On Łukasiewicz's four-valued modal logic

Josep Maria Font
University of Barcelona

Petr Hájek
Academy of Sciences, Prague

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Abstract. Łukasiewicz's four-valued modal logic is surveyed and analyzed, together with Łukasiewicz's motivations to develop it. A faithful interpretation of it into classical (non-modal) two-valued logic is presented, and some consequences are drawn concerning its classification and its algebraic behaviour. Some counter-intuitive aspects of this logic are discussed under the light of the presented results, Łukasiewicz's own texts, and related literature.

1 Introduction

The Polish philosopher and logician Jan Łukasiewicz (Lwów, 1878 – Dublin, 1956) is one of the fathers of modern many-valued logic, and some of the systems he introduced are presently a topic of deep investigation. In particular his infinitely-valued logic belongs to the core systems of mathematical fuzzy logic as a logic of comparative truth, see [5, 15, 14, 16]. However, it must be stressed here that Łukasiewicz's logical work bears also a special relationship to modal logic. Actually, modal notions were part of Łukasiewicz's motivation from the start, and these notions presented naturally to him through his critical study of Aristotelian logic. In his "Farewell lecture", considered as the earliest reference to three-valued logic[1], he said:

> In 1910 I published a book on the principle of contradiction in Aristotle's work, in which I strove to demonstrate that that principle is not so self-evident as it is believed to be. Even then I strove to construct

[1] March 7th, 1918. See Słupecki's introduction to Łukasiewicz's selected works [28, p. viii].

> non-Aristotelian logic, but in vain. Now I believe to have suceeded in this. [...] I have proved that in addition to true and false propositions there are *possible* propositions, to which objective possibility corresponds as a third in addition to being and non-being. This gave rise to a system of *three-valued logic*, which I worked out last summer. That system is as coherent and self-consistent as Aristotle's logic, and much richer in laws and formulae. (Quoted from [28], p. 86)

In 1930, he and Tarski published their famous paper [29], which laid the foundations of the modern algebraic study of propositional logics and of matrix semantics. Its Section 3 is devoted to many-valued logics, and in the same volume of the same journal Łukasiewicz published his paper [24] in order to "clarify the origin and significance of those systems from a philosophical point of view". It is then interesting to notice that this paper of his is almost entirely devoted to the study of modal propositions.

On the other hand, as he tells in the preface to the second edition [27] of his book on Aristotle, it was his aim of formalizing (and correcting) Aristotle's modal syllogistic what made him develop his system of modal logic:

> The first edition of this book did not contain an exposition of Aristotle's modal syllogistic. I was not able to examine Aristotle's ideas of necessity and possibility from the standpoint of the known systems of modal logic, as none of them was in my opinion correct. In order to master this difficult subject I had to construct for myself a system of modal logic [which] is different from any other such system, and from this standpoint I was able to explain the difficulties and correct the errors of the Aristotelian modal syllogistic.

In the paper [26] and in Chapter VII of [27] he introduces a propositional logic with two modalities $\Box$ and $\Diamond$ (necessity and possibility)[2] acting as unary connectives. His study of Aristotle and other philosophical reasons, whose investigation goes far beyond the aims of the present paper, lead him to assume that these connectives should be *extensional*, in particular the formulas

$$(1) \qquad (\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \quad \text{and} \quad (\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$$

should be valid. He took a generalized form of the extensionality principle as a basis of a syntactical presentation of his logic; the generalization was

[2]Lukasiewicz uses L or Γ for necessity and M or Δ for possibility; he also uses his "Polish" notation. Actually, it seems that he was among the first to use L and M in this sense, see [17, Appendix 4]. We use the usual symbolism here.

achieved by the use of "variable functors", a device taken from Leśniewski's protothetics which he had already adapted and perfected in his 1921 presentation of classical two-valued logic. But in Łukasiewicz's conception the real definition of a logic must be semantical; indeed, in the same paper on two-valued logic he says (italics and quotation marks are his):

> By *logic* I mean the science of logical values. Conceived in this way, logic has its own subject-matter of research, with which no other discipline is concerned. Logic is not a science of propositions, since that belongs to grammar; it is not a science of judgements or convictions, since that belongs to psychology; it is not a science of contents expressed by propositions, since that, according to the content involved, is the concern of various detailed disciplines; it is not a science of "objects in general", since that belongs to ontology. Logic is the science of objects of a specific kind, namely a science of *logical values*.
>
> (Quoted from [28], p. 90)

Thus, when trying to define a new logic for some specific purposes one has to find the logical values it deals with; and its propositional functors are to be *truth functions* operating on these logical values. Łukasiewicz appears here as a truth-functionalist in a very strict sense: he believes each logic has, or should have, a single truth-table able to define it, i.e., able to define its connectives or propositional functors and also able to obtain its valid laws and to reject the invalid ones. Clearly, two-valued functions cannot define one-argument functions for necessity and possibility satisfying the minimal requirements for them to be *modal*, which we reproduce as items (23) to (26) in Section 4. This made him turn to the many-valued logics he had introduced years before by putting *possible* in a par with *true* and *false*. But he said:

> When I had discovered in 1920 a three-valued system of logic, I called the third value, which I denoted by $^1/_2$, "possibility". Later on, having found my n-valued modal systems[3], I thought that only two of them may be of philosophical importance, viz., the 3-valued and the $\aleph_0$-valued system. [...] This opinion, as I see it today, was wrong.
>
> ([26], quoted from [28], pp. 370–371)

and also:

[3] He calls here "modal" his many-valued systems because of the interpretation of the non-classical truth values in terms of "possibility" and because in each of them the modal operators were definable, by $\Box\varphi = \neg(\varphi \to \neg\varphi)$ and $\Diamond\varphi = \neg\varphi \to \varphi$, as suggested by Tarski in 1921 when he was Łukasiewicz's student; see [24] or [28, p. 167].

> I see today that this system [the 3-valued one] does not satisfy all our intuitions concerning modalities and should be replaced by the system described below. ([27], pp. 166–167)

and he presented his four-valued truth-tables, claimed that they completely characterize his logic, and went as far as declaring:

> No serious objection can be maintained against this system. We shall see that this system refutes all false inferences drawn in connexion with modal logic, explains the difficulties of the Aristotelian modal syllogistic, and reveals some unexpected logical facts which are of the greatest importance for philosophy. ([27], p. 169)

However, as we shall discuss, both the semantics and many theorems of the logic have a difficult interpretation in terms of any intuitive notions of possibility and of necessity. In view of the enormous development of modal logic since Łukasiewicz's book appeared[4], cf. [2, 3], it is legitimate to ask whether his system still has some value (except as historical rarity) as *modal* logic, and whether it is *natural* in some sense. But it is one thing to say, as we claim, that Łukasiewicz's system is rather a dead end from an intuitive or applied point of view, and quite a different one to deny that some lessons can be learned from our enquiry.

The paper is organized as follows: In Section 2 we present Łukasiewicz's system and offer some mathematical results concerning its classification under several standard criteria and its semantical and algebraic properties. In Section 3 we refer to the literature containing related observations and state some connections with our own results. Finally, Section 4 contains a discussion of the meaningfulness and (un)naturality of Łukasiewicz's system, partly based on our own results, and on the literature analyzed here and in the preceding section, as well as on Łukasiewicz's own texts.

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[4] Relational semantics of several kinds were elaborated in full only after Łukasiewicz's death. Actually, [27], the second edition of his [25], was published a few months after he passed away, and he could not proof-read it; this was done by his disciple Cz. Lejewski. Thus, one can consider the chapters on Aristotle's modal logic in [27] as Łukasiewicz's logical testament.

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2 Łukasiewicz's four-valued modal logic

Modulo inessential notational changes (see Note [2]), Łukasiewicz's system, which he denoted by *Ł*[5], is presented as a propositional calculus with propositional variables, the usual non-modal connectives $\rightarrow, \neg, \&, \vee$ and the specific modal connectives $\Box, \Diamond$ (unary). For convenience we will also use two constant symbols $\top, \bot$ representing truth and falsehood; since we assume classical logic, they are definable as any chosen theorem (for $\top$) or contradiction (for $\bot$) of CPC. Łukasiewicz introduced his system syntactically in a non-standard way, using both inference and rejection rules and functorial variables, and claimed that it was characterized by the four-element truth-table he also introduced. The rather obscure argument he offered to this effect was later criticized by Smiley, who in [38] proved that this truth-table does characterize the system when presented in a more usual axiomatic style (i.e., with no rejection part, but still with functorial variables); Lemmon in [21] fully re-proved this for the (now completely standard) syntactical presentation we will briefly use at the end of the section. We start by taking the semantics as our working definition of the logic.

There are four truth-values denoted by $11, 10, 01, 00$ (pairs of zeros and ones; Łukasiewicz used $1, 2, 3, 0$). The truth tables of $\rightarrow$, $\neg$, $\Box, \Diamond$ are as follows, $\&$ and $\vee$ being defined as in classical logic (i.e., $\varphi \& \psi = \neg(\varphi \rightarrow \neg\psi)$ and $\varphi \vee \psi = \neg\varphi \rightarrow \psi$).

$\rightarrow$	11	10	01	00
11	11	10	01	00
10	11	11	01	01
01	11	10	11	10
00	11	11	11	11

	$\neg$	$\Diamond$	$\Box$
11	00	11	10
10	01	11	10
01	10	01	00
00	11	01	00

Let us call it $\mathbf{B}_4$. In this set of truth-values, *truth* is represented by the value 11; that is, $\top$ is always interpreted as 11 (and $\bot$ as 00). Hence the *tautologies* of the *Ł*-system are the formulas constantly equal to 11 under all evaluations.

[5] Warning: The symbol *Ł* is also used in the literature, often with subscripts, to denote Łukasiewicz's infinitely-valued logic or his usual, linearly-ordered, finitely-valued systems.

It is immediately seen that according to implication and negation (and hence to conjunction and disjunction as well) this algebra of truth functions is just the cartesian product of the two-element Boolean algebra $\mathbf{B}_2$ with universe $B_2 = \{0, 1\}$ with itself (i.e. operations act coordinatewise) if we identify uv with the pair (u, v). From this it follows that modality-free formulas are tautologies of L if and only if they are tautologies of CPC (classical propositional calculus), that is:

Proposition 1. *The logic L is a conservative expansion of CPC.* ∎

This fact is not explicitly mentioned by Łukasiewicz, who only says that L contains CPC (in the sense that every CPC-tautology is also a tautology of L). Moreover, the given truth functions of $\Box$ and $\Diamond$ satisfy (for $u, v = 0, 1$)

$$\text{(2)} \qquad \Box(uv) = 10 \,\&\, uv = u0 \quad , \quad \Diamond(uv) = 01 \vee uv = 10 \rightarrow uv = u1.$$

Also observe that $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$, and dually $\Box\varphi$ is equivalent to $\neg\Diamond\neg\varphi$; hence only one of the operators needs to be taken as primitive (actually, this is what Łukasiewicz himself does). The equations (2) immediately suggest:

Definition. *For each formula φ of L we define its translation φ^* using a new propositional variable L in the following recursive way: $p^* = p$ for each propositional variable p of L, * commutes with $\rightarrow$ and $\neg$, and*

$$\text{(3)} \qquad (\Box\varphi)^* = L \,\&\, \varphi^* \qquad \textit{and} \qquad (\Diamond\varphi)^* = L \rightarrow \varphi^*$$

Readers who do not feel at ease with this use of a "new" variable and prefer that the two logics to be compared share a common set of variables should consider the following alternative translation: If the set of variables is ordered as $\{p_0, p_1, \ldots, p_n, \ldots\}$ then put $(p_i)^* = p_{i+1}$ for all i and use p_0 instead of L in (3). With minor adjustments in the proofs they will get the same results.

Theorem 1. *A formula φ is a tautology of L if and only if φ^* is a tautology of the classical propositional calculus CPC.*

Proof. Let us assume that φ^* is a tautology of CPC, and let e be any evaluation of the variables of φ with values in B_4. Since L does not appear in φ, we can safely extend it to an evaluation e' such that $e'(L) = 10$. Then it is easy to show recursively, using (2) and (3), that $e(\psi) = e'(\psi^*)$ for any formula ψ, in particular for φ. Since $\mathbf{B}_4$ is a Boolean algebra and φ^* is a tautology of CPC, $e'(\varphi^*) = 11$, and therefore $e(\varphi) = 11$ as well. This shows that φ is a tautology of L.

Conversely, assume that φ is an L-tautology and let e_0 be an evaluation of $L, p_0, p_1, \ldots$ with values in $B_2 = \{0,1\}$. Then define $e_1(p_i) = 11$ if $e_0(p_i) = 1$ and $e_1(p_i) = 00$ if $e_0(p_i) = 0$. Thus e_1 is an evaluation of the variables of φ with values in B_4. Then it is easy to show recursively, using (2) and (3) again, that if $e_0(L) = 1$, then for any ψ, $e_0(\psi^*)$ is the projection of $e_1(\psi)$ on the first coordinate, while if $e_0(L) = 0$, then for any ψ, $e_0(\psi^*)$ is the projection of $e_1(\psi)$ on the second coordinate. Since by assumption $e_1(\varphi) = 11$, in both cases we get $e_0(\varphi^*) = 1$. This shows that φ^* is a tautology of CPC. ∎

Corollary 1. *The mapping* * *is a faithful interpretation of* L *into* CPC*.* ∎

A natural enquiry concerning Łukasiewicz's modal logic is to look for a Kripke-style semantics. A Kripke frame is a structure $K = \langle W, R\rangle$ where $W \neq \emptyset$ is a set (the so-called "possible worlds"), and R is a binary relation on W (the so-called "accessibility relation"). A Kripke model is a structure $K = \langle W, R, e\rangle$ that is a Kripke frame endowed with a mapping e assigning to each $w \in W$ and each propositional variable p a truth value $e(p, w) \in B_2 = \{0,1\}$. This mapping is used in the truth conditions for modalized formulas: $e(\Box\varphi, w) = 1$ if and only if $e(\varphi, w') = 1$ for all $w' \in W$ such that $(w, w') \in R$; from this it follows that $e(\Diamond\varphi, w) = 1$ if and only if there is some $w' \in W$ such that $(w, w') \in R$ and $e(\varphi, w') = 1$. The truth conditions for non-modal connectives are the usual ones of classical logic for each fixed world w. If the evaluation e is clear from context then it is customary to write $w \Vdash \varphi$ for $e(w, \varphi) = 1$.

Let $\mathcal{L}$ be the class of all Kripke models where R is a subrelation of identity: $(w, w') \in R$ implies $w = w'$. Observe that then the general truth conditions become $w \Vdash \Box\varphi$ if and only if $(w, w) \notin R$ or $w \Vdash \varphi$, and $w \Vdash \Diamond\varphi$ if and only if $(w, w) \in R$ and $w \Vdash \varphi$.

We are going to see that $\Box$ behaves in the class $\mathcal{L}$ as $\Diamond$ behaves in the logic L and conversely. To this end, let us consider, for each modal formula φ, the formula $\widetilde{\varphi}$ resulting from φ by changing $\Box$ to $\Diamond$ and $\Diamond$ to $\Box$, and leaving everything else untouched.

Theorem 2. *A formula* φ *is an* L*-tautology if and only if the formula* $\widetilde{\varphi}$ *is true in all models of the class* $\mathcal{L}$*.*

Proof. Observe that the satisfaction of φ in w depends only on the evaluation of variables in w and on whether $(w, w) \in R$ or not. It is straightforward to show that $\Diamond\varphi \equiv (\Diamond\top \mathbin{\&} \varphi)$ and $\Box\varphi \equiv (\Diamond\top \to \varphi)$ hold in $\mathcal{L}$. After comparing these expressions with (3) we see that $\widetilde{\varphi}$ is $\mathcal{L}$-equivalent to the substitution instance of φ^* where $\Diamond\top$ is substituted for L; and these are the only occurrences of modal operators in $\widetilde{\varphi}$. Since $\Diamond\top$ holds in w if and only if $(w,w)\in R$, it follows that $\widetilde{\varphi}$ is true in all models in $\mathcal{L}$ if and only if φ^* is true in all Boolean evaluations, that is, if and only if it is a CPC-tautology. By our previous Theorem 1, this happens if and only if φ is an L-tautology. ∎

Certainly $\mathcal{L}$ is a rather unusual class of Kripke models; e.g. $\Diamond\varphi\to\varphi$ and $\varphi\to\Box\varphi$ hold in $\mathcal{L}$ while $\varphi\to\Diamond\varphi$ and $\Box\varphi\to\varphi$ don't. As we see, only after interchanging the roles of $\Box$ and $\Diamond$ does it make sense modally speaking. However, this would amount to considering a semantics based on the same class of structures but where the truth conditions of the modal operators are interchanged, that is, truth of $\Box\varphi$ at w would no longer mean that φ is true at all worlds accessible (i.e., R-related) from w but only in some of them, and dually for $\Diamond$. In this case the tautologies of L would be the formulas that are true in this semantics; this would sound as a completeness-like result, but the main virtue of Kripke semantics, which is the natural interpretation of the notions of necessity and possibility in terms of the accessibility relation R, would be lost. Instead, we will see that relational models of a more general kind, but where a subrelation of identity is also used, can give a satisfactory semantics for Ł.

The real reason why L does not have a Kripke semantics in the standard sense lies in the following result, which classifies this logic with respect to standard criteria:

Proposition 2. *The logic L is a regular modal logic and is neither a quasi-normal nor a normal modal logic.*

Proof. Regular modal logics can be defined (see [4]) as the logics in the modal language we are using that contain CPC, prove (or define) $\Diamond\varphi$ equivalent to $\neg\Box\neg\varphi$, and are closed under Modus Ponens and under the following rule:

$$\frac{\varphi\ \&\ \psi\to\theta}{\Box\varphi\ \&\ \Box\psi\to\Box\theta} \tag{4}$$

All this is straightforward to check. Quasi-normal modal logics can be defined (see [37]) as the logics that contain the minimal normal modal logic K and are closed under Modus Ponens. We have already seen that L satisfies this last condition. However, K contains CPC and is moreover closed under the so-called *Rule of Necessitation*, that is, the rule:

$$\frac{\varphi}{\Box\varphi} \tag{5}$$

Thus in particular L should have all formulas $\Box\varphi$, for φ a CPC-tautology, as theorems. But L has no theorem of the form $\Box\varphi$ at all, so that it cannot contain all K-theorems, and hence it is not quasi-normal. Finally, normal modal logics are those containing K and being closed under both Modus Ponens and Necessitation. The previous reasoning also shows that L is not normal. ∎

Any logic characterised by a class of Kripke frames is normal; this explains why L is in fact not a candidate to having a good Kripke semantics. Regular modal logics, first studied by Lemmon in 1957, belong to the larger group of *classical* modal logics, a kind of logics for which Scott and Montague introduced in 1968 *neighbourhood semantics*, based on more complicated structures than Kripke frames. However, for regular logics it can be simplified so as to coincide with the semantics based on Kripke models augmented with a subset of possible worlds labelled as "normal", introduced and studied by Kripke in [20]. A similar semantics is studied by Lemmon in [21], where a complete semantics for L is indirectly determined in the form of an algebraic semantics[6].

Let us call here a *model structure* an ordered triple $\langle W, R, Q\rangle$ where $\langle W, R\rangle$ is a Kripke frame in the above sense, supplemented with a subset $Q \subseteq W$, which, according to Lemmon ([21], part I, p. 57) represents "the universes in which anything (including a contradiction) is possible". In this semantics the evaluation of modalized formulas is not the standard one, but the following: $w \mathrel{||\!-} \Diamond\varphi$ iff either $w \in Q$ or there is some $w' \in W$ such that $(w, w') \in R$ and $w' \mathrel{||\!-} \varphi$. Accordingly, $w \mathrel{||\!-} \Box\varphi$ iff both $w \notin Q$ and $w' \mathrel{||\!-} \varphi$ for all $w' \in W$ such that $(w, w') \in R$. Then Lemmon proved that L is characterized by the class of *discrete epistemic model structures*, which are those model structures such that: (1) R is a subrelation of identity; and (2) R is reflexive outside Q. This semantics justifies why L does not have any theorem of the form $\Box\varphi$: its models admit worlds where nothing is necessary and everything is possible.

Lemmon's proof used actually an algebraic semantics and the suitable representation theorems. Let us consider Boolean algebras endowed with a unary operator $\Box$, from which $\Diamond = \neg\Box\neg$ is derived, and consider the following equations:

(6) $$\forall x, y \in A, \quad \Box(x \,\&\, y) = \Box x \,\&\, \Box y$$
(7) $$\forall x \in A, \quad \Box x \leq x$$
(8) $$\forall x \in A, \quad \Box x = \Box 1 \,\&\, x$$

[6] More details of historical kind can be obtained in Segerberg's monograph [37].

In what has become standard terminology, Lemmon introduced *modal algebras* as those satisfying (6). To study certain classical systems he also introduced *epistemic algebras*, which are modal algebras satisfying (7); and finally he proved completeness of L with respect to the class of matrices of the form $\langle \mathbf{B}, \{1\} \rangle$ where $\mathbf{B}$ is a *discrete epistemic algebra*, which is an epistemic algebra satisfying equation (8); that is, the tautologies of L are the formulas that always evaluate to 1 in all discrete epistemic algebras. We have already encountered condition (8) as the property (2) that suggested our translation (3). Notice that (8) implies (6) and (7); hence discrete epistemic algebras are just Boolean algebras where one element is selected as $\Box 1$, and where (8) defines $\Box$. That is, discrete epistemic algebras can be presented as Boolean algebras with a distinguished element; but since this element can take any value on any Boolean algebra, it can be treated as the interpretation of a propositional variable into Boolean algebras. From the algebraic point of view these models of L are less odd than the relational ones, and probably more useful. For instance, let us show how we can use our Theorem 1 to obtain a direct proof of the Smiley-Lemmon's[7] completeness result:

Theorem 3. *The logic L is complete with respect to the class of all discrete epistemic algebras. That is, φ is an L-tautology if and only if $v(\varphi) = 1$ for all evaluations v in all discrete epistemic algebras.*

Proof. We use the following translation $^\circ$ from CPC-formulas to L-formulas, which is inverse to * modulo the logic L: Put $p^\circ = p$ for each CPC variable p save L, put $L^\circ = \Box\top$, and let $^\circ$ commute with $\neg$ and $\rightarrow$. Taking the above description of discrete epistemic algebras and of their relationship to Boolean algebras into account, it is easy to show that a CPC-formula ψ evaluates to 1 in all Boolean algebras if and only if its translation ψ° evaluates to 1 in all discrete epistemic algebras. Now let φ be any L-formula. By our Theorem 1, φ is an L-tautology if and only if φ^* is a CPC-tautology, that is, if and only if φ^* evaluates to 1 in all Boolean algebras. By the above observation, this is equivalent to $\left(\varphi^*\right)^\circ$ evaluating to 1 in all discrete epistemic algebras. But observe that $\left(\varphi^*\right)^\circ$ is the transformation of L-formulas obtained by recursively replacing $\Box\psi$ by $\Box\top \,\&\, \psi$, and hence by (8) the formulas $\left(\varphi^*\right)^\circ$ and φ are equivalent in the variety of discrete epistemic algebras. This completes the proof. ∎

As a consequence, we can show:

[7] Smiley showed the completeness of L with respect to his presentation of the axiomatization suggested by Łukasiewicz, and Lemmon showed the completeness of this syntactically defined logic with respect to the class of discrete epistemic algebras.

Theorem 4. *The class of discrete epistemic algebras is a variety, and it is generated by the four-element algebra* $\mathbf{B}_4$.

Proof. This class is defined by a set of equations, thus it is a variety. It is trivial to check that $\mathbf{B}_4$ is a discrete epistemic algebra. Now let $\varphi \approx \psi$ be an equation in the modal language that holds in $\mathbf{B}_4$. Since this is a Boolean algebra, we know that this is equivalent to the equations $\varphi \to \psi \approx \top$ and $\psi \to \varphi \approx \top$ holding in $\mathbf{B}_4$. By definition of L, this means that the formulas $\varphi \to \psi$ and $\psi \to \varphi$ are theorems of L; by our Theorem 3 this holds iff $v(\varphi) \to v(\psi) = 1$ and $v(\psi) \to v(\varphi) = 1$ for all evaluations v in all discrete epistemic algebras. Since these are Boolean, this is equivalent to saying that the equation $\varphi \approx \psi$ holds in every discrete epistemic algebra. We have thus shown that $\mathbf{B}_4$ generates the whole variety. ∎

Another application of these algebras would be to obtain an alternative and straightforward proof of our Theorem 1 starting from Theorem 3 (relying on an independent proof of this one, such as Smiley-Lemmon's).

To close this section, let us consider the problem of extending the logic L to a *consequence relation* between a set of formulas and a formula, that is, a logic in Tarski's sense, something not present among Łukasiewicz's aims. There are two natural ways to do it, the semantical and the syntactical one. Let us begin with the *semantical* one, which consists in taking Łukasiewicz's original truth-table, with 11 still representing truth, and use it in the so-called "preservation of truth" scheme: If Γ is a set of formulas and φ is any formula, then we say that φ follows from Γ, denoted by $\Gamma \models_L \varphi$, if and only if φ is evaluated to 11 whenever all formulas in Γ are evaluated to 11, taking all possible evaluations of the set of variables into the set of truth-values. It is easy to see that in this way L becomes a sentential logic in the sense of Tarski, which is finitary (compact) because the defining matrix is finite (see [44], Theorem 4.1.7). An obvious question is now the extension of our result on the faithful translation to consequence from assumptions in L. As we show, this is only possible in one direction.

Proposition 3. *Modus Ponens is a rule of* L, *that is,* $\{\varphi, \varphi \to \psi\} \models_L \psi$. ∎

Proposition 4. *Let* Γ *be a set of formulas and put* $\Gamma^* = \{\psi^* : \psi \in \Gamma\}$ *and let* φ *be any formula. If* $\Gamma^* \models_{CPC} \varphi^*$ *then* $\Gamma \models_L \varphi$.

Proof. If $\Gamma^* \models_{CPC} \varphi^*$ and $\Gamma \neq \emptyset$ then by Compactness and the Deduction Theorem for CPC there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\psi_1^* \to \big(\cdots \to (\psi_n^* \to \varphi^*)\big)$ is a tautology of CPC. By the previous Theorem 1, $\psi_1 \to \big(\cdots \to (\psi_n \to \varphi)\big)$ is a tautology of L, and by Proposition 3 $\{\psi_1, \ldots, \psi_n\} \models_L \varphi$, which shows that $\Gamma \models_L \varphi$. ∎

However, the converse implication is not true: If p and q are distinct variables, then trivially $\Box p \models_L q$, because no evaluation gives to $\Box p$ the value 11; but it is certainly not the case that $(\Box p)^* = L \,\&\, p \models_{CPC} q = q^*$. The same example shows that the Deduction Theorem in its ordinary form cannot hold for L, because $\Box p \to q$ is not a tautology of L (evaluate p to 10 and q to 01).

These negative results make it more desirable to explore the alternative path of defining a consequence relation associated with L: the *syntactical* approach. Let us take the axiomatization of L shown to be complete for L by Lemmon in [21]. It is presented as an expansion of classical, two-valued propositional calculus CPC, hence one assumes any set of axioms for it, plus the following three axioms:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{9}$$

$$\Box\varphi \to \varphi \tag{10}$$

$$\Box\varphi \to (\psi \to \Box\psi) \tag{11}$$

Let $\vdash_L$ be the consequence relation obtained by the usual notion of proof from assumptions in the formal system having the preceding axioms (including those of CPC) and having *Modus Ponens* as its only rule of inference: If Γ is a set of formulas and φ is any formula, then we say that φ follows from Γ in this logic, denoted by $\Gamma \vdash_L \varphi$, if and only if there is a finite sequence of formulas ending in φ and where every formula is either an instance of one of the axioms or belongs to Γ or is obtained by Modus Ponens from previous formulas in the sequence. This logic has a much better behaviour than $\models_L$. By Lemmon's (or Smiley's) completeness proof, the two logics have the same set of theorems; but *these two logics are different* as consequence relations. As one reason for this, note that $\vdash_L$ satisfies the Deduction Theorem, because it is an axiomatic expansion of CPC, while $\models_L$ does not, as we have just noticed. Also, thanks to the Deduction Theorem, the extension of our Theorem 1 to the consequence $\vdash_L$ is straightforward:

Theorem 5. *If Γ is a set of formulas and φ is any formula, then $\Gamma \vdash_L \varphi$ if and only if $\Gamma^* \models_{CPC} \varphi^*$. Hence the mapping * is a faithful interpretation of L into CPC at the level of consequence.* ∎

Algebraically speaking $\vdash_L$ has a very good behaviour:

Theorem 6. *The logic $\vdash_L$ is strongly complete with respect to the class of discrete epistemic algebras.*

Proof. The proof follows a standard Tarski-Lindenbaum procedure. Let Γ be a set of formulas and φ be any formula. In Theorem 56 of [21, part II] it is shown that for any discrete epistemic algebra $\mathbf{B}$, the matrix $\langle \mathbf{B}, \{1\} \rangle$ satisfies all axioms of $\vdash_L$ and is closed under Modus Ponens. Hence if $\Gamma \vdash_L \varphi$ and all formulas of Γ are evaluated to 1 then also φ is evaluated to 1. Now assume that $\Gamma \nvdash_L \varphi$. It is easy to check that the following binary relation between formulas

$$\alpha \sim_\Gamma \beta \iff \Gamma \vdash_L \alpha \leftrightarrow \beta$$

is a congruence of the formula algebra such that the quotient algebra is a discrete epistemic algebra and the projection mapping is in fact an evaluation sending exactly the formulas of the theory generated by Γ to 1 in this discrete epistemic algebra. Hence φ is not evaluated to 1. This shows completeness. ∎

One consequence of this is an alternative proof of Theorem 4; the argument is parallel to the one we used after Theorem 4 to supply a proof of Theorem 1, hence we omit it. The next result classifies $\vdash_L$ according to recent criteria of abstract algebraic logic, as introduced in [1] and further developed in [11] and [12].

Theorem 7. *The logic $\vdash_L$ is finitely, strongly and regularly algebraizable and its equivalent algebraic semantics is the class of discrete epistemic algebras.*

Proof. $\vdash_L$ is an expansion of CPC, a logic that is finitely algebraizable and has the class of all Boolean algebras as its equivalent algebraic semantics. The equivalence formulas are $\{ p \to q, q \to p \}$ and the defining equation is $p \approx \top$. By the syntactic characterization of algebraizability given in Theorem 4.7 of [1], in order to check that $\vdash_L$ is also finitely algebraizable with the same equivalence formulas and defining equation, it is enough to check that the property of congruentiality also holds for the extra language; in the present case we should show that

$$\{ p \to q, q \to p \} \vdash_L \Box p \to \Box q .$$

But this is a consequence of the formula $(p \to q) \to (\Box p \to \Box q)$, which is a tautology of L as is easy to check. By this we know that $\vdash_L$ is finitely algebraizable. By Corollary 5.3 of [1] its equivalent algebraic semantics is the class of the algebra reducts of all its reduced matrices. With the help of Theorem 56 of [21, part II] it is easy to check that this is the class of discrete epistemic algebras. Since this class is a variety, the algebraizability is "strong", and since the filter of the reduced matrices is a one-element subset $\{1\}$, the algebraizability is "regular". This completes the proof of the Theorem. ∎

Thus from the point of view of abstract algebraic logic, Łukasiewicz's modal four-valued logic enjoys the best possible behaviour.

It is interesting, though simply anecdotic, to compare the present situation with that of Łukasiewicz's infinitely-valued logic (see [16]). In both cases he suggested a logic defined by a truth-table and conjectured its finite axiomatization. His conjecture was fully proved by other researchers, and in both cases the consequence relations defined by the original matrix and by the formal system do not concide. And, again in both cases, it seems that the most standard behaviour is found in the syntactically defined one: it is algebraizable and enjoys strong completeness with respect to the class of algebras intuitively associated with the original truth-table (here, discrete epistemic algebras; there, MV-algebras). One big difference is that in the infinitely-valued case the semantic consequence is not even finitary (compact); see [16, p. 75] or [44, p. 281].

3 Related results

Our main result in the previous section, Theorem 1, can be related to some results already found in the literature. For instance, as a corollary, we can obtain the "two-formulas interpretation" of L first observed by Prior in [36]. For each L-formula φ, let φ' result from φ by omitting all modalities, and let φ'' result from φ by recursively defining $p'' = p$ for all variables p, letting $''$ commute with $\neg$ and $\rightarrow$, and putting $(\Box\varphi)'' = \bot$ and $(\Diamond\varphi)'' = \top$. We have:

Corollary 2. *A formula φ is a tautology of L if and only if both φ' and φ'' are tautologies of CPC.*

Proof. Observe that φ' is equivalent, modulo CPC, to the formula that results from φ^* after substituting $\top$ for L, while φ'' is equivalent, modulo CPC, to the formula that results from φ^* after substituting $\bot$ for L. The (constant) values of $\top$ and $\bot$ are the two possible values of the variable L in B_2. Hence φ^* is a tautology of CPC if and only if both φ' and φ'' are. Now the result follows as a consequence of Theorem 1. ∎

This fact has been paraphrased in a different way in the works of Kripke [20, p. 210], Lemmon [21, part II, Theorem 60] and Segerberg [37, p. 210]; of these, only Lemmon devotes certain space to L, while the other two just mention the result in passing (and attribute to Prior). They say that L is the intersection of two systems obtained from it by adding the axiom $\varphi \rightarrow \Box\varphi$ in one case, and the axiom $\neg\Box\varphi$ in the other case. It is easy to recognise that in the first system each formula φ is equivalent to φ' while in the second one φ is equivalent to φ''. This gives the connection between Prior's formulation and theirs.

After having written the first version of this paper, we found several papers in the literature that had more or less approached (some of) our results.

Smiley, in the context of his investigation [40] of several formalisations of a notion of necessity relative to some statement, considers modal logics with a unary operator O meaning "it is obligatory that" and translates them into ordinary modal logics by taking the translation of $O\varphi$ to be $\Box(T \to \varphi)$, where T is *a propositional constant* that has to be added to the ordinary modal vocabulary. In Section 5 he considers his strongest system, called OPC, and shows it to be another presentation of L if O is read as the $\Diamond$ of L, and is unrelated to the $\Box$ mentioned before. Independently of this fact, he also shows that after deleting the previous uses of $\Box$ in the translations and taking the translation of $O\varphi$ to be $T \to \varphi$ then a formula φ is a theorem of OPC if and only if its translation is a theorem of CPC. This is very close to our Theorem 1, but observe that the translated formula contains the new constant T and the interpretation of this T in CPC is not specified (the proofs in this section are syntactic). The way the paper is presented, specially the mixing of $\Box$ and $\Diamond$ in the interpretations of O, make it difficult to recognise the result.

Porte describes in [34] a presentation of the L system that has a propositional constant as a primitive operator, instead of any tipically modal ones. He shows that the logic so obtained as an extension of CPC is precisely the system L, if one defines the modal operators $\Box$ and $\Diamond$ similarly to the formulas (3) we used in defining our translation. He even formulates the equivalence between L and his system through two translations, but does not establish the relation with the CPC-tautologies. According to [35], this work is independent of Smiley's.

A similar situation to that with Smiley's paper happens in Vakarelov's papers [42, 43]. In the first one Vakarelov defines and studies a propositional calculus with functors for "probably" and "hardly", denoted by L and D; for reasons explained later let us write $\Diamond$ for Vakarelov's L. His axioms, besides axioms of CPC, are:

$$\Diamond(\Diamond p \to p)$$
$$(p \to q) \to (\Diamond p \to \Diamond q)$$
$$(p \to q) \to (Dp \to Dq)$$
$$(\Diamond p \to q) \to DDq$$
$$Dp \to (p \to (\Diamond q \to q))$$

He shows his axioms to be consistent and independent; to prove decidability he presents a faithful interpretation $*$ of his calculus in the classical propositional calculus using a new propositional variable s, and putting $p_i^* = p_i$, $*$ commutes with non-modal connectives, $(\Diamond\varphi)^*$ is $s \to \varphi^*$, and $(D\varphi)^*$ is $\varphi^* \to s$. This is in Part I, written apparently in ignorance of Łukasiewicz's [26].

In Part II [43] he shows a normal form. It says the following: each formula is equivalent to a formula not containing $\Diamond$ and containing D only in the context $D\top$. This follows by observing that $\Diamond\varphi$ is equivalent to $D\top \to \varphi$ and $D\varphi$ is equivalent to $\varphi \to D\top$. Analogously one could eliminate D and allow $\Diamond$ only in the context $\Diamond\bot \vee \varphi$. Then he presents the relation to Łukasiewicz's L, i.e., he gives the four-valued truth tables and shows that his logic is equivalent to what we have called L. Indeed, his $\Diamond$ is our $\Diamond$ and his D can be defined in L by setting $D\varphi$ to be $\Diamond\bot \to \neg\varphi$. He says that the equivalence of his logic to Łukasiewicz's four-valued modal logic was unexpected. He explicitly proves completeness of his system showing that provable formulas coincide with 11-tautologies.

Connecting the results in the two papers our Theorem 1 follows immediately. This fact is not explicitly mentioned by Vakarelov, and since his papers were published in Bulgarian they have remained rather unknown.

Finally, we find a similar (possibly more general) result hidden in Curry's 1952 paper [8] on general conditions on Gentzen-style rules so that the cut-elimination theorem holds. As an example (indeed, the one that motivates the issue) Curry discusses several rules governing possibility, and selects the following three (here Γ, Δ, Π are arbitrary finite sequences of formulas; the sequent calculi are of the 'multiple-conclusion' kind):

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \Diamond\varphi, \Delta} \tag{12}$$

$$\frac{\Gamma, M \vdash \Diamond\varphi, \Delta}{\Gamma \vdash \Diamond\varphi, \Delta} \tag{13}$$

$$\frac{\Gamma \vdash M, \Delta \qquad \Gamma, \varphi \vdash \Delta, \Pi}{\Gamma, \Diamond\varphi \vdash \Delta, \Pi} \tag{14}$$

M is an arbitrary, but fixed proposition; Curry interprets it as a finite axiomatization of what he calls "the outer system", a (non-modal) deductive system whose metalogic the modal system is supposed to describe[8]. However the formal results are not affected by this interpretation and hold for

[8] Curry's work is done in a proof-theoretical context; however, his interpretation is not far from Smiley's idea of "relative" necessity.

any choice of M; so that one can treat it as a variable. Then he formulates a translation, which amounts to our * but with M for our L, and states and proves that, given a Gentzen system LX of a certain kind, if we form its modal expansion LXZ by adding to it the rules (12)–(14), then a modal sequent is derivable (he says "valid") in LXZ if and only if its translation is derivable in LX. The proof is entirely syntactic, Gentzen-style, and seems to hold (although this is not made completely clear) for any of the systems LX presented in his 1950 monograph [9]. It is remarkable that in his well-known 1963 book [10] he rejects these rules and considers a more standard set giving the usual modal systems $S4$ and $S5$. Curry's paper is earlier than Łukasiewicz's publication of his L system. Kripke in [20, p. 210] says that these rules amount to a formulation of L, but offers no proof. One has to assume that he is talking about the case where the non-modal base LX is Curry's Gentzen-style presentation of CPC. In this case, it is straightforward to find proofs of the axioms of L in LXZ, and using our Theorem 4 it is easy to see that $\vdash_L$ satisfies the rules (12)–(14), because after the translation they become valid rules of CPC. This shows that the derivable sequents of LXZ correspond to the inferences of L, and once we know this we realize that Curry's result is a generalization of our Theorem 4.

4 Discussion

Regardless its certainly good algebraic behaviour, it is obvious that Łukasiewicz's four-valued system of modal logic is very difficult to interpret as a modal logic; this has been recognized by many scholars, either philosophers or logicians[9].

Witness to this is the difficulty one has to find any influence of L in the mainstream development of modern modal logic. For instance, this system is nowhere mentioned in the historical introduction of Lemmon's [22], originally written in 1966, nor in the historical part of Bull and Segerberg's [2], published in 1984, nor in the historical notes to Chapter 3 of Chagrov and Zacharyaschev's [3], published in 1997; it is also noticeable that while it is discussed (as an extremely non-standard, yet legitimate system) in the first book by Hughes and Cresswell [17], published in 1968, it no longer appears in their more recent book [18] published in 1996. The strange character of L was highlighted in 1957 by Prior, who on page 3 of his [36] says:

[9] We have already noticed that only one of the papers where it is mentioned, Lemmon's [21], does really consider it with some detail; but he does not attempt to even discuss any intuitive reading of his relational or of his algebraic semantics.

> Ever since this system was put forward in 1953 logicians, including Łukasiewicz himself, have been finding new oddities in it.

Let us look at some of them, by way of example and as an approximation to Łukasiewicz's own views. From direct verification in the four- valued tables, or by use of the faithful interpretation, it is straightforward to check the next result.

Proposition 5. *The following formulas are L-tautologies:*

(15) $(\Box\varphi \,\&\, \Box(\varphi \to \psi)) \to \Box\psi$

(16) $\Box\Box\varphi \equiv \Box\varphi$

(17) $(\Box\varphi \,\&\, \Box\psi) \equiv \Box(\varphi \,\&\, \psi)$

(18) $(\Diamond\varphi \,\&\, \Diamond\psi) \equiv \Diamond(\varphi \,\&\, \psi)$

(19) $(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(20) $(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$

(21) $\Box\varphi \to (\psi \equiv \Box\psi)$

(22) $\Box\varphi \to (\Diamond\psi \equiv \Box\psi)$ ∎

Here (15)–(17) look familiar, and in fact they hold in a host of modal logical systems; but (18)–(22) do not. This mere fact would not be disastrous; but apparently there is no satisfactory natural intuitive notion of necessity for which (18)–(22) would be tautologies. Tautologicity of (18)–(20) is known to Łukasiewicz and discussed by him. Formula (18), already rejected by Lewis in [23], would account for a very strange notion of "possible", and Łukasiewicz [27, pp. 177–178] takes pains to justify it through some examples and argumentations about two different, yet indistinguishable operators of possibility or contingency, which do not appear to the modern reader's eyes as clearly concluding. As we said in the Introduction, (19) and (20) are called by him "laws of extensionality", and he devotes four sections [27, §39–§42] to study them, starting from his finding them in Aristotle's own writings, discussing whether the "→" in the antecedent should be interpreted as a material or as a strict implication, and deciding this in favour of the material one based on the careful analysis of Aristotle's exact words and argumentations. Notice that solving this in favour of the strict implication would have produced the formulas

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \quad \text{and} \quad \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)\,,$$

two intuitively more acceptable laws that hold in all normal modal logics, i.e., in any class of Kripke models. But, besides their Aristotelian roots,

(19) and (20) are also consequences of the strong "extensionalist" approach to modalities taken by Łukasiewicz, hence they are, in the present form or in some variant, to some extent unavoidable. As to (21), it seems not to have been stated explicitly by Łukasiewicz with this generality, but a particular case of it is discussed in [27, Chapter VIII, formula 113] in the context of his analysis of a controversy concerning modal syllogisms (we refer to it below). Note that in (21) φ and ψ are arbitrary formulas; thus (using Łukasiewicz's own examples) e.g. the following is true: "If necessarily each man is an animal, then if the number drawn from the box is divisible by 4 then necessarily the number drawn from the box is divisible by 4". Since Łukasiewicz did not accept that the truth of some proposition might entail that it was necessarily true, he argued that from the particular case of (21) he was considering, which holds in his system, one should draw as a consequence that its premiss should be rejected; but since the premiss states that something is necessary, and in his system no formula of the form $\Box\varphi$ is a tautology, he found such consequence entirely acceptable. One can wonder whether Aristotle would accept (21) as a logical truth, for the premiss of our example is a proposition explicitly asserted to hold by him, which would lead him to accept that "if the number drawn from the box is divisible by 4 then necessarily the number drawn from the box is divisible by 4", a plainly counterintuitive sentence. (22) is similar but apparently still worse (modify the example just given!). The fact that formulas like (21) and (22) are tautologies of the system is the main reason for our claim that as a modal logic of necessity and possibility it is a dead end.

Still, one can try to look for some reasons in favour of Łukasiewicz's system. The first thing one can do for this is to join Prior in his [36] and Hughes and Cresswell in their [17] to acknowledge that the system L satisfies certain principles that were regarded by Łukasiewicz himself as minimal requirements for a system of propositional logic to count as a *modal* one. These are:

(23) The system must contain the ordinary, two-valued propositional logic.

(24) It must contain two unary functors $\Box$ and $\Diamond$ not definable in terms of the non-modal ones.

(25) The following should hold: $\Box\varphi \to \varphi$, $\varphi \to \Diamond\varphi$, $\Box\varphi \equiv \neg\Diamond\neg\varphi$, and $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

(26) The following should not hold: $\varphi \to \Box\varphi$, $\Diamond\varphi \to \varphi$, $\neg\Box\varphi$, $\Diamond\varphi$.

Another justification for L would be to find some senses of the modal notions which may be in accordance with his results. Actually, Łukasiewicz closed his 1953 paper [26] by saying:

> I am fully aware that other systems of modal logic are possible based on different concepts of necessity and possibility. I firmly believe that we shall never be able to decide which of them is true. Systems of logic are instruments of thought, and the more useful a logical system is, the more valuable it is. I hope that the L-modal system expounded above will be a useful instrument, and deserves a further investigation and development. (Quoted from [28], pp. 378–379)

While Łukasiewicz's interpretation of the modal operators revolved only around so-to-speak ontological ideas of 'possibility' and 'necessity', we should bear in mind the large variety of interpretations and applications of modalities that have been developed, including some in the field of theoretical computer science. The following remark by Hughes and Cresswell may be relevant here:

> So $\Diamond$ and $\Box$ in the L-modal system do seem to express senses of 'possibility' and 'necessity' considerably different from any of the usual ones; and perhaps, if by a 'modal logic' we mean a logic of possibility and necessity, this system takes us to the limit of what we should regard as a modal logic at all. [17, p. 310]

The idea of possibility seems better suited than that of necessity to exploit our faithful interpretation. Recall that "$\Diamond p$" is translated as "$L \to p$" (for a non-modalized "p"), and observe that in it "L" is a propositional variable and may be given any suggestive reading, e.g. "Alice is in the wonderland"; thus $\Diamond p$ is understood as "If Alice is in the wonderland then p". More generally, we can read "p is possible" as "If something happens then p", that is, as a way of asserting that the truth of p is not actual but may depend on some other, unspecified fact. This interpretation seems to fit well with Prior's idea that one can consider the possibility operator in Łukasiewicz's system as a kind of variable operator, whose value ranges between plain assertion and plain truth:

> I do not know whether anyone has ever used the words 'Possibly' and 'Necessarily' in the way which I am suggesting that the $\Diamond$ and $\Box$ of the L-modal system could be used. It would be rash and indeed definitely erroneous to say that no one ever has [...] Sometimes when a man says 'Possibly p' it does look as if he is trying to convey to some people the idea that he is assenting to the proposition p, and to others that he is not really committing himself to anything at all, and I suppose this would be something like using 'Possibly' as a variable operator capable of taking these two values, and the L-modal calculus would serve to show what propositions such a man could assent to without giving the game away. [36, p. 5]

We remark here that it was this idea what led Prior to the "two-formulas interpretation" of L, a result we have obtained in Corollary 2 as a by-product of our translation.

We believe that most of the oddities one can find in Łukasiewicz's system should be atributed directly to Aristotle, and to Łukasiewicz's efforts to obtain a system that could faithfully represent Aristotle's ideas about necessity and possibility (obviously reinforced by his commitment to truth-functionality). We would like to recall that Łukasiewicz's analysis of the classical, non-modal Aristotelian theory of the syllogism is a remarkably fine piece of historical reconstruction based on purely scientific grounds. In Chapters I to III of [25, 27] he expounds Aristotle's doctrines following his texts and those of his contemporaries and disciples, and highlights many errors made by later commentators. Then Chapters IV and V contain a brief and elementary, but sufficient exposition of modern propositional logic and the development of the theory of the syllogism as a formalized theory; the particular atoms of this theory are the four kinds of expressions appearing in Aristotle's syllogisms, which he symbolises using the letters A, E, I, O as medieval logicians did, thus finding the four expressions:

Aab	meaning	'all a is b'
Eab	"	'no a is b'
Iab	"	'some a is b'
Oab	"	'some a is not b'

He uses quantifiers only to better explain the laws of conversion, which justify taking only Aab and Iab as primitive; but since Aristotle did not use quantifiers in his works he keeps his formal theory strictly inside propositional logic. With the obvious definitions $Eab = \neg Iab$ and $Oab = \neg Aab$ and four axioms (the two laws of identity Aaa and Iaa and the two syllogistic moods known as *Barbara* and *Datisi*) he completely succeeds in proving all 24 valid moods of the syllogism, besides showing that his system is consistent and independent. He also sets up a formal theory for rejection and claims it shows rejection of all 232 invalid moods. Finally, going beyond Aristotle, he investigates the propositional expressions in the new atoms that do not correspond to syllogisms, shows that they cannot be decided on the basis of the formal systems for acceptance and for rejection he has given, and presents a decision procedure based on the reduction to normal forms, essentially due to his pupil Słupecki.

By contrast with this success, the situation concerning Aristotle's modal logic is quite different; in Łukasiewicz's own words:

> There are two reasons why Aristotle's modal logic is so little known. The first is due to the author himself: in contrast to the assertoric

syllogistic which is perfectly clear and nearly free of errors, Aristotle's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies. [...]

The second reason is that modern logicians have not as yet been able to construct a universally acceptable system of modal logic which would yield a solid basis for the interpretation and appreciation of Aristotle's work. I have tried to construct such a system, different from those hitherto known, and built up upon Aristotle's ideas.

[27, p. 133]

We clearly see here the two main facts that support our claim that the oddities in the Ł system should be attributed to Aristotle rather than to Łukasiewicz himself: The inconsistencies already present in Aristotle, and Łukasiewicz's firm determination of shaping his modal logic upon Aristotle's work as closely as possible. As we have already found, complete agreement was not always possible, since following Aristotle at some places would have led to a contradiction.

He divided his analysis of Aristotle's modal logic into two parts, namely the propositional modal logic and the modal syllogistic, located respectively in Chapters VI and VIII of [27], and having a different status:

It is possible to speak of an Aristotelian modal logic of propositions, as some of his theorems are general enough to comprise all kinds of proposition, and some others are expressly formulated by him with propositional variables. I shall begin with Aristotle's modal logic of propositions, which is logically and philosophically far more important than his modal syllogistic of terms.

[...]

Aristotle's modal syllogistic has, in my opinion, less importance in comparison with his assertoric syllogistic or his contributions to propositional modal logic. This system looks like a logical exercise which in spite of its seeming subtlety is full of careless mistakes and does not have any useful application to scientific problems.

[27, pp. 133 and 181]

One of the points where he finds more difficulties to agree with Aristotle is the controversy concerning syllogisms with one assertoric and one apodeictic premiss. Here 'assertoric' means non-modal, and 'apodeictic' means modalized with the necessity operator. The two controversial syllogisms are:

$$\big(\Box(Aba) \,\&\, Acb\big) \to \Box(Aca) \tag{27}$$

$$(28) \qquad \big(Aba \,\&\, \Box(Acb)\big) \to \Box(Aca)$$

Aristotle accepts (27) but rejects (28), which has given rise to many philosophical discussions over the centuries. Łukasiewicz shows, syntactically, that both should be accepted by use of his L modal system combined with his formal system of non-modal syllogisms. It is in discussing this that he finds the particular case of formula (21) referred to above.

It does not make sense to ask whether (27) and (28) are theses of any modern system of modal logic, unless we re-write the propositional atom Aab, where a, b are terms, as the non-atomic predicate formula $(\forall x)(a(x) \to b(x))$, where $a(x)$ would mean "x is an a". According to Łukasiewicz's views [27, p. 130], this would be contrary to Aristotle's understanding of universal propositions, and it is a very common mistake. Nevertheless, it is clear that the resulting formulas

$$\big(\Box(\forall x)(b(x) \to a(x)) \,\&\, (\forall x)(c(x) \to b(x))\big) \to \Box\big(\forall x)(c(x) \to a(x)\big)$$
$$\big((\forall x)(b(x) \to a(x)) \,\&\, \Box\big(\forall x)(c(x) \to b(x))\big) \to \Box(\forall x)(c(x) \to a(x)\big)$$

are not theorems of any of the normal modal logics, as they are not theorems of predicate S5.

Clearly, it should still be possible, and even advisable, to revise Łukasiewicz's analysis of Aristotle's modal logic under the light and with the now powerful mathematical techniques of intensional logic[10]. The intensional standpoint was explicitly disregarded by him when he wrote:

> This [his formulations of the laws of extensionality] seems perfectly evident, unless modal functions are regarded as intensional functions, i.e. as functions whose truth-values do *not* depend solely on the truth-values of their arguments. But what in this case the necessary and the possible would mean, is for me a mystery as yet. [27, p. 140]

He also rejected, and with no explicit reason, Quine's objections to the substitutivity of singular terms for variables in modal contexts, which is precisely one of the distinctive features of intensionality; see [27, p. 150].

Anyway, Łukasiewicz's L remains a remarkable witness of the development of modal logic before it has become "modern modal logic" in our

[10] Actually several aspects of Łukasiewicz's reading of Aristotle's syllogistics (both the non-modal and the modal one) have been much revised and critiziced, but mainly from the philosophical standpoint; see for instance [6, 7, 30, 31, 33, 39, 41]. Of particular interest is the discussion of extensionality vs. truth-functionality contained in [19]. We warn also the reader that [32] states (without proof) that L is a normal modal logic, which is wrong as shown in our Proposition 2.

present sense. Due to this development, the following advice of Łukasiewicz himself, originally motivated by his assessment of XIXth-century historians of logic, may be even more in order at the dawn of the XXIst century:

> [...] perhaps it would not be impossible to persuade living philosophers that they should cease to write about logic or its history before having acquired a solid knowledge of what is called 'mathematical logic'. It would otherwise be a waste of time for them as well as for their readers. It seems to me that this point is of no small practical importance. [27, p. 47]

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Faculty of Mathematics, University of Barcelona
Gran Via 585, 08007 Barcelona, Spain
font@mat.ub.es

Institute of Computer Science, Academy of Sciences
Pod vodarenskou vezi 2, 182 07 Prague, Czech Republic
hajek@cs.cas.cz