)_y$. Since $C_S(y)$ is a Sylow p-subgroup of $\mathrm{Map}(B\mathbb{Z}/p, X)_y$ by (ii), the natural map $BC_S(x) \longrightarrow \mathrm{Map}(B\mathbb{Z}/p, X)_x$ factors through $BC_S(y)$. In other words, there is some $\psi \in \mathrm{Hom}(C_S(x), C_S(y))$ such that the following square commutes up to homotopy:

$$\begin{array}{ccc} BC_S(x) \times B\mathbb{Z}/p & \xrightarrow{f\circ B(\mathrm{incl} \times i_x)} & X \\ \Big\downarrow{\scriptstyle B\psi\times \mathrm{Id}} & & \Big\| \\ BC_S(y) \times B\mathbb{Z}/p & \xrightarrow{f\circ B(\mathrm{incl} \times i_y)} & X \end{array} \tag{1}$$

Thus $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$. If $\rho, \rho' \in \mathrm{Hom}(C_S(x) \times \mathbb{Z}/p, S)$ denote the homomorphisms $\rho(g, t) = gx^t$ and $\rho'(g, t) = \psi(g)y^t$, then $f \circ B\rho \simeq f \circ B\rho'$ by (1), and hence $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho')$ by Proposition 5(d). And this implies that $\psi(x) = y$.

(c) Fix some $x \in \mathfrak{X}$; we must show that $C_{\mathcal{F}}(x)$ is a saturated Frobenius system. By (iii), the Frobenius system $\mathcal{F}' \stackrel{\text{def}}{=} \mathcal{F}_{C_S(x)}(\mathrm{Map}(B\mathbb{Z}/p, X)_x)$ is saturated, so it suffices to show that these two Frobenius systems over $C_S(x)$ are equal.

To see this, fix $P, Q \le C_S(x)$, and let $\varphi \in \mathrm{Hom}(P, Q)$ be any monomorphism. Set $\overline{P} = P{\cdot}\langle x\rangle$ and $\overline{Q} = Q{\cdot}\langle x\rangle$. Let $\rho \in \mathrm{Hom}(P \times \mathbb{Z}/p, S)$ and $\rho' \in \mathrm{Hom}(Q \times \mathbb{Z}/p, S)$ be defined by $\rho(g, t) = gx^t$ and $\rho'(g, t) = gx^t$. Then $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ if and only if the following square commutes up to homotopy:

$$\begin{array}{ccc} BP \times B\mathbb{Z}/p & \xrightarrow{f\circ B\rho} & X \\ \Big\downarrow{\scriptstyle B\varphi\times \mathrm{Id}} & & \Big\| \\ BQ \times B\mathbb{Z}/p & \xrightarrow{f\circ B\rho'} & X. \end{array} \tag{2}$$

By Proposition 5(d), this holds if and only if $K \stackrel{\text{def}}{=} \operatorname{Ker}(\rho) = \operatorname{Ker}(\rho' \circ (\varphi \times \mathrm{Id}))$ and the induced maps from $B((P \times \mathbb{Z}/p)/K)$ to X are homotopic. The kernels are equal if and only if φ extends to a monomorphism $\bar{\varphi}$ from $\overline{P}$ to $\overline{Q}$ which sends x to itself. And in this case, the induced maps on $B((P \times \mathbb{Z}/p)/K)$ are homotopic if and only if $f|_{B\overline{P}} \simeq f|_{B\overline{Q}} \circ B\bar{\varphi}$, if and only if $\varphi \in \operatorname{Hom}_{C_{\mathcal{F}}(x)}(P, Q)$. □

The following conjecture is an attempt to formulate more general conditions for constructing saturated Frobenius systems. Note that conditions (a) and (b) in Proposition 8 follow easily from the hypotheses below; the problem is to prove condition (c).

Conjecture 8.3. *Fix the following data:*

- *a prime p, a finite group G, and $S \in \mathrm{Syl}_p(G)$;*
- *a unique maximal abelian subgroup $S_0 \lhd S$;*
- *$H = N_G(S_0)$ [and $H_0 = C_G(S_0)$];*
- *a subgroup $\Gamma \leq \operatorname{Aut}(H)$ which leaves S_0 [and H_0?] invariant, and which contains $\operatorname{Inn}(H)$ with index prime to p.*
- *[a Γ-invariant set $\mathcal{A}$ of order p-subgroups $A \leq S_0$.]*

Let $\mathcal{F}$ be the Frobenius system over S generated by $\operatorname{Hom}_G(-,-)$ and $\operatorname{Hom}_\Gamma(-,-)$ via composition. Assume the following:

(a) *Every elementary abelian subgroup in S is G-conjugate to a subgroup of S_0.*

(b) *Two elements of order p in S_0 are $\mathcal{F}$-conjugate if and only if they are Γ-conjugate.*

Then the Frobenius system $\mathcal{F}$ is saturated.

[Is $N_{\mathcal{F}}(T) = \mathcal{F}_{S_0}(N_G(T))$ in this situation??]

9. Examples at the prime 2: Solomon's groups

The main result of this section is the following:

Theorem 9.1. *Let q be an odd prime power, and fix $S \in \mathrm{Syl}_2(\mathrm{Spin}(7,q))$. Then there is a saturated Frobenius system $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ which satisfies the following conditions:*

(a) *For all $P, Q \leq S$ which contain z, if $\alpha \in \operatorname{Hom}(P,Q)$ is such that $\alpha(z) = z$, then $\alpha \in \operatorname{Hom}_{\mathcal{F}}(P,Q)$ if and only if $\alpha \in \operatorname{Hom}_{\mathrm{Spin}(7,q)}(P,Q)$.*

(b) *All involutions of S are $\mathcal{F}$-conjugate.*

Furthermore, there is a unique $\mathcal{L}$-system $\mathcal{L} = \mathcal{L}_{\mathrm{Sol}}(q)$ associated to $\mathcal{F}$.

We write $B\mathrm{Sol}(q) = |\mathcal{L}_{\mathrm{Sol}}(q)|_2^\wedge$ to denote the classifying space. By a theorem of Solomon [Sol], $\mathcal{F}_{\mathrm{Sol}}(q)$ is not the Frobenius system of any finite group. So by Theorem 5, $B\mathrm{Sol}(q)$ does not have the homotopy type of $BG_2^\wedge$ for any finite group G.

Note that $S \in \mathrm{Syl}_2(\mathrm{Spin}(7,q))$ has order 2^{3k+1}, where 2^k is the largest power which divides $q^2 - 1$.

The proof of Theorem 9 will be based on the following proposition.

Proposition 9.2. *Fix the following data:*

– *a prime* p*, a finite group* G*, and* $S \in \mathrm{Syl}_p(G)$*;*

– *a normal subgroup* $Z \lhd G$ *of order* p*;*

– *an elementary abelian subgroup* $T \lhd S$ *of rank two such that* $Z \le T$ *and* $S_0 \stackrel{\text{def}}{=} C_S(T) \in \mathrm{Syl}_p(C_G(T))$*;*

– *a subgroup* $H \lhd C_G(T)$ *normal in* $N_G(T)$ *which contains* S_0*; and*

– *a subgroup* $\Gamma \le \mathrm{Aut}(H)$ *which leaves* T *invariant.*

For all $P, Q \le S$*, set*

$$\mathrm{Hom}_\Gamma(P,Q) = \begin{cases} \{\alpha|_P \,\big|\, \alpha \in \Gamma,\ \alpha(P) \le Q\} & \text{if } P \le S_0 \\ \emptyset & \text{if } P \not\le S_0, \end{cases}$$

and let $\mathcal{F}$ *be the Frobenius system over* S *generated by the sets* $\mathrm{Hom}_G(-,-)$ *and* $\mathrm{Hom}_\Gamma(-,-)$ *via composition. Assume the following:*

(a) *All subgroups of order* p *in* S *not in* Z *are* G*-conjugate.*

(b) Γ *permutes transitively the subgroups of order* p *in* T*.*

(c) $\{\varphi \in \Gamma \,|\, \varphi(Z) = Z\} = \mathrm{Aut}_{N_G(T)}(H)$*.*

(d) *for any* $T \le E \le S$ *such that* E *is elementary abelian of rank three and* $C_S(E) \in \mathrm{Syl}_p(C_G(E))$*,*

$$\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(E)) \,|\, \alpha(Z) = Z\} = \mathrm{Aut}_G(C_S(E)).$$

Then the Frobenius system $\mathcal{F}$ *is saturated. Also, for any* $P \le S$ *such that* $Z \le P$*,*

$$\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S) \,|\, \varphi(Z) = Z\} = \mathrm{Hom}_G(P,S). \tag{1}$$

Proof. Note first that $Z \le Z(S)$, since it is a normal subgroup of order p in a p-group. We first claim that for any $P \le S$, and any central subgroup $Z' \le Z(P)$ of order p,

$$Z \ne Z' \le T \Longrightarrow \exists \varphi \in \mathrm{Hom}_\Gamma(P, S_0) \text{ such that } \varphi(Z') = Z \tag{2}$$

and

$$Z' \not\le T \Longrightarrow \exists \varphi \in \mathrm{Hom}_G(P, S_0) \text{ such that } \varphi(Z') \le T. \tag{3}$$

If $Z' \le T$, then by (b) there is $\alpha \in \Gamma$ such that $\alpha(Z') = Z$. Since $S_0 \in \mathrm{Syl}_p(H)$, there is $h \in H$ such that $g(\alpha(P))g^{-1} \le S_0$, and $c_h \in \Gamma$ by (c). So $c_h \circ \alpha \in \mathrm{Hom}_\Gamma(P, S_0)$ and sends Z' to Z. If $Z' \not\le T$, then by (a), there is $g \in G$ such that $gZ'g^{-1} \le T \smallsetminus Z$. Then $[T, gPg^{-1}] = 1$, so $gPg^{-1} \le H \lhd C_G(T)$. Since $S_0 \in \mathrm{Syl}_p(H)$, there is $h \in H$ such that $h(gPg^{-1})h^{-1} \le S_0$; and we can take $\varphi = c_{hg} \in \mathrm{Hom}_G(P, S_0)$.

We now show in Step 1 that $N_{\mathcal{F}}(Z) = \mathcal{F}_S(G)$, and in Step 2 that $\mathcal{F}$ is saturated.

Step 1: We must show, for any $Z \le P, P' \le S$ and any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, P')$ such that $\varphi(Z) = Z$, that $\varphi \in \mathrm{Hom}_G(P, P')$. Upon replacing S by $\varphi(P) \le S$, we can assume that φ is an isomorphism, and as a composite of isomorphisms $P_{i-1} \xrightarrow[\cong]{\varphi_i} P_i$ for $1 \le i \le k$, each in Hom_G or Hom_Γ. Let $Z_i \le Z(P_i)$ be the subgroups of order p such that $Z_0 = Z = Z_k$ and $Z_i = \varphi_i(Z_{i-1})$. We can assume that $P_i \le S_0$ for each i, since otherwise there are no Γ-isomorphisms with source or target P_i.

For each i, choose some $\psi_i \in \mathrm{Hom}_{\mathcal{F}}(P_iT, S_0)$ such that $\psi_i(Z_i) = Z$ as in (2) and (3) (and letting $\psi_i = \mathrm{Id}$ if $Z_i = Z$), and set $P_i' = \psi_i(P_i)$. Thus, φ factors as a composite of morphisms

$$(P'_{i-1},Z) \xrightarrow{\psi_{i-1}^{-1}} (P_{i-1},Z_{i-1}) \xrightarrow{\varphi_i} (P_i,Z_i) \xrightarrow{\psi_i} (P'_i,Z).$$

In most cases, this reduces to a composite of two homomorphisms in Hom_Γ and one in Hom_G, and this is handled in Case 1 below. The exception occurs when φ_i is in Hom_Γ and $Z_{i-1}, Z_i \nleq T$. If this occurs, then we can replace P_{i-1} by $P_{i-1}T$ and similarly for the other groups, and are reduced to the situation handled in Case 2 below.

Case 1: Assume that φ is a composite of isomorphisms of the following form:

$$(P_0,Z) \xrightarrow[(\Gamma)]{\varphi_1} (P_1,Z_1) \xrightarrow[(G)]{\varphi_2} (P_2,Z_2) \xrightarrow[(\Gamma)]{\varphi_3} (P_3,Z).$$

If $Z_1 = Z$, then $Z_2 = Z$, φ_1 and φ_3 are in Hom_G by (c), and the result follows. Otherwise, $ZZ_1 = T = ZZ_2$, so φ_2 is in Hom_Γ by (c), $\varphi \in \mathrm{Iso}_\Gamma(P_0,P_3)$ sends Z to itself, and hence lies in Hom_G by (c) again.

Case 2: Assume that φ is a composite of isomorphisms of the following form:

$$(P_0,Z) \xrightarrow[(\Gamma)]{\varphi_1} (P_1,Z_1) \xrightarrow[(G)]{\varphi_2} (P_2,Z_2) \xrightarrow[(\Gamma)]{\varphi_3} (P_3,Z_3) \xrightarrow[(G)]{\varphi_4} (P_4,Z_4) \xrightarrow[(\Gamma)]{\varphi_5} (P_5,Z),$$

and that $T \le P_2, P_3$. We can assume that $P_2, P_3 \le S_0$, since otherwise $\mathrm{Hom}_\Gamma(P_2,P_3) = \emptyset$. We can also assume that $Z_2, Z_3 \notin T$, since otherwise φ_2 and φ_4 are in Hom_Γ. Let $E_i \le P_i$ be the rank three elementary abelian subgroups such that $E_2 = TZ_2$, $E_3 = TZ_3$, and $\varphi_i(E_{i-1}) = E_i$. Then $T = ZZ_4 \le \varphi_4(E_3) = E_4$ since $\varphi_4(Z) = Z$, and thus $T = \varphi_5(T) \le E_5$. Via similar considerations for E_0 and E_1, we see that $T \le E_i$ for all i.

Let $\mathcal{E}'_3$ be the set of all elementary abelian subgroups $E \le S$ of rank three which contain T, and with the property that $C_S(E) \in \mathrm{Syl}_p(C_H(E))$. If $E \le S$ is any rank three elementary abelian subgroup which contains T, then there is some $a \in C_H(E)$ such that $E' = aEa^{-1} \in \mathcal{E}'_3$, and $c_a \in \mathrm{Iso}_{G\cap\Gamma}(E,E')$ by (c). So by composing with such isomorphisms, we can assume that $E_i \in \mathcal{E}'_3$ for all i, and also that $\varphi_i(C_S(E_{i-1})) = C_S(E_i)$ for each i.

In this way, φ can be assumed to extend to an $\mathcal{F}$-isomorphism $\bar\varphi$ from $C_S(E_0)$ to $C_S(E_5)$ which sends Z to itself. By (d), $\bar\varphi$ is in Hom_G, and hence $\varphi \in \mathrm{Iso}_G(P_0,P_5)$.

Step 2: It remains to prove that $\mathcal{F}$ is saturated as a Frobenius system. We apply Proposition 8, by letting $\mathfrak{X}$ be the set of generators of Z. Condition (a) of the proposition (every $x \in S$ of order p is $\mathcal{F}$-conjugate to an element of $\mathfrak{X}$) holds by (2) and (3) above. If z, z' are two generators of Z which are $\mathcal{F}$-conjugate, then $z' = gzg^{-1}$ for some $g \in G$ by (1), and we can choose $g \in N_G(S)$, so $c_g \in \mathrm{Iso}_G(C_S(z), C_S(z'))$. Together with (2) and (3), this shows that for any $\mathcal{F}$-conjugate $x, z \in S$ with $z \in \mathfrak{X}$, there is $\psi \in \mathrm{Hom}_\mathcal{F}(C_S(x), C_S(z))$ such that $\psi(x) = z$. Condition (b) thus holds, and (c) follows since $C_\mathcal{F}(Z)$ is the Frobenius system of the group $C_G(Z)$ by (1) again, and hence is saturated. □

Background results needed for computations in $\mathrm{Spin}(V,\mathfrak{q})$ have been collected in Appendix C. We focus attention here on $SO(7,q)$ and on $\mathrm{Spin} \overset{\mathrm{def}}{=} \mathrm{Spin}(7,q)$, and also on these groups over the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$.

In order to make certain computations more explicit, we set

$$\widehat{V} = M_2(\bar{\mathbb{F}}_q) \oplus M_2^0(\bar{\mathbb{F}}_q) \cong (\bar{\mathbb{F}}_q)^7 \qquad \text{and} \qquad \mathfrak{q} = \det \oplus \det,$$

and let $V = M_2(\mathbb{F}_q) \oplus M_2^0(\mathbb{F}_q) \subseteq \widehat{V}$. Identify $SO(7,\bar{\mathbb{F}}_q) = SO(\widehat{V},\mathfrak{q})$ and $SO(7,q) = SO(V,\mathfrak{q})$, and similarly for $\mathrm{Spin}(7,q) \le \mathrm{Spin}(7,\bar{\mathbb{F}}_q)$. If $\alpha \in \mathrm{Spin}(M_2(\bar{\mathbb{F}}_q),\det)$ and $\beta \in \mathrm{Spin}(M_2^0(\bar{\mathbb{F}}_q),\det)$, we write $\alpha\oplus\beta$ for their image in $\mathrm{Spin}(7,\bar{\mathbb{F}}_q)$ under the natural inclusion

$$\mathrm{Spin}(4,\bar{\mathbb{F}}_q) \times_{C_2} \mathrm{Spin}(3,\bar{\mathbb{F}}_q) \longrightarrow \mathrm{Spin}(7,\bar{\mathbb{F}}_q).$$

Let $z \in Z(\mathrm{Spin}(7,q))$ denote the central involution, set

$$z_1 = \widetilde{\rho}_4([-I,I]) \oplus 1 \in \mathrm{Spin}(7,q),$$

and define $T = \langle z, z_1\rangle$. Here, $\widetilde{\rho}_4\colon SL(2,q)\times SL(2,q) \xrightarrow{\cong} \mathrm{Spin}(4,q)$ is the isomorphism of Proposition C.

In general, for any element $x \in \mathrm{Spin}(7,\bar{\mathbb{F}}_q)$, we let $\bar{x}$ denote its image in $SO(7,\bar{\mathbb{F}}_q)$. For example,

$$\bar{z}_1 = -\,\mathrm{Id}_{M_2(\mathbb{F}_q)} \oplus \mathrm{Id}_{M_2^0(\mathbb{F}_q)} \in \Omega(7,q).$$

Notation 9.3. *Define*

$$\begin{aligned}\widehat{H} &= SL(2,\bar{\mathbb{F}}_q)^3/\langle[-I,-I,-I]\rangle \\ &\cong \bigl\{(A,B,C) \in GL(2,\bar{\mathbb{F}}_q)^3 \bigm| \det(A)=\det(B)=\det(C)\bigr\}/\bigl\{(\lambda I,\lambda I,\lambda I) \bigm| \lambda\in\bar{\mathbb{F}}_q^*\bigr\},\end{aligned}$$

let $\psi^q \in \mathrm{Aut}(\widehat{H})$ be the automorphism induced by the field automorphism $(a \mapsto a^q)$ of $\bar{\mathbb{F}}_q$, and set

$$\begin{aligned}H &= \{x \in \widehat{H} \mid \psi^q(x) = x\} \\ &\cong \bigl\{(A,B,C) \in GL(2,q)^3 \bigm| \det(A)=\det(B)=\det(C)\bigr\}/\bigl\{(\lambda I,\lambda I,\lambda I) \bigm| \lambda\in\mathbb{F}_q^*\bigr\}.\end{aligned}$$

Let $H_0 \lhd H$ be the subgroup of classes of triples in $SL(2,q)$, and set

$$z' = [I,I,-I], \qquad z_1' = [-I,I,I], \qquad \text{and} \qquad T' = \langle z', z_1'\rangle = Z(H).$$

Define

$$\widehat{\Gamma} = \mathrm{Inn}(\widehat{H}) \rtimes \Sigma_3 \le \mathrm{Aut}(\widehat{H}) \qquad \text{and} \qquad \Gamma = \mathrm{Inn}(H) \rtimes \Sigma_3 \le \mathrm{Aut}(H),$$

where Σ_3 has the obvious permutation action. Let $\tau' \in \Gamma$ be the transposition which switches the first two factors $SL(2,q)$.

Note in particular that $H_0 = SL(2,q)^3/\langle[-I,-I,-I]\rangle$, and that $[H:H_0]=2$.

The idea of the construction is now to identify $\widehat{H}$ with $C_{\mathrm{Spin}(7,\bar{\mathbb{F}}_q)}(T)$ and H with $C_{\mathrm{Spin}(7,q)}(T)$, choose $S \in \mathrm{Syl}_2(\mathrm{Spin}(7,q))$ contained in the normalizer of H, and let $\mathcal{F}_{\mathrm{Sol}}(q)$ be the Frobenius system over S generated by conjugation in $\mathrm{Spin}(7,q)$ together with Γ.

Proposition 9.4. *There is a monomorphism*

$$\omega\colon \widehat{H} \longrightarrow \mathrm{Spin}(7,\bar{\mathbb{F}}_q),$$

and an involution $\tau \in N_{\mathrm{Spin}(7,q)}(T)$, with the following properties:

(a) *$\omega(z') = z$ and $\omega(z_1') = z_1$.*

(b) *$\omega\tau'\omega^{-1} \in \mathrm{Aut}(\omega H)$ is conjugation by τ.*

(c) *$C_{\mathrm{Spin}(7,q)}(T) = \omega H$ and $N_{\mathrm{Spin}(7,q)}(T) = \langle \omega H, \tau\rangle$.*

(d) *$N_{\mathrm{Spin}(7,q)}(T)$ contains a Sylow 2-subgroup of* $\mathrm{Spin}(7,q)$.

(e) *For each elementary abelian 2-subgroup $E \le H_0$ of rank three which contains T', and each $Q \in \mathrm{Syl}_2(C_\Gamma(E))$, there is a subgroup $\mathcal{K} \le \mathrm{Aut}(Q)$ such that*

$$\{\alpha \in \mathcal{K} \mid \alpha(T') = T'\} = \mathrm{Aut}_\Gamma(Q) \tag{1}$$

and

$$\{\alpha \in \mathcal{K} \mid \alpha(z) = z\} = \omega^{-1}\,\mathrm{Aut}_{\mathrm{Spin}(7,q)}(\omega Q)\omega. \tag{2}$$

Proof. The construction of ω and the proof of (b) will be carried out in Step 1. Point (e) will be proven in Step 2, and points (a), (c), and (d) in Step 3.

Step 1: Define

$$\omega\colon \widehat{H} \longrightarrow \mathrm{Spin}(7, \bar{\mathbb{F}}_q)$$

by setting, in the notation of Proposition C,

$$\omega([A,B,C]) = \widetilde{\rho}_4([A,B]) \oplus \widetilde{\rho}_3(C).$$

Thus ω is the unique lifting to $\mathrm{Spin}(7, \bar{\mathbb{F}}_q)$ of the homomorphism $\bar{\omega}$ from $\widehat{H}$ to $SO(7, \bar{\mathbb{F}}_q)$, defined by

$$\bar{\omega}([A,B,C])(X,Y) = (AXB^{-1}, CYC^{-1}).$$

Let $\theta \in \mathrm{Aut}_{\mathbb{F}_p}(M_2(\bar{\mathbb{F}}_q))$ be the orthogonal automorphism of $M_2(\bar{\mathbb{F}}_q)$ defined by $\theta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Then $X{\cdot}\theta(X) = \det(X){\cdot}I$ for all X. In particular, for all $A \in GL(2, \bar{\mathbb{F}}_q)$ and $X, Y \in M_2(\bar{\mathbb{F}}_q)$,

$$\theta(A) = \det(A){\cdot}A^{-1} \qquad \text{and} \qquad \theta(XY) = \theta(Y){\cdot}\theta(X). \tag{1}$$

Let $\bar{\tau} \in SO(7,q) \le SO(7, \bar{\mathbb{F}}_q)$ be the automorphism

$$\bar{\tau}(X,Y) = (-\theta(X), -Y)$$

for $(X,Y) \in \widehat{V} = M_2(\bar{\mathbb{F}}_q) \oplus M_2^0(\bar{\mathbb{F}}_q)$. By Lemma C, $\bar{\tau} \in \Omega(7,q)$ (its (-1)-eigenspace on V is $\langle I \rangle \oplus M_2^0(\mathbb{F}_q)$ which has discriminant 1), and thus lifts to an involution $\tau \in \mathrm{Spin}(7,q)$. By (1), for any $A, B, C \in SL(2, \bar{\mathbb{F}}_q)$,

$$\bar{\tau}{\cdot}\bar{\omega}([A,B,C]){\cdot}\bar{\tau}^{-1} = \bar{\omega}([B,A,C]).$$

Since $\widehat{H}$ is perfect, each automorphism of $\widehat{H}/\langle z' \rangle$ lifts to a unique automorphism of $\widehat{H}$, and this shows that τ satisfies condition (b).

Step 2: We next prove (e). Any $E \le H_0$ which is elementary abelian of rank three and contains T' is of the form $E = \langle T', \alpha \rangle$, where $\alpha = [A_1, A_2, A_3]$ for some $A_i \in SL(2,q)$ of order 4. Since all elements of order four in $SL(2,q)$ are conjugate, this shows that any two such subgroups E are conjugate in H_0, and hence that (e) holds for all of them if it holds for one. So we fix some $A \in SL(2,q)$ of order 4, set $\alpha = [A,A,A]$, and work with the subgroup $E = \langle T', \alpha \rangle$.

Now, $C_H(\alpha)$ is generated by elements of the form $[X_1, X_2, X_3]$, where either $X_i \in C_{SL(2,q)}(A)$, or $X_1 = X_2 = X_3 \in C_{GL(2,q)}(A)$ and has nonsquare determinant, or $X_1 = X_2 = X_3 \in SL(2,q)$ and $\langle X_1, A \rangle \cong Q_8$. Also, $C_{SL(2,q)}(A)$ is cyclic and $C_{PGL(2,q^n)}([A])$ is dihedral, both of order $q \pm 1$ (whichever is a multiple of 4), and $N_{SL(2,q)}(\langle A \rangle)$ is quaternion of order $2(q \pm 1)$. So if we let $2^k \ge 4$ be the largest power which divides $q \pm 1$, then $Q \in \mathrm{Syl}_2(C_\Gamma(E))$ has the form

$$Q \cong (C_{2^k})^3 \rtimes \langle \xi \rangle$$

where $\xi^2 = 1$ and ξ acts on $(C_{2^k})^3$ via $(g \mapsto g^{-1})$. Let $Q_0 \cong (C_{2^k})^3$ be the (unique) index two abelian subgroup.

Fix generators $\beta_1, \beta_2, \beta_3 \in Q_0$ as follows. Set $\beta_1 = [I, I, X]$ and $\beta_2 = [X, I, I]$ for some $X \in C_{SL(2,\bar{\mathbb{F}}_q)}(A_3)$ of order 2^k. Thus, $\beta_1^{2^{k-1}} = z$ and $\beta_2^{2^{k-1}} = z_1$. Set $\beta_3 = [Y, Y, Y]$, where $[Y]$ has order 2^k in $C_{PGL(2,\bar{\mathbb{F}}_q)}([A])$; then $\beta_3^{2^{k-1}} = \alpha$. We identify $\mathrm{Aut}(E) \cong GL(3, 2)$ via the basis $\{z, z_1, \alpha\}$ and $\mathrm{Aut}(Q_0) \cong GL(3, \mathbb{Z}/2^k)$ via the basis $\{\beta_1, \beta_2, \beta_3\}$. Under these identifications,

$$\mathrm{Aut}_{\mathrm{Spin}(7,q)}(E) = P_1 \stackrel{\mathrm{def}}{=} \{(a_{ij}) \mid a_{21} = a_{31} = 0\}$$

and

$$\mathrm{Aut}_{\Gamma}(E) = P_2 \stackrel{\mathrm{def}}{=} \{(a_{ij}) \mid a_{31} = a_{32} = 0\}.$$

Since the only elements of $\mathrm{Aut}_{\mathrm{Spin}}(Q_0)$ and $\mathrm{Aut}_{\widehat{\Gamma}}(Q_0)$ which are the identity on E are conjugation by ξ (the only additional element in the centralizer of E), these automorphism groups define sections

$$P_i \longrightarrow GL(3, \mathbb{Z}/2^k)/\{\pm I\},$$

which lift to sections

$$\psi_i \colon P_i \longrightarrow SL(3, \mathbb{Z}/2^k)$$

which agree on the group $P_0 = P_1 \cap P_2$ of upper triangular matrices.

By carrying out this same procedure over the field $\mathbb{F}_{q^2}$, we see that both of these sections can be lifted further to $SL(3, \mathbb{Z}/2^{k+1})$. So by Lemma C, there is a section ψ defined on $GL(3, 2)$ which extends both ψ_1 and ψ_2. Let $\mathcal{K}_0 \le \mathrm{Aut}(Q_0)$ be the image of ψ (together with $-\mathrm{Id}$), and define

$$\mathcal{K} = \mathrm{Inn}(Q)\cdot\{\gamma \in \mathrm{Aut}(Q) \mid \gamma(\xi) = \xi,\ \gamma|_{Q_0} \in \mathcal{K}_0\}.$$

Then (1) and (2) hold since for $G = \Gamma$ or $G = \mathrm{Spin}(7, q)$,

$$N_G(Q) = H \cdot \big(N_G(E) \cap N_G(\langle E, \xi\rangle)\big).$$

And this finishes the proof of point (e).

Step 3: Point (a) is clear: $\omega(z') = z$ and $\omega(z_1') = z_1$ by construction. The centralizer of $\bar{z}_1 \in \Omega(7, q)$ is the subgroup of those orthogonal automorphisms of the form $\alpha_- \oplus \alpha_+$ for $\alpha_\pm \in O(V_\pm)$, and this is easily seen to be $\langle \bar{\omega}(H), \tau\rangle$. Thus $\langle \omega(H), \tau\rangle = N_{\mathrm{Spin}}(T)$, and hence $\omega(H) = C_{\mathrm{Spin}}(T)$. This proves (c).

If $N_{\mathrm{Spin}}(T)$ did not contain a Sylow 2-subgroup of $\mathrm{Spin}(7, q)$, then since every noncentral involution of $\mathrm{Spin}(7, q)$ is conjugate to z_1, the Sylow 2-subgroups of $\mathrm{Spin}(7, q)$ would have no normal subgroup isomorphic to C_2^2. By a theorem of Hall (cf. [Go, Theorem 5.4.10]), this would imply that they are cyclic or have an index two cyclic subgroup. This is clearly not the case, so $N_{\mathrm{Spin}}(T)$ must contain a Sylow 2-subgroup of $\mathrm{Spin}(7, q)$, and this proves point (c). □

In order to simplify notation throughout the rest of this section, we now identify $H = C_{\mathrm{Spin}}(T)$ via the isomorphism ω. In particular, we now write $T' = T$, $z' = z$, and $z_1' = z_1$.

We now let S be a Sylow 2-subgroup of $\mathrm{Spin}(7, q)$ which is contained in $\langle H, \tau\rangle = N_{\mathrm{Spin}}(T)$. More precisely:

Definition 9.5. *Fix some $S_{GL} \in \mathrm{Syl}_2(GL(2, q))$. Define S to be the subgroup generated by the elements $[A, B, C]$ for $A, B, C \in S_{GL}$ such that either all have square determinant or none do, together with the element $\tau \in N_{\mathrm{Spin}}(T)$ whose conjugation action on H exchanges the first two coordinates.*

As usual, for $P, Q \le S$, $\mathrm{Hom}_{\mathrm{Spin}}(P,Q)$ and $\mathrm{Hom}_\Gamma(P,Q)$ denote the sets of homomorphisms from P to Q induced by conjugation by some element of $\mathrm{Spin} = \mathrm{Spin}(7,q)$ or by an element of Γ, respectively. Let $\mathrm{Hom}_{\mathrm{Spin}\cap\Gamma}(P,Q)$ denote the intersection of these two sets.

Let $\mathcal{F} = \mathcal{F}_{\mathrm{Sol}}(q)$ be the Frobenius system over S generated by Spin and Γ. In other words, for each $P, Q \le S$, $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the set of all composites

$$P = P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-1} \xrightarrow{\varphi_k} P_k = Q,$$

where each φ_i lies in $\mathrm{Hom}_{\mathrm{Spin}}(P_{i-1},P_i)$ or $\mathrm{Hom}_\Gamma(P_{i-1},P_i)$. This clearly defines a Frobenius system.

Proposition 9.6. *Assume that $T \le E \le S$, where E is elementary abelian of rank 3 and $C_S(E) \in \mathrm{Syl}_2(C_{\mathrm{Spin}}(E))$. Then $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}(E)$, and*

$$\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(C_S(E)) \,|\, \alpha(z) = z\} = \mathrm{Aut}_{\mathrm{Spin}}(C_S(E)). \tag{1}$$

Proof. Assume first that E has type II (see Definition C). By Proposition C(f,e), $C_S(E)$ is elementary abelian of rank four and type II, and $\mathrm{Aut}_{\mathrm{Spin}}(C_S(E))$ is the group of all automorphisms which are the identity on z and on one other element $z_* \in C_S(E) \smallsetminus T$. In particular, $\mathrm{Aut}_{\mathrm{Spin}\cap\Gamma}(C_S(E))$ contains an involution which fixes z_* and zz_*, and exchanges the other two elements of z_*T. Since $\mathrm{Aut}_\Gamma(C_S(E))$ leaves T invariant, it has one orbit of order 3 in z_*T, and hence fixes either z_* or zz_*. Thus (after replacing z_* by zz_* if necessary), $\varphi(z_*) = z_*$ for every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(C_S(E))$. So if we also assume $\varphi(z) = z$, then φ is a Spin-automorphism of $C_S(E)$, and this proves (1).

Now assume that E has type I. Then $E \le H_0$. So by Proposition 9(d), there is a subgroup $\mathcal{K} \in \mathrm{Aut}(C_S(E))$ such that

$$\mathrm{Aut}_{\mathrm{Spin}}(C_S(E)) = \{\alpha \in \mathcal{K} \,|\, \alpha(z) = z\} \quad \text{and} \quad \mathrm{Aut}_\Gamma(C_S(E)) = \{\alpha \in \mathcal{K} \,|\, \alpha(T) = T\}.$$

In particular, $\mathrm{Aut}_{\mathcal{F}}(C_S(E)) \le \mathcal{K}$, and (1) follows. □

We will also need information about the elementary abelian subgroups of S of rank 4, when showing the existence and uniqueness of $\mathcal{L}$-systems associated to the $\mathcal{F}_{\mathrm{Sol}}(q)$.

There are precisely two conjugacy classes of monomorphisms $Q_8 \longrightarrow SL(2,q)$, which differ by conjugation by an element of $GL(2,q)$ of nonsquare determinant. If $q \equiv \pm 3 \pmod 8$, then representatives of these classes can be chosen to have the same image, and differ by an automorphism of order 2. If $q \equiv \pm 1 \pmod 8$, then there are actually two distinct conjugacy classes of subgroups isomorphic to C_2^2.

Fix $\rho_1, \rho_2 \colon Q_8 \longrightarrow SL(2,q)$ which represent the two conjugacy classes, chosen such that $\mathrm{Im}(\rho_i)$ is contained in the Sylow subgroup $S_{GL} \le GL(2,q)$ chosen in Definition 9. Fix a pair of generators of Q_8, let $A, B \in SL(2,q)$ be their images under ρ_1, and let A', B' be their images under ρ_2. In particular, $\langle A, B\rangle \cong \langle A', B'\rangle \cong Q_8$, but there is no inner automorphism of $SL(2,q)$ which sends $A \mapsto A'$ and $B \mapsto B'$.

Define elementary abelian subgroups of rank 4 as follows:

$$E_4 = \big\langle z, z_1, [A,A,A], [B,B,B]\big\rangle, E_4' = \big\langle z, z_1, [A,A,A'], [B,B,B']\big\rangle,$$
$$\text{and} \qquad E_4'' = \big\langle z, z_1, [A',A,A], [B',B,B]\big\rangle.$$

By construction, these are all contained in S (see Definition 9).

Lemma 9.7. *E_4 and E_4' represent the two conjugacy classes of rank four elementary abelian subgroups of $\mathrm{Spin}(7,q)$ of type I, while E_4'' represents the unique conjugacy class* *of type II subgroups. There are two $\mathcal{F}$-conjugacy classes of rank 4 elementary abelian 2-subgroups of S, one containing E_4 and the other E_4' and E_4''. Furthermore,*

$$\mathrm{Aut}_{\mathcal{F}}(E_4) = \mathrm{Aut}(E_4) \qquad \textit{while} \qquad \mathrm{Aut}_{\mathcal{F}}(E_4') = \{\alpha \in \mathrm{Aut}(E_4') \mid \alpha(z) = z\}.$$

Proof. By Proposition C(d), if $T \le E \le \mathrm{Spin}$, where E is elementary abelian of rank 4 and type I, then $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of all automorphisms which fix z. If $T \le E \le H_0$ and E has rank 4 and type II, then by Proposition C(c,e), $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of all automorphisms which fix z and z_1. Thus, in both cases, any two such subgroups which are conjugate in Spin are conjugate by an element of $N_{\mathrm{Spin}}(T) = \langle H, \tau\rangle$. Since no two of the groups E_4, E_4', and E_4'' are conjugate in $\langle H, \tau\rangle$, this shows that they represent the three classes of rank 4 subgroups of Spin.

Clearly, E_4' and E_4'' are in the same orbit of Γ, while neither is in the Γ-orbit of E_4. Thus, there are exactly two $\mathcal{F}$-conjugacy classes of elementary abelian groups of rank 4, represented by E_4 and E_4'. Both of these are of type I in Spin, and hence their Spin-automorphism groups contain all automorphisms which fix z. Clearly, z is fixed by all Γ-automorphisms of E_4', and so $\mathrm{Aut}_{\mathcal{F}}(E_4') = \mathrm{Aut}_{\mathrm{Spin}}(E_4')$.

Finally, E_4 contains automorphisms (induced by permuting the three coordinates of H) which permute the three elements z, z_1, zz_1; and these together with $\mathrm{Aut}_{\mathrm{Spin}}(E_4)$ generate $\mathrm{Aut}(E_4)$. □

We are now ready to construct the spaces $B\mathrm{Sol}(q)$.

Proof of Theorem 9. We apply Proposition 9, where $G = \mathrm{Spin}(7, q)$, T and $H = C_G(T)$ are as defined above, $Z = \langle z\rangle = Z(G)$, and $S_0 = C_S(T) = S \cap H$. Also, $\Gamma \le \mathrm{Aut}(H)$ is as defined above. We have seen that condition (a) in Proposition 9 holds (all noncentral involutions in G are conjugate), and condition (b) holds by definition of Γ. Condition (c) holds since

$$\{\gamma \in \Gamma \mid \gamma(z) = z\} = \mathrm{Inn}(H)\cdot\langle\tau'\rangle$$

by definition, and (since $H = C_G(T)$) this is equal to

$$\mathrm{Aut}_{N_G(T)}(H) = \mathrm{Inn}(H)\cdot\langle c_\tau\rangle$$

by Proposition 9(b). Condition (d) was shown in Proposition 9. So by Proposition 9, $\mathcal{F}$ is a saturated Frobenius system, and $C_{\mathcal{F}}(Z) = \mathcal{F}_S(\mathrm{Spin}(7, q))$.

It remains to show that there is a unique $\mathcal{L}$-system associated to $\mathcal{F}$. By Proposition 2, this means showing that $\varprojlim^i_{\mathcal{O}^c(\mathcal{F})}(\mathcal{Z}) = 0$ for $i = 2, 3$, where $\mathcal{Z}$ is the functor which sends P to $Z(P)$. By Lemma 3, it suffices to show, for each $\mathcal{F}$-centric subgroup $P \le S$, that

$$\Lambda^i(\mathrm{Out}_{\mathcal{F}}(P), Z(P)) = 0$$

for all $i = 2, 3$. By [BLO, Proposition 5.8], this holds whenever $\mathrm{rk}(Z(P)) < 4$. Since all elementary abelian subgroups of rank four in S are self-centralizing (Proposition C(c)), it remains only to show that $\Lambda^2(\mathrm{Out}_{\mathcal{F}}(E), E) = 0$ for $E \le S$ elementary abelian of rank 4.

If E is $\mathcal{F}$-conjugate to E_4 in the notation of Lemma 9, then $\mathrm{Out}_{\mathcal{F}}(E) \cong GL(4, 2)$, and $\Lambda^2(GL(4, 2), \mathbb{F}_2^4) = 0$ by [JMO, Proposition 6.3]. If E is $\mathcal{F}$-conjugate to E_4', then by Lemma 9 again, the $\mathrm{Out}_{\mathcal{F}}(E)$-action on E has a rank one invariant subspace E_0, and $\Lambda^2(\mathrm{Out}_{\mathcal{F}}(E); E_0)$ and $\Lambda^2(\mathrm{Out}_{\mathcal{F}}(E); E/E_0)$ both vanish by [BLO, Proposition 5.8] again. So $\Lambda^2(\mathrm{Out}_{\mathcal{F}}(E); E) = 0$ in this case. □

By construction (and Proposition 9), the inclusions $\mathrm{Spin}(7,q) \le \mathrm{Spin}(7,q^n)$ for all n induce inclusions of Frobenius systems $\mathcal{F}_{\mathrm{Sol}}(q) \subseteq \mathcal{F}_{\mathrm{Sol}}(q^n)$, and hence (by their uniqueness) inclusions of associated $\mathcal{L}$-systems $\mathcal{L}_{\mathrm{Sol}}(q) \subseteq \mathcal{L}_{\mathrm{Sol}}(q^n)$. We can thus take the union of the $B\mathrm{Sol}(q^n)$, considered as the nerve of a certain "$\mathcal{L}$-category" $\mathcal{L}_{\mathrm{Sol}}(\bar{\mathbb{F}}_q)$. This union commutes with the centralizer decompositions of the $B\mathrm{Sol}(q^n)$ (Theorem 4), and hence $|\mathcal{L}_{\mathrm{Sol}}(\bar{\mathbb{F}}_q)|$ is the homotopy colimit over the category of elementary abelian 2-groups of rank ≤ 4 of a functor which sends groups of rank 1, 2, 3, and 4 to $B\mathrm{Spin}(7,\bar{\mathbb{F}}_q)$, $B\widehat{H}$, $B\big((\bar{\mathbb{F}}_q^*)^3 \rtimes C_2\big)$, and $B(C_2)^4$, respectively. Since $B\mathrm{Spin}(7,\bar{\mathbb{F}}_q)^\wedge_2 \simeq B\mathrm{Spin}(7)^\wedge_2$, and similarly for the other spaces, this shows that $\bigcup_{n=1}^\infty B\mathrm{Sol}(q^n)$ has the homotopy type of the space $BDI(4)$ constructed by Dwyer and Wilkerson in [DW2].

[Check details!! Add details?? Prove that these $B\mathrm{Sol}(q)$ have the homotopy type of Dave's spaces?]

Appendix

Appendix A. Properties of saturated Frobenius systems

We collect here some results on saturated Frobenius systems which are needed elsewhere in the paper. All of the results presented here are due to Lluis Puig [Pu, §1].

Let G be any group, and let $H \le G$ be a subgroup. For any group of automorphisms $K \le \mathrm{Aut}(H)$, the *K-normalizer of H in G* is the subgroup

$$N_G^K(H) = \left\{\, x \in N_G(H) \,\middle|\, c_x \in K \,\right\}.$$

In particular, $N_G^{\mathrm{Aut}(H)}(H) = N_G(H)$ is the usual normalizer, and $N_G^{\{\mathrm{Id}\}}(H) = C_G(H)$ is the centralizer. Also, if $\alpha \in \mathrm{Hom}(H,H')$ is any monomorphism, we write

$$\alpha K \alpha^{-1} = \{\alpha\chi\alpha^{-1} \,|\, \chi \in K\} \le \mathrm{Aut}(\alpha H).$$

[Define $\mathrm{Aut}_{\mathcal{F}}^K(P)$, $\mathrm{Aut}_S^K(P)$ somewhere??]

Definition A.1. *Let $\mathcal{F}$ be a saturated Frobenius system over S. For any $P \le S$ and any $K \le \mathrm{Aut}(P)$, we say that P is* K-saturated *in $\mathcal{F}$ if P is saturated in $\mathcal{F}$, and*

$$N_S^K(P) \in \mathrm{Syl}_p(N_{\mathcal{F}}^K(P)).$$

In particular, we say that P is N-saturated in $\mathcal{F}$ if it is $\mathrm{Aut}(P)$-saturated. This definition of a K-saturated subgroup is more restrictive than Puig's definition [Pu, 1.11] of subgroups having "$\mathcal{F}$-saturated K-normalizer", but it is equivalent to his definition in the case of saturated Frobenius systems.

For example, if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with Sylow p-subgroup S, and if $P \le S$ is a p-subgroup of S and $K \le \mathrm{Aut}(P)$ a subgroup of automorphisms, then P is K-saturated in $\mathcal{F}_S(G)$ if and only if $N_S^K(P) \in \mathrm{Syl}_p(N_G^K(P))$.

Proposition A.2. *Let $\mathcal{F}$ be a saturated Frobenius system over a p-group S. Fix subgroups $P \le S$ and $K \le \mathrm{Aut}(P)$. Then*

(a) *P is K-saturated if and only if $|N_S^K(P)| \ge |N_S^{\varphi K\varphi^{-1}}(\varphi P)|$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$; and*

(b) *for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $P' \overset{\mathrm{def}}{=} \varphi P$ is $K' \overset{\mathrm{def}}{=} \varphi K \varphi^{-1}$-saturated in $\mathcal{F}$, there are homomorphisms $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S^K(P){\cdot}P, S)$ and $\chi \in K$ such that $\bar\varphi|_P = \varphi \circ \chi$.*

Proof. By definition, P is K-saturated if and only if $\mathrm{Aut}_S^K(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^K(P))$ and $|C_S(P)| \geq |C_S(P')|$ for all P' $\mathcal{F}$-conjugate to P. These conditions clearly imply that $|C_S^K(P)| \geq |C_S^{\varphi K\varphi^{-1}}(\varphi P)|$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Conversely, if there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that φP is $\varphi K\varphi^{-1}$-saturated and if $|C_S^K(P)| \geq |C_S^{\varphi K\varphi^{-1}}(\varphi P)|$, then clearly P must be K-saturated. So to prove (a), it suffices to find some φ such that φP is $\varphi K\varphi^{-1}$-saturated.

By condition (I) in Definition 1, there exists $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $P' \stackrel{\text{def}}{=} \varphi P$ is saturated and $\mathrm{Aut}_S(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$. Then there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P')$ such that

$$\varphi \,\mathrm{Aut}_S^K(P)\varphi^{-1} \leq (\alpha^{-1}\,\mathrm{Aut}_S(P')\alpha) \cap (\varphi\,\mathrm{Aut}_{\mathcal{F}}^K(P)\varphi^{-1}) \in \mathrm{Syl}_p(\varphi\,\mathrm{Aut}_{\mathcal{F}}^K(P)\varphi^{-1}).$$

Upon replacing φ by $\alpha \circ \varphi$ and setting $K' = \varphi K\varphi^{-1}$, this shows that

$$\varphi\,\mathrm{Aut}_S^K(P)\varphi^{-1} \leq \mathrm{Aut}_S^{K'}(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^{K'}(P')).$$

In particular, P' is K'-saturated in $\mathcal{F}$, and this finishes the proof of (a).

Now assume that $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ is such that $P' \stackrel{\text{def}}{=} \varphi P$ is $K' \stackrel{\text{def}}{=} \varphi K\varphi^{-1}$-saturated. Then clearly P' is $K'{\cdot}\,\mathrm{Inn}(P')$-saturated, and so upon replacing K by $K{\cdot}\,\mathrm{Inn}(P)$ we can assume that $P \leq N_S^K(P)$. Since $\mathrm{Aut}_S^{K'}(P')$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}^{K'}(P')$, there is some $\chi \in \mathrm{Aut}_{\mathcal{F}}^K(P)$ such that

$$\varphi\big(\chi\,\mathrm{Aut}_S^K(P)\chi^{-1}\big)\varphi^{-1} \leq \mathrm{Aut}_S^{K'}(P').$$

And since P' is saturated, condition (II) now applies to show that $\varphi \circ \chi$ extends to a homomorphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S^K(P),S)$. □

Condition (I) in Definition 1, applied with $P = S$, says that $\mathrm{Out}_{\mathcal{F}}(S)$ has order prime to p for any saturated Frobenius system over S. Together with Proposition A, this implies that any Frobenius system which is saturated according to Definition 1 is also saturated according to the definition in [Pu]. Conversely, if $\mathcal{F}$ is a saturated Frobenius system over S under the definition in [Pu], then for each $P \leq S$ and each $K \leq \mathrm{Aut}(P)$, [Pu, Proposition 1.21] says that P has $\mathcal{F}$-saturated K-normalizer if and only if P is K-saturated in the sense of our Definition A; and using this one sees that $\mathcal{F}$ is also saturated in the sense of our Definition 1. So all definitions are equivalent.

We have already defined the K-normalizer in S of a subgroup $Q \leq S$, for any $K \leq \mathrm{Aut}(Q)$, to be a certain subgroup $N_S^K(Q)$. We now consider the K-normalizer of Q in $\mathcal{F}$, defined to be a Frobenius system over $N_S^K(Q)$.

Definition A.3. *Let $\mathcal{F}$ be a Frobenius system over S. For each $Q \leq S$ and each $K \leq \mathrm{Aut}(Q)$, let $N_{\mathcal{F}}^K(Q)$ (the* K-normalizer of Q in $\mathcal{F}$*) be the Frobenius system over $N_S^K(Q)$ defined by setting, for all $P, P' \leq N_S^K(Q)$,*

$$\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,P') = \big\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,P') \,\big|\, \exists\psi \in \mathrm{Hom}_{\mathcal{F}}(PQ,P'Q),\ \psi|_P = \varphi,\ \psi|_Q \in K\big\}.$$

In particular, we write $N_{\mathcal{F}}(P) = N_{\mathcal{F}}^{\mathrm{Aut}(P)}(P)$ and $C_{\mathcal{F}}(P) = N_{\mathcal{F}}^{\{\mathrm{Id}\}}(P)$: the normalizer and centralizer of P in $\mathcal{F}$. For example, if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G and some $S \in \mathrm{Syl}_p(G)$, then for any $Q \leq S$,

$$N_{\mathcal{F}_S(G)}(Q) = \mathcal{F}_{N_S(Q)}(N_G(Q)).$$

Proposition A.4. *Let $\mathcal{F}$ be any saturated Frobenius system over S. Fix $Q \leq S$ and $K \leq \mathrm{Aut}(Q)$ such that Q is K-saturated in $\mathcal{F}$. Then $N_{\mathcal{F}}^K(Q)$ is saturated as a Frobenius system over $N_S^K(Q)$.*

Proof. For each $P \leq N_S^K(Q)$ and each $I \leq \mathrm{Aut}(P)$, set

$$I{*}K = \{\alpha \in \mathrm{Aut}(PQ) \mid \alpha|_P \in I,\ \alpha|_Q \in K\} \leq \mathrm{Aut}(PQ).$$

Then

$$N^I_{N_S^K(Q)}(P) = N_S^{I*K}(PQ) \leq N_S^K(Q), \tag{1}$$

and the restriction map

$$\mathrm{Aut}_{\mathcal{F}}^{I*K}(PQ) \xrightarrow{\text{restr}} \mathrm{Aut}^I_{N_{\mathcal{F}}^K(Q)}(P) \tag{2}$$

is surjective.

We write $K_P = \mathrm{Aut}(P){*}K$ for short. Using Proposition A(a), choose $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$ such that $\varphi(PQ)$ is $\varphi(K_P)\varphi^{-1}$-saturated. Then $N_S^{\varphi(K_P)\varphi^{-1}}(\varphi(PQ)) \leq N_S^{\varphi K\varphi^{-1}}(\varphi Q)$, and since Q is K-saturated Proposition A(b) applies to show that there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S^{\varphi K_P\varphi^{-1}}(\varphi(PQ))\cdot Q, S)$ such that $\psi\circ\varphi|_Q \in K$. Then $\psi\varphi(PQ)$ is $(\psi\varphi)K_P(\psi\varphi)^{-1}$-saturated, and upon replacing φ by $\psi\circ\varphi$ we can assume that $\varphi(Q) = Q$ and $\varphi|_Q \in K$. In particular, $\varphi(PQ)$ is $\varphi 1{*}K\varphi^{-1}$-saturated, since it is $\varphi K_P\varphi^{-1}$-saturated and $1{*}K \lhd K_P$, and hence by (1) φP is saturated in $N_{\mathcal{F}}^K(Q)$. This also shows that

$$\mathrm{Aut}_S^{\varphi(K_P)\varphi^{-1}}(\varphi(PQ)) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^{\varphi(K_P)\varphi^{-1}}(\varphi(PQ))),$$

and hence by (2) that

$$\mathrm{Aut}_{N_S^K(Q)}(\varphi P) \in \mathrm{Syl}_p(\mathrm{Aut}_{N_{\mathcal{F}}^K(Q)}(\varphi P)).$$

This finishes the proof of (I) in Definition 1.

It remains to prove condition (II). We first claim that

$$P \text{ is } I\text{-saturated in } N_{\mathcal{F}}^K(Q) \Longrightarrow PQ \text{ is } (I{*}K)\text{-saturated in } \mathcal{F}. \tag{3}$$

To see this, assume P is I-saturated in $N_{\mathcal{F}}^K(Q)$, and choose $\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$ such that $\varphi(PQ)$ is $\varphi(I{*}K)\varphi^{-1}$-saturated in $\mathcal{F}$. Since Q is K-saturated in $\mathcal{F}$, there are homomorphisms

$$\psi \in \mathrm{Hom}_{\mathcal{F}}\big(N_S^{\varphi K\varphi^{-1}}(\varphi Q)\cdot\varphi Q, N_S^K(Q)\cdot Q\big) \qquad \text{and} \qquad \chi \in K$$

such that $\psi(\varphi Q) = Q$ and $\psi|_{\varphi Q} = \chi \circ (\varphi|_Q)^{-1}$. In particular, $(\psi \circ \varphi)|_Q = \chi \in K$; and $\psi\varphi(PQ)$ is $(\psi\varphi)(I{*}K)(\psi\varphi)^{-1}$-saturated in $\mathcal{F}$ since

$$|N_S^{(\psi\varphi)(I*K)(\psi\varphi)^{-1}}(\psi\varphi(PQ))| \geq |N_S^{\varphi(I*K)\varphi^{-1}}(\varphi(PQ))|.$$

So upon replacing φ by $\psi \circ \varphi$, we can assume that $\varphi|_Q \in K$, and hence that $\varphi|_P \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P, N_S^K(Q))$. Thus

$$\begin{aligned}|N_S^{I*K}(PQ)| = |N^I_{N_S^K(Q)}(P)| \geq |N^{\varphi I\varphi^{-1}}_{N_S^K(Q)}(\varphi P)| \\ = |N_S^{(\varphi I\varphi^{-1})*K}(\varphi P\cdot Q)| = |N_S^{\varphi(I*K)\varphi^{-1}}(\varphi(PQ))|\end{aligned}$$

the three equalities by (1) and since $\varphi|_Q \in K$, and the inequality by the assumption on P. So PQ is $(I{*}K)$-saturated in $\mathcal{F}$ by Proposition A(a) and since $\varphi(PQ)$ is $\varphi(I{*}K)\varphi^{-1}$-saturated in $\mathcal{F}$, and this finishes the proof of (3).

Now fix $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P, N_S^K(Q))$, and assume that φP is saturated in $N_{\mathcal{F}}^K(Q)$. Set

$$N = \{g \in N_{N_S^K(Q)}(P) \mid \varphi c_g\varphi^{-1} \in \mathrm{Aut}_{N_S^K(Q)}(\varphi P)\} \quad \text{and} \quad I = \mathrm{Aut}_N(P),$$

and

$$I = \{\alpha \in \mathrm{Aut}_{N_S^K(Q)}(P) \mid \varphi\alpha\varphi^{-1} \in \mathrm{Aut}_{N_S^K(Q)}(\varphi P)\}, \quad \text{and} \quad N = N^I_{N_S^K(Q)}(P).$$

In particular,

$$I = \mathrm{Aut}_N(P) \qquad \text{and} \qquad N = N^I_{N^K_S(Q)}(P).$$

Set $I' = \varphi I \varphi^{-1} \le \mathrm{Aut}_{N^K_S(Q)}(\varphi P)$. Then φP is I'-saturated since it is saturated and $\mathrm{Aut}^{I'}_{C^K_S(Q)}(P) = I' = \mathrm{Aut}^{I'}_{C^K_{\mathcal{F}}(Q)}(P)$; and hence $\varphi P{\cdot}Q$ is $I'{*}K$-saturated by (3). By definition of $N^K_{\mathcal{F}}(Q)$, there exists $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, S)$ such that $\varphi = \widehat{\varphi}|_P$ and $\widehat{\varphi}|_Q \in K$. Also, $I'{*}K = \widehat{\varphi}(I * K)\widehat{\varphi}^{-1}$, and hence by Proposition A(b), there are homomorphisms

$$\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N^{I*K}_S(PQ){\cdot}PQ, S) \qquad \text{and} \qquad \chi \in I * K$$

such that $\bar{\varphi}|_{PQ} = \widehat{\varphi} \circ \chi$. Set $\varphi_0 = \bar{\varphi}|_N$. Then $\mathrm{Im}(\varphi_0) \le N^K_S(Q)$ since $\varphi_0|_Q \in K$. And since $\chi|_P \in I$, there is some $g \in N$ such that $\chi|_P = c_g$; and φ thus extends to

$$\varphi_0 \circ c_g \in \mathrm{Hom}_{N^K_{\mathcal{F}}(Q)}(N, N^K_S(Q)).$$

And this finishes the proof of condition (II). □

Let $\mathcal{F}$ be any Frobenius system over S. Recall that a subgroup $P \le S$ is called $\mathcal{F}$-centric if $C_S(\varphi P) = Z(\varphi P)$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. In other words, $P \le S$ is $\mathcal{F}$-centric if each subgroup in the $\mathcal{F}$-conjugacy class of P contains its S-centralizer. In particular, if P is $\mathcal{F}$-centric, then every $Q \ge P$ contained in S is $\mathcal{F}$-centric; and in fact every $Q \le S$ such that $\mathrm{Hom}_{\mathcal{F}}(P, Q) \ne \emptyset$ is $\mathcal{F}$-centric.

Clearly, every $\mathcal{F}$-centric subgroup of S is saturated in $\mathcal{F}$. Conversely, if $P \le S$ is saturated in $\mathcal{F}$, then $C_S(P){\cdot}P$ is $\mathcal{F}$-centric.

If $\mathcal{F}$ is the Frobenius system of a finite group G over a Sylow p-subgroup $S \le G$, then a subgroup $P \le S$ is $\mathcal{F}$-centric if and only if P is p-centric in G; i.e., if and only if $Z(P) \in \mathrm{Syl}_p(C_G(P))$. This follows immediately from the observation that $C_S(P') \in \mathrm{Syl}_p(C_G(P'))$ for some P' conjugate to P in G.

The following proposition gives one important property of $\mathcal{F}$-centric subgroups.

Proposition A.5. *Let $\mathcal{F}$ be a saturated Frobenius system over the p-group S. Then for each $\mathcal{F}$-centric subgroup $P \le S$, each $P \le Q \le S$, and each $\varphi, \varphi' \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$ such that $\varphi|_P = \varphi'|_P$, there is some $g \in Z(P)$ such that $\varphi' = \varphi \circ c_g$.*

Proof. Assume first that $P \lhd Q$. Then for each $x \in Q$, $c_{\varphi(x)}$ and $c_{\varphi'(x)}$ are equal on $\varphi P = \varphi' P$, and thus $\varphi(x) \equiv \varphi'(x)$ modulo $C_S(\varphi P) \le \varphi P$ (since P is $\mathcal{F}$-centric). In particular, this shows that $\varphi(Q) = \varphi'(Q)$. So upon replacing φ by $(\varphi')^{-1} \circ \varphi$, we can assume that $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\varphi|_P = \mathrm{Id}_P$.

Set $K = \{\varphi \in \mathrm{Aut}(Q) \,|\, \varphi|_P = \mathrm{Id}_P\}$. Since $C_Q(P) \le P$, each $\psi \in K$ must induce the identity on Q/P, and hence K is a p-group (cf. [Go, Corollary 5.3.3]). We can assume that Q is K-saturated in $\mathcal{F}$: otherwise replace it by some other subgroup in the same $\mathcal{F}$-conjugacy class. Then by Proposition A(a), $\mathrm{Aut}^K_S(Q) = \mathrm{Aut}^K_{\mathcal{F}}(Q)$ since K is a p-group. So $\varphi = c_g$ for some $g \in N_S(P)$, and $g \in C_S(P) = Z(P)$ since $\varphi|_P = \mathrm{Id}$.

If P is not normal in Q, then there is a subnormal sequence $P = P_0 \lhd P_1 \lhd \cdots \lhd P_k = Q$, and hence elements $g_i \in Z(P_{i-1}) \le Z(P)$ such that $\varphi'|_{P_i} = (\varphi \circ c_{g_1} \circ \cdots \circ c_{g_i})|_{P_i}$ for each $1 \le i \le k$. □

Radical subgroups can also be defined in the context of Frobenius systems.

Definition A.6. *For any Frobenius system $\mathcal{F}$ over a p-group S, a subgroup $P \le S$ is called* $\mathcal{F}$-radical *if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced; i.e., if $O_p\big(\mathrm{Out}_{\mathcal{F}}(P)\big) = 1$.*

The following is one version of Alperin's fusion theorem for saturated Frobenius systems; one which suffices for our purposes here. A stronger version has been shown by Puig [Pu].

Theorem A.7 (Alperin's fusion theorem for saturated Frobenius systems)**.** *Let $\mathcal{F}$ be a saturated Frobenius system over S. Then for any $P, P' \le S$ and any $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, there exist sequences of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \qquad \text{and} \qquad Q_1, Q_2, \ldots, Q_k,$$

and elements $\varphi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

(a) *Q_i is $\mathcal{F}$-radical and $\mathcal{F}$-centric for each i,*

(b) *$P_{i-1}, P_i \le Q_i$ and $\varphi_i(P_{i-1}) = P_i$ for each i, and*

(c) *$\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$.*

Proof. By downward induction on the order of P. The claim is clear for $P = S$.

Assume $P \lneqq S$. Let $P^* \le S$ be any subgroup which is $\mathcal{F}$-conjugate to P and N-saturated in $\mathcal{F}$, and fix $\psi \in \mathrm{Iso}_{\mathcal{F}}(P, P^*)$. The theorem holds for $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$ if it holds for ψ and for $\psi \circ \varphi^{-1} \in \mathrm{Iso}_{\mathcal{F}}(P', P^*)$. So we are thus reduced to proving the theorem when the target group P' is N-saturated in $\mathcal{F}$.

Since $\mathcal{F}$ is N-saturated, there are homomorphisms $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ and $\chi \in \mathrm{Aut}_{\mathcal{F}}(P')$ such that $\bar\varphi(P) = P'$ and $\varphi = \chi \circ (\bar\varphi|_P)$ (Proposition A(b)). Since $N_S(P) \gneqq P$ (since $P \lneqq S$), the theorem holds for $\bar\varphi$ (as an isomorphism to its image) by the induction hypothesis. So it now remains only to prove it when $P = P'$ is N-saturated in $\mathcal{F}$ and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$.

In particular, P is saturated. So if P is not $\mathcal{F}$-centric, then by condition (II) in Definition 1, φ extends to an automorphism $\bar\varphi \in \mathrm{Aut}_{\mathcal{F}}(C_S(P){\cdot}P)$. Since $C_S(P){\cdot}P \gneqq P$, the theorem holds for φ by the induction hypothesis.

Now assume that φ is not $\mathcal{F}$-radical. Set $K = O_p(\mathrm{Aut}_{\mathcal{F}}(P)) \gneqq \mathrm{Inn}(P)$. Since P is N-saturated, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and so $K \le \mathrm{Aut}_S(P)$. In particular, $N_S^K(P) \gneqq P$ since $K \gneqq \mathrm{Inn}(P)$. Also, for each $g \in N_S^K(P)$, $\varphi c_g \varphi^{-1} \in K$ (since $K \lhd \mathrm{Aut}_{\mathcal{F}}(P)$), and hence $\varphi c_g \varphi^{-1} = c_h$ for some $h \in N_S^K(P)$. So by condition (II) in Definition 1, φ extends to an automorphism of $N_S^K(P) \gneqq P$, and the theorem again holds for φ by the induction hypothesis.

Finally, if $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ and P is an $\mathcal{F}$-centric $\mathcal{F}$-radical subgroup of S, then the theorem holds for trivial reasons. □

Appendix B. The Dwyer-Kan obstructions to diagram lifting

Let Top denote the category of spaces, and hoTop the homotopy category. In [DK] and [DK2], Dwyer and Kan identify the obstructions to lifting a functor $F : \mathcal{C} \longrightarrow \mathsf{hoTop}$ to a functor $\widetilde{F} : \mathcal{C} \longrightarrow \mathsf{Top}$; and also describe the space of such liftings. The idea of the alternative approach outlined here is to first construct a space which resembles a "homotopy colimit" of F — doing so is usually the main motivation for lifting F — and then note that this homotopy colimit automatically induces a lifting $\widetilde{F}$.

Fix a small category $\mathcal{C}$. Recall that the nerve of $\mathcal{C}$ is defined by setting

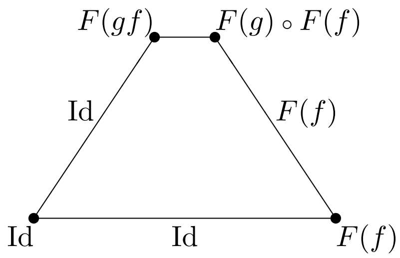
$$B\mathcal{C} = \Bigl(\coprod_{n\geq 0}\ \coprod_{X_0\to\cdots\to X_n} \Delta^n\Bigr)\Big/\sim;$$

and that the homotopy colimit of any functor $F : \mathcal{C} \longrightarrow \mathsf{Top}$ is the space

$$\underset{\mathcal{C}}{\underrightarrow{\mathrm{hocolim}}}(F) = \Bigl(\coprod_{n\geq 0}\ \coprod_{X_0\to\cdots\to X_n} F(X_0)\times\Delta^n\Bigr)\Big/\sim.$$

Let $p_F : \underrightarrow{\mathrm{hocolim}}(F) \longrightarrow B\mathcal{C}$ be the projection. It will be convenient to refer to the "skeleta" of the homotopy colimit: let $\underrightarrow{\mathrm{hocolim}}^{(n)}(F)$ denote the union of the $F(X_0)\times\Delta^i$ for all $i \geq n$ (and all $X_0 \to \cdots \to X_i$ in $\mathcal{C}$).

Now let $F : \mathcal{C} \longrightarrow \mathsf{hoTop}$ be a functor to the homotopy category. Assume that for each $f : X_0 \longrightarrow X_1$ in $\mathcal{C}$, a map $F(f) : F(X_0) \longrightarrow F(X_1)$ (not just a homotopy class) has been chosen. The 1-skeleton $\underrightarrow{\mathrm{hocolim}}^{(1)}(F)$ is defined in the same way as before: it is the union of the mapping cylinders of the $F(f)$ taken over all $f \in \mathrm{Mor}(\mathcal{C})$. It is also straightforward to define the 2-skeleton; but it is convenient at this stage to replace Δ^2 by a truncated triangle Δ^2_t. More precisely, for each sequence $X_0 \xrightarrow{\ f\ } X_1 \xrightarrow{\ g\ } X_2$, $F(X_0)\times\Delta^2_t$ is attached to $\underrightarrow{\mathrm{hocolim}}^{(1)}(F)$ via the following picture:



where the small segment at the top is mapped using some homotopy between $F(gf)$ and $F(g)\circ F(f)$.

The first obstructions arise when constructing the 3-skeleton. For each $X_0 \to \cdots \to X_3$, we want to attach $F(X_0)\times\Delta^3_t$ to $\underrightarrow{\mathrm{hocolim}}^{(2)}(F)$, where Δ^3_t (the "truncated 3-simplex") is the cone over Δ^2_t with its vertex cut off. The attachment map is easily defined, except on the "top" surface resulting from truncating the cone vertex. Hence, the obstruction to defining the attachment map lies in the group

$$\pi_1\Bigl(\mathrm{Map}(F(X_0),F(X_3))\,,\ F(X_0\to X_3)\Bigr).$$

At this point, it becomes necessary to switch from the intuitive picture to formal definitions. First, the truncated simplices Δ^n_t are replaced by cubes I^n. Simplices are then regarded as cubes modulo certain identifications, using the projection maps $\sigma_n : I^n \longrightarrow \Delta^n$ defined by

$$\sigma_n(t_1,\ldots,t_n) = \Bigl(\prod_{i=1}^n(1-t_i),\, t_1{\cdot}\prod_{i=2}^n(1-t_i),\ \ldots,\ t_{n-1}{\cdot}(1-t_n), t_n\Bigr).$$

In this context, it will be useful to define

$$\mu : I^2 \longrightarrow I \qquad \text{by setting} \qquad \mu(s,t) = s+t-st = 1-(1-s)(1-t).$$

For each $n \geq 0$, define $\mathrm{Mor}^n = \mathrm{Mor}^n(\mathcal{C})$ to be the set of all sequences $X_0 \to X_1 \to \cdots \to X_n$ of composable morphisms in $\mathcal{C}$. In particular, $\mathrm{Mor}^0(\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ and

$\mathrm{Mor}^1(\mathcal{C}) = \mathrm{Mor}(\mathcal{C})$. For each $X \in \mathrm{Mor}^n(\mathcal{C})$, we let X_i ($0 \le i \le n$) be the i-th object in the sequence, write

$$X_{ij} = (X_i \to \cdots \to X_j) \in \mathrm{Mor}^{i-j}(\mathcal{C}), \qquad \text{(all } 0 \le i \le j \le n)$$

let $\mathring{X}_{ij} \in \mathrm{Mor}_{\mathcal{C}}(X_i, X_j)$ denote the composite of this sequence of maps; and set

$$\partial_i X = (X_0 \to \cdots \to X_{i-1} \to X_{i+1} \to \cdots \to X_n) \in \mathrm{Mor}^{n-1}(\mathcal{C}) \qquad \text{(all } 0 \le i \le n)$$

and

$$s_i X = \bigl(X_0 \to \cdots \to X_i \xrightarrow{\mathrm{Id}} X_i \to \cdots \to X_n\bigr) \in \mathrm{Mor}^{n+1}(\mathcal{C}). \qquad \text{(all } 0 \le i \le n)$$

Definition B.1. *Fix a functor $F : \mathcal{C} \longrightarrow \mathsf{hoTop}$. An "$A_\infty$-structure" $\overline{F}$ on F consists of maps*

$$\overline{F}(X) \colon I^{n-1} \longrightarrow \mathrm{Map}\bigl(\overline{F}(X_0), \overline{F}(X_n)\bigr),$$

defined for each $n \ge 1$ and each $X \in \mathrm{Mor}^n$, which satisfy the following relations.

(a) $\overline{F}(X_0 \xrightarrow{f} X_1)$ *is in the homotopy class of $F(f)$, and* $\overline{F}(\mathrm{Id}_X) = \mathrm{Id}_{\overline{F}(X)}$.

(b) *For all $n \ge 2$, $X \in \mathrm{Mor}^n$, and $1 \le i \le n-1$,*

$$\overline{F}(X)(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-2}) = \overline{F}(\partial_i X)(t_1, \dots, t_{n-2})$$
$$\overline{F}(X)(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{n-2}) = \overline{F}(X_{in})(t_i, \dots, t_{n-2}) \circ \overline{F}(X_{0i})(t_1, \dots, t_{i-1}).$$

(c) *For all $n \ge 1$, $X \in \mathrm{Mor}^n$, and $0 \le i \le n$,*

$$\overline{F}(s_i X)(t_1, \dots, t_n) = \begin{cases} \overline{F}(X)(t_2, \dots, t_n) & \text{if } i = 0 \\ \overline{F}(X)(t_1, \dots, \mu(t_i, t_{i+1}), \dots, t_n) & \text{if } 1 \le i \le n-1 \\ \overline{F}(X)(t_1, \dots, t_{n-1}) & \text{if } i = n. \end{cases}$$

Note for example that by (c),

$$\overline{F}(X_0 \xrightarrow{\mathrm{Id}} X_0 \xrightarrow{f} X_1) \qquad \text{and} \qquad \overline{F}(X_0 \xrightarrow{f} X_1 \xrightarrow{\mathrm{Id}} X_1)$$

are the constant maps to $\overline{F}(f)$.

Note in particular that when $n = 2$, condition (b) says that $\overline{F}\bigl(X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2\bigr)$ is a homotopy from $\overline{F}(g \circ f)$ to $\overline{F}(g) \circ \overline{F}(f)$. More generally, when $X \in \mathrm{Mor}^n$ for $n \ge 2$,

$$\overline{F}(X)(0, \dots, 0) = \overline{F}(\mathring{X}_{0n}) \quad \text{and} \quad \overline{F}(X)(1, \dots, 1) = \overline{F}(X_{n-1,n}) \circ \cdots \circ \overline{F}(X_{12}) \circ \overline{F}(X_{01}).$$

At the other vertices of I^{n-1}, we get all other possible composites of the $\overline{F}(\mathring{X}_{ij})$. An A_∞-structure on F is thus a series of higher homotopies connecting given homotopies $F(gf) \simeq F(g) \circ F(f)$.

Proposition B.2. *Assume that F is a centric diagram: that*

$$\mathrm{Aut}(F(X_0))_1 \overset{\mathrm{def}}{=} \mathrm{Map}\bigl(F(X_0), F(X_0)\bigr)_{\mathrm{Id}} \xrightarrow{F(f)\circ -} \mathrm{Map}\bigl(F(X_0), F(X_1)\bigr)_{F(f)}$$

is a (weak) homotopy equivalence for each $f : X_0 \to X_1$ in $\mathcal{C}$. For $i \ge 1$, let $\alpha_i : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ denote the functor $\alpha_i(X) = \pi_i\bigl(\mathrm{Aut}(F(X))_{\mathrm{Id}}\bigr)$. Then the obstructions to the existence of an A_∞-structure on F lie in the groups $\varprojlim_{\mathcal{C}}^{i+2}(\alpha_i)$, and the obstructions to uniqueness (up to homotopy) in the groups $\varprojlim_{\mathcal{C}}^{i+1}(\alpha_i)$.

Proof. This follows immediately from the definitions, together with the description of the higher limits as homology groups of a cochain complex

$$\varprojlim_{\mathcal{C}}{}^i(\alpha) \cong H^i(C^*(\mathcal{C};\alpha),\delta), \qquad \text{where} \qquad C^n(\mathcal{C};\alpha) = \prod_{X_0\to\cdots\to X_n} \alpha(X_0)$$

(cf. [GZ, Appendix II, Proposition 3.3]). □

Assume now that $F:\mathcal{C}\longrightarrow \mathsf{hoTop}$ has been given, and that $\overline{F}$ is an A_∞-structure on F. Define the "homotopy colimit" of $\overline{F}$ by setting

$$\underrightarrow{\mathrm{hocolim}}_{\mathcal{C}}(\overline{F}) = \Bigl(\coprod_{n\ge 0}\ \coprod_{X_0\to\cdots\to X_n} \overline{F}(X_0)\times I^n\Bigr)\Big/\sim;$$

where the following face and degeneracy identifications are made for all $X\in\mathrm{Mor}^n$, $1\le i\le n$, and $\alpha\in\overline{F}(X_0)$:

$$\begin{aligned}
\bigl(\alpha;t_1,\dots,t_{i-1},0,t_i,\dots,t_{n-1}\bigr)_{[X]} &\sim \bigl(\alpha;t_1,\dots,t_{n-1}\bigr)_{[\partial_i X]}\\
\bigl(\alpha;t_1,\dots,t_{i-1},1,t_i,\dots,t_{n-1}\bigr)_{[X]} &\sim \bigl(\overline{F}(X_{0i})(t_1,\dots,t_{i-1})(\alpha);t_i,\dots,t_{n-1}\bigr)_{[X_{in}]}\\
\bigl(\alpha;t_1,\dots,t_{n+1}\bigr)_{[s_0X]} &\sim \bigl(\alpha;t_2,\dots,t_{n+1}\bigr)_{[X]}\\
\bigl(\alpha;t_1,\dots,t_{n+1}\bigr)_{[s_iX]} &\sim \bigl(\alpha;t_1,\dots,\mu(t_i,t_{i+1}),\dots,t_{n+1}\bigr)_{[X]}.
\end{aligned}$$

Define $p_{\overline{F}}:\underrightarrow{\mathrm{hocolim}}(\overline{F})\longrightarrow B\mathcal{C}$ to be the union of the maps

$$\sigma_n\circ\mathrm{pr}_2\colon \overline{F}(X_0)\times I^n \longrightarrow \Delta^n.$$

Note that the second identification above, when $i=1$, reduces to

$$\bigl(\alpha;1,t_1,\dots,t_{n-1}\bigr)_{[X]}\sim\bigl(\alpha;t_1,\dots,t_{n-1}\bigr)_{[\partial_0X]}.$$

If $\overline{F}(X)$ is constant for all $X\in\mathrm{Mor}^n$ and all $n\ge 2$, then one easily checks that $\underrightarrow{\mathrm{hocolim}}(\overline{F})$ is homeomorphic to the homotopy colimit of $\overline{F}$ regarded as a functor from $\mathcal{C}$ to Top.

Now, given an A_∞-structure $\overline{F}$ on $F:\mathcal{C}\longrightarrow\mathsf{hoTop}$, define a lifting $\widetilde{F}:\mathcal{C}\longrightarrow\mathsf{Top}$ as follows. For each object X in $\mathcal{C}$, let $\mathcal{C}{\downarrow}X$ be the category whose objects are morphisms $Y\to X$ in $\mathcal{C}$. Define $\widetilde{F}(X)$ to be the pullback of $B(\mathcal{C}{\downarrow}X)$ and $\underrightarrow{\mathrm{hocolim}}(\overline{F})$ over $B\mathcal{C}$. A morphism $f:X\to Y$ induces a map $B(\mathcal{C}{\downarrow}f):B(\mathcal{C}{\downarrow}X)\longrightarrow B(\mathcal{C}{\downarrow}Y)$ by composition, and hence a map $\widetilde{F}(f):\widetilde{F}(X)\longrightarrow\widetilde{F}(Y)$. This clearly makes $\widetilde{F}$ into a functor from $\mathcal{C}$ to Top.

For each X, $\overline{F}(X)$ can be identified with the fiber of the map $\widetilde{F}(X)\longrightarrow B(\mathcal{C}{\downarrow}X)$ over the vertex $\widehat{X}$ corresponding to the identity map $X\xrightarrow{\mathrm{Id}}X$. The composite $F(X)\underset{\simeq}{\xrightarrow{\psi(X)}}\overline{F}(X)\subseteq\widetilde{F}(X)$ defines a natural transformation $F\longrightarrow\mathsf{ho}\circ\widetilde{F}$ of functors $\mathcal{C}\longrightarrow\mathsf{hoTop}$. The following proposition now shows that this is a natural equivalence, and hence that $\widetilde{F}$ is a homotopy lifting of F.

Proposition B.3. *Fix a small category $\mathcal{C}$ and a centric functor $F\colon\mathcal{C}\longrightarrow\mathsf{hoTop}$. Then for any A_∞-structure $\overline{F}$ on F, the functor $\widetilde{F}\colon\mathcal{C}\longrightarrow\mathsf{Top}$ defined above is a homotopy lifting of F. In particular, for each $X\in\mathrm{Ob}(\mathcal{C})$, $\overline{F}(X)$ is a deformation retract of $\widetilde{F}(X)$.*

Proof. Define the deformation retraction directly, by homotoping along the last coordinate. □

Appendix C. Spinor groups over finite fields

Let F be any field of characteristic $\neq 2$. Let V be a vector space over F, and let $\mathfrak{q}\colon V \longrightarrow F$ be a nonsingular quadratic form. As usual, $O(V,\mathfrak{q})$ denotes the group of isometries of $(V,\mathfrak{q})$, and $SO(V,\mathfrak{q})$ the subgroup of isometries of determinant 1. We will be particularly interested in elementary abelian 2-subgroups of such orthogonal groups.

Lemma C.1. *Fix an elementary abelian 2-subgroup $T \leq O(V,\mathfrak{q})$. For each irreducible character $\chi \in T^* = \mathrm{Hom}(T,\{\pm1\})$, let $V_\chi \subseteq V$ denote the corresponding eigenspace: the subspace of elements $v \in V$ such that $g(v) = \chi(g)\cdot v$ for all $g \in T$. Then each V_χ is a nonisotropic subspace of V, and V is the orthogonal direct sum of the V_χ.*

Proof. Elementary. □

Let $\theta_{V,\mathfrak{q}}\colon O(V,\mathfrak{q}) \longrightarrow F/F^{*2}$ denote the spinor norm. As described in [Art, §V.5], $\theta_{V,\mathfrak{q}}$ is characterized by the property that it sends any involution α to the discriminant of the (-1)-eigenspace of α (well defined modulo squares). Here, by the discriminant of a nonisotropic subspace $W \subseteq V$, we mean the discriminant (determinant) of $\mathfrak{q}|_W$.

Set $\Omega(V,\mathfrak{q}) = \mathrm{Ker}(\theta|_{SO(V,\mathfrak{q})})$. Note in particular that $\Omega(V,\mathfrak{q})$ has index 2 in $SO(V,\mathfrak{q})$ if F is a finite field, and that these groups are equal if F is algebraically closed (all units are squares). In general, $\mathrm{Spin}(V,\mathfrak{q})$ is a 2-fold cover of $\Omega(V,\mathfrak{q})$.

Lemma C.2. *Fix an involution $x \in SO(V,\mathfrak{q})$, and let $V = V_+ \oplus V_-$ be its eigenspace decomposition. Then the following hold.*

(a) *$x \in \Omega(V,\mathfrak{q})$ if and only if the discriminant of V_- is a square.*

(b) *If $x \in \Omega(V,\mathfrak{q})$, then it lifts to an element of order 2 in $\mathrm{Spin}(V,\mathfrak{q})$ if and only if $\dim(V_-) \in 4\mathbb{Z}$.*

(c) *If $x \in \Omega(V,\mathfrak{q})$, and if $\alpha \in \Omega(V,\mathfrak{q})$ is such that $[x,\alpha]=1$, then $\alpha = \alpha_+ \oplus \alpha_-$, where $\alpha_\pm \in O(V_\pm,\mathfrak{q})$, and the liftings of x and α commute in $\mathrm{Spin}(V,\mathfrak{q})$ if and only if $\det(\alpha_-) = 1$.*

Proof. Point (a) is part of the definition of $\Omega(V,\mathfrak{q})$. Point (b) is shown in [Sol, Lemma 3.1], and follows from the definition of $\mathrm{Spin}(V,\mathfrak{q})$ via Clifford algebras (cf. [Art]).

It suffices to prove (c) when α is an involution. If $x,y \in \Omega(V,\mathfrak{q})$ are commuting involutions and lift to $\widetilde{x},\widetilde{y} \in \mathrm{Spin}(V,\mathfrak{q})$, then $[\widetilde{x},\widetilde{y}] = 1$ if and only if the number of elements of order four among $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{x}\widetilde{y}$ is even — if and only if the intersection of the (-1)-eigenspaces for x and y is odd dimensional. □

Note that points (b) and (c) above *don't* apply when x or α is in $SO(V,\mathfrak{q}){\smallsetminus}\Omega(V,\mathfrak{q})$.

We will need explicit isomorphisms which describe $\mathrm{Spin}(3,F)$ and $\mathrm{Spin}(4,F)$ in terms of $SL(2,F)$. These are constructed in the following proposition, where $M_2^0(F)$ denotes the vector space of matrices of trace zero. Note that the determinant is a nonsingular quadratic form on $M_2(F)$ and on $M_2^0(F)$, in both cases with square discriminant.

Proposition C.3. *Define*

$$\rho_3\colon PGL(2,F) \longrightarrow SO(M_2^0(F),\det) \cong SO(3,F)$$

and

$$\rho_4\colon \frac{\{(A,B)\in GL(2,F)^2 \mid \det(A)=\det(B)\}}{\{(\lambda I,\lambda I)\mid \lambda\in F^*\}} \longrightarrow SO(M_2(F),\det) \cong SO^+(4,F)$$

by setting

$$\rho_3(A)(X) = AXA^{-1} \qquad \text{and} \qquad \rho_4([A,B])(X) = AXB^{-1}.$$

Then ρ_3 and ρ_4 are both isomorphisms, and restrict to isomorphisms

$$PSL(2,F) \cong \Omega(M_2^0(F),\det) \qquad \text{and} \qquad SL(2,F)^2/\langle(-I,-I)\rangle \cong \Omega(M_2(F),\det).$$

And these in turn lift to unique isomorphisms

$$SL(2,F) \xrightarrow[\cong]{\widetilde{\rho}_3} \underset{\cong \mathrm{Spin}(3,F)}{\mathrm{Spin}(M_2^0(F),\det)} \text{ and } SL(2,F)\times SL(2,F) \xrightarrow[\cong]{\widetilde{\rho}_4} \underset{\cong \mathrm{Spin}^+(4,F)}{\mathrm{Spin}(M_2(F),\det)}.$$

Proof. [**Find reference??**] Note that left or right multiplication by any $A\in GL(2,F)$ is an endomorphism of determinant $\det(A)^2$ on $M_2(F)$. Hence ρ_3 and ρ_4 are monomorphisms to the special orthogonal groups. And since $SL(2,F)$ is perfect, or (if $q=3$) has abelianization of odd order, $\rho_3(PSL(2,F)) \le \Omega(M_2^0(F),\det)$ and similarly for ρ_4.

To see that ρ_3 and ρ_4 are surjective, it suffices to show that each involution in the target groups lies in the images. For example, each involution in $SO(M_2^0(F),\det)$ is of the form S_X (symmetry with respect to X) for some nonisotropic $X\in M_2^0(F)$. Then $X\in GL(2,F)$ and X^2 is a multiple of the identity, and $S_X=\rho_3(X)$. Similarly, for any 2-dimensional nonisotropic subspace $W\subseteq V$ with orthogonal basis $\{Y,Z\}$, then $Y,Z\in GL(2,F)$, and $S_W=\rho_4(ZY^{-1},Y^{-1}Z)$. These observations also show that ρ_3 and ρ_4 both commute with obvious epimorphisms to F/F^{*2}, and hence restrict to isomorphisms on $PSL(2,F)$ and $SL(2,F)^2/\langle(-I,-I)\rangle$ as described.

The liftings to $\mathrm{Spin}(-)$ now follow since the groups in question are universal central extensions (or are universal among central extensions by 2-groups if $F=\mathbb{F}_3$). □

We now restrict to the case $F=\mathbb{F}_q$ where q is an odd prime power. As usual, we write $SO(n,q)=SO(\mathbb{F}_q^n,\mathfrak{q})$ when n is odd, $SO^+(n,q)=SO(\mathbb{F}_q^n,\mathfrak{q})$ if n is even and $\mathfrak{q}$ has discriminant $(-1)^{n/2}$ (mod $\mathbb{F}_q^{*2}$), and $SO^-(n,q)=SO(\mathbb{F}_q^n,\mathfrak{q})$ if n is even and $\mathfrak{q}$ has discriminant $(-1)^{n/2}\cdot u$ for $u\in\mathbb{F}_q^*\smallsetminus\mathbb{F}_q^{*2}$. Note that if n is odd and $\mathfrak{q}$ has nonsquare discriminant, then $u\mathfrak{q}$ has square discriminant for any $u\in\mathbb{F}_q^*\smallsetminus\mathbb{F}_q^{*2}$, and these two forms have the same automorphism groups. This notation extends in the obvious way to $\Omega^\pm(n,q)$ and $\mathrm{Spin}^\pm(n,q)$.

We must classify the conjugacy classes of those elementary abelian 2-subgroups of $\mathrm{Spin}(7,q)$ which contain z. The following definition will be useful when doing this.

Definition C.4. *An elementary abelian 2-subgroup of $SO(7,q)$ will be called of* type I *if its eigenspaces all have square discriminant, and of* type II *otherwise. Let $\mathcal{E}_n$ be the set of elementary abelian 2-subgroups of* $\mathrm{Spin}(7,q)$ *which contain z, and let $\mathcal{E}_n^I$ and $\mathcal{E}_n^{II}$ be the subsets of those of types I and II, respectively.*

Proposition C.5. (a) *All noncentral involutions in* $\mathrm{Spin}(7,q)$ *are conjugate to z_1, and all subgroups in $\mathcal{E}_2$ are conjugate to $T=\langle z,z_1\rangle$.*

(b) *There are two conjugacy classes in* $\mathcal{E}_3$*, one of type I and the other of type II. For* $E \in \mathcal{E}_3$*, its four eigenspaces (the eigenspaces of its image in* $\Omega(7,q)$*) either all have nonsquare discriminant (if* $E \in \mathcal{E}_3^{II}$*) or all have square discriminant (if* $E \in \mathcal{E}_3^{I}$*).*

(c) $\mathcal{E}_4$ *contains two conjugacy classes of subgroups of type I, and one conjugacy class of subgroups of type II, all of which are self-centralizing. If* $E \in \mathcal{E}_4^{II}$*, then there is an element* $z_* \in E$ *such that for* $\chi \in (E)^*$*,* V_χ *has nonsquare discriminant when* $\chi(z_*) = -1$ *and* V_χ *has square discriminant when* $\chi(z_*) = +1$*. If, in addition,* $E \le \mathrm{Spin}(4) \times_{C_2} \mathrm{Spin}(3)$ *(where the decomposition is taken with respect to the eigenspaces of* z_1*), then* $z_* = z_1$ *or* zz_1*.*

(d) *If* $E \in \mathcal{E}_3$ *or* $E \in \mathcal{E}_4^I$*, then* $\mathrm{Aut}_{\mathrm{Spin}}(E)$ *contains all automorphisms of* E *which send* z *to itself.*

(e) *If* $E \in \mathcal{E}_4^{II}$*, then* $\mathrm{Aut}_{\mathrm{Spin}}(E)$ *is the group of those automorphisms which are the identity on* $\langle z, z_* \rangle$*.*

(f) *If* $E \in \mathcal{E}_3^{II}$*, then the Sylow 2-subgroups of* $C_{\mathrm{Spin}}(E)$ *are elementary abelian of rank 4.*

(g) $\mathcal{E}_n = \emptyset$ *for* $n \ge 5$*.*

Proof. Fix an elementary abelian subgroup $E \le \mathrm{Spin}$ such that $z \in E$. Using Lemma C, we see that every involution in $\bar{E}$ has a 4-dimensional (-1)-eigenspace, every rank 2 subgroup of $\bar{E}$ has 2-dimensional eigenspaces for its characters $\chi \ne 1$, and that every rank 3 subgroup of $\bar{E}$ has 1-dimensional eigenspaces for its nontrivial characters. In particular, $\mathrm{rk}(E) \le 4$, and $C_{\mathrm{Spin}}(E) = E$ if $\mathrm{rk}(E) = 4$.

Now assume that E has type II. For each $x \in E \smallsetminus \langle z \rangle$, $\bar{x} \in \Omega(7,q)$ and hence its (-1)-eigenspace of $\bar{x}$ has square discriminant. Thus, $\mathrm{rk}(E) \ge 3$, and if $\mathrm{rk}(E) = 3$ then each of the four eigenspaces of $\bar{E}$ has nonsquare discriminant.

Using this, one sees that if E has rank 4 and type II, then there must be some $z_* \in E$ such that for $\chi \in (\bar{E})^*$, V_χ has nonsquare discriminant exactly when $\chi(z_*) = -1$. If $z_* \notin \langle z, z_1 \rangle$, then for any $x \in E \smallsetminus \langle z, z_1, z_* \rangle$, the intersection of the (-1)-eigenspaces of $\bar{x}$ and $\bar{z}_1$ has nonsquare discriminant, so the components of $\bar{x}$ under the decomposition $SO(4,q) \times SO(3,q)$ do not lie in $\Omega(-,q)$, and thus x is not in $\mathrm{Spin}(4,q) \times_{C_2} (3,q)$.

¿From the above description, we see immediately that if E and E' have the same rank and type, then their images in $SO(7,q)$ are conjugate by an element of $O(7,q)$. Also, each automorphism $\alpha \in \mathrm{Aut}(\bar{E})$, such that $\alpha(z_*) = z_*$ if E has rank 4 and type II, is given by conjugation by some element of $O(7,q)$. And since the central element $-I \in O(7,q)$ has determinant (-1), in both cases we can conjugate by elements of $SO(7,q)$.

In all cases except where $\mathrm{rk}(E) = 4$ and E has type I, there is an element $\gamma \in SO(7,q) \smallsetminus \Omega(7,q)$ which centralizes $\bar{E}$: γ can be chosen to be (-1) on two 1-dimensional nonisotropic summands of two distinct eigenspaces (and $(+1)$ on the orthogonal complement), where one of these summands has nonsquare discriminant and the other has square discriminant. Thus, in these cases, E is Spin-conjugate to all other subgroups of the same rank and type, and the image of $\mathrm{Aut}_{\mathrm{Spin}}(E)$ in $\mathrm{Aut}(\bar{E})$ is as described.

If E has rank 4 and type I, then this also shows that E and $\gamma E \gamma^{-1}$, for any $\gamma \in SO(7,q) \smallsetminus \Omega(7,q)$, are representatives for the two distinct conjugacy classes. And since

$\mathrm{Aut}(\overline{E})$ is simple in this case, all of its elements are induced by conjugation by elements of $\Omega(7,q)$.

We have now determined in all cases the number of conjugacy classes, and the image of $\mathrm{Aut}_{\mathrm{Spin}}(E)$ in $\mathrm{Aut}(\overline{E})$. If $\mathrm{rk}(E) \leq 3$ or $E \in \mathcal{E}_4^I$, then one easily sees that all elements of $\mathrm{Aut}(E)$ which send z to itself and induce the identity on $\overline{E}$ are induced by conjugation by elements of Spin. So in all of these cases, $\mathrm{Aut}_{\mathrm{Spin}}(E)$ is the group of all automorphisms which send z to itself.

It remains only to prove (e) and (f).

(f) Now assume $E \in \mathcal{E}_3^{II}$. Let $1 = \chi_1, \chi_2, \chi_3, \chi_4$ be the four characters of $\overline{E}$, and set $V_i = V_{\chi_i}$. Then V_i has nonsquare discriminant for each i, $\dim(V_1) = 1$, and $\dim(V_i) = 2$ for $i = 2, 3, 4$. Thus $O(V_1, \mathfrak{q}_1) = \{\pm \mathrm{Id}\}$, while for $i = 2, 3, 4$, $O(V_i, \mathfrak{q}_i) \cong O^{\epsilon}(2, q)$ where $\epsilon = (-1)^{(q+1)/2}$. So by **[REF???]**, $O(V_i, \mathfrak{q}_i)$ is dihedral for $i = 2, 3, 4$, of order $2(q+1)$ if $q \equiv 1 \pmod 4$, or of order $2(q-1)$ if $q \equiv 3 \pmod 4$. In either case, $SO(V_i, \mathfrak{q}_i)$ is the cyclic subgroup of index 2 and its only 2-power torsion is $-\mathrm{Id}$, while $\Omega(V_i, \mathfrak{q}_i)$ has odd order. So any element $\alpha \in C_{\mathrm{Spin}}(E) \smallsetminus E$ of 2-power order is of the form $\bar{\alpha} = \bigoplus_{i=1}^4 \alpha_i$ where $\alpha_1 = -\mathrm{Id}$ and $\alpha_i \in O(V_i, \mathfrak{q}_i)$ has determinant (-1). Also, if α' is any other such element, then $\alpha'_i = -\alpha_i$ for exactly two of the $i = 2, 3, 4$ (since otherwise $\bar{\alpha}'$ has nontrivial Spinor norm). And thus the (unique) Sylow 2-subgroup of $C_{\mathrm{Spin}}(E)$ is elementary abelian of rank 4.

(e) It remains to consider the case where E has rank 4 and type II. Fix elements $z_2, z_3 \in E$ such that $\{z, z_*, z_2, z_3\}$ is a basis, write $z_1 = z_*$, and let $\chi_{ijk} \in (\overline{E})^*$ (for $i, j, k \in \{0, 1\}$) be the character $\chi_{ijk}(z_1) = (-1)^i$, etc. Choose some nonsquare $u \in \mathbb{F}_q^*$; and for each $\chi \in (\overline{E})^*$, fix $v_\chi \in V_\chi$ such that $\mathfrak{q}(v_\chi) = u$ if $\chi(z_*) = -1$ and $\mathfrak{q}(v_\chi) = 1$ if $\chi(z_*) = 1$. Write $v_{\chi_{ijk}} = v_{ijk}$ for short.

Define $\alpha \in SO(7, q)$ by setting $\alpha(v_{i01}) = v_{i10}$, $\alpha(v_{i10}) = v_{i01}$, $\alpha(v_{i11}) = -v_{i11}$, and $\alpha(v_{100}) = v_{100}$. The (-1)-eigenspace of α has basis the set

$$\{v_{001} - v_{010},\ v_{101} - v_{110},\ v_{011},\ v_{111}\}.$$

The discriminant of this subspace is thus $(2)(2u)(1)(u) = (2u)^2$, so $\alpha \in \Omega(7, q)$. Also, the intersection of this eigenspace with the (-1)-eigenspace of $z_1 = z_*$ is 2-dimensional, and so α lifts to an element $\widetilde{\alpha} \in \mathrm{Spin}(7, q)$ such that $\alpha(z_*) = z_*$ (Lemma C). And modulo z, the action of $\widetilde{\alpha}$ on E switches z_2 and z_3.

There is clearly an element of order 3 in $N_{\Omega(7,q)}(E)$ which centralizes z and z_* and permutes z_2, z_3, and $z_2 z_3$ cyclically. Also, one easily constructs elements in $N_{\mathrm{Spin}}(E)$ which are the identity on $\langle z, z_* \rangle$ and on $E/\langle z, z_* \rangle$. So it remains only to show that if $\alpha \in N_{\mathrm{Spin}}(E)$ is the identity on $\overline{E} = E/\langle z \rangle$, then $[\alpha, z_*] = 1$. Any such α must act as ± 1 on each V_χ, and since the discriminant of its (-1)-eigenspace is a square, that eigenspace must have even dimensional intersection with the (-1)-eigenspace of z_*. And this shows that $[\alpha, z_*] = 1$. □

The following lemma is much more technical, and is also needed when constructing a Frobenius system for the Solomon groups.

Lemma C.6. *Fix $k \geq 2$. Let $e_{13}(2^{k-1}) \in GL(3, \mathbb{Z}/2^k)$ be the elementary matrix $A = I + X$ which has off diagonal entry 2^{k-1} in position $(1,3)$. Let P_1 and P_2 be the two maximal parabolic subgroups*

$$P_1 = P(1,2) = \{(a_{ij}) \mid a_{21} = a_{31} = 0\} \quad \text{and} \quad P_2 = P(2,1) = \{(a_{ij}) \mid a_{31} = a_{32} = 0\},$$

and set $P_0 = P_1 \cap P_2$, the group of upper triangular matrices. Assume that

$$\psi_i \colon P_i \longrightarrow SL(3, \mathbb{Z}/2^k)$$

are sections (for $i = 1, 2$) such that $\psi_1|_{P_0} = \psi_2|_{P_0}$. Then there is a homomorphism

$$\psi \colon GL(3, 2) \longrightarrow SL(3, \mathbb{Z}/2^k)$$

such that $\psi|_{P_1} = \psi_1$, and either $\psi|_{P_2} = \psi_2$, or $\psi|_{P_2} = c_A \circ \psi_2$.

Proof. We first claim that any two sections $\sigma, \sigma' \colon P_2 \longrightarrow SL(3, \mathbb{Z}/2^k)$ are conjugate by an element of $SL(3, \mathbb{Z}/2^k)$. This clearly holds when $k = 1$, and so we can assume inductively that $\sigma \equiv \sigma' \pmod{2^{k-1}}$. Define $\rho \colon P_2 \longrightarrow M_3^0(\mathbb{F}_2)$ via the formula

$$\sigma'(A) = \sigma(A) \cdot (I + 2^{k-1}\rho(A))$$

for $A \in P_2$. Then ρ is a 1-cocycle. Also, $H^1(P_2; M_3^0(\mathbb{F}_2)) = 0$ by [DW2, Lemma 4.3] (the module is $\mathbb{F}_2[P_2]$-projective), so ρ is the coboundary of some $X \in M_3^0(\mathbb{F}_2)$, and σ and σ' differ by conjugation by $I + 2^{k-1}X$.

By [DW2, Theorem 4.1], there exists a section ψ defined on $GL(3, 2)$ such that $\psi|_{P_1} = \psi_1$. Let $A \in SL(3, \mathbb{Z}/2^k)$ be such that $\psi|_{P_2} = c_A \circ \psi_2$. Since $\psi|_{P_0} = \psi_2|_{P_0}$, A must commute with all elements in $\psi(P_0)$. And one easily checks that the only elements which commute with $\psi(P_0)$ are $e_{13}(2^{k-1})$ and the identity. □

References

[Alp] J. Alperin, Local representation theory, Cambridge Univ. Press (1986)

[Art] E. Artin, Geometric algebra, Interscience (1957)

[Be] D. Benson, Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants, Geometry and cohomology in group theory, London Math. Soc. Lecture notes ser. 252, Cambridge Univ. Press (1998), 10–23

[BK] P. Bousfield & D. Kan, Homotopy limits, completions, and localizations, Lecture notes in math. 304, Springer-Verlag (1972)

[BrK] C. Broto & N. Kitchloo, Classifying spaces of Kac-Moody groups, preprint

[BL] C. Broto, R. Levi, On spaces of self homotopy equivalences of p-completed classifying spaces of finite groups and homotopy group extensions, Topology (to appear)

[BLO] C. Broto, R. Levi, & B. Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, preprint

[BrM] C. Broto & J. Møller, [title to be announced]

[Car] R. Carter, Simple groups of Lie type, Wiley (1989)

[Dw] W. Dwyer, The centralizer decomposition of BG, in: Algebraic topology: new trends in localization and periodicity, Progr. Math. 136, Birkhäuser (1996), 167–184

[DK] W. Dwyer & D. Kan, Realizing diagrams in the homotopy category by means of diagrams of simplicial sets, Proc. Amer. Math. Soc. 91 (1984), 456–460

[DK2] W. Dwyer & D. Kan, Centric maps and realizations of diagrams in the homotopy category, Proc. Amer. Math. Soc. 114 (1992), 575–584

[DW1] W. Dwyer & C. Wilkerson, A cohomology decomposition theorem, Topology 31 (1992), 433–443

[DW2] W. Dwyer & C. Wilkerson, A new finite loop space at the prime two, J. Amer. Math. Soc. 6 (1993), 37–64

[DW3] W. Dwyer & C. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces, Annals of Math. 139 (1994), 395–442

[GZ] P. Gabriel & M. Zisman, Calculus of fractions and homotopy theory, Springer-Verlag (1967)

[GJ] P. Goerss & J. Jardine, Simplicial homotopy theory, Birkhäuser (1999)

[Go] D. Gorenstein, Finite groups, Harper & Row (1968)

[HLS] H.-W. Henn, J. Lannes, & L. Schwartz, The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects, Amer. J. Math. 115 (1993), 1053–1106

[HV] J. Hollender & R. Vogt, Modules of topological spaces, applications to homotopy limits and E_∞ structures, Arch. Math. 59 (1992), 115–129

[JM] S. Jackowski & J. McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups, Topology 31 (1992), 113–132

[JMO] S. Jackowski, J. McClure, & B. Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math. 135 (1992), 184–270

[JMO2] S. Jackowski, J. McClure, & B. Oliver, Homotopy theory of classifying spaces of compact Lie groups, Algebraic topology and its applications, M.S.R.I. Publ. 27, Springer-Verlag (1994), 81–123

[La] J. Lannes, Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire, Publ. I.H.E.S. 75 (1992)

[LS] J. Lannes & L. Schwartz, Sur la structure des A-modules instables injectifs, Topology 28 (1989), 153–169

[Lk] M. Linckelmann, Transfer in Hochschild cohomology of blocks of finite groups, Algebr. Repres. Theory 2 (1999), 107–135

[LW] M. Linckelmann & P. Webb,

[Ol] B. Oliver, Higher limits via Steinberg representations, Comm. in Algebra 22 (1994), 1381-1393

[Pu] L. Puig, Unpublished notes.

[Qu] D. Quillen, The spectrum of an equivariant cohomology ring: I, Annals of Math. 94 (1971), 549–572

[Se] G. Segal, Categories and cohomology theories, Topology 13 (1974), 293–312

[Sol] R. Solomon, Finite groups with Sylow 2-subgroups of type .3, J. Algebra 28 (1974), 182–198