ON TOPOLOGICAL INVARIANTS ASSOCIATED TO A POLYNOMIAL WITH COMPACT CRITICAL SET

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ABSTRACT. In this paper, we consider a polynomial $G : W \to \mathbb{R}$ where $W \subset \mathbb{R}^n$ is a smooth algebraic variety of dimension $n-k$ defined by the vanishing of k polynomials. Under the assumption that the set of critical points of G is compact, we express $\chi(G^{-1}(0))$ and $\chi(\{G \geq 0\}) - \chi(\{G \leq 0\})$ in terms of topological degrees of polynomials mappings. For the generic case of a finite set of critical points, using the theory of Frobenius algebras, we express $\chi(G^{-1}(0))$ and $\chi(\{G \geq 0\}) - \chi(\{G \leq 0\})$ in terms of signatures of bilinear symmetric forms.

1. INTRODUCTION

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial mapping and let $W = F^{-1}(0)$. Let $G_1, \ldots, G_l$ be polynomials. An interesting problem is the computation $\chi(W)$ and $\chi\left(W \cap \{G_1 \geq 0, \ldots, G_l \geq 0\}\right)$ in terms of the polynomials F_i and G_j.

When W is compact, Bruce [Bru] and Szafraniec [Sz1] proved that there exists a polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ with an algebraically isolated critical point at the origin such that

$$\chi(W) = \frac{1}{2}((-1)^n - \deg_0 \nabla P),$$

where $\deg_0 \nabla P$ is the topological degree at the origin of the gradient of P. They also obtained a formula for the non-compact case, but, unfortunately, this formula is impractical, so one has to find an other method for the case W non-compact.

This has already been done in [DLNS, Sz3, Sz4], when $1 \leq k < n$ and W is a smooth manifold of dimension $n-k$. In [Sz3], Szafraniec constructed a polynomial map $H : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ and proved that $H^{-1}(0) \subset B_R^{n+k}$, where B_R^{n+k} is a ball in $\mathbb{R}^{n+k}$ centered at the origin with sufficiently big radius R, and that $\chi(W) = (-1)^k \deg\, h$, where $h = H/||H|| : S_R^{n+k-1} \to S^{n+k-1}$ and

Date: 05-12-00 .
Mathematics Subject Classification (2000) : 14P25 .

S_R^{n+k-1} is the sphere of radius R. In [DLNS], they considered a polynomial algebra A and they got, assuming that $\dim_{\mathbb{R}} A < +\infty$,

$$\chi(W) \equiv \dim_{\mathbb{R}} A \bmod 2. \tag{1}$$

Using the theory of Frobenius algebras, Szafraniec [Sz4] refined this formula and proved that there are two bilinear symmetric forms Φ and Φ_M on A such that

$$\begin{array}{lccc} \text{if } n-k \text{ is odd} & \chi(W) & = & (-1)^k \text{signature } \Phi, \\ \text{if } n-k \text{ is even} & \chi(W) & = & \text{signature } \Phi_M. \end{array} \tag{2}$$

We generalized formulas (1) and (2) above in two different ways. In [Dut1] and [Dut3], keeping the assumption that W is a smooth manifold of codimension k, we considered a polynomial $g : W \to \mathbb{R}$, such that $g_{|W}$ is proper and $g^{-1}(0) \cap W$ has a finite number of singularities in [Dut1], and such that $g^{-1}(0) \cap W$ is smooth and non compact in [Dut3]. We expressed the Euler characteristic of $W \cap \{g \geq 0\}$ and of $W \cap \{g \leq 0\}$ in terms of signatures of suitable bilinear symmetric forms.

In [Dut4], we investigated the case when W admits a finite number of singularities. We first generalized formula (1) above and we obtained

$$\chi(W) + \Sigma_\mu \equiv \dim_{\mathbb{R}} A \bmod 2,$$

where Σ_μ is the sum of the Milnor numbers at the singularities of W. Then we generalized formulas (2) but only in the cases of curves ($k = n-1$) and of odd-dimensional hypersurfaces ($k = 1$ and n is even).

The aim of this paper is to give formulas for $\chi(W \cap \{G = 0\})$, $\chi(W \cap \{G \geq 0\})$ and $\chi(W \cap \{G \leq 0\})$, where W is smooth and $G : \mathbb{R}^n \to \mathbb{R}$ is a polynomial such that $G_{|W}$ has a compact set of critical points. Taking $W = \mathbb{R}^n$, we will recover the case of all hypersurfaces with isolated singularities.

First, let us describe the situation and give some notations. Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial mapping and let $W = F^{-1}(0)$. We assume that W is a smooth $(n-k)$-dimensional manifold. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let Σ_G be the set of critical points of $G_{|W}$. We assume that Σ_G is compact and we denote $\Sigma = \Sigma_G \cap G^{-1}(0)$.

Let $(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k, \mu) = (x; \lambda; \mu)$ be a coordinate system in $\mathbb{R}^{n+k+1}$ and let H, K, L_1, L_2, $M_1 : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ and L_0 and $M_0 : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be defined in the following way :

$$H(x;\lambda;\mu) = \left(\mu x + \sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \nabla G(x), F(x), G(x) \right),$$

$$K(x;\lambda;\mu) = \left(\mu x + \sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \nabla G(x), F(x), \mu G(x) \right),$$

$$L_1(x;\lambda;\mu) = \left(\sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \nabla G(x), F(x), \mu G(x) - 1\right),$$

$$L_2(x;\lambda;\mu) = \left(\sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \nabla G(x), F(x), \mu G(x)^2 - 1\right),$$

$$M_1(x;\lambda;\mu) = \left(\sum_{i=1}^{k} \lambda_i \nabla F_i(x) + x, F(x), \mu G(x) - 1\right),$$

$$L_0(x;\lambda) = \left(\sum_{i=1}^{k} \lambda_i \nabla F_i(x) + \nabla G(x), F(x)\right),$$

$$M_0(x;\lambda) = \left(\sum_{i=1}^{k} \lambda_i \nabla F_i(x) + x, F(x)\right).$$

We will adopt the following notations.

- For all $R > 0$ and all integers $N > 0$, $B_R^N = \{x \in \mathbb{R}^N \mid \|x\| < R^2\}$ and $S_R^{N-1} = \partial \overline{B_R^N}$. When $R = 1$, we shall omit it.
- For all integer $N > 0$, for all function $f : \mathbb{R}^N \to \mathbb{R}$, ∇f is the gradient of f, and for all $i, j \in \{1, \ldots, N\}$, $f_{x_i} = \frac{\partial f}{\partial x_i}$, $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_i}$.
- For all integer $N > 0$, for all function $f : \mathbb{R}^N \to \mathbb{R}$, $\{f \geq 0\} = \{x \in \mathbb{R}^N \mid f(x) \geq 0\}$.
- For all integer $N > 0$ and for all map $H : \mathbb{R}^N \to \mathbb{R}^N$, JH denotes the Jacobian of H.
- For all integer $N > 0$, for all N-dimensional smooth manifold M with boundary ∂M and for all mapping $H : M \to \mathbb{R}^N$ such that H never vanishes on ∂M, deg (H, M) is the topological degree of

$$\frac{H}{\|H\|} : \partial M \to S^{N-1}.$$

  When $H : \mathbb{R}^N \to \mathbb{R}^N$ and $M = \overline{B_R^N}$, where $R \gg 0$ is such that $H^{-1}(0) \subset B_R^N$, we shall write deg H. When $M = B_\varepsilon^N$, $0 < \varepsilon \ll 1$, is such that $H^{-1}(0) \cap B_\varepsilon^N = \{0\}$, we shall write $\text{deg}_0 H$.

We will prove that there exists $R \gg 0$ such that $H^{-1}(0) \subset B_R^{n+k+1}$, $K^{-1}(0) \subset B_R^{n+k+1}$, $L_1^{-1}(0) \subset B_R^{n+k+1}$, $L_2^{-1}(0) \subset B_R^{n+k+1}$, and $M_1^{-1}(0) \subset B_R^{n+k+1}$. Since Σ_G is supposed to be compact, we can also choose R such that $L_0^{-1}(0) \subset B_R^{n+k}$. Under the assumption that $G^{-1}(0) \cap W \cap \{T_0^{-1}\} \cap \mathbb{R}^n = \emptyset$, we will show that (see Theorem 4.7 and Corollary 4.10), if $n - k$ is even

- $\chi(W \cap G^{-1}(0)) - (-1)^k (\text{deg } L_0 - \text{deg } L_2) = (-1)^k \text{deg } H$,

- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) - (-1)^k \text{deg } L_1 =$
$(-1)^k \text{deg } M_1 - (-1)^k \text{deg } K,$

if $n-k$ is odd

- $\chi(W \cap G^{-1}(0)) + (-1)^k \text{deg } L_1 = (-1)^k \text{deg } K,$
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) + (-1)^k (\text{deg } L_0 - \text{deg } L_2) =$
$(-1)^k \text{deg } M_1 - (-1)^k \text{deg } H.$

As a corollary (see Corollary 4.11), we will obtain that there exists polynomials mappings H', K', L'_1, L'_2, $M'_1 : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ and $L'_0 : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, defined in a explicit way in terms of F and G, with an isolated singularity at the origin such that, if $n-k$ is even

- $\chi(W \cap G^{-1}(0)) - (-1)^k (\text{deg}_0 L'_0 - \text{deg}_0 L'_2) = (-1)^k \text{deg}_0 H',$
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) - (-1)^k \text{deg}_0 L'_1 =$
$(-1)^k \text{deg}_0 M'_1 - (-1)^k \text{deg}_0 K',$

if $n-k$ is odd

- $\chi(W \cap G^{-1}(0)) + (-1)^k \text{deg}_0 L'_1 = (-1)^k \text{deg}_0 K',$
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) + (-1)^k (\text{deg}_0 L'_0 - \text{deg}_0 L'_2) =$
$(-1)^k \text{deg}_0 M'_1 - (-1)^k \text{deg}_0 H'.$

Unfortunately, these last formulas are difficult to implement because the polynomials involved consist of a large number of monomials. However, when the polynomials H, K, L_0, L_1, L_2, M_1 all have isolated zeros in $\mathbb{C}^{n+k+1}$ or $\mathbb{C}^{n+k}$, it is possible to express the different topological degrees in terms of signatures of bilinear symmetric forms defined over Frobenius algebras (see Corollary 4.12). These formulas are really more effective.

In the last section, we will focus on the case $W = \mathbb{R}^n$. We will define two numerical invariants $\nu_G^{0,+}$ and $\nu_G^{0,-}$ associated to the fiber $G^{-1}(0)$ and we will prove (see Corollary 5.3), that for R sufficiently big,

$$\chi(\{G \leq 0\} \cap S_R^{n-1}) = 1 - \text{deg } \nabla G + \nu_G^{0,+} - \nu_G^{0,-}.$$

Using the previous results, we will be able to express $\chi(\{G \leq 0\} \cap S_R^{n-1})$, $\nu_G^{0,+}$ and $\nu_G^{0,-}$ in terms of topological degrees.

It is proper to mention that this work was motivated by previous papers on topological invariants of real semi-analytic sets. Germs of analytic curves were studied in [AFS], [AFN1], [AFN2], [Da1], [Da2], [Da3], [Mvs], [Sz2]. Germs of hypersurfaces were studied in [Ar], [Fu1], [Fu2], [GM1], [GM2],

[Kh] and [Wa]. One can also find results on germs of complete intersections in [ASV], [DLNS], [Dut5], [Dut6], [EG], [Wa] and [Sz6].

The paper is organized as follows : in Section 2, we recall some facts about Morse theory for manifolds with boundary ; Section 3 and Section 4 are devoted to the proof of our main result. In Section 5, we study the case $W = \mathbb{R}^n$. Some computations are given along the paper. They have been done with a program written by Andrzej Lecki. The author is very grateful to him and Zbigniew Szafraniec for giving him this program. He also thanks Karim Bekka for his helpful remarks and comments.

This research has been supported by a Marie Curie Fellowship of The European Community programme *Improving Human Research Potential and the Socio-economic Knowledge Base* under contract number *HPMFCT-2000-00645*. It was done at the *Centre de Recerca Matemàtica* in Barcelona. The author wants to thank this institution for its hospitality.

2. Morse theory for manifolds with boundary

We recall the results of Morse theory for manifolds with boundary. Our reference is [HL] where the results are given for a C^∞ manifold M with boundary ∂M. For simplicity we will present the results for manifolds with boundary of type $M \cap \{g * 0\}$, $* \in \{\geq, \leq\}$, where M is a C^∞ manifold and $g : M \to \mathbb{R}$ a C^∞ function such that $M \cap g^{-1}\{0\}$ is smooth. In fact this is the case we need in the following sections.

Let M be a C^∞ manifold of dimension n. Let $g : M \to \mathbb{R}$ be a C^∞ function such that $\nabla g(x) \neq 0$ for all $x \in g^{-1}(0)$. This implies that $M \cap g^{-1}(0)$ is a smooth manifold of dimension $n-1$ and that $M \cap \{g \geq 0\}$ and $M \cap \{g \leq 0\}$ are smooth manifolds with boundary. Let $f : M \to \mathbb{R}$ be a smooth function. A critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) is a critical point of $f_{|M\cap\{g>0\}}$ (resp. $f_{|M\cap\{g<0\}}$) or a critical point of $f_{|M\cap g^{-1}(0)}$.

Definition 2.1. *Let $q \in M \cap g^{-1}(0)$. We say that q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) if q is a critical point of $f_{|M\cap g^{-1}(0)}$ and q is not a critical point of $f_{|M}$.*

We say that q is a correct non-degenerate critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) if q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) and q is a non-degenerate critical point of $f_{|M\cap g^{-1}(0)}$.

If q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ (resp. $f_{|M\cap\{g\leq 0\}}$) then $\nabla f(q) \neq 0$, $\nabla f(q)$ and $\nabla g(q)$ are colinear and there is $\tau(q) \in \mathbb{R}^*$ with $\nabla f(q) = \tau(q) \cdot \nabla g(q)$.

Definition 2.2. *If q is a correct critical point of $f_{|M\cap\{g\geq 0\}}$ then*

- *$\nabla f(q)$ points inwards if and only if $\tau(q) > 0$,*
- *$\nabla f(q)$ points outwards if and only if $\tau(q) < 0$.*

If q is a correct critical point of $f_{|M\cap\{g\leq 0\}}$ then

- *$\nabla f(q)$ points inwards if and only if $\tau(q)<0$,*
- *$\nabla f(q)$ points outwards if and only if $\tau(q)>0$.*

Definition 2.3. *A C^∞ function $f: M\cap\{g\geq 0\}\to\mathbb{R}$ (resp. $M\cap\{g\leq 0\}\to\mathbb{R}$) is a correct function if all critical points of $f_{|M\cap g^{-1}(0)}$ are correct. A C^∞ function $f: M\cap\{g\geq 0\}\to\mathbb{R}$ (resp. $M\cap\{g\leq 0\}\to\mathbb{R}$) is a Morse correct function if $f_{|M\cap\{g>0\}}$ (resp. $f_{|M\cap\{g<0\}}$) admits only non-degenerate critical points and if f admits only non-degenerate correct critical points.*

Proposition 2.4. *For any C^∞ manifold M and for any function $g: M\to\mathbb{R}$ such that $\nabla g(x)\neq 0$ for all $x\in g^{-1}(0)$, the set of C^∞ functions $f: M\to\mathbb{R}$ such that $f_{|M\cap\{g\geq 0\}}$ and $f_{|M\cap\{g\leq 0\}}$ are Morse correct functions is dense in $C^\infty(M,\mathbb{R})$.*

We will denote $\chi\Big(M\cap\{g*0\}\cap\{f?0\}\Big)$ by $\chi_{*,?}$ and we will use the following result.

Theorem 2.5. *Let M be a C^∞ manifold of dimension n and let $g: M\to\mathbb{R}$ be a C^∞ function such that $\nabla g(x)\neq 0$ for all $x\in g^{-1}(0)$. Let $f: M\to\mathbb{R}$ be a C^∞ function such that $f_{|M}$ is proper, and that $f_{|M\cap\{g\geq 0\}}$ and $f_{|M\cap\{g\leq 0\}}$ are Morse correct. Let $\{p_i\}$ be the set of critical points of $f_{|M}$ and $\{\lambda_i\}$ be the set of their respective indices. Let $\{q_j\}$ be the set of critical points of $f_{|M\cap g^{-1}(0)}$ and $\{\mu_j\}$ be the set of their respective indices. Assume that $\{p_i\}$ and $\{q_j\}$ are finite. Then we have*

$$\chi_{\geq,\geq}-\chi_{\geq,=}=\sum_{\substack{i/f(p_i)>0\\ g(p_i)>0}}(-1)^{\lambda_i}+\sum_{\substack{j/f(q_j)>0\\ \tau(q_j)>0}}(-1)^{\mu_j},$$

$$\chi_{\geq,\leq}-\chi_{\geq,=}=(-1)^n\sum_{\substack{i/f(p_i)<0\\ g(p_i)>0}}(-1)^{\lambda_i}+(-1)^{n-1}\sum_{\substack{j/f(q_j)<0\\ \tau(q_j)<0}}(-1)^{\mu_j},$$

and

$$\chi_{\leq,\geq}-\chi_{\leq,=}=\sum_{\substack{i/f(p_i)>0\\ g(p_i)<0}}(-1)^{\lambda_i}+\sum_{\substack{j/f(q_j)>0\\ \tau(q_j)<0}}(-1)^{\mu_j},$$

$$\chi_{\leq,\leq}-\chi_{\leq,=}=(-1)^n\sum_{\substack{i/f(p_i)<0\\ g(p_i)<0}}(-1)^{\lambda_i}+(-1)^{n-1}\sum_{\substack{j/f(q_j)<0\\ \tau(q_j)>0}}(-1)^{\mu_j}.$$

□

For the following result we assume that $f\geq 0$. For each $a\in\mathbb{R}$, we denote $\chi\left(M\cap\{g*0\}\cap\{f?a\}\right)$ by $\chi_{*,?a}$.

Corollary 2.6. *Under the same assumptions and the fact that $f \geq 0$, we have that $\{f=0\} \cap \{g=0\}$ is empty and for each $a>0$*

$$\chi_{\geq,\leq a} = \sum_{\substack{i/f(p_i)<a \\ g(p_i)>0}} (-1)^{\lambda_i} + \sum_{\substack{j/f(q_j)<a \\ \tau(q_j)>0}} (-1)^{\mu_j},$$

$$\chi_{\leq,\leq a} = \sum_{\substack{i/f(p_i)<a \\ g(p_i)<0}} (-1)^{\lambda_i} + \sum_{\substack{j/f(q_j)<a \\ \tau(q_j)<0}} (-1)^{\mu_j}.$$

Proof. The level $\{f=0\}$ is a critical level of f since $f \geq 0$. So $\{f = 0\} \cap \{g=0\}$ must be empty, for otherwise f would not be Morse correct. This proves the first assertion. For the second one, we have that, on the one hand, since $f \geq 0$,

$$\chi(\{g \geq 0\}) = \sum_{i/g(p_i)>0} (-1)^{\lambda_i} + \sum_{j/\tau(q_j)>0} (-1)^{\mu_j}.$$

On the other hand,

$$\chi(\{g \geq 0\}) = \chi_{\geq,\geq a} + \chi_{\geq,\leq a} - \chi_{\geq,=a}.$$

To get the result, it enough to compute $\chi_{\geq,\geq a} - \chi_{\geq,=a}$ using the above theorem applied to g and $f-a$. □

3. Preliminaries

In this section, we give some preliminaries lemmas. First we characterize a non-degenerate correct critical point of an analytic function defined on an analytic manifold with boundary. We also relate its Morse index to a local topological degree.

Let $(f,g) = (f_1,\ldots,f_k,g) : \mathbb{R}^n \to \mathbb{R}^{k+1}$, $0 \leq k < n-1$, be an analytic mapping. Let $p \in \mathbb{R}^n$ be such that $f_1,\ldots,f_k$ and g vanish at p and that rank $(\nabla f_1,\ldots,\nabla f_p,\nabla g) = k+1$ at p. From the implicit function theorem, $f^{-1}(0)$ is a smooth $(n-k)$-manifold in the neighborhood of p and $f^{-1}(0) \cap g^{-1}(0)$ is a smooth $(n-k-1)$-manifold in the neighborhood of p. Let $\omega : (\mathbb{R}^n,p) \to (\mathbb{R},\omega(p))$ be an analytic function defined in the neighborhood of p.

Let $(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_k;\mu) = (x;\lambda;\mu)$ be a coordinate system in $\mathbb{R}^n \times \mathbb{R}^{k+1}$ and let $H : \mathbb{R}^n \times \mathbb{R}^{k+1} \to \mathbb{R}^n \times \mathbb{R}^{k+1}$ be the analytic mapping given by

$$H(x;\lambda;\mu) = \left(\mu\nabla\omega(x) + \nabla g(x) + \sum_{i=1}^{k} \lambda_i f_i(x); f(x); g(x)\right).$$

We shall study the situation at the point p.

Lemma 3.1. *The function* $\omega_{|f^{-1}(0)\cap\{g\geq 0\}}$ *(or* $\omega_{|f^{-1}(0)\cap\{g\leq 0\}}$*) admits a correct critical point at* p *if and only if there exists a unique* $(\lambda_1,\ldots,\lambda_k,\mu) = (\lambda;\mu) \in \mathbb{R}^k \times \mathbb{R}^*$ *such that* $H(p,\lambda,\mu)=0$.

Proof. If there exists a unique $(\lambda,\mu)\in\mathbb{R}^{k+1}$ with $\mu\neq 0$ and $H(p,\lambda,\mu) = 0$ then

$$\nabla\omega = -\frac{1}{\mu}\nabla g - \sum_{i=1}^{k}\frac{\lambda_i}{\mu}\nabla F_i \text{ at } p.$$

Hence p is a critical point of $\omega_{|f^{-1}(0)\cap g^{-1}(0)}$. Furthermore, since $\frac{1}{\mu}\neq 0$, p is a correct critical point of $\omega_{|f^{-1}(0)\cap\{g\geq 0\}}$ (or $\omega_{|f^{-1}(0)\cap\{g\leq 0\}}$).

Conversely, if p is a correct critical point of $\omega_{|f^{-1}(0)\cap\{g\geq 0\}}$ (or $\omega_{|f^{-1}(0)\cap\{g\leq 0\}}$) then p is a critical point of $\omega_{|f^{-1}(0)\cap g^{-1}(0)}$ and there exists a unique $(\lambda;\nu)\in \mathbb{R}^k\times\mathbb{R}^*$ such that

$$\nabla\omega + \nu\nabla g + \sum_{i=1}^{k}\lambda_i\nabla g_i = 0 \text{ at } p.$$

It is clear that $H(p;\lambda/\nu;1/\nu) = 0$ and that $(\lambda/\nu;1/\nu)$ is uniquely determined. □

Lemma 3.2. *The function* $\omega_{|f^{-1}(0)\cap\{g\geq 0\}}$ *(or* $\omega_{|f^{-1}(0)\cap\{g\leq 0\}}$*) admits a Morse correct critical point at* p *if and only if there exists a unique* $(\lambda_1,\ldots,\lambda_k,\mu) = (\lambda;\mu)\in\mathbb{R}^k\times\mathbb{R}^*$ *such that* $H(p;\lambda;\mu)=0$ *and* $JH(p;\lambda;\mu)\neq 0$. *Furthermore, if* s *is the Morse index of* $\omega_{|f^{-1}(0)\cap g^{-1}(0)}$ *at* p,

$$(-1)^s = (-1)^k \times \text{sign } (\mu)^{n-k} \times \text{sign } JH(p;\lambda;\mu).$$

Proof. Let $\tilde{H}:\mathbb{R}^{n+k+1}\to\mathbb{R}^{n+k+1}$ be defined by

$$\tilde{H}(x;\lambda;\mu) = \left(\nabla\omega(x) + \mu\nabla g(x) + \sum_{i=1}^{k}\lambda_k\nabla f_k(x), f(x), g(x)\right).$$

In [Sz3], Szafraniec proves in Lemma 1.4 that $\omega_{|f^{-1}(0)\cap g^{-1}(0)}$ has a Morse critical point at p if and only if there is a unique $(\tilde{\lambda},\tilde{\mu})$ such that $\tilde{H}(p,\tilde{\lambda},\tilde{\mu}) = 0$ and $J\tilde{H}(p,\tilde{\lambda},\tilde{\mu})\neq 0$. In that case, $(-1)^{s+k+1} = \text{sign } J\tilde{H}(p,\tilde{\lambda},\tilde{\mu})$. Now,

$$J\tilde{H}(p,\tilde{\lambda},\tilde{\mu}) = \det\begin{pmatrix} \tilde{a}_{ij} & \frac{\partial f_i}{\partial x_j} & \frac{\partial g}{\partial x_j} \\ \frac{\partial f_i}{\partial x_j} & 0 & 0 \\ \frac{\partial g}{\partial x_j} & 0 & 0 \end{pmatrix},$$

where for all $i,j\in\{1,\ldots,n\}$,

$$\tilde{a}_{ij} = \frac{\partial^2\omega}{\partial x_i\partial x_j} + \tilde{\mu}\frac{\partial^2 g}{\partial x_i\partial x_j} + \sum_{i=1}^{k}\tilde{\lambda}_i\frac{\partial^2 f_i}{\partial x_i\partial x_j}.$$

Then

$$J\tilde{H}(p,\tilde{\lambda},\tilde{\mu}) = (\tilde{\mu})^n \det \begin{pmatrix} \frac{1}{\tilde{\mu}}\tilde{a}_{ij} & \frac{1}{\tilde{\mu}}\frac{\partial f_i}{\partial x_j} & \frac{1}{\tilde{\mu}}\frac{\partial g}{\partial x_j} \\ \frac{\partial f_i}{\partial x_j} & 0 & 0 \\ \frac{\partial g}{\partial x_j} & 0 & 0 \end{pmatrix}.$$

Putting $\lambda_i = \frac{\tilde{\lambda}_i}{\tilde{\mu}}$ and $\mu = \frac{1}{\tilde{\mu}}$, we get

$$J\tilde{H}(p,\tilde{\lambda},\tilde{\mu}) = (\frac{1}{\mu})^{n-k+1} \det \begin{pmatrix} a_{ij} & \frac{\partial f_i}{\partial x_j} & \frac{\partial g}{\partial x_j} \\ \frac{\partial f_i}{\partial x_j} & 0 & 0 \\ \frac{\partial g}{\partial x_j} & 0 & 0 \end{pmatrix},$$

where for all $i,j \in \{1,\ldots,n\}$,

$$a_{ij} = \mu\frac{\partial^2 \omega}{\partial x_i \partial x_j} + \frac{\partial^2 g}{\partial x_i \partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial^2 f_i}{\partial x_i \partial x_j}.$$

Since at (p,λ,μ), $\nabla g = -\mu\nabla\omega - \sum_{i=1}^k \lambda_i \nabla F_i$, we see that

$$J\tilde{H}(p,\tilde{\lambda},\tilde{\mu}) = -\left(\frac{1}{\mu}\right)^{n-k} JH(p,\lambda,\mu),$$

which enables us to conclude. □

The remainder of this section is devoted to the study of the topological degree, around a big sphere of a polynomial mapping. Let $(x_1,\ldots,x_N)$ be a coordinate system in $\mathbb{R}^N$. Let $H = (H_1,\ldots,H_N) : \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial mapping such that $H^{-1}(0)$ is a compact set and let $R \gg 0$ be such that $H^{-1}(0) \subset B_R^N$. We shall recall two methods for computing deg H given by Szafraniec in [Sz3] and [Sz5]. The first one enables us to reduce the computation of this degree to the computation of a local degree at the origin. Let $I : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}$ be the inversion defined by $I(x) = x/\|x^2\|$, let d_i denote the degree of the polynomial H_i for each $i \in \{1,\ldots,N\}$ and let

$$H'(x) = \left(\|x\|^{2d_1} \cdot H_1 \circ I(x), \ldots, \|x\|^{2d_1} \cdot H_N \circ I(x)\right) \text{ for } x \neq 0.$$

Then H' can be extended to a polynomial map $\mathbb{R}^N \to \mathbb{R}^N$ such that $H'(0) = 0$ and $0 \in \mathbb{R}^N$ is isolated in $H'^{-1}(0)$. Let $r = 1/R$ then the map

$$\begin{array}{ccc} S_R^{N-1} & \to & S_R^{N-1} \\ x & \mapsto & I(x) \end{array}$$

is of degree $+1$. Clearly, the maps $H' : S_r \to \mathbb{R}^N \setminus \{0\}$ and $H \circ I : S_R \to \mathbb{R}^N \setminus \{0\}$ are homotopic, and so, if r is small,

Lemma 3.3. $$\deg H = \deg\,(H', S_r^{N-1}) = \deg_0 H'.$$ □

For the second method, we need the assumption that the polynomial factor algebra $A_H = \frac{\mathbb{R}[x_1,\ldots,x_N]}{(H_1,\ldots,H_N)}$ is finite dimensional as a vector space over $\mathbb{R}$. In that case, $H^{-1}(0)$ is a finite set of points. Let $\phi : A_H :\to \mathbb{R}$ be the Kronecker symbol or global residue on A_H. For a description of the global residue, the reader may refer to one of the following papers : [BCRS], [Ca], [Dut1], [Dut2], [Dut3], [Dut4], [SS], [Sz4] and [Sz5]. It is a linear functional and let us define the following bilinear symmetric form :

$$\Phi : A_H \times A_H \to \text{ defined by } \Phi(f,g) = \phi(fg).$$

Thus we have (see [Sz5, Theorem 1.5])

Theorem 3.4. *Assume that* $\dim_{\mathbb{R}} A_H < +\infty$*, then* Φ *is non-degenerate and*

$$\deg H = \text{signature } \Phi.$$

□

Now let $C_1,\ldots,C_t$ be the connected components of $H^{-1}(0)$. For each $i \in \{1,\ldots,t\}$, let T_i be a tubular neighborhood of C_i. It is a smooth manifold with boundary. Let $P : \mathbb{R}^N \to \mathbb{R}$ be a polynomial such that for all $i \in \{1,\ldots,t\}$, either $P \equiv 0$ on C_i or $P^{-1}(0) \cap C_i = \emptyset$. We write $C_1,\ldots,C_s$ for the components of $H^{-1}(0)$ on which P vanishes, $C_{s+1},\ldots,C_r$ for the ones on which $P > 0$ and $C_{r+1},\ldots,C_t$ for the ones on which $P < 0$. Let $(x_1,\ldots,x_N,\mu) = (x;\mu)$ be a coordinate system in $\mathbb{R}^{N+1}$ and let $H_1 : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be given by $H_1(x;\mu) = (H;\mu P - 1)$. Then

Lemma 3.5. *The set* $H_1^{-1}(0)$ *is compact and*

$$\sum_{i=s+1}^{r} \deg\,(H,T_i) - \sum_{i=r+1}^{t} \deg(H,T_i) = \deg\, H_1.$$

Proof. A point (x,μ) belongs to $H_1^{-1}(0)$ if and only if $H(x) = 0$ and $P(x) \neq 0$. So

$$H_1^{-1}(0) = \{(x, 1/P(x)) \mid x \in \cup C_i, i = s+1,\ldots,t\}.$$

Each function $C_i \ni x \mapsto 1/P(x)$ is continuous and, since each C_i is compact, $H_1^{-1}(0)$ is compact. Let ε be a small regular value of H and for each $i \in \{1,\ldots,t\}$, let $\{p_{ij}\}$, $j = 1,\ldots,\sigma(i)$ be the set of preimages of ε lying in T_i. We have that for each $i \in \{1,\ldots,t\}$,

$$\deg\,(H,T_i) = \sum_{j=1}^{\sigma(i)} \text{sign } JH(p_{ij}).$$

It is easy to see that $H_1^{-1}((\varepsilon,0)) = \{(p_{ij}, 1/P(p_{ij})) \mid i = s+1,\dots,t\}$ and that for each $i \in \{s+1,\dots,t\}$ and that for each $j \in \{1,\dots,\sigma(i)\}$,

$$\text{sign } JH_1(p_{ij}, 1/F(p_{ij})) = \text{sign } F(p_{ij}).\text{sign } JH(p_{ij}).$$

This implies that $(\varepsilon, 0)$ is a small regular value of H_1 and that

$$\deg H_1 = \sum_{i=s+1}^{t} \sum_{j=1}^{\sigma(i)} \text{sign } F(p_{ij}) \cdot \text{sign } JH(p_{ij}).$$

Taking ε sufficiently small, sign $F(p_{ij})$ is exactly the sign of F on C_i and this gives the result. □

4. Degree formulas

In this section, we prove our main formulas. Recall that $F = (F_1,\dots,F_k) : \mathbb{R}^n \to \mathbb{R}^k$ is a polynomial map such that $W = F^{-1}(0)$ is a smooth manifold of dimension $n-k$, $G : \mathbb{R}^n \to \mathbb{R}$ is a polynomial such that that Σ_G, the set of critical points of $G_{|W}$, is compact. We denote $\Sigma = \Sigma_G \cap G^{-1}(0)$. Let $\Omega(x_1,\dots,x_n) = \frac{1}{2}(x_1^2 + \ldots + x_n^2)$. Let $(x_1,\dots,x_n;\lambda_1,\dots,\lambda_k;\mu) = (x;\lambda;\mu)$ be a coordinate system in $\mathbb{R}^{n+k+1}$ and let H and K be defined by

$$H(x,\lambda,\mu) = \left(\mu x + \nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), G(x)\right),$$

and

$$K(x,\lambda,\mu) = \left(\mu x + \nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), \mu G(x)\right).$$

Let $M_0 : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be given by

$$M_0(x,\lambda) = \left(x + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x)\right).$$

Let $\Pi_x : \mathbb{R}^{n+k+1} \to \mathbb{R}^n$ be the projection on the n first coordinates. For convenience, we also denote Π_x the projection on the n first coordinates in $\mathbb{R}^{n+k}$. We assume that

$$G^{-1}(0) \cap W \cap \Pi_x(T_0^{-1}(0)) = \emptyset \qquad (A)$$

We first shall describe the set $H^{-1}(0)$. Let $M = (W \cap G^{-1}(0)) \setminus \Sigma$. It is clear that M is either empty or a smooth manifold of dimension $n-k-1$. The polynomial function $\Omega_{|M}$ has a finite number of critical values (see [Mi], corollary 2.8), which implies that the set C of critical points of $\Omega_{|M}$ is bounded. We have

Lemma 4.1. *A point p belongs to C (resp. Σ) if and only if there exists a unique $(\lambda,\mu)\in\mathbb{R}^{k+1}$ such that $H(p,\lambda,\mu)=0$. Furthermore $\mu\neq 0$ (resp. $\mu=0$).*

Proof. By condition (A), each critical point of $\Omega_{|M}$ is a correct critical point. Hence by Lemma 3.1, $p\in C$ if and only if there exists a unique $(\lambda,\mu)\in\mathbb{R}^{k+1}$ such that $H(p,\lambda,\mu)=0$ and $\mu\neq 0$.

If there exists a unique $(\lambda,0)\in\mathbb{R}^{k+1}$ such that $H(p,\lambda,0)=0$ then it is clear that $p\in W\cap G^{-1}(0)$ and that $p\in\Sigma_G$, so that $p\in\Sigma$. Conversely, if $p\in\Sigma$ then there exists a unique $\lambda\in\mathbb{R}^k$ such that $\nabla G(p)+\sum_{i=1}^k\lambda_i\nabla F_i(p)=0$ and, hence, $H(p,\lambda,\mu)=0$. We must show that there is no other $(\tilde\lambda,\tilde\mu)$ with $H(p,\tilde\lambda,\tilde\mu)=0$. If there were one, then $\tilde\mu\neq 0$ and $\tilde\mu p+\nabla G(p)+\sum_{i=1}^k\tilde\lambda_i\nabla F_i(p)=0$. This would give that $p+\frac{1}{\tilde\mu}\sum_{i=1}^k(\tilde\lambda_i-\lambda_i)\nabla F_i(p)=0$, which contradicts condition (A). □

Corollary 4.2. *The set $\Pi_x(H^{-1}(0))$ is equal to $C\sqcup\Sigma$.*

Proof. It is clear by the previous lemma. □

Lemma 4.3. *The set $H^{-1}(0)$ is compact.*

Proof. Since C is bounded and Σ is compact, $\Pi_x(H^{-1}(0))$ is bounded. Moreover, it is closed because it is the algebraic set defined by the vanishing of F, G and all the $(k+2)\times(k+2)$ minors of the Jacobian matrix of the map (F,G,Ω), so $\Pi_x(H^{-1}(0))$ is compact. For all $p\in\Pi_x(H^{-1}(0))$, there exists a unique $(\lambda(p),\mu(p))\in H^{-1}(0)$ such that

$$\mu(p)p+\nabla G(p)+\sum_{i=1}^k\lambda_i(p)\nabla F_i(p)=0.$$

By condition (A), rank $(p,\nabla F_1(p),\ldots,\nabla F_k(p))=k+1$ and, using Cramer's rule, the map $\Pi_x(H^{-1}(0))\ni p\mapsto((\lambda(p),\mu(p))$ is continuous, and so, $H^{-1}(0)$ is bounded. □

Now we describe the set $K^{-1}(0)$.

Lemma 4.4. *A point p belongs to C (resp. Σ_G) if and only if there exists a unique $(\lambda,\mu)\in\mathbb{R}^{k+1}$ such that $K(p,\lambda,\mu)=0$. Furthermore $\mu\neq 0$ (resp. $\mu=0$).*

Proof. We only prove the equivalence concerning points in Σ_G, the other being an obvious consequence of Lemma 4.1. If there is a unique $(\lambda,0)\in\mathbb{R}^{k+1}$ such that $K(p,\lambda,0)=0$ then p belongs to Σ_G. Conversely, if $p\in\Sigma_G$ then there exists an unique $\lambda\in\mathbb{R}^k$ with $\nabla G(p)+\sum_{i=1}^k\lambda_i\nabla F_i(p)=0$ and, hence, $K(p,\lambda,0)=0$. If there were another $(\tilde\lambda,\tilde\mu)\in\mathbb{R}^{k+1}$ such that

$K(p, \tilde{\lambda}, \tilde{\mu}) = 0$ then necessarily $\tilde{\mu} \neq 0$. If $G(p) = 0$ we have the same contradiction as in Lemma 4.1. If $G(p) \neq 0$ then $\tilde{\mu}.G(p) \neq 0$ and $K(p, \tilde{\lambda}, \tilde{\mu}) \neq 0$. □

Corollary 4.5. *The set* $\Pi_x(K^{-1}(0)$ *is equal to* $C \sqcup \Sigma_G$.

Proof. It is clear. □

Lemma 4.6. *The set* $K^{-1}(0)$ *is compact.*

Proof. Since Σ_G is compact, by a similar argument as in Lemma 4.3, the set $\{(x, \lambda, 0) \in \mathbb{R}^{n+k+1} \mid K(x, \lambda, 0) = 0\}$ is compact. But $K^{-1}(0)$ is the union of this set with $H^{-1}(0)$, which is compact. □

Now we introduce two new polynomial mappings L_0 and M_1. The map $L_0 : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is given by

$$L_0(x, \lambda) = \left(\nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x) \right).$$

We have just seen that $L_0^{-1}(0)$ is a compact algebraic set and so consists of a finite number of compact connected components. Let $C_1, \ldots, C_s$ be the ones lying in $\{G = 0\}$, $C_{s+1}, \ldots, C_r$ be the ones lying in $\{G > 0\}$ and $C_{r+1}, \ldots, C_t$ be the ones lying in $\{G < 0\}$. For each $i \in \{1, \ldots, t\}$, let T_i be a tubular neighborhood of C_i. It is a compact manifold with boundary.

The map $M_1 : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ is given by

$$M_1(x, \lambda, \mu) = \left(x + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), \mu G(x) - 1 \right).$$

Let $N_1, \ldots, N_v$ be the connected components of $M_0^{-1}(0)$. By condition (A), for all $i \in \{1, \ldots, v\}$ $N_i \cap G^{-1}(0) = \emptyset$. Applying Lemma 3.5, we get that $M_1^{-1}(0)$ is compact and, if we write $N_1, \ldots, N_\beta$ the components of $M_1^{-1}(0)$ lying in $\{G > 0\}$ and $N_{\beta+1}, \ldots, N_\alpha$ the ones lying in $\{G < 0\}$,

$$\deg M_1 = \sum_{i=1}^{\beta} \deg(M_0, N_i) - \sum_{i=\beta+1}^{\alpha} \deg(M_0, N_i). \qquad (*)$$

We can choose $R \gg 0$ such that $H^{-1}(0) \subset B_R^{n+k+1}$, $K^{-1}(0) \subset B_R^{n+k+1}$, $M_1^{-1}(0) \subset B_R^{n+k+1}$ and $L_0^{-1}(0) \subset B_R^{n+k}$. We can also choose this R in such a way that $W \cap G^{-1}(0) \cap B_R^n$ (resp. $W \cap \{G \geq 0\} \cap B_R^n$, $W \cap \{G \leq 0\} \cap B_R^n$) is a deformation retract of $W \cap G^{-1}(0)$ (resp. $W \cap \{G \geq 0\}$, $W \cap \{G \leq 0\}$) (see [BR], [BCR]). We are now ready to state :

Theorem 4.7. *Assume that* Σ_G *is compact and that* $W \cap G^{-1}(0) \cap \Pi_x$ $(T_0^{-1}(0))$ *is empty. Then, if* $n - k$ *is even,*

- $\frac{1}{2}\chi(W \cap G^{-1}(0) \cap S_R^{n-1}) = \chi(W \cap G^{-1}(0)) - (-1)^k \sum_{i=1}^{s} \deg(L_0, T_i) = (-1)^k \deg H$,
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) - (-1)^k \sum_{i=s+1}^{r} \deg(L_0, T_i) + (-1)^k \sum_{i=r+1}^{t} \deg(L_0, T_i) = (-1)^k \deg T_1 - (-1)^k \deg K$,

if $n-k$ is odd,

- $\chi(W \cap G^{-1}(0)) + (-1)^k \sum_{i=s+1}^{r} \deg(L_0, T_i) - (-1)^k \sum_{i=r+1}^{t} \deg(L_0, T_i) = (-1)^k \deg K$,
- $\frac{1}{2}\chi(W \cap \{G \geq 0\} \cap S_R^{n-1}) - \frac{1}{2}\chi(W \cap \{G \leq 0\} \cap S_R^{n-1}) = \chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) + (-1)^k \sum_{i=1}^{s} \deg(L_0, T_i) = (-1)^k \deg M_1 - (-1)^k \deg H$.

Proof. We give the proof for $n-k$ odd. Let $\delta \neq 0$ be a small regular value of $G_{|W}$ and let $\tilde{H}, \tilde{K} : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ be defined by

$$\tilde{H}(x, \lambda, \mu) = \left(\mu x + \nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), G(x) - \delta\right)$$

and

$$\tilde{K}(x, \lambda, \mu) = \left(\mu x + \nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), \mu(G(x) - \delta)\right).$$

We first study $\deg(\tilde{H}, B_{R'}^{n+k+1})$ and $\deg(\tilde{K}, B_{R'}^{n+k+1})$. We prove that they are equal to deg H and deg K. Let $m = \{\min \|K(x, \lambda, \mu)\| \mid (x, \lambda, \mu) \in S_{R'}^{n+k}\}$. On $S_{R'}^{n+1}$, $\|K - \tilde{K}\| = |\mu\delta|$. Taking δ such that $|\delta R'| < \frac{m}{2}$ then, $\|K - \tilde{K}\| < \frac{m}{2}$ on $S_{R'}^{n+k}$ and $\|\tilde{K}\| > \frac{m}{2}$, so that $\deg(\tilde{K}, B_{R'}^{n+k+1})$ is well defined. Now,if there is a point $z = (p, \lambda, \mu) \in S_{R'}^{n+k}$ such that $K(z, \lambda, \mu)$ and $\tilde{K}(z, \lambda, \mu)$ point in opposite direction, then $\mu p + \nabla G(p) + \sum_{i=1}^{k} \lambda_i \nabla F_i(p) = 0$ and for every $i \in \{1, \ldots, k\}$, $F_i(p) = 0$ because $\tilde{K}(z)$ and $K(z)$ have the same $n+k$ first components. So $\mu G(p)$ and $\mu(G(p) - \delta)$ have different sign. This can happen only if $|G(p)| < |\delta|$. In that case, we find that $\|K(p, \lambda, \mu)\| \leq |\delta R'| < m$, a contradiction. We have proved that $\deg(\tilde{K}, B_{R'}^{n+k+1}) = \deg K$. Similarly, we can prove that $\deg(\tilde{H}, B_{R'}^{n+k+1}) = \deg H$.

Now let search the zeroes of $\tilde{H}^{-1}(0) \cap B_{R'}^{n+k+1}$. Let $(p, \lambda, \mu) \in \tilde{H}^{-1}(0) \cap B_{R'}^{n+k+1}$, then clearly p is a critical point of $\Omega_{|W \cap G^{-1}(\delta)}$. We have that $\|p\| \leq R'$. Since on $\{x \in \mathbb{R}^n \mid R \leq \|x\| \leq R'\}$ $\Omega_{|f^{-1}(0)}$ do not admit critical points, taking δ sufficiently small, $\Omega_{|f^{-1}(\delta)}$ will not admit critical points on $\{x \in \mathbb{R}^n \mid R \leq \|x\| \leq R'\}$, so that $\|p\| < R$. Conversely, let p be a critical point of $\Omega_{|W \cap G^{-1}(\delta) \cap B_R^n}$. Then there exist (λ, μ) such that $\tilde{H}(p, \lambda, \mu) = 0$. Taking δ sufficiently small, p will lie close to $C \sqcup \Sigma$ and then, by continuity, (p, λ, μ) will be close to $H^{-1}(0)$, hence $(p, \lambda, \mu) \in B_{R'}^{n+k+1}$. So, $\Pi_x(\tilde{H}^{-1}(0) \cap B_{R'}^{n+k+1})$ is exactly the set of critical points of $\Omega_{|W \cap G^{-1}(\delta) \cap B_{R'}^n}$, which we denote $\tilde{C}$. Similarly, $\Pi_x(\tilde{K}^{-1}(0) \cap B_R^{n+k+1})$ is $\tilde{C} \sqcup \Sigma_G$. We will write from now on $\deg(\tilde{H}, B_{R'}^{n+k+1}) = \deg \tilde{H}$ and $\deg(\tilde{K}, B_{R'}^{n+k+1}) = \deg \tilde{K}$.

Let us now compute $\deg \tilde{H}$. We choose a function $\tilde{\Omega}\colon \mathbb{R}^n \to \mathbb{R}$ which uniformly approximates Ω in the Whitney C^2-topology and such that $\tilde{\Omega}_{|W \cap G^{-1}(\delta)}$ has only non-degenerate critical points in B_R^n. Let $\{p_1, \ldots, p_m\}$ be the set of those critical points and let $\{\sigma_1, \ldots, \sigma_m\}$ be the set of their respective Morse indices. Choosing $\tilde{\Omega}$ sufficiently close to Ω, we can assume, thanks to condition (A), that $\tilde{\Omega}_{|W}$ has no critical point lying in $G^{-1}(\delta)$ so that $\tilde{\Omega}_{|W \cap \{G \geq \delta\}}$ and $\tilde{\Omega}_{|W \cap \{G \leq \delta\}}$ are correct. By Lemma 3.1, for all $j \in \{1, \ldots, m\}$, there exists $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jk})$ and $\mu_j \neq 0$ such that

$$\mu_j \nabla \tilde{\Omega}(p_j) + \nabla G(p_j) + \sum_{i=1}^{k} \lambda_{ji} \nabla F_i(p_j) = 0.$$

By Lemma 3.2, each $(p_j, \lambda_{j1}, \ldots, \lambda_{jk}, \mu_j)$ is a non-degenerate zero of $\tilde{H}$ and

$$(-1)^{\sigma_j} = (-1)^k \cdot \text{sign } \mu_j \cdot \text{sign } J\tilde{H}(p_j, \lambda_j, \mu_j).$$

Hence

$$\deg \tilde{H} = (-1)^k \left(\sum_{j/\mu_j > 0} (-1)^{\sigma_j} - \sum_{j/\mu_j < 0} (-1)^{\sigma_j} \right).$$

We can also assume that $\tilde{\Omega}_{|W}$ is a Morse function in B_R^n. Let $\{q_1, \ldots, q_l\}$ be the set of non-degenerate critical points of $\tilde{\Omega}_{|W}$ lying in B_R^n and let $\{\tau_1, \ldots, \tau_l\}$ be the set of their respective indices. By condition (A), for all $i \in \{1, \ldots, l\}$, $G(q_i) \neq 0$. Then we have, applying Corollary 2.6,

$$\chi(\{G \geq \delta\} \cap W \cap B_R^n) = \sum_{i/G(q_i) > 0} (-1)^{\tau_i} + \sum_{j/\mu_j < 0} (-1)^{\sigma_j},$$

$$\chi(\{G \leq \delta\} \cap W \cap B_R^n) = \sum_{i/G(q_i) < 0} (-1)^{\tau_i} + \sum_{j/\mu_j > 0} (-1)^{\sigma_j},$$

which leads to

$$\chi(\{G \geq \delta\} \cap W \cap B_R^n) - \chi(\{G \leq \delta\} \cap W \cap B_R) = \sum_{i/G(q_i)>0} (-1)^{\tau_i} -$$

$$\sum_{i/G(q_i)<0} (-1)^{\tau_i} + \sum_{j/\mu_j<0} (-1)^{\sigma_j} - \sum_{j/\mu_j>0} (-1)^{\sigma_j}.$$

From Szafraniec's results [Sz3],

$$\sum_{i/G(q_i)>0} (-1)^{\tau_i} + \sum_{i/G(q_i)<0} (-1)^{\tau_i} = (-1)^k \text{deg } M_0.$$

Using $(*)$, one easily gets that

$$\sum_{i/G(q_i)>0} (-1)^{\tau_i} - \sum_{i/G(q_i)<0} (-1)^{\tau_i} = (-1)^k \text{deg } M_1,$$

and hence

$$\chi(\{G \geq \delta\} \cap W \cap B_R^n) - \chi(\{G \leq \delta\} \cap W \cap B_R^n) = (-1)^k \left(\text{deg } M_1 - \text{deg } H\right).$$

We also have to compute deg $\tilde{K}$. First it is easy to notice that for all $j \in \{1, \ldots, m\}$,

$$J\tilde{K}(p_j, \lambda_j, \mu_j) = \mu_j \times J\tilde{H}(p_j, \lambda_j, \mu_j),$$

so that

$$(-1)^{\sigma_j} = (-1)^k \text{sign } J\tilde{K}(p_j, \lambda_j, \mu_j). \tag{1}$$

We then have to consider the points $(p, \lambda, 0)$ where $\tilde{K}(p, \lambda, 0) = 0$ and $p \in \Sigma_G$. The set of these points is exactly $\cup C_i \times \{0\}$, $i = 1, \ldots, t$. For each $i \in \{1, \ldots, t\}$, let Γ_i be a tubular neighborhood of $C_i \times \{0\}$ in $\mathbb{R}^{n+k+1}$. It remains to calculate $\deg(\tilde{K}, \Gamma_i)$. We choose a function $\tilde{G} : \mathbb{R}^n \to \mathbb{R}$ which uniformly approximates G in the Whitney C^2-topology and such that $\tilde{G} = G$ outside a neighborhood of Σ_G and $\tilde{G}$ has only non-degenerate critical points in a neighborhood of Σ_G. For each $i \in \{1, \ldots, t\}$, let $\{v_{i1}, \ldots, v_{i\gamma(i)}\}$ be the set of those critical points lying near $\Pi_x(C_i)$. Let

$$\tilde{\tilde{K}}(x, \lambda, \mu) = \left(\mu x + \nabla\tilde{G}(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), (G(x) - \delta)\mu\right).$$

For each $i \in \{1, \ldots, t\}$ and each $l \in \{1, \ldots, \gamma(i)\}$, there exists $\lambda_{il} \in \mathbb{R}^k$ such that $\tilde{\tilde{K}}(v_{il}, \lambda_{il}, 0) = 0$ and

$$J\tilde{\tilde{K}}(v_{il}, \lambda_{il}, 0) = (G(v_{il}) - \delta) \cdot J\tilde{L_0}(v_{il}, \lambda_{il}),$$

where, as usual, $\tilde{L_0}(x,\lambda) = (\nabla\tilde{G}(x) + \sum_{i=1}^k F_i(x), F(x))$. From this, we deduce that for every $i \in \{1,\ldots,l\}$ and every $l \in \{1,\ldots,\gamma_i\}$, $(v_{il},\lambda_{il},0)$ is a non-degenerate zero of $\tilde{\tilde{K}}$ and that

$$\deg(\tilde{K},\Gamma_i) = \sum_{l=1}^{\gamma(i)} \text{sign} J\tilde{\tilde{K}}(v_{il},\lambda_{il},0) = \sum_{l=1}^{\gamma(i)} \text{sign}\,(G(v_{il})-\delta)\cdot \text{sign} J\tilde{L_0}(v_{il},\lambda_{il}).$$

If $i \in \{s+1,\ldots,r\}$, taking δ sufficiently small, $G(v_{il})-\delta>0$ and then for each $i \in \{s+1,\ldots,r\}$,

$$\deg(\tilde{K},\Gamma_i) = \deg(\tilde{L_0},T_i) = \deg(L_0,T_i). \tag{2}$$

Similarly for all $i \in \{r+1,\ldots,t\}$,

$$\deg(\tilde{K},\Gamma_i) = -\deg(\tilde{L_0},T_i). \tag{3}$$

If $i \in \{1,\ldots,s\}$ then taking $\tilde{G}$ sufficiently close to G, we have that for $l \in \{1,\ldots,\gamma(i)\}$, $|G(v_{il})|<\delta$ so that sign $(G(v_{il})-\delta) = -\text{sign}\ \delta$ and then

$$\deg(\tilde{K},\Gamma_i) = -(\text{sign}\ \delta)\deg(L_0,T_i). \tag{4}$$

Combining (1), (2), (3) and (4) gives

$$(-1)^k \deg\ K = \chi(W\cap G^{-1}(\delta)\cap B_R^n) - (\text{sign}\ \delta)(-1)^k \sum_{i=1}^s \deg\ (L_0,T_i) +$$
$$(-1)^k \sum_{i=s+1}^r \deg(L_0,T_i) - (-1)^k \sum_{i=r+1}^t \deg(L_0,T_i).$$

The followings lemmas will end the proof. □

Lemma 4.8. *Let δ be a small regular value of $G_{|W}$. Then*

- $\chi(G^{-1}(\delta)\cap W\cap B_R^n) = \chi(G^{-1}(0)\cap W\cap B_R^n)$
$$-\text{sign}\ (-\delta)^{n-k}(-1)^k \sum_{i=1}^s \deg\ (L_0,T_i),$$
- $\chi(\{G\geq\delta\}\cap W\cap B_R^n) - \chi(\{G\leq\delta\}\cap W\cap B_R^n) = \chi(\{G\geq 0\}\cap W\cap B_R^n) -$
$$\chi(\{G\leq 0\}\cap W\cap B_R^n) + \text{sign}\ (-\delta)^{n-k+1}(-1)^k \sum_{i=1}^s \deg(L_0,T_i).$$

Proof. We prove the case $n-k$ odd. Keeping the notations introduced above , let ρ_{il} be the Morse index of the point v_{il}. Assume that $\delta>0$. From Theorem 2.5, we have

$$\chi(G^{-1}[-\delta,\delta]\cap W\cap B_R^n) - \chi(G^{-1}(-\delta)\cap W\cap B_R^n) = \sum_{i=1}^s \sum_{l=1}^{\gamma(i)} (-1)^{\rho_{il}}.$$

Since $G^{-1}[-\delta,\delta]\cap W\cap B_R^n$ retracts to $G^{-1}(0)\cap W\cap B_R^n$, this leads to

$$\chi(G^{-1}(-\delta)\cap W\cap B_R^n)=\chi(G^{-1}(0)\cap W\cap B_R^n)-\sum_{i=1}^{s}\sum_{l=1}^{\gamma(i)}(-1)^{\rho_{il}}.$$

By Szafraniec's results, we get

$$\chi(G^{-1}(-\delta)\cap W\cap B_R^n)=\chi(G^{-1}(0)\cap W\cap B_R^n)-(-1)^k\sum_{i=1}^{s}\deg(L_0,T_i).$$

Applying this method to $-G$ gives the result because the index at a critical points for $-G$ is $(-1)^{n-k}$ times the index of this point for G.

In order to prove the second point, for $\delta>0$, we use the fact that

$$\begin{aligned}\chi(\{G\geq 0\}\cap W\cap B_R^n)&=\chi(\{G\geq\delta\}\cap W\cap B_R^n)\\ &+\chi(\{0\leq G\leq\delta\}\cap W\cap B_R^n)-\chi(G^{-1}(\delta)\cap W\cap B_R^n),\end{aligned}$$

and

$$\begin{aligned}\chi(\{G\leq\delta\}\cap W\cap B_R^n)&=\chi(\{G\leq 0\}\cap W\cap B_R^n)\\ &+\chi(\{0\leq G\leq\delta\}\cap W\cap B_R^n)-\chi(G^{-1}(0)\cap W\cap B_R^n),\end{aligned}$$

and the fact that $\{0\leq G\leq\delta\}\cap W\cap B_R^n$ retracts to $G^{-1}(0)\cap W\cap B_R^n$. Applying this to $-G$ gives the result for $\delta<0$. □

Lemma 4.9. *Let δ be a small regular value of $G_{|W}$. If $n-k$ is odd,*

$$\frac{1}{2}\chi(\{G\geq 0\}\cap W\cap S_R^{n-1})-\frac{1}{2}\chi(\{G\leq 0\}\cap W\cap S_R^{n-1})=$$
$$\chi(\{G\geq\delta\}\cap W\cap B_R^n)-\chi(\{G\leq\delta\}\cap W\cap B_R^n).$$

Proof. Since $G^{-1}(0)\cap W$ intersects S_R^{n-1} transversally, $\chi(\{G\leq 0\}\cap W\cap S_R^{n-1})=\chi(\{G\leq\delta\}\cap W\cap S_R^{n-1})$. Let $X=\{G\leq\delta\}\cap W\cap B_R^n$. It is a manifold with corners. Let $\partial X=X\setminus\operatorname{int}(X)$. Smoothing the corners there exists a smooth manifold with boundary $\tilde{X}$ such that $\chi(X)=\chi(\tilde{X})$ and $\chi(\partial X)=\chi(\partial\tilde{X})$. So we have $\chi(\partial X)=2\chi(X)$. By Mayer-Vietoris sequence,

$$\chi(\partial X)=\chi(G^{-1}(\delta)\cap W\cap B_R^n)+\chi(\{G\leq\delta\}\cap W\cap S_R^{n-1}).$$

Similarly,

$$2\chi(\{G\geq\delta\}\cap W\cap B_R^n)=\chi(G^{-1}(\delta)\cap W\cap B_R^n)+\chi(\{G\geq\delta\}\cap W\cap S_R^{n-1}),$$

and it is easy to conclude. □

Now let L_1 and $L_2 : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ be defined by

$$L_1(x,\lambda,\mu) = \left(\nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), \mu G(x) - 1\right),$$

$$L_2(x,\lambda,\mu) = \left(\nabla G(x) + \sum_{i=1}^{k} \lambda_i \nabla F_i(x), F(x), \mu G(x)^2 - 1\right).$$

By Lemma 3.5, $L_1^{-1}(0)$ and $L_2^{-1}(0)$ are compact so we can choose $R' \gg 0$ such that $L_1^{-1}(0) \subset B_{R'}^{n+k+1}$ and $L_2^{-1}(0) \subset B_{R'}^{n+k-1}$. We thus have

Theorem 4.10. *Assume that Σ_G is compact and $W \cap G^{-1}(0) \cap \Pi_x(T_0^{-1}(0))$ is empty, then if $n-k$ is even,*

- $\frac{1}{2}\chi(W \cap G^{-1}(0) \cap S_R^{n-1}) = \chi(W \cap G^{-1}(0)) - (-1)^k (\deg L_0 - \deg L_2) = (-1)^k \deg H,$
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) - (-1)^k \deg L_1 = (-1)^k \deg M_1 - (-1)^k \deg K,$

if $n-k$ is odd,

- $\chi(W \cap G^{-1}(0)) + (-1)^k \deg L_1 = (-1)^k \deg K,$
- $\frac{1}{2}\chi(\{G \geq 0\} \cap W \cap S_R^{n-1}) - \frac{1}{2}\chi(\{G \leq 0\} \cap W \cap S_R^{n-1}) = \chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) + (-1)^k (\deg L_0 - \deg L_2) = (-1)^k \deg M_1 - (-1)^k \deg H.$

□

Here we give an example. Let $G : \mathbb{R}^3 \to \mathbb{R}$ be defined by $G(x_1,x_2,x_3) = (x_1-1)(x_2^2+x_3^2-1)$. Then $G^{-1}(0)$ is the union of a plane and a cylinder which intersect transversally and Σ is a circle. We have $G_{x_1} = x_2^2+x_3^2-1$, $G_{x_2} = 2x_2(x_1-1)$ and $G_{x_3} = 2x_3(x_3-1)$. It is easy to see that $\Sigma_G = \Sigma$ and that condition (A) is checked. Then applying Theorem 4.7 :

$$\chi(G^{-1}(0)) = \deg K,$$

where $K(\lambda,x_1,x_2,x_3) = ((x_1-1)(x_2^2+x_3^2-1)\lambda, \lambda x_1 + x_2^2 + x_3^2 - 1, \lambda x_2 + 2x_2(x_1-1), \lambda x_3 + 2x_3(x_1-1))$. This is not exactly the same map as in Theorem 4.7 but since we apply the same permutation to the sink and to the target, the topological degree is preserved. We write $K = (K_1,K_2,K_3,K_4)$ and we consider the following mapping $\theta(\lambda,x_1,x_2,x_3) = (\lambda^2 + x_1^2 + x_2^2 +$

$x_3^2 - R^2, K_1, K_2, K_3)$. If $\theta^{-1}(0) \cap \{K_4 = 0\}$ is void and if $\theta^{-1}(0) \cap \{K_4 \neq 0\}$ consists of a finite number of points $p_1, \ldots, p_k$ then (see [Sz5], Section 3),

$$\deg K = -\frac{1}{2} \sum_{i=1}^{k} \text{sign } K_4(p_i) \cdot \deg_{p_i} \theta.$$

The remainder of the computation is rather technical and we shall not give the details. In order to apply the last formula, we have to find $\theta^{-1}(0)$. Distinguishing the case $x_2^2 + x_3^2 - 1 = 0$ and $x_2^2 + x_3^2 - 1 \neq 0$ and taking R sufficiently big, the reader may find that $\theta^{-1}(0)$ consists of the three following set of points :

$$A = \left\{ \pm\sqrt{R^2-1}, 0, 0, \pm 1 \right\},$$

$$B = \left\{ 0, \pm\sqrt{R^2-1}, 0, \pm 1 \right\},$$

$$C = \left\{ \frac{1-\sqrt{4R^2-7}}{2}, 1, 0, \pm\sqrt{\frac{1+\sqrt{4R^2-7}}{2}} \right\}.$$

For each $p \in A$, one finds that p is a non-degenerate zero of θ and that $\deg_p \theta = \text{sign } (-4\lambda^3 x_3(-2+\lambda))$ at p. We see that $\sum_{p \in A} \text{sign } K_4(p) \cdot \deg_p \theta = 0$.

For each $p \in C$, one also finds that p is a non-degenerate zero of θ. We get that $\deg_p \theta = \text{sign } (x_3)$ and then $\sum_{p \in C} \text{sign } K_4(p) \deg_p \theta = -2$. For all $p \in B$, it is easy to see that p is a degenerate point of θ so it is more difficult to compute $\deg_p \theta$. We shall show that for all $p \in B$, $\deg_p \theta = 0$. Let us do it, for example, for $p = (0, \sqrt{R^2-1}, 0, 1)$. Let ε be a small constant and let us find the points of $\theta^{-1}((0,0,\varepsilon,0))$ lying near p. We have to solve the system

$$\left\{ \begin{array}{rcll} \lambda^2 + x_1^2 + x_2^2 + x_3^2 - R^2 & = & 0 & (1) \\ (x_1 - 1)(x_2^2 + x_3^2 - 1)\lambda & = & 0 & (2) \\ x_2^2 + x_3^2 - 1 + \lambda x_1 & = & \varepsilon & (3) \\ 2x_2(x_1 - 1) + \lambda x_2 & = & 0 & (4) \end{array} \right.$$

Since $x_1(p) = \sqrt{R^2-1}$ then near p, $x_1 - 1 > 0$. Hence two cases are possible
<u>First case:</u> $\lambda = 0$ then $2x_2(x_1 - 1) = 0$ which implies that $x_2 = 0$ and then $x_3^2 = 1 + \varepsilon$. Finally we find that, near p,

$$\theta^{-1}\left((0,0,\varepsilon,0)\right) \cap \{\lambda = 0\} = (0, \sqrt{R^2-1-\varepsilon}, 0, \sqrt{1+\varepsilon}).$$

<u>Second case:</u> $\lambda \neq 0$ then $x_2^2 + x_3^2 - 1 = 0$ and $\lambda = \frac{\varepsilon}{x_1}$. By (1), this gives that $x_1^4 - (R^2-1)x_1^2 + \varepsilon^2 = 0$. The only solution of this equation lying near

$\sqrt{R^2-1}$ is

$$R' = \sqrt{\frac{R^2-1+\sqrt{(R^2-1)^2-4\varepsilon^2}}{2}}.$$

Taking R sufficiently big and $\varepsilon > 0$, (4) will imply that $x_2 = 0$. So, near p,

$$\theta^{-1}\left((0,0,\varepsilon,0)\right) \cap \{\lambda \neq 0\} = \left(\frac{\varepsilon}{R'}, R', 0, 1\right).$$

Let us call $\tilde{p} = (0, \sqrt{R^2-1-\varepsilon}, 0, \sqrt{1+\varepsilon})$ and $\tilde{q} = (\frac{\varepsilon}{R'}, R', 0, 1)$. An easy computation gives that $\tilde{p}$ is a non degenerate zero of θ and that $\deg_{\tilde{p}}\theta =$ sign $(8(x_1-1)^2\varepsilon x_1 x_3)$ at $\tilde{p}$, so that $\deg_{\tilde{p}}\theta = +1$. An other computation gives that $\tilde{q}$ is a non-degenerate zero of θ and that $\deg_{\tilde{q}}\theta =$ sign $(2x_3(x_1-1)\lambda(2x_1-2+\lambda)(2\lambda^2-2x_1^2))$. Since $x_1 \gg \lambda$, this leads to $\deg_{\tilde{q}}\theta = -1$, so finally $\deg_p\theta = 0$. Similarly one can prove that for all $p \in B$, $\deg_p\theta = 0$. Hence, we have found that

$$\deg K = 1 = \chi(W).$$

□

This example, which is one of the simplest, show how difficult it is to compute a topological degree around a big sphere. However, using Lemma 3.3, we can state

Corollary 4.11. *There exists polynomial mappings H', K', L_1', L_2' and $M_1' : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ and $L_0 : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, defined in terms of $F_1, \ldots, F_k$ and G, with an isolated critical point at the origin, such that, if $n-k$ is even,*

- $\frac{1}{2}\chi(W \cap G^{-1}(0) \cap S_R^{n-1}) = \chi(W \cap G^{-1}(0)) - (-1)^k\left(\deg_0 L_0' - \deg_0 L_2'\right) = (-1)^k \deg_0 H',$
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) - (-1)^k \deg_0 L_1' = (-1)^k \deg_0 M_1' - (-1)^k \deg_0 K',$

if $n-k$ is odd,

- $\chi(W \cap G^{-1}(0)) + (-1)^k \deg_0 L_1' = (-1)^k \deg_0 K',$
- $\frac{1}{2}\chi(W \cap \{G \geq 0\} \cap S_R^{n-1}) - \frac{1}{2}\chi(W \cap \{G \leq 0\} \cap S_R^{n-1}) = \chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) + (-1)^k\left(\deg_0 L_0' - \deg_0 L_2'\right) = (-1)^k \deg_0 M_1' - (-1)^k \deg_0 H'.$ □

These local topological degrees can be computed, when the origin is an algebraically isolated critical point, using the well-known Eisenbud-Levine formula and Lecki's program. Unfortunately, because of the large number of monomials involved in the polynomials above, computations are not efficient. That is why, in order to end this section, we will present special cases where computations are efficient.

Let $A_H = \frac{\mathbb{R}[x,\lambda,\mu]}{(H)}$, where (H) is the ideal generated by the components of H and let us assume that $\dim_{\mathbb{R}} A_H < +\infty$. We can define a bilinear symmetric form Φ_H on A_H, as we did in Section 3. Similarly, for all polynomials involved in Theorem 4.7, we define an algebra that we suppose to be finite dimensional and we construct the corresponding bilinear symmetric form. Thanks to Theorem 3.4, we can state

Theorem 4.12. *Assume that A_H, A_K, A_{L_1}, A_{L_2}, A_{T_1} and A_{L_0} are finite dimensional and that*

$$\dim_{\mathbb{R}} \frac{\mathbb{R}[x,\lambda]}{(x+\sum_{i=1}^k \lambda_i \nabla F_i, F_1, \ldots, F_k, G)} = 0,$$

then, if $n-k$ is even,

- $\frac{1}{2}\chi(W \cap G^{-1}(0) \cap S_R^{n-1}) =$
  $$\chi(W \cap G^{-1}(0)) - (-1)^k (\text{signature } \Phi_{L_0} - \text{signature } \Phi_{L_2}) = (-1)^k \text{signature } \Phi_H,$$
- $\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) - (-1)^k \text{signature } \Phi_{L_1} =$
  $$(-1)^k \text{signature } \Phi_{T_1} - (-1)^k \text{signature } \Phi_K,$$

if $n-k$ is odd,

- $\chi(W \cap G^{-1}(0)) + (-1)^k \text{signature } \Phi_{L_1} = (-1)^k \text{signature } \Phi_K,$ □
- $\frac{1}{2}\chi(W \cap \{G \geq 0\} \cap S_R^{n-1}) - \frac{1}{2}\chi(W \cap \{G \leq 0\} \cap S_R^{n-1}) =$

$$\chi(W \cap \{G \geq 0\}) - \chi(W \cap \{G \leq 0\}) + (-1)^k (\text{signature } \Phi_{L_0} - \text{signature } \Phi_{L_2}) = (-1)^k \text{signature } \Phi_{T_1} - (-1)^k \text{signature } \Phi_H.$$

Proof. Since $\dim A_{L_0} < +\infty$, Σ_G is a finite set of points. Since

$$\dim_{\mathbb{R}} \frac{\mathbb{R}[x,\lambda]}{(x+\sum_{i=1}^k \lambda_i \nabla F_i, F_1, \ldots, F_k, G)} = 0,$$

$W \cap G^{-1}(0) \cap \Pi_x(T_0^{-1}(0)) = \emptyset$. We can apply Theorem 4.7 and Theorem 3.4. □

Remark 4.13. *Using the fact that*

$$\chi(W) = \chi(W \cap \{G \geq 0\}) + \chi(W \cap \{G \leq 0\}) - \chi(W \cap \{G = 0\}),$$

and Szafraniec's results about the computations of $\chi(W)$ *([Sz3,Sz4]), we can calculate* $\chi(W \cap \{G \geq 0\})$ *and* $\chi(W \cap \{G \leq 0\})$.

5. The special case of $\mathbb{R}^n$

In this section, we investigate the topology of a polynomial $G : \mathbb{R}^n \to \mathbb{R}$ whose set of critical points is compact (generically it is a finite set of points). So let $G : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let Σ_G be its set of critical points, assumed to be compact. Let $C_1, \ldots, C_s$ be the connected components of Σ_G lying in $G^{-1}(0)$, $C_{s+1}, \ldots, C_r$ the ones lying in $\{G > 0\}$ and $C_{r+1}, \ldots, C_t$ the ones lying in $\{G < 0\}$. For each $i \in \{1, \ldots, t\}$, let T_i be a tubular neighborhood of C_i. First we define two numerical invariants associated to the fiber $G^{-1}(0)$. We choose a radius $R > 0$ sufficiently big so that $G^{-1}(0)$ (resp. $\{G \geq 0\}$, $\{G \leq 0\}$) retracts to $G^{-1}(0) \cap B_R^n$ (resp. $\{G \geq 0\} \cap B_R^n$, $\{G \leq 0\} \cap B_R^n$), $G^{-1}(0)$ and S_R^{n-1} intersect transversally and $\Sigma_G \subset B_R^n$. Now let $\tilde{G} : \mathbb{R}^n \to \mathbb{R}$ be an approximation of G such that $\tilde{G}_{|S_R^{n-1}}$ is a Morse function. Let $\{p_1, \ldots, p_m\}$ be the critical points of $\tilde{G}_{|S_R^{n-1}}$ and let $\{\sigma(1), \ldots, \sigma(m)\}$ be the set of their respective Morse indices. For each $i \in \{1, \ldots, m\}$, there is μ_i with $\nabla \tilde{G}(p_i) = \mu_i . p_i$.

Definition 5.1. *The numbers* $\nu_{G,0}^+$ *and* $\nu_{G,0}^-$ *are defined as follows :*

$$\nu_{G,0}^+ = \sum_{\substack{i / G(q_i) > 0 \\ \mu_i < 0}} (-1)^{\mu_i} \text{ and } \nu_{G,0}^- = \sum_{\substack{i / G(q_i) < 0 \\ \mu_i > 0}} (-1)^{\mu_i}.$$

These numbers are related to $\chi(G^{-1}(0))$ in the following way :

Theorem 5.2. *Assume that* Σ_G *is compact then, if* n *is even,*

- $\chi(G^{-1}(0)) = 1 - \sum_{i=s+1}^{t} \deg(\nabla G, T_i) - (\nu_{G,0}^+ - \nu_{G,0}^-)$,

- $\chi(\{G \geq 0\}) - \chi(\{G \leq 0\}) = \sum_{i=s+1}^{r} \deg(\nabla G, T_i) - \sum_{i=r+1}^{t} \deg(\nabla G, T_i) + (\nu_{G,0}^+ + \nu_{G,0}^-)$, *if n is odd,*

- $$\chi(G^{-1}(0)) = 1 - \sum_{i=s+1}^{r} \deg(\nabla G, T_i) + \sum_{i=r+1}^{t} \deg(\nabla G, T_i) - (\nu^+_{G,0} + \nu^-_{G,0}),$$

- $$\chi(\{G \geq 0\}) - \chi(\{G \leq 0\}) = \sum_{i=s+1}^{t} \deg(\nabla G, T_i) + (\nu^+_{G,0} - \nu^-_{G,0}).$$

Proof. We use Theorem 2.5, which gives

$$\chi(\{G \geq 0\} \cap B^n_R) - \chi(\{G = 0\} \cap B^n_R) = \sum_{i=s+1}^{r} \deg(\nabla G, T_i) + \nu^+_{G,0},$$

$$\chi(\{G \leq 0\} \cap B^n_R) - \chi(\{G = 0\} \cap B^n_R) = (-1)^n \sum_{i=r+1}^{t} \deg(\nabla G, T_i) + (-1)^{n-1} \nu^-_{G,0},$$

and the relation $1 = \chi(\{G \geq 0\}) + \chi(\{G \leq 0\}) - \chi(\{G = 0\}$. □

Corollary 5.3. *Assume that Σ_G is compact, then*

$$\chi(\{G \leq 0\} \cap S^{n-1}_R) = 1 - \deg \nabla G - (\nu^+_{G,0} - \nu^-_{G,0}).$$

Proof. If n is even, we have

$$\chi(\{G \leq 0\} \cap S^{n-1}_R) = \frac{1}{2} \chi(\{G = 0\} \cap S^{n-1}_R),$$

which is also, by Theorem 4.7, $\chi(G^{-1}(0)) - \sum_{i=1}^{s} \deg\ (\nabla G, T_i)$, and it is easy to conclude with the above theorem.

If n is odd, we have

$$\chi(\{G \geq 0\} \cap S^{n-1}_R) - \chi(\{G \leq 0\} \cap S^{n-1}_R) = 2\chi(\{G \geq 0\}) - 2\chi(\{G \leq 0\}) + 2 \sum_{i=1}^{s} \deg\ (\nabla G, T_i),$$

which is also $2 \deg \nabla G + 2(\nu^+_{G,0} - \nu^-_{G,0})$. It is enough to use then the fact that $2 = \chi(S^{n-1}_R) = \chi(\{G \geq 0\} \cap S^{n-1}_R) + \chi(\{G \leq 0\} \cap S^{n-1}_R)$. □

Remark 5.4. *These formulas are to be compared with the Arnol'd, Khimshiashvili, Wall's formula ([Ar], [Kh], [Wa]) which asserts that*

$$\chi(f^{-1}(\delta) \cap B^n_\varepsilon) = 1 - \text{sign}\ (-\delta)^n \deg_0 \nabla f,$$

$$\chi(\{f \leq 0\} \cap S^{n-1}_\varepsilon) = 1 - \deg_0 \nabla f,$$

for an analytic function-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *with an isolated critical point at the origin,* δ *a small regular value and* B_ε^n *a small sphere.*

In our formulas the two numbers $\nu_{G,0}^+$ *and* $\nu_{G,0}^-$ *appear to describe the topology at infinity.*

With our previous work, we will be able to compute $\nu_{G,0}^+$, $\nu_{G,0}^-$ and $\chi(\{G \leq 0\} \cap S_R^{n-1})$. Let $(x_1, \ldots, x_n, \mu)$ be a coordinate system in $\mathbb{R}^{n+1}$ and let H and $K : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be defined by $H(\lambda, x) = (\lambda x + \nabla G, G)$ and $K(\lambda, x) = (\lambda x + \nabla G, \lambda G)$. In order to apply the results of Section 4, it is necessary to assume condition (A). It is easy to see that in the case of $W = \mathbb{R}^n$ this condition is equivalent to $G(0) \neq 0$. Actually we will assume that $G(0) > 0$. In that case Theorem 4.7 becomes

Theorem 5.5. *Assume that* $G(0) > 0$. *If* n *is even,*

- $$\frac{1}{2}\chi(G^{-1}(0) \cap S_R^{n-1}) = \chi(G^{-1}(0)) - \sum_{i=1}^{s} \deg(\nabla G, T_i) = \deg H,$$
- $$\chi(\{G \geq 0\}) - \chi(\{G \leq 0\}) - \sum_{i=s+1}^{r} \deg(\nabla G, T_i) + \sum_{i=r+1}^{t} \deg(\nabla G, T_i) = 1 - \deg K,$$

if n *is odd,*

- $$\frac{1}{2}\chi(\{G \geq 0\} \cap S_R^{n-1}) - \frac{1}{2}\chi(\{G \leq 0\} \cap S_R^{n-1}) = \chi(\{G \geq 0\}) - \chi(\{G \leq 0\}) + \sum_{i=1}^{s} \deg(\nabla G, T_i) = 1 - \deg H,$$
- $$\chi(G^{-1}(0)) + \sum_{i=s+1}^{r} \deg(\nabla G, T_i) - \sum_{i=r+1}^{t} \deg(\nabla G, T_i) = \deg K.$$

□

Corollary 5.6. *Assume that* $G(0) > 0$. *Then*

- $\nu_{G,0}^+ - \nu_{G,0}^- = 1 - \deg \nabla G - \deg H$,
- $\nu_{G,0}^+ + \nu_{G,0}^- = 1 - \deg K$,
- $\chi(\{G \leq 0\} \cap S_R^{n-1}) = \deg H$.

Proof. It is a combination of the previous theorems and corollaries. □

Examples

• Let $G(x_1,x_2) = -x_1^2x_2^5 + x_1^4x_2^3 + 5x_2^3 - 5x_1^2x_2 - 4x_2^2 + 4x_1^2$. The computer gives that dim $\mathbb{R}[x_1,x_2]/(G,G_{x_1},G_{x_2}) = 13$ so $G^{-1}(0)$ may admit singularity. Let us consider H and $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$H(x_1,x_2,x_3) = (x_1x_3 + G_{x_1}, x_3(x_2-1) + G_{x_2}, G)\,,$$

$$K(x_1,x_2,x_3) = (x_1x_3 + G_{x_1}, x_3(x_2-1) + G_{x_2}, Gx_3)\,.$$

Here we use the following distance function $\Omega(x_1,x_2) = \frac{1}{2}(x_1^2 + (x_2-1)^2)$. Since $G(0,1) = 1 > 0$, we can apply the previous theorems. We find deg $H = 5$, deg $K = 1$, deg $\nabla G = -4$, deg $L_1 = -1$ and deg $L_2 = 1$. So by our theorems, we have :

$$\frac{1}{2}\chi(W \cap G^{-1}(0) \cap S_R^1) = 10 = \chi(\{G \leq 0\} \cap S_R^1,$$

$$\chi(G^{-1}(0)) = 5 + (-4) - 1 = 0,$$

$$\chi(\{G \geq 0\}) - \chi(\{G \leq 0\}) = 1 + (-1) + (-1) = -1.$$

• Let $G(x_1,x_2,x_3) = x_1^2x_2^3 + x_1x_2^4 - 2x_1^3 - 2x_1^2x_2 - x_2^3 - x_1x_2 + 2x_1x_3 + x_3^2 + 2x_1 + 1$. First dim $\mathbb{R}[x_1,x_2,x_3]/(G,G_{x_1},G_{x_2},G_{x_3}) = 6$, so $G^{-1}(0)$ may have singularities. We find that deg $H = 3$, deg $K = 5$, deg $\nabla G = -2$ and deg $L_1 =$ deg $L_2 = 0$. So,

$$\chi(G^{-1}(0)) = 5 - 0 = 5,$$

$$\chi(\{G \geq 0\} \cap S_R^2) = 3,$$

$$\frac{1}{2}\chi(\{G \geq 0\} \cap S_R^2) - \frac{1}{2}\chi(\{G \leq 0\} \cap S_R^2) = 1 - 3 = -2,$$

$$\chi(\{G \geq 0\}) - \chi(\{G \leq 0\}) = 1 - 3 - (-2) = 0.$$

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