

Axiomatic definition of the topological entropy on the interval

Ll. Alsedà, S. Kolyada, J. Llibre and Ľ. Snoha

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Abstract The aim of this paper is to give an axiomatic definition of the topological entropy for continuous interval maps and, in such a way, to shed some more light on the importance of the different properties of the topological entropy in this setting. We give two closely related axiomatic definitions of topological entropy and an axiomatic characterization of the topological chaos.

1 Introduction

The *topological entropy* as a measure of the complexity of a continuous self-map of a compact topological space was introduced by Adler, Konheim and McAndrew [1] and has been studied by many authors. In particular, in compact metric spaces an equivalent definition has been found by Bowen [8] and Dinaburg [10]. For the definition and main properties of topological entropy we refer the reader to [2] and [17] (the book [2] is particularly focused on interval maps and very often, when we use well known facts, we refer the reader to this book instead to the original paper).

The idea of giving an *axiomatic* definition of entropy belongs to Rohlin who gave in [14] an axiomatic definition of the measure-theoretic entropy of an automorphism of a Lebesgue space. Later an analogue of the Rohlin's result for a \mathbb{Z}^d -action for every $d \geq 2$ was proved by Kamiński in [13]. An axiomatic definition of topological entropy for endomorphisms on compact groups was found by Stojanov [16]. For completeness we recall that there

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are also many papers dealing with axiomatic characterizations of various kinds of entropy in the context of information theory.

Our main concern in this paper is to give an axiomatic definition of the topological entropy for continuous interval maps and, in such a way, to shed some more light on the importance of the different properties of the topological entropy in this setting. We hope that the axiomatic definitions will enable one to understand better the role of topological entropy in topological dynamics (at least on the interval). We give two closely related axiomatic definitions of topological entropy (Theorems A and B), and an axiomatic characterization of topological chaos (Theorem C).

In what follows, $\mathcal{C}(I)$ will denote the class of all continuous maps from a real compact interval I (say $I = [0, 1]$) into itself endowed with the metric of the uniform convergence. For each $f \in \mathcal{C}(I)$, $\text{Top}(f)$ will denote the topological entropy of f .

Before stating the main results of the paper in detail we will recall some of the basic properties of the topological entropy for interval maps. They will be the candidates for the axioms of our definitions.

First of all we recall that $\text{Top}: \mathcal{C}(I) \rightarrow [0, +\infty]$ is lower semicontinuous (see for instance [2, Theorem 4.5.2]).

If $f, g \in \mathcal{C}(I)$ then we say that g is obtained from f by *pouring water*, denoted by $g \in \text{PW}(f)$, if there exists an open set $G \subset I$ (in the relative topology) such that g is constant on each component of G and $g(x) = f(x)$ for all $x \in I \setminus G$. If $g \in \text{PW}(f)$ then $\text{Top}(g) \leq \text{Top}(f)$ (see for instance [3, Lemma 2.2]). We remark that, contrary to the lower semicontinuity of $\text{Top}(\cdot)$, the fact that $\text{Top}(g) \leq \text{Top}(f)$ whenever $g \in \text{PW}(f)$ is true even for compact metric spaces (see [12, Lemma 5]).

Let P be a finite subset of I . A map $f \in \mathcal{C}(I)$ will be called *P-monotone* (respectively *P-linear*) if it is constant on $[0, \min P]$ and $[\max P, 1]$, and f is (not necessarily strictly) monotone (respectively, affine) on the closure of each connected component of $I \setminus P$. Also, a set $P \subset I$ is called *weakly f-invariant* if $f(P) \subset P$.

Let $f, g \in \mathcal{C}(I)$. Recall that g is a *factor* of f or, equivalently, f is *semiconjugate* to g if there is a surjective map $\phi \in \mathcal{C}(I)$ such that $\phi \circ f = g \circ \phi$. In such a case we have $\text{Top}(g) \leq \text{Top}(f)$ (this holds for self-maps of compact topological spaces, see [2, Lemma 4.1.3]). If, additionally, ϕ is non-decreasing we will say that g is a *strong factor* of f . The class of all strong factors of a map f will be denoted by $\text{SF}(f)$.

In what follows we will denote by τ the *standard tent map*. That is, τ is the map from $I = [0, 1]$ to itself defined by $\tau(x) = 1 - |2x - 1|$ for each $x \in [0, 1]$.

The piecewise linear maps such that the absolute value of their slopes is constant play an important role in the theory of the topological entropy on the interval. For $\lambda > 0$ we denote

$$\mathcal{CS}_\lambda := \{f \in \mathcal{C}(I) : f \text{ is piecewise linear with slopes either } \lambda \text{ or } -\lambda\}.$$

Recall that if $f \in \mathcal{CS}_\lambda$ then $\text{Top}(f) = \max\{0, \log \lambda\}$ (see, for instance, [2, Corollary 4.3.13]). In particular we have $\text{Top}(\tau) = \log 2$. Also, from [2, Theorem 4.6.8] it follows that if f is a piecewise monotone map then there exists $g \in \mathcal{CS}_{\exp(\text{Top}(f))} \cap \text{SF}(f)$ (and, hence, $\text{Top}(g) = \text{Top}(f)$).

We will also use the notion of the (*Sharkovskii*) *type* of a map (which is closely related to the topological entropy). Given a map $f \in \mathcal{C}(I)$ we will denote by $\text{Per}(f)$ the set of periods of all periodic orbits of f . The set $\text{Per}(f)$ can be characterized in terms of the *Sharkovskii ordering* \preceq , which is the following ordering defined on the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \succ 5 \succ 7 \succ \dots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \dots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ 4 \cdot 7 \succ \dots \succ \dots \succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \dots \succ 2^\infty \succ \dots \succ 2^n \succ \dots \succ 16 \succ 8 \succ 4 \succ 2 \succ 1.$$

For $t \in \mathbb{N} \cup \{2^\infty\}$ we denote by $S(t)$ the set $\{k \in \mathbb{N} : k \preceq t\}$ (observe that $S(2^\infty) = \{1, 2, 4, \dots, 2^k, \dots\}$). The first part of the Sharkovskii Theorem [15] (see also [2, Theorem 2.1.1]) states that for every $f \in \mathcal{C}(I)$ there exists a $t \in \mathbb{N} \cup \{2^\infty\}$ such that $\text{Per}(f) = S(t)$.

Let $t \in \mathbb{N} \cup \{2^\infty\}$ and let $f \in \mathcal{C}(I)$ be such that $\text{Per}(f) = S(t)$. Then f is said to be *of type* t . When speaking of types we consider them to be ordered by the Sharkovskii ordering. The class of all maps $f \in \mathcal{C}(I)$ such that $\text{type}(f) = t$ (respectively, $\text{type}(f) \prec t$, $\text{type}(f) \preceq t$, $\text{type}(f) \succ t$, $\text{type}(f) \succcurlyeq t$) will be denoted by $T(t)$ (respectively, $T(\prec t)$, $T(\preceq t)$, $T(\succ t)$, $T(\succcurlyeq t)$).

Now we are ready to state the main results of the paper. The first two of them display two groups of axioms which characterize the topological entropy on the interval.

Theorem A. *Let $\text{Ax} : \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

- (LSC) *Ax is lower semicontinuous.*
- (PW) *Ax(g) ≤ Ax(f) whenever g ∈ PW(f).*
- (SF) *Ax(g) ≤ Ax(f) whenever g ∈ SF(f).*
- (CSF) *If f is a P-linear map, where P is a weakly f-invariant set and Ax(f) > 0, then f has a piecewise linear (not necessarily strong) factor g ∈ CS_λ for some λ such that Ax(f) = Ax(g).*
- (≤LOG) *Ax(f) ≤ log λ whenever f ∈ CS_λ is a P-linear map, where P is weakly f-invariant and λ ≥ 1.*

(\geq LOG) $\text{Ax}(f) \geq \log \lambda$ whenever $f \in \mathcal{CS}_\lambda$ is a P -linear map, where P is a periodic orbit of f and $\lambda > 1$.

Then $\text{Ax} \equiv \text{Top}$. That is, Top is the only map satisfying the above properties.

A natural question is whether some of the axioms in the theorem are not superfluous. In Remark 4.6 (see Section 4) we will show that the axioms (LSC), (SF), (\leq LOG) and (\geq LOG) are necessary in the sense that if one removes any of them then the theorem will no longer hold. We conjecture that also (PW) and (\mathcal{CSF}) are necessary.

There is also the possibility of replacing the two axioms (\leq LOG) and (\geq LOG) by a combination of other axioms with the effect that the number of axioms will be larger but Ax and Top in the axioms will be (implicitly) compared on a much smaller set of maps than in Theorem A. Such a possibility is shown in the next result.

Theorem B. Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:

- (LSC) Ax is lower semicontinuous.
- (PW) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.
- (SF) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{SF}(f)$.
- (\mathcal{CSF}) If f is a P -linear map, where P is a weakly f -invariant set and $\text{Ax}(f) > 0$, then f has a piecewise linear (not necessarily strong) factor $g \in \mathcal{CS}_\lambda$ for some λ such that $\text{Ax}(f) = \text{Ax}(g)$.
- (COMP) $\text{Ax}(f \circ g) = \text{Ax}(f) + \text{Ax}(g)$ whenever $f \in \mathcal{CS}_{\lambda_1}$ and $g \in \mathcal{CS}_{\lambda_2}$ where $\lambda_1, \lambda_2 \geq 1$.
- (TENT) $\text{Ax}(\tau) = \log 2$.
- ($\mathcal{CS}_1\text{-T}_1$) $\text{Ax}(f) = 0$ whenever $f \in \mathcal{CS}_1$ is of type 1.

Then $\text{Ax} \equiv \text{Top}$. That is, Top is the only map satisfying the above properties.

In Remark 5.4, after the proof of this theorem (see Section 5), we will show that the axioms (LSC), (SF), (COMP) and (TENT) are necessary in the sense that, if one removes any of them, Theorem B will no longer hold. We do not know whether ($\mathcal{CS}_1\text{-T}_1$) is superfluous or not but we conjecture that (PW) and (\mathcal{CSF}) are necessary.

The question of which of the two axiomatic definitions of the topological entropy given by Theorems A and B is better is more philosophical than mathematical and the answer to it depends on the ‘‘taste’’ of the reader. More precisely, it depends on whether one prefers a small number of axioms or a set as small as possible where we prescribe the values of the axiomatic

entropy (or where we implicitly compare them with the values of topological entropy).

We recall that the notion of topological entropy is sometimes used to define chaos. Namely, f is called (*topologically*) *chaotic* if $\text{Top}(f) > 0$ (see [5, p. 218]). Besides of finding an axiomatic definition of the topological entropy we will also be concerned with finding an axiomatic characterization of chaos in the above sense. To this end we will say that an axiomatic entropy *characterizes chaos* if $\text{Ax}(f) > 0$ is equivalent to $\text{Top}(f) > 0$. In this context it is natural to find which axioms are sufficient for an axiomatic entropy to characterize chaos (without necessarily coinciding with the topological entropy). The answer to this question is given by the next result. To state it we first need to recall some well known properties of the topological entropy:

- $\text{Top}(f^n) = n \text{Top}(f)$ whenever n is a nonnegative integer and f is a continuous self-map of a compact topological space (see, e.g., [2, Lemma 4.1.2]).
- For a map $f \in \mathcal{C}(I)$ it follows that $\text{Top}(f) > 0$ if and only if $f \in T(> 2^\infty)$ (see, e.g., [2, Theorem 4.4.19]).

Theorem C. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(ITER) $\text{Ax}(f^n) = n \text{Ax}(f)$ for $n \geq 0$.

(T₁) $\text{Ax}(f) = 0$ whenever f is of type 1.

(T₃) $\text{Ax}(f) > 0$ whenever f is the second iterate of a map of type 3.

Then Ax characterizes chaos.

In Remark 6.4 (see Section 6) we will show that all axioms in the above theorem are necessary in the sense that, if one removes any of them, Theorem C will no longer be true. The axiom (T₃) is somehow surprising and needs, perhaps, an explanation: First of all realize that for any map from $T(> 2^\infty)$ there exists some iterate which is of type 3. Hence the axiom (ITER) guarantees that $\text{Ax}(f) > 0$ for all maps from $T(> 2^\infty)$ provided we assume that $\text{Ax}(f) > 0$ for all maps from $T(3)$. Nevertheless, this assumption is still too strong because $T(3)$ is residual in $\mathcal{C}(I)$ (see for instance [11]). Fortunately, it is easy to weaken the assumption of positiveness of Ax on the whole $T(3)$. In fact, by [6], the set of all iterates of all maps from $\mathcal{C}(I)$ is nowhere dense in $\mathcal{C}(I)$. Thus, in axiom (T₃) we assume positiveness of Ax only on the nowhere dense set $T(3)^2$ (if F is a family of maps and $k \in \mathbb{N}$ then we denote the set $\{f^k: f \in F\}$ by F^k). Moreover, (T₃) and (ITER) imply positiveness of Ax on the whole $T(3)$ (and hence on $T(> 2^\infty)$).

Remark 1.1. Observe that $T(3) \supset T(3)^2$. Hence,

$$T(3) \supset T(3)^2 \supset \dots \supset T(3)^{2^n} \supset \dots$$

So, in axiom (T_3) we can replace $T(3)^2$ by any of the smaller sets $T(3)^{2^n}$. \square

Note also that all maps of the form $\mu \text{Top}(\cdot)$ satisfy the above theorem and, hence, characterize chaos. However, there are also maps which are not of this form and still satisfy Theorem C. For such examples see Remark 6.4.

The paper is organized as follows. In Section 2 we give a list of maps from $\mathcal{C}(I)$ to $[0, +\infty]$ which will be used in subsequent sections to show the necessity of some axioms or to illustrate Theorem C. In Section 3 we prove some preliminary results as well as some propositions which are of their own interest. In Section 4 we prove Theorem A, in Section 5 Theorem B and in Section 6 Theorem C. Finally, in Section 7 we formulate some open problems.

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2 Some useful maps from $\mathcal{C}(I)$ to $[0, +\infty]$

In this section we define eleven maps from $\mathcal{C}(I)$ to $[0, +\infty]$, that will be used later as examples and/or counterexamples. They will be labeled as $\text{Ax}_1, \text{Ax}_2, \dots, \text{Ax}_{11}$. We start by defining the first three of these maps.

$$\begin{aligned} \text{Ax}_1(f) &= 0 \quad \text{for all } f \in \mathcal{C}(I). \\ \text{Ax}_2(f) &= +\infty \quad \text{for all } f \in \mathcal{C}(I). \\ \text{Ax}_3(f) &= \begin{cases} \text{Top}(f) & \text{if } f \in \mathcal{C}(I) \text{ is piecewise monotone,} \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Now we choose two maps from $\mathcal{C}(I)$ as follows. Let $\varphi \in \mathcal{C}(I)$ be such that (recall that we are taking $I = [0, 1]$):

- (i) $\varphi(0) = \varphi(1/2) = 0$ and $\varphi(1/4) = 1/2$,

- (ii) φ is affine on the intervals $[0, 1/4]$ and $[1/4, 1/2]$,
- (iii) φ is not piecewise monotone.

Also, let $\psi \in \mathcal{C}(I)$ be such that it is not monotone in any subinterval of I and has positive topological entropy. Then we define

$$\text{Ax}_4(f) = \begin{cases} \text{Top}(f) & \text{if } f \neq \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Ax}_5(f) = \begin{cases} \text{Top}(f) & \text{if } f(0) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Ax}_6(f) = \begin{cases} \text{Top}(f) & \text{if } f \notin \text{PW}(\psi), \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Ax}_7(f) = \begin{cases} \log 2 & \text{if } \text{Top}(f) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Ax}_8(f) = \begin{cases} +\infty & \text{if } \text{Top}(f) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Ax}_9(f) = \begin{cases} +\infty & \text{if } f \in T(\succcurlyeq 2^\infty), \\ 0 & \text{otherwise.} \end{cases}$$

We say that $f \in \mathcal{C}(I)$ is *type-stable* if $\text{type}(g) \succcurlyeq \text{type}(f)$ for any $g \in \mathcal{C}(I)$ sufficiently close to f . Given $t \in \mathbb{N} \cup \{2^\infty\}$ we will denote by $T_S(t)$ the class of all maps from $T(t)$ which are type-stable and by $T_U(t)$ the complement of $T_S(t)$ in $T(t)$.

$$\text{Ax}_{10}(f) = \begin{cases} +\infty & \text{if } f \in T(\succcurlyeq 2^\infty), \\ 2^n & \text{if } f \in T_S(2^n) \cup T_U(2^{n+1}) \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } f \in T(1) \cup T_U(2). \end{cases}$$

Given $f \in \mathcal{C}(I)$ we define the *orbit of f* , denoted by $\text{Orb}(f)$, as the set $\{f^n : n \in \mathbb{N}\}$. We also define the *full orbit of f* , which we denote by $\text{Forb}(f)$, as $\{g \in \mathcal{C}(I) : g^n \in \text{Orb}(f) \text{ for some } n \in \mathbb{N}\}$. Observe that $\text{Forb}(f) \supset \text{Orb}(f)$.

Let $\bar{\tau} \in \mathcal{C}(I)$ be the map defined by $\bar{\tau}(x) = |2x - 1|$. In the terminology of [7] this map is called a *strict horseshoe of type (2, -)*. Moreover, from [7, Theorem C] it follows that $\text{Forb}(\bar{\tau}) = \text{Orb}(\bar{\tau})$. We claim that the set $\text{Forb}(\bar{\tau})$ is closed in $\mathcal{C}(I)$. To prove this, take $k < n$ and denote by z the

largest element of I such that $\bar{\tau}^k(z) = 0$. Clearly, $\bar{\tau}^{k+1}(z) = 1$ and, hence, $\bar{\tau}^n(z) = 1$. Thus, the distance between $\bar{\tau}^k$ and $\bar{\tau}^n$ is one. This shows that $\text{Forb}(\bar{\tau}) = \text{Orb}(\bar{\tau})$ is closed in $\mathcal{C}(I)$.

Now we are ready to define the map Ax_{11} :

$$\text{Ax}_{11}(f) = \begin{cases} \frac{1}{2} \text{Top}(f) & \text{if } f \in \text{Forb}(\bar{\tau}), \\ \text{Top}(f) & \text{otherwise.} \end{cases}$$

It is worth noticing that for the tent map $\tau = 1 - \bar{\tau}$, which is considered to be a prototype of a map with positive topological entropy, we have $\text{Ax}_{11}(\tau) = \text{Top}(\tau) = \log 2$.

Remark 2.1. All maps $\text{Ax}_1, \text{Ax}_2, \dots, \text{Ax}_{11}$ defined in this section are different from Top . \square

3 Preliminary results

We start this section with a number of technical auxiliary lemmas. In what follows, the expression $f_n \rightrightarrows f$ will mean that $f \in \mathcal{C}(I)$ and that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{C}(I)$ which converges uniformly to f .

Lemma 3.1. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(PW) *$\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.*

Assume that $f_n \rightrightarrows f$ and $f_n \in \text{PW}(f)$ for $n \in \mathbb{N}$. Then we have $\text{Ax}(f) = \lim_{n \rightarrow \infty} \text{Ax}(f_n)$.

Proof. From (PW) it follows that $\limsup_{n \rightarrow \infty} \text{Ax}(f_n) \leq \text{Ax}(f)$ and, from (LSC), $\text{Ax}(f) \leq \liminf_{n \rightarrow \infty} \text{Ax}(f_n)$. \blacksquare

Lemma 3.2. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(PW) *$\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.*

(C1) *$\text{Ax}(f) \leq \text{Top}(f)$ whenever $f \in \mathcal{C}(I)$ is piecewise monotone.*

Then $\text{Ax}(f) \leq \text{Top}(f)$ for each $f \in \mathcal{C}(I)$.

Proof. Let $f \in \mathcal{C}(I)$ and take a sequence $\{f_n\}_{n \in \mathbb{N}}$ of piecewise monotone maps obtained from f by pouring water such that $f_n \rightrightarrows f$ (it exists in view of [12]). By Lemma 3.1, $\text{Ax}(f) = \lim_{n \rightarrow \infty} \text{Ax}(f_n)$. Since the topological entropy also has the properties (LSC) and (PW), Lemma 3.1 gives also that $\text{Top}(f) = \lim_{n \rightarrow \infty} \text{Top}(f_n)$. By (C1) we have $\text{Ax}(f_n) \leq \text{Top}(f_n)$. Thus, we get $\text{Ax}(f) \leq \text{Top}(f)$. \blacksquare

In what follows we denote by $\text{Orb}_f(x)$ the orbit of a point x under f .

Lemma 3.3. *Assume that $\{0, 1\} \subset P \subset I$ is finite, $f \in \mathcal{C}(I)$ is P -monotone and $d_f = \text{Card}\{x \in P : \text{Orb}_f(x) \text{ is infinite}\} > 0$. Then, for each $\varepsilon > 0$ there exists a P -monotone map $g \in \mathcal{C}(I)$ such that $f|_P = g|_P$, $\text{Card}\{x \in P : \text{Orb}_g(x) \text{ is infinite}\} < d_f$ and $\text{Top}(g) < \text{Top}(f) + \varepsilon$.*

The proof of Lemma 3.3 is analogous to the proof of [3, Lemma 3.3] in this setting with the obvious changes of notation. In particular one has to take the set A from [3, Lemma 3.3] as the empty set and replace the set of vertices of the graph G by the set P appearing in the statement of Lemma 3.3.

Lemma 3.4. *Let $f \in \mathcal{C}(I)$ be piecewise monotone and let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers converging to 0. Then, there exist sequences $\{P_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \rightrightarrows f$ and, for each $n \in \mathbb{N}$, $f_n(P_n) \subset P_n$, f_n is P_n -linear and $\text{Top}(f_n) < \text{Top}(f) + \varepsilon_n$.*

Remark 3.5. In the assumptions of the above lemma, since $f_n \rightrightarrows f$, $\text{Top}(f_n) < \text{Top}(f) + \varepsilon_n$ and the topological entropy is lower semicontinuous, we have $\lim_{n \rightarrow \infty} \text{Top}(f_n) = \text{Top}(f)$. \square

Proof of Lemma 3.4. Fix $n \in \mathbb{N}$ and take $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{1}{n}$. Now let $\{0, 1\} \subset P \subset I$ be a finite set such that f is P -monotone and for each consecutive $x, y \in P$ it follows that $|x - y| < \delta$ (such a set clearly exists since f is piecewise monotone).

Assume that $d_f > 0$. Then, by using iteratively Lemma 3.3, we obtain P -monotone maps $g_0 = f, g_1, \dots, g_l$ with $l \leq d_f$ such that for $i = 1, 2, \dots, l$ we have $f|_P = g_i|_P$,

$$\text{Card}\{x \in P : \text{Orb}_{g_i}(x) \text{ is infinite}\} < \text{Card}\{x \in P : \text{Orb}_{g_{i-1}}(x) \text{ is infinite}\}$$

and $\text{Top}(g_i) < \text{Top}(g_{i-1}) + \frac{\varepsilon_n}{d_f}$. Moreover,

$$\text{Card}\{x \in P : \text{Orb}_{g_l}(x) \text{ is infinite}\} = 0.$$

Observe that $\text{Top}(g_l) < \text{Top}(f) + \frac{l\varepsilon_n}{d_f} \leq \text{Top}(f) + \varepsilon_n$. Therefore, by setting

$$\tilde{f}_n = \begin{cases} g_l & \text{if } d_f > 0, \\ f & \text{otherwise,} \end{cases}$$

we get a P -monotone map \tilde{f}_n such that $\tilde{f}_n|_P = f|_P$, $\text{Top}(\tilde{f}_n) < \text{Top}(f) + \varepsilon_n$ and $\text{Card}\{x \in P : \text{Orb}_{\tilde{f}_n}(x) \text{ is infinite}\} = 0$.

Set $P_n = \cup_{x \in P} \text{Orb}_{\tilde{f}_n}(x)$. Clearly, P_n is finite and $\tilde{f}_n(P_n) \subset P_n$. Also, since $P \subset P_n$, \tilde{f}_n is P_n -monotone. Now let f_n be the P_n -linear map such that $f_n|_{P_n} = \tilde{f}_n|_{P_n}$. Clearly we have $f_n(P_n) \subset P_n$. Moreover, in view of [2, Theorem 4.4.5] it follows that $\text{Top}(f_n) = \text{Top}(\tilde{f}_n) < \text{Top}(f) + \varepsilon_n$.

To end the proof of the lemma it is enough to show that $\|f - f_n\| < \frac{1}{n}$. To see this, note that since \tilde{f}_n is P -monotone, f_n is P_n -linear and $\tilde{f}_n|_{P_n} = f_n|_{P_n}$ it follows that f_n is also P -monotone. Moreover, since $P \subset P_n$, we have $f_n|_P = \tilde{f}_n|_P = f|_P$. Recall that if x and y are two consecutive points of P then $|f(x) - f(y)| < \frac{1}{n}$. Therefore, $\|f - f_n\| < \frac{1}{n}$ because f is also P -monotone. ■

Lemma 3.6. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(C2) *$\text{Ax}(f) \leq \text{Top}(f)$ whenever there exists $P \subset I$ such that $f(P) \subset P$ and f is P -linear.*

Then (C1) of Lemma 3.2 holds.

Proof. Let $f \in \mathcal{C}(I)$ be piecewise monotone. By Lemma 3.4 and Remark 3.5, there exist sequences $\{P_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \rightrightarrows f$ and, for each $n \in \mathbb{N}$, $f_n(P_n) \subset P_n$, f_n is P_n -linear and $\lim_{n \rightarrow \infty} \text{Top}(f_n) = \text{Top}(f)$. From (C2) we have $\text{Top}(f_n) \geq \text{Ax}(f_n)$. So, $\text{Top}(f) \geq \limsup_{n \rightarrow \infty} \text{Ax}(f_n)$. Further, by (LSC) we have $\liminf_{n \rightarrow \infty} \text{Ax}(f_n) \geq \text{Ax}(f)$. This ends the proof of the lemma. ■

From Lemma 3.2 and Lemma 3.6 we immediately obtain:

Proposition 3.7. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(PW) *$\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.*

(C2) *$\text{Ax}(f) \leq \text{Top}(f)$ whenever there exists $P \subset I$ such that $f(P) \subset P$ and f is P -linear.*

Then $\text{Ax}(f) \leq \text{Top}(f)$ for all maps $f \in \mathcal{C}(I)$.

The next proposition is a kind of converse of the previous one.

Proposition 3.8. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(PW) *$\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.*

(C3) $Ax(f) \geq \text{Top}(f)$ whenever f is P -monotone and P is a periodic orbit of f .

Then $Ax(f) \geq \text{Top}(f)$ for all maps $f \in \mathcal{C}(I)$.

Proof. Let $f \in \mathcal{C}(I)$ and let P be a periodic orbit of f . We will denote by f_P the map from $\mathcal{C}(I)$ which is constant on $[0, \min P]$ and $[\max P, 1]$, and for each $a, b \in P$ such that $a < b$ and $(a, b) \cap P = \emptyset$ we have

$$f_P(x) = \begin{cases} \min\left\{\sup\{f(y) : y \in [a, x]\}, f(b)\right\} & \text{if } f(a) < f(b), \\ \max\left\{\inf\{f(y) : y \in [a, x]\}, f(b)\right\} & \text{if } f(b) < f(a), \end{cases}$$

for each $x \in [a, b]$. Observe that $f_P|_P = f|_P$, f_P is P -monotone and $f_P \in \text{PW}(f)$.

By Theorem 4.4.10 and Corollary 4.4.7 of [2] it follows that

$$\text{Top}(f) = \sup\{\text{Top}(f_P) : P \text{ is a periodic orbit of } f\}.$$

By (C3) we have $\text{Top}(f_P) \leq Ax(f_P)$ and so,

$$\text{Top}(f) \leq \sup\{Ax(f_P) : P \text{ is a periodic orbit of } f\}.$$

Since $f_P \in \text{PW}(f)$, $Ax(f_P) \leq Ax(f)$. Thus, $\text{Top}(f) \leq Ax(f)$. ■

Now we are ready to prove the following proposition which is not far from being an axiomatic definition of topological entropy. Its advantage is that it contains only four axioms. Unfortunately, the axioms (C2) and (C3) seem to be too strong and, what is worse, they *explicitly* compare the values of axiomatic entropy and topological entropy (see Remark 3.15).

Proposition 3.9. *Let $Ax: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) Ax is lower semicontinuous.

(PW) $Ax(g) \leq Ax(f)$ whenever $g \in \text{PW}(f)$.

(C2) $Ax(f) \leq \text{Top}(f)$ whenever there exists $P \subset I$ such that $f(P) \subset P$ and f is P -linear.

(C3) $Ax(f) \geq \text{Top}(f)$ whenever f is P -monotone and P is a periodic orbit of f .

Then $Ax \equiv \text{Top}$. That is, Top is the only map satisfying the above properties.

Proof. It follows from Proposition 3.7 and Proposition 3.8. ■

Remark 3.10. All axioms in Proposition 3.9 are necessary in the sense that if one removes any of them the proposition will no longer be true. To see this, we will show that for each of the four axioms there is a map Ax satisfying the other three axioms but different from Top (consequently, such a map Ax will not satisfy the fourth axiom — otherwise it has to coincide with Top).

We consider the maps $\text{Ax}_1, \text{Ax}_2, \text{Ax}_3, \text{Ax}_4$ from Section 2 (recall that they are different from Top). The map Ax_1 satisfies (LSC), (PW) and (C2). The map Ax_2 satisfies (LSC), (PW) and (C3). The map Ax_3 obviously satisfies (C2) and (C3). It satisfies also (PW). To see this, it is sufficient to realize that both Top and the constant map which is equal to $+\infty$ satisfy (PW) and that if f is piecewise monotone then any $g \in \text{PW}(f)$ is also piecewise monotone. Finally, the map Ax_4 satisfies (C2) and (C3) since for piecewise monotone maps Ax_4 coincides with Top . But it satisfies also (LSC) since Top satisfies (LSC) and Ax_4 differs from Top only in one map φ for which $\text{Ax}_4(\varphi) < \text{Top}(\varphi)$. \square

Lemma 3.11. *Let $f \in \mathcal{C}(I)$ be a P -monotone map, where P is weakly f -invariant. Assume that f is semiconjugate to a map $g \in \mathcal{C}(I)$ via a nondecreasing map ψ (hence $g \in \text{SF}(f)$). Then $\psi(P)$ is weakly g -invariant and g is $\psi(P)$ -monotone. Moreover, if P is a periodic orbit of f then $\psi(P)$ is a periodic orbit of g .*

Proof. From [2, Lemma 4.6.1] we get that g is $\psi(P)$ -monotone. Further, let $b \in \psi(P)$. We need to prove that $g(b) \in \psi(P)$. Take $a \in P$ with $\psi(a) = b$. Then the semiconjugacy gives $g(b) = g(\psi(a)) = \psi(f(a))$. Since $f(a) \in P$ we are done. Finally suppose that P is a periodic orbit of f . Take $b_1, b_2 \in \psi(P)$. To see that $\psi(P)$ is a periodic orbit of g it is sufficient to show that there exists $k \in \mathbb{N}$ such that $g^k(b_1) = b_2$. To this end, take $a_1, a_2 \in P$ with $\psi(a_i) = b_i$, $i = 1, 2$ and $k \in \mathbb{N}$ with $f^k(a_1) = a_2$. Then the semiconjugacy gives $g^k(b_1) = b_2$. \blacksquare

The following result shows that in (C3) of Proposition 3.8 we can replace the P -monotonicity by P -linearity provided we simultaneously add the axiom (SF).

Proposition 3.12. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

- (PW) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.
- (SF) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{SF}(f)$.

(C4) $\text{Ax}(f) \geq \text{Top}(f)$ whenever f is P -linear and P is a periodic orbit of f .

Then $\text{Ax}(f) \geq \text{Top}(f)$ for all maps $f \in \mathcal{C}(I)$.

Proof. Let $f \in \mathcal{C}(I)$. As in the proof of Proposition 3.8 we get

$$\text{Top}(f) = \sup\{\text{Top}(f_P) : P \text{ is a periodic orbit of } f\}.$$

Thus, to end the proof it is sufficient to show that $\text{Top}(f_P) \leq \text{Ax}(f)$ whenever P is a periodic orbit of f . Since $f_P \in \text{PW}(f)$ we have $\text{Ax}(f_P) \leq \text{Ax}(f)$ and thus it is sufficient to show that $\text{Top}(f_P) \leq \text{Ax}(f_P)$.

If $\text{Top}(f_P) = 0$ this is trivial. So, assume that $\text{Top}(f_P) > 0$. Then, as it has been said in the introduction, there exists $g \in \mathcal{CS}_{\text{exp}(\text{Top}(f_P))} \cap \text{SF}(f)$ (and $\text{Top}(g) = \text{Top}(f_P)$). Recall that f_P is P -monotone and P is a periodic orbit of f_P . Hence, by Lemma 3.11, $\phi(P)$ is a periodic orbit of g (where ϕ denotes the non-decreasing semiconjugacy from f_P to g), and g is $\phi(P)$ -monotone. In particular, g is constant on $[0, \min \phi(P)]$ and on $[\max \phi(P), 1]$. But we know that g is a piecewise linear map such that the absolute value of the slopes is constant. Hence, $\min \phi(P) = 0$, $\max \phi(P) = 1$ and g is $\phi(P)$ -linear. Then, by (C4), $\text{Top}(f_P) = \text{Top}(g) \leq \text{Ax}(g)$. Since $g \in \text{SF}(f_P)$, the assumption (SF) gives $\text{Ax}(g) \leq \text{Ax}(f_P)$, which finishes the proof. ■

From Proposition 3.7 and Proposition 3.12 we obtain the following analogue of Proposition 3.9.

Proposition 3.13. *For a map $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ consider the following properties:*

(LSC) Ax is lower semicontinuous.

(PW) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{PW}(f)$.

(SF) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{SF}(f)$.

(C2) $\text{Ax}(f) \leq \text{Top}(f)$ whenever there exists $P \subset I$ such that $f(P) \subset P$ and f is P -linear.

(C4) $\text{Ax}(f) \geq \text{Top}(f)$ whenever f is P -linear and P is a periodic orbit of f .

Then $\text{Ax} \equiv \text{Top}$. That is, Top is the only map satisfying the above properties.

Remark 3.14. All axioms in Proposition 3.13 are necessary in the sense that if one removes any of them, then the statement will no longer be true. To see this we will show that, for each of the above five axioms, there is a map Ax , different from Top , satisfying the other four axioms (consequently,

by the above proposition, such a map will not satisfy the remaining axiom). Recall that all maps from Section 2 are different from Top.

Necessity of (LSC). Take the map Ax_3 . In Remark 3.10 we showed that it satisfies (C2) and (PW). Also, it obviously satisfies (C4). Finally, Ax_3 satisfies (SF) because by [2, Corollary 4.6.3], a strong factor of a piecewise monotone map is again a piecewise monotone map, and Top verifies (SF).

Necessity of (PW). Take the map Ax_5 . Since Top is lower semicontinuous and the set of all maps having zero as their fixed point is closed in $\mathcal{C}(I)$, Ax_5 is also lower semicontinuous. Further, if $f(0) = 0$ and $g \in SF(f)$ then also $g(0) = 0$. Consequently, Ax_5 satisfies (SF). The map Ax_5 trivially satisfies (C2). To see that it also fulfills (C4) it is sufficient to realize that if f is P -linear and P is a periodic orbit of f then either $f(0) \neq 0$ or f is identically zero. In both cases $Ax(f) \geq Top(f)$.

Necessity of (SF). Take Ax_6 which, trivially, satisfies (C2). To see that it satisfies also (C4) take any P -linear map f . First assume that $f \in PW(\psi)$. Since no map from $PW(\psi)$ can be linear and nonconstant on any subinterval of I , f is constant and so $Ax_6(f) \geq Top(f) = 0$. Now assume that $f \notin PW(\psi)$. Then $Ax_6(f) = Top(f)$. Thus, (C4) is satisfied. Next we are going to prove that Ax_6 satisfies (PW). Let $h \in PW(g)$. We need to show that $Ax_6(h) \leq Ax_6(g)$. If $h \in PW(\psi)$ this is trivial. If $h \notin PW(\psi)$ then also $g \notin PW(\psi)$ and it is sufficient to use that Top satisfies (PW). Finally, we claim that Ax_6 is lower semicontinuous. To prove this it is obviously sufficient to prove that $PW(\psi)$ is closed in $\mathcal{C}(I)$. To this end take any $f \notin PW(\psi)$. Then there is a connected component J of the set $D = \{x \in I: f(x) \neq \psi(x)\}$ such that $f|_J$ is nonconstant. Take $a < b$ in J with $f(a) \neq f(b)$. Then all maps \tilde{f} sufficiently close to f verify $\tilde{f}(a) \neq \tilde{f}(b)$ and $\tilde{f}(x) \neq \psi(x)$ for all $x \in [a, b]$. Hence $\tilde{f} \notin PW(\psi)$ and we are done.

Necessity of (C2) and (C4). It is enough to take Ax_2 and Ax_1 , respectively. \square

Remark 3.15. Propositions 3.9 and 3.13 formally could be called axiomatic definitions of the topological entropy, since each of them says that $Ax \equiv Top$ is the only map satisfying some group of axioms. Nevertheless, they do not deserve this name since each of them contains axioms which *explicitly* compare the values of axiomatic entropy and topological entropy for some kind of maps. In a “good” axiomatic definition of topological entropy it is clearly undesirable that the symbol “Top” appears in the axioms. \square

4 Proof of Theorem A

We start this section with some auxiliary lemmas. The proof of Theorem A will follow easily from them and from the last proposition of the previous section.

Lemma 4.1. *Let $h \in \mathcal{CS}_1$ be of type 1. Then the set of all fixed points of h is a nondegenerate closed interval $[a, b]$ and, for some $n \in \mathbb{N}$, $h^n(I) = [a, b]$.*

Proof. Since $h \in \mathcal{CS}_1$ it is obvious that the set of fixed points of h is connected. Suppose that there is only one fixed point. Then it is necessarily an interior point of I and h has slope -1 in a neighborhood of it. Hence, h has a periodic orbit of period 2; a contradiction. So, the fact that the set of all fixed points of h is a nondegenerate closed interval $[a, b]$ is proved.

The slope of h is -1 both in a left neighborhood of a and in a right neighborhood of b (if a and b are interior points of I , respectively). Hence, a sufficiently small open neighborhood U of $[a, b]$ is mapped by h onto $[a, b]$.

Now take any point $x \in I$. Since h is of type 1, the trajectory of x converges to a fixed point (see, e.g., [9]). So, $h^{k(x)}(x) \in U$ for some $k(x) \in \mathbb{N}$. By a standard compactness argument, there is some $k \in \mathbb{N}$ such that $h^k(I) \subset U$. Then, $[a, b] \subset h^{k+1}(I) \subset [a, b]$. ■

Lemma 4.2. *Let $f \in \mathcal{CS}_\lambda$ be with $\lambda < 1$. Then arbitrarily close to f there is a map $g \in \mathcal{CS}_1$ such that g is P -linear, where P is weakly g -invariant.*

Proof. Fix $\varepsilon > 0$. Within an ε -neighborhood of f one can obviously construct a map $g \in \mathcal{CS}_1$. Observe that a map from \mathcal{CS}_1 has no periodic orbit of period 4 and so it is of type at most 2. Note that g^2 belongs to \mathcal{CS}_1 and is of type 1. Therefore, by Lemma 4.1, the orbit under g^2 (and, hence, under g) of any point from Q , the set of all turning points of g , is finite. So, the set $P = \bigcup_{n=0}^{\infty} g^n(Q)$ is finite and $P \supset Q$. Since g is Q -linear, it is also P -linear. Obviously, P is weakly g -invariant. ■

Lemma 4.3. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(SF) $\text{Ax}(g) \leq \text{Ax}(f)$ whenever $g \in \text{SF}(f)$.

(\geq LOG) $\text{Ax}(f) \geq \log \lambda$ whenever $f \in \mathcal{CS}_\lambda$ is a P -linear map, where P is a periodic orbit of f and $\lambda > 1$.

Then Ax satisfies the axiom (C4) of Propositions 3.12 and 3.13.

Proof. Let f be P -linear where P is a periodic orbit of f . We need to prove that $\text{Ax}(f) \geq \text{Top}(f)$. If $\text{Top}(f) = 0$ there is nothing to prove. So, assume that $\text{Top}(f) > 0$. There exists $g \in \mathcal{CS}_{\exp(\text{Top}(f))} \cap \text{SF}(f)$ with $\text{Top}(g) = \text{Top}(f)$. By (SF) we have $\text{Ax}(f) \geq \text{Ax}(g)$ and, by Lemma 3.11,

$\psi(P)$ is a periodic orbit of g and g is $\psi(P)$ -monotone (where ψ denotes the non-decreasing semiconjugacy from f to g). Since $g \in \mathcal{CS}_{\exp(\text{Top}(f))}$ it follows that g is $\psi(P)$ -linear with $\{0,1\} \subset \psi(P)$. Thus we can apply (\geq LOG) to get $\text{Ax}(g) \geq \log \exp(\text{Top}(f)) = \text{Top}(f)$. ■

Lemma 4.4. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(CSF) *If f is a P -linear map, where P is a weakly f -invariant set and $\text{Ax}(f) > 0$, then f has a piecewise linear (not necessarily strong) factor $g \in \mathcal{CS}_\lambda$ for some λ such that $\text{Ax}(f) = \text{Ax}(g)$.*

(\leq LOG) *$\text{Ax}(f) \leq \log \lambda$ whenever $f \in \mathcal{CS}_\lambda$ is a P -linear map, where P is weakly f -invariant and $\lambda \geq 1$.*

Then Ax satisfies the axiom (C2) of Propositions 3.9 and 3.13.

Proof. Let f be a P -linear map where P is weakly f -invariant. We need to prove that $\text{Ax}(f) \leq \text{Top}(f)$. This is trivial if $\text{Ax}(f) = 0$. So, we suppose that $\text{Ax}(f) > 0$. By (CSF), f has a piecewise linear factor $g \in \mathcal{CS}_\lambda$ such that $\text{Ax}(f) = \text{Ax}(g)$. Moreover, since g is a factor of f , $\text{Top}(g) \leq \text{Top}(f)$. Finally, recall that $\text{Top}(g) = \max\{0, \log \lambda\}$. Therefore, it is sufficient to prove that $\text{Ax}(g) \leq \max\{0, \log \lambda\}$.

By Lemma 3.11, g is $\psi(P)$ -linear where $\psi(P)$ is weakly g -invariant (ψ denotes the semiconjugacy from f to g). If $\lambda \geq 1$, then we can apply (\leq LOG) to get $\text{Ax}(g) \leq \log \lambda$, which proves the desired inequality.

Now assume that $\lambda < 1$. From Lemma 4.2, (\leq LOG) and (LSC) it follows that $\text{Ax}(g) = 0$, which ends the proof of the lemma. ■

Proof of Theorem A. It follows from Proposition 3.13 and Lemmas 4.3 and 4.4. ■

Remark 4.5. As we said in Remark 3.15, despite of the fact that Propositions 3.9 and 3.13 formally characterize topological entropy, they do not deserve the name of axiomatic definitions of topological entropy since they *explicitly* compare the values of axiomatic entropy and topological entropy for some kind of maps. In other words, in a “good” definition of axiomatic entropy, the axioms (C2), (C3) and (C4) should be replaced by “weaker” axioms. To achieve this we are ready, even, to pay the price of increasing the number of axioms. This is the difference between Theorems A and B and Propositions 3.9 and 3.13. In Theorem A this strategy provokes the appearance of axiom (CSF) that assumes the existence of a factor with constant slopes and in a sense is similar to that of the “principal factor-automorphism” axiom in the Rohlin’s paper [14]. □

Remark 4.6. To show the necessity of the axioms (LSC), (SF), (\leq LOG) and (\geq LOG) in Theorem A one can use the maps Ax_3 , Ax_6 , $2 \cdot \text{Top}$ and $1/2 \cdot \text{Top}$, respectively. Moreover, we conjecture that the axioms (PW) and (CSF) are also necessary. \square

5 Proof of Theorem B

For $\lambda > 0$ let f_λ denote the unique map in \mathcal{CS}_λ such that $f_\lambda(0) = 0$ and $f_\lambda(q) \in \{0, 1\}$ whenever q is a turning point of f_λ . Observe that $f_\lambda \circ f_\mu \in \mathcal{CS}_{\lambda\mu}$ but in general it is not true that $f_\lambda \circ f_\mu = f_{\lambda\mu}$. Note also that $f_2 = \tau$.

Lemma 5.1. *For each $f \in \mathcal{CS}_\lambda$ with $\lambda \geq 1$ there exists $g \in \mathcal{CS}_1$ such that $f = f_\lambda \circ g$.*

Proof. Fix a map $f \in \mathcal{CS}_\lambda$. For any $y \in I$ let $\hat{y} = \min f_\lambda^{-1}(f(y))$ (i.e., $\hat{y} = f(y)/\lambda$). Now we define g as:

$$g(x) = \begin{cases} x - a + \hat{a} & \text{if } f \text{ is increasing on } [a, b], \\ -x + a + \hat{a} & \text{if } f \text{ is decreasing on } [a, b], \end{cases}$$

where $[a, b]$ denotes the lap of f containing x . One can check that g belongs to $\mathcal{C}(I)$ and has the desired properties. \blacksquare

Proposition 5.2. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(COMP) $\text{Ax}(f \circ g) = \text{Ax}(f) + \text{Ax}(g)$ whenever $f \in \mathcal{CS}_{\lambda_1}$ and $g \in \mathcal{CS}_{\lambda_2}$ where $\lambda_1, \lambda_2 \geq 1$.

(TENT) $\text{Ax}(\tau) = \log 2$.

and one of the following three properties:

(\mathcal{CS}_1 -T₁) $\text{Ax}(f) = 0$ whenever $f \in \mathcal{CS}_1$ is of type 1.

(FIN- \mathcal{CS}_λ) $\text{Ax}(f)$ is finite whenever $f \in \mathcal{CS}_\lambda$, $\lambda \geq 1$,

(EQ- \mathcal{CS}_1) Ax is constant on \mathcal{CS}_1 .

Then $\text{Ax}(f) = \text{Top}(f)$ for all maps $f \in \mathcal{CS}_\lambda$, $\lambda \geq 1$.

Proof. We start by proving that if either (\mathcal{CS}_1 -T₁) or (FIN- \mathcal{CS}_λ) or (EQ- \mathcal{CS}_1) holds, then Ax satisfies:

(\mathcal{CS}_1) $\text{Ax}(g) = 0$ for all $g \in \mathcal{CS}_1$.

To prove this we fix a map $g \in \mathcal{CS}_1$ and we will show that $\text{Ax}(g) = 0$. Observe that g^2 also belongs to \mathcal{CS}_1 and it is of type 1.

We start by assuming that Ax satisfies $(\mathcal{CS}_1\text{-T}_1)$. By (COMP) we have

$$\text{Ax}(g) = \frac{1}{2}(\text{Ax}(g) + \text{Ax}(g)) = \frac{1}{2}\text{Ax}(g \circ g) = 0.$$

Now let Ax satisfy $(\text{EQ-}\mathcal{CS}_1)$. Again by (COMP) we have

$$\text{Ax}(\tau) = \text{Ax}(\tau \circ \text{Id}) = \text{Ax}(\tau) + \text{Ax}(\text{Id}).$$

Hence, $\text{Ax}(\text{Id}) = 0$ by (TENT). Then, by $(\text{EQ-}\mathcal{CS}_1)$, $\text{Ax}(g) = 0$.

Finally, let Ax satisfy $(\text{FIN-}\mathcal{CS}_\lambda)$. By Lemma 4.1 the map g^2 is the identity on an interval $[a, b]$ and there exists $n \in \mathbb{N}$ such that $g^{2n}(I) = [a, b]$. Now we define a new map $\xi \in \mathcal{C}(I)$ as follows:

$$\xi(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in [a, b], \\ \xi(g^{2n}x) & \text{otherwise.} \end{cases}$$

Then ξ is surjective and $\xi \circ g^{2n} = \xi \circ \text{Id} \circ \xi$ (i.e., Id is a factor of g^{2n}). It is also obvious that $\xi \in \mathcal{CS}_\lambda$ for $\lambda = \frac{1}{b-a} \geq 1$. By (COMP),

$$\text{Ax}(\xi) + \text{Ax}(g^{2n}) = \text{Ax}(\xi \circ g^{2n}) = \text{Ax}(\text{Id} \circ \xi) = \text{Ax}(\xi).$$

By $(\text{FIN-}\mathcal{CS}_\lambda)$, $\text{Ax}(\xi)$ is finite and we get $\text{Ax}(g^{2n}) = 0$. By (COMP), $\text{Ax}(g^{2n}) = 2n \text{Ax}(g)$. Thus $\text{Ax}(g) = 0$.

So, from now on we assume that Ax satisfies the axioms (COMP), (TENT) and (\mathcal{CS}_1) .

Let $f \in \mathcal{CS}_\lambda$ with $\lambda \geq 1$. We need to prove that $\text{Ax}(f) = \text{Top}(f)$. By Lemma 5.1, $f = f_\lambda \circ g$ for some $g \in \mathcal{CS}_1$. Then (COMP) and (\mathcal{CS}_1) give

$$\text{Ax}(f) = \text{Ax}(f_\lambda \circ g) = \text{Ax}(f_\lambda) + \text{Ax}(g) = \text{Ax}(f_\lambda).$$

Thus $\text{Ax}(f)$ depends only on λ (that is, all maps from \mathcal{CS}_λ have the same axiomatic entropy). Hence there exists a map $\phi: [0, +\infty) \rightarrow [0, +\infty]$ such that:

$$\text{Ax}(f) = \phi(\log \lambda) \text{ whenever } f \in \mathcal{CS}_\lambda \text{ and } \lambda \geq 1.$$

We are going to show that ϕ is additive and monotone. Let $\lambda_x, \lambda_y \geq 1$. Then

$$\phi(\log \lambda_x + \log \lambda_y) = \phi(\log(\lambda_x \lambda_y)) = \text{Ax}(f_{\lambda_x \lambda_y}) = \text{Ax}(f_{\lambda_x} \circ f_{\lambda_y})$$

(though in general $f_{\lambda_x \lambda_y} \neq f_{\lambda_x} \circ f_{\lambda_y}$, the last equality holds true due to the fact that both $f_{\lambda_x \lambda_y}$ and $f_{\lambda_x} \circ f_{\lambda_y}$ belong to $\mathcal{CS}_{\lambda_x \lambda_y}$). Therefore, by (COMP) we get that

$$\phi(\log \lambda_x + \log \lambda_y) = \text{Ax}(f_{\lambda_x}) + \text{Ax}(f_{\lambda_y}) = \phi(\log \lambda_x) + \phi(\log \lambda_y),$$

i.e., ϕ is additive.

Now assume that $1 \leq \lambda_x < \lambda_y$. Observe that $f_{\lambda_y}, f_{\lambda_x} \circ f_{\lambda_y/\lambda_x} \in \mathcal{CS}_{\lambda_y}$. Therefore both maps have the same axiomatic entropy. Thus, by (COMP) we get

$$\begin{aligned} \phi(\log \lambda_y) &= \text{Ax}(f_{\lambda_y}) = \text{Ax}(f_{\lambda_x} \circ f_{\lambda_y/\lambda_x}) = \text{Ax}(f_{\lambda_x}) + \text{Ax}(f_{\lambda_y/\lambda_x}) \\ &\geq \text{Ax}(f_{\lambda_x}) = \phi(\log \lambda_x), \end{aligned}$$

i.e., ϕ is monotone.

Since the map ϕ is additive and monotone, by (TENT), we get that all values of ϕ are finite. So, it is of the form

$$\phi(t) = k \cdot t \quad \text{for some } k \in \mathbb{R}.$$

However, since $\phi(\log 2) = \text{Ax}(\tau) = \log 2$, it follows that $k = 1$ and, hence, ϕ is the identity. Thus, $\text{Ax}(f) = \log \lambda = \text{Top}(f)$ whenever $f \in \mathcal{CS}_\lambda$ and $\lambda \geq 1$. \blacksquare

Proof of Theorem B. It follows from Theorem A and Proposition 5.2. \blacksquare

Remark 5.3. Some variations of Theorem B can be easily deduced from Theorem A and Proposition 5.2. Namely, the axiom $(\mathcal{CS}_1\text{-T}_1)$ can be replaced by $(\text{FIN-}\mathcal{CS}_\lambda)$ or by $(\text{EQ-}\mathcal{CS}_1)$ from Proposition 5.2. \square

Remark 5.4. To show the necessity of the axioms (LSC), (SF), (COMP) and (TENT) in Theorem B one can use the maps $\text{Ax}_3, \text{Ax}_6, \text{Ax}_7$ and Ax_8 , respectively. We do not know whether $(\mathcal{CS}_1\text{-T}_1)$ is not superfluous but we conjecture that (PW) and (\mathcal{CSF}) are necessary. \square

6 Proof of Theorem C

To prove Theorem C we will use the following proposition:

Proposition 6.1. *Let $\text{Ax}: \mathcal{C}(I) \rightarrow [0, +\infty]$ satisfy the following properties:*

(LSC) *Ax is lower semicontinuous.*

(ITER) *$\text{Ax}(f^n) = n \text{Ax}(f)$ for $n \geq 0$.*

(T₁) $Ax(f) = 0$ whenever f is of type 1.

Then $Ax(f) = \text{Top}(f) = 0$ for all maps f of type at most 2^∞ .

Proof. Let $f \in T(\preceq 2^\infty)$ (that is, $\text{Top}(f) = 0$). We have to prove that $Ax(f) = 0$. If $f \in T(1)$ then there is nothing to prove by (T₁). If $f \in T(2^n)$ then $f^{2^n} \in T(1)$. Hence, by (ITER) and (T₁), $Ax(f) = \frac{1}{2^n} Ax(f^{2^n}) = 0$. If $f \in T(2^\infty)$ then, by [12], f belongs to the closure of the set of maps of type less than 2^∞ (that is, to the closure of the set of maps whose axiomatic entropy is 0). By (LSC), $Ax(f) = 0$. ■

Remark 6.2. The above proposition is formulated for an axiomatic entropy map defined on $\mathcal{C}(I)$ but, from its proof it follows clearly that the map Ax could have been defined just in $T(\preceq 2^\infty)$. Moreover, as in previous results, some of its axioms can be weakened. For instance, it is sufficient to formulate (ITER) for maps of type less than 2^∞ and (LSC) for maps of type 2^∞ . □

Remark 6.3. All axioms in the above proposition are necessary in the sense that if one removes any of them then the proposition will no longer be true. To prove the necessity of (T₁) one can use the map Ax_2 . Further, Ax_9 shows that (LSC) is necessary (use the fact that both $T(2^\infty)$ and $T(\succ 2^\infty)$ are closed under iteration). Finally, to prove the necessity of (ITER) we are going to use Ax_{10} .

Observe that, by definition, Ax_{10} satisfies (T₁). So, we only have to prove that Ax_{10} is lower semicontinuous. If $f \in T(1) \cup T_U(2)$ then Ax_{10} is clearly lower semicontinuous at f . If $f \in T(\succ 2^\infty)$ then Ax_{10} is lower semicontinuous at f by using the stability theorem of Block (see [4]). Now assume that $f \in T(2^\infty)$ and let $k \in \mathbb{N}$. Again from [4] we get that there exists a neighborhood U_k of f such that each map in U_k has type at least 2^k . Thus, for each $g \in U_k$ we have $Ax(g) \geq 2^{k-1}$. Consequently, Ax_{10} is lower semicontinuous at f . Now assume that $f \in T_S(2^k)$ for some $k \in \mathbb{N}$. All maps sufficiently close to f belong to $T_S(2^k) \cup T(\succ 2^k)$. Therefore, they have axiomatic entropy larger than or equal to $2^k = Ax_{10}(f)$ and, as above, Ax_{10} is lower semicontinuous at f .

Now suppose that $f \in T_U(2^k)$ for some $k > 1$ (that is, $Ax_{10}(f) = 2^{k-1}$). We claim that there exists a neighborhood of f contained in $T_S(2^{k-1}) \cup T(\succ 2^k)$. To prove the claim suppose, on the contrary, that there is a sequence $\{f_n\}_{n \in \mathbb{N}}$, converging uniformly to f , such that $f_n \in T_U(2^{k-1}) \cup T(\preceq 2^{k-2})$ for each $n \in \mathbb{N}$. Take $n \in \mathbb{N}$ such that $f_n \in T_U(2^{k-1})$. By the definition of $T_U(2^{k-1})$ there exists a map $\tilde{f}_n \in T(\preceq 2^{k-2})$ such that $\|f_n - \tilde{f}_n\| < \frac{1}{n}$. Consequently, there exists a new sequence $\{\tilde{f}_n\}_{n \in \mathbb{N}}$ still converging uniformly to f and such that $\tilde{f}_n \in T(\preceq 2^{k-2})$ for $n \in \mathbb{N}$. But

this contradicts the stability theorem of Block (see [4]) because $f \in T(2^k)$. This ends the proof of the claim.

By the claim, for each map g in a neighborhood of f we have $Ax_{10}(g) \geq 2^{k-1} = Ax_{10}(f)$, which shows that Ax_{10} is lower semicontinuous at f . \square

Proof of Theorem C. It follows from Proposition 6.1 and the discussion right after the statement of Theorem C in the Introduction. \blacksquare

Remark 6.4. The necessity of axioms (LSC), (ITER) and (T_1) in Theorem C follows from Remark 6.3 since the maps Ax_2 , Ax_9 and Ax_{10} satisfy (T_3) . The necessity of (T_3) can be proved by using Ax_1 .

Observe also that there exist trivial examples of maps which characterize chaos. The first easy kind of examples are the maps of the form $\mu \text{Top}(\cdot)$ with $\mu > 0$. Another kind of examples are maps defined by

$$Ax(f) = \begin{cases} 0 & \text{if } f \in T(\preceq 2^\infty), \\ c & \text{otherwise,} \end{cases}$$

where $c \in (0, +\infty]$. However there exist also “true” examples of axiomatic entropies characterizing chaos. One of such examples is Ax_{11} . Indeed, it satisfies (T_1) and (T_3) since Top satisfies them. Also, it satisfies (ITER) because Top satisfies it and both $\text{Forb}(\bar{\tau})$ and $\mathcal{C}(I) \setminus \text{Forb}(\bar{\tau})$ are closed under iteration. Finally, the lower semicontinuity of Ax_{11} follows from the fact that it differs from the lower semicontinuous map Top only on a closed set, where it is $\frac{1}{2} \text{Top}$ (which is also lower semicontinuous). \square

7 Open problems

In this section we present some open problems on the axiomatic definition of topological entropy.

- (I) Prove (or disprove) the necessity of the axioms (PW), (\mathcal{CSF}) in Theorem A and (PW), (\mathcal{CSF}) , (\mathcal{CS}_1-T_1) in Theorem B.
- (II) Find an axiomatic definition of topological entropy for the class of the \mathcal{C}^r -smooth maps of the interval, with $r \in \mathbb{N} \cup \{\infty, \omega\}$ (Proposition 3.9, Proposition 3.13, Theorem A and Theorem B cannot be used in this case since each of them contains some axioms dealing with piecewise linear maps).
- (III) Find an axiomatic definition of topological entropy for general topological dynamical systems or, at least, an axiomatic definition of topological entropy on the *interval* consisting only of such axioms which hold for the topological entropy of continuous selfmaps in any *compact topological space*.

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ADDRESS OF LL. ALSEDA: DEPARTAMENT DE MATEMÀTIQUES, EDIFICI CC, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08913 CERDANYOLA DEL VALLÈS, BARCELONA, SPAIN.

On leave at: DEPARTAMENT DE MATEMÀTICA APLICADA I, UNIVERSITAT POLITÈCNICA DE CATALUNYA, DIAGONAL 647, 08028 BARCELONA, SPAIN.

E-mail address: alseda@mat.uab.es

ADDRESS OF S. KOLYADA: INSTITUTE OF MATHEMATICS, UKRAINIAN ACADEMY OF SCIENCES, TERESHCHENKIVS'KA 3, 252601 KIEV – 4, UKRAINE.

E-mail address: skolyada@imath.kiev.ua

ADDRESS OF J. LLIBRE: DEPARTAMENT DE MATEMÀTIQUES, EDIFICI CC, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08913 CERDANYOLA DEL VALLÈS, BARCELONA, SPAIN.

E-mail address: jllibre@mat.uab.es

ADDRESS OF L. SNOHA: DEPARTMENT OF MATHEMATICS, FPV, MATEJ BEL UNIVERSITY, TAJOVSKÉHO 40, 974 01 BANSKÁ BYSTRICA, SLOVAKIA.

E-mail address: snoha@fpv.umb.sk