A NEW CRITERIUM TO CONTROL THE NUMBER OF LIMIT CYCLES OF SOME GENERALIZED LIÉNARD EQUATIONS

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Abstract. We consider a class of planar differential equations which include the Liénard differential equations. By applying the Bendixson-Dulac Criterium for ℓ-connected sets we reduce the study of the number of limit cycles for such equations to the condition that a certain function in just one variable does not change sign. As an application, this method is used to give a sharp upper bound for the number of limit cycles of some Liénard differential equations. In particular, we present a polynomial Liénard system with exactly three limit cycles.

1. Main results

This paper deals with the problem of finding upper bounds for the number of limit cycles of planar $\mathcal{C}^1$ differential equations of the form

$$\dot{x} = P(x,y),\ \dot{y} = Q(x,y),$$

for P and Q with a special shape which will be described in the sequel. The criterium that we obtain can be applied to give sharp upper bounds for the number of limit cycles of some Liénard systems, see Section 3. For instance we prove the following result

Theorem 1.1. *The Liénard system*

$$\dot{x} = y - x(x^2-2)(x^2-1)(x^2-\frac{1}{4}), \qquad \dot{y} = -x \tag{1.1}$$

has exactly three limit cycles. Furthermore they are concentric and hyperbolic.

We remark that most criteria in the literature can only be applied to systems with at most 1 or 2 limit cycles, see for instance [C1, Ye]. The criterium that we present has not this restriction.

In order to state our main result we introduce some notation.

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Given an open subset U of $\mathbb{R}^2$ with smooth boundary, it is said that it is *ℓ-connected* if its fundamental group, $\pi_1(U)$ is $\mathbb{Z}*\overset{(\ell)}{\cdots}*\mathbb{Z}$, or in other words if U has ℓ gaps. Given an open subset W with smooth boundary and a smooth function $f : W \to \mathbb{R}$ we denote by $\ell(W, f)$ the sum of $\ell(U)$ where U ranges over all the connected components of $W \setminus \{f = 0\}$. Finally, we denote by $c(W, f)$ the number of closed ovals of $\{f = 0\}$ contained in W. See Figure 1 for an illustration of these definitions.

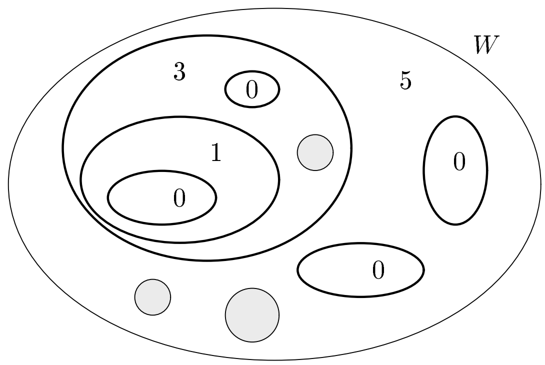

Figure 1. Open set W with $\ell(W) = 3$. The grey cercles are holes in W and the thick lines correspond to $\{f = 0\}$. The numbers displayed are the values $\ell(U)$ for each connected component U of $W \setminus \{f = 0\}$. For this example $c(W, f) = 6$ and $\ell(W, f) = 9$.

Our main result is given in the following Theorem.

Theorem A. *Consider a $\mathcal{C}^1$ system of the form*

$$\begin{aligned}\dot{x} &= p_0(x) + p_1(x)y = P(x, y),\\ \dot{y} &= q_0(x) + q_1(x)y + q_2(x)y^2 = Q(x, y),\end{aligned} \tag{1.2}$$

with $p_1(x) \not\equiv 0$. For each $s \in \mathbb{R}$ and for each $n \in \mathbb{N}$ it is possible to associate to it a $(n+1)$-parameter family of functions $f_n(x, y; c_0, c_1, \ldots, c_n) := f_n(x, y)$ of the form

$$f_n(x, y) = g_0(x) + g_1(x)y + g_2(x)y^2 + \cdots + g_n(x)y^n,$$

such that for each one of them

$$\langle \nabla f_n, (P, Q)\rangle + s f_n \operatorname{div}(P, Q) := M_{s,n}(x)$$

is a function just of the x-variable. Furthermore if for some $s = \bar{s}$ and some $n = \bar{n}$ the corresponding function $M_{\bar{s},\bar{n}}(x)$ does not change sign in the strip

$$S_{a,b} = \{(x, y) \in \mathbb{R}^2 \, : \, a < x < b\},$$

then the number of limit cycles of (1.2)*, totally contained in* $S_{a,b}$*, is bounded above by*

(i) $c(S_{a,b}, f_{\bar{n}}) + \ell((S_{a,b}, f_{\bar{n}})$*, if* $\bar{s} < 0$*, and by*
(ii) $c((S_{a,b}, f_{\bar{n}})$ *if* $\bar{s} \geq 0$.

Moreover, all the limit cycles in $S_{a,b}$ *which are not contained in* $\{f_{\bar{n}} = 0\}$ *are hyperbolic.*

Remark 1.2. *Observe that the generalized Liénard system*

$$\dot{x} = P(x,y) = y - F(x)\,, \qquad \dot{y} = Q(x,y) = -g(x)$$

is included in the system studied in Theorem A. It corresponds to the case $p_0 = -F$, $p_1 = 1$, $q_0 = -g$ *and* $q_1 = q_2 = 0$.

Remark 1.3. *In the definition of* $S_{a,b}$ *above, the values* $a = -\infty$ *and* $b = +\infty$ *are also allowed.*

From the above theorem follows a relation between the first part of Hilbert's sixteenth problem (number and distribution of the closed components of an algebraic planar curve) and the second part of the Hilbert's sixteenth problem (number and distribution of the limit cycles of a planar polynomial vector field of the form (2.1). A lot of useful information about the first part is given in the paper of Wilson [Wi].

To prove Theorem A we use the generalized Bendixson-Dulac Criterium for ℓ-connected regions. For sake of completeness we include a different proof of it in next section, see Proposition 2.1. This method has been already used for several authors, see for instance [C2, L, Y, CG1, CG2]. Our main contribution is to reduce the problem to the control of the sign of a family of functions, just in one variable, see also Propositions 2.3 and 2.5.

In Section 3 we apply Theorem A to some examples of Liénard differential equations. Finally we recall in an appendix how to test if a polynomial does not change sign.

One of the main disadvantages of the method suggested by Theorem A, is that when it applies, all the limit cycles (except the ones contained in $\{f_n = 0\}$) are hyperbolic. Therefore if we study a parametric family in which, for some values of the parameters, a non hyperbolic limit cycle appears, the whole family never can be under the hypotheses of the theorem, unless its non hyperbolic limit cycles were contained in $\{f_n = 0\}$.

2. Preliminary results and proof of Theorem A

Consider the $\mathcal{C}^1$ differential system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{2.1}$$

and set $X = (P, Q)$.

The following proposition gives a generalization of the Bendixson-Dulac Criterium. It is already proved in several papers, see for instance [L, Y]. Here we give a short and different proof.

Proposition 2.1. *Let U be an open ℓ-connected subset of $\mathbb{R}^2$ with smooth boundary. Let $g\colon U \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that*

$$M := \operatorname{div}(gX) = \frac{\partial g}{\partial x}P + \frac{\partial g}{\partial y}Q + g(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) = \langle \nabla g, X\rangle + g\operatorname{div}(X)$$

does not change sign in U and vanishes only on a null measure Lebesgue set, such that $\{M=0\}\cap\{g=0\}$ does not contain periodic orbits of (2.1). Then the maximum number of periodic orbits of (2.1) contained in U is ℓ. Furthermore each one of them is a hyperbolic limit cycle that does not cut $\{g=0\}$ and its stability is given by the sign of gM over it.

Remark 2.2. *If in the above proposition we remove the hypothesis that $\{M=0\}\cap\{g=0\}$ does not contain periodic orbits of (2.1), the proposition also works by adding to its statement these new periodic orbits, which need not to be hyperbolic limit cycles.*

Proof of Proposition 2.1. Observe that $M|_{\{g=0\}} = \langle \nabla g, X\rangle|_{\{g=0\}} \geq 0$ does not change sign in U. Since, by hypothesis, there are not periodic orbits of (2.1) contained in $\{M=0\}\cap\{g=0\}$ we have that the periodic orbits of (2.1) does not cut $\{g=0\}$.

If U is simply connected ($\ell = 0$) then by the well known Bendixson-Dulac Criterion we have that (2.1) has no periodic orbits in U. We give now a proof for an arbitrary ℓ. Assume that system (2.1) has $\ell+1$ different periodic orbits γ_i, included in U. These orbits induce $\ell+1$ elements $\overline{\gamma}_i$ in the first homology group of $U, H_1(U) = \mathbb{Z}\oplus \overset{(\ell)}{\cdots} \oplus\mathbb{Z}$. Since this group has at most ℓ linearly independent elements it follows that there is a non trivial linear combination of them giving $0 \in H_1(U)$. Then $\sum_{i=1}^{\ell+1} m_i\overline{\gamma}_i = 0$, with $(m_1,\ldots,m_{\ell+1}) \neq 0$.

This last fact means that the curve $\sum_{i=1}^{\ell+1} m_i\gamma_i$ is the boundary of a two cell C for which Stokes Theorem can be applied. Then

$$\iint_C \operatorname{div}(gX) = \int_{\sum_{i=1}^{\ell+1} m_i\gamma_i} \langle gX, \mathbf{n}\rangle.$$

Note that the right hand term in this equality is zero because gX is tangent to the curves γ_i and that the left one is not null by our hypothesis. This fact gives a contradiction. So ℓ is the maximum number of periodic orbits of (2.1) in U.

Let us prove their hyperbolicity. Fix one periodic orbit $\gamma = \{(x(t), y(t)),$ $t \in [0,T]\} \subset U$, where T is its period. Remember that $\gamma\cap\{g=0\} = \emptyset$. In order to study its hyperbolicity and stability we have to compute

$\int_0^T \operatorname{div} X(x(t), y(t))dt$, and to prove that it is not zero. This fact follows by integrating the equality

$$\operatorname{div} X = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\operatorname{div}(gX)}{g} - \frac{\frac{\partial g}{\partial x}P + \frac{\partial g}{\partial y}Q}{g},$$

because the last term of the right hand side of the above equality coincides with $\dfrac{d}{dt}\ln|g(x(t), y(t))|$. □

In order to apply the above theorem, we consider a function $g(x, y)$ of the form $|f(x, y)|^m$ where f is a smooth function in two variables in $\mathbb{R}^2$ and m is a real number. The application of Proposition 2.1 to this particular g is given in the following result.

Proposition 2.3. *Assume that there exist a real number s and an analytic function f in $\mathbb{R}^2$ such that*

$$M_s := \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + sf(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) = \langle \nabla f, X\rangle + sf\operatorname{div}(X)$$

does not change sign in a open region $W \subset \mathbb{R}^2$ with regular boundary and vanishes only in a null measure Lebesgue set. Then, system (2.1) has two types of limit cycles in W,

(a) Limit cycles totally contained in $\{f = 0\}$, and
(b) Limit cycles which does not cut $\{f = 0\}$.

Furthermore, the following hold:

(c) The number of limit cycles described in (a) is at most $c(W, f)$.
(d) The number of limit cycles described in (b) is at most $\ell(W)$ if $s > 0$ and zero if $s = 0$. When $s < 0$, this number is bounded above by $\ell(W, f)$. Moreover, for any value of s, all the limit cycles are hyperbolic.

Proof. First observe that since M_s does not change sign we have that on the analytic curves $\{f = 0\}$, $\langle \nabla f, X\rangle$ does not change sign. Therefore these curves are either solutions of (2.1) or curves crossed by the flow generated by (2.1) just in one direction. Hence all limit cycles in W are either contained in the connected components of $W \setminus \{f = 0\}$) or in $\{f = 0\}$. This fact implies assertions (a) and (b) of the Theorem. In order to bound the number of limit cycles of (2.1) we apply Theorem 2.1 to each one of the connected components U of $W \setminus \{f = 0\}$. The fact that when $g = |f|^m$,

$$\operatorname{div}(gX) = \langle \nabla g, X\rangle + g\operatorname{div}(X) = \operatorname{sign}(f)m|f|^{m-1}\left[\langle \nabla f, X\rangle + \frac{1}{m}f\operatorname{div}(X)\right],$$

gives the theorem by taking $m = 1/s$. Observe that the difference between the cases $s > 0$ and $s < 0$ comes from the fact that in the first case the function g is well defined in the whole plane. For the case $s = 0$ the proof is easier because $M_0 = df/dt = \langle \nabla f, X \rangle$. □

Remark 2.4. *By using a Dulac function of the form $f^m(x,y)e^{\psi(x,y)}$ it is possible to improve Proposition 2.3 by taking the following $M_{s,\psi}$ instead of M_s,*

$$M_{s,\psi} := \langle \nabla f, X \rangle + sf \left[\langle \nabla \psi, X \rangle + \operatorname{div}(X) \right],$$

for any smooth function ψ. Proposition 2.3 corresponds to the case where ψ is a constant function.

In most cases we will use special algebraic functions of the form

$$f(x,y) = g_0(x) + g_1(x)y + g_2(x)y^2 + \cdots + g_n(x)y^n,$$

with $g_i(x)$ polynomials. For these functions the number $\ell(\mathbb{R}^2, f)$ is sometimes not difficult to be computed, and therefore Proposition 2.3 is easy to apply.

In the next result we prove a key point of our paper. We consider a family of planar differential equations for which the functions M_s that appear in Proposition 2.3 are functions just in one variable. For these functions the condition that they do not change sign is easier to be tested. In the appendix we study a way to check this condition when the functions M_s are polynomials.

Proposition 2.5. *Consider the $\mathcal{C}^1$ system*

$$\begin{aligned} \dot{x} &= p_0(x) + p_1(x)y = P(x,y), \\ \dot{y} &= q_0(x) + q_1(x)y + q_2(x)y^2 = Q(x,y), \end{aligned}$$

with $p_1(x) \not\equiv 0$. Then for each $s \in \mathbb{R}$ and for each $n \in \mathbb{N}$ there exists a $(n+1)$-parameter family of functions $f_n(x,y;c_0,c_1,\ldots,c_n) := f_n(x,y)$ of the form

$$f_n(x,y) = g_0(x) + g_1(x)y + g_2(x)y^2 + \cdots + g_n(x)y^n, \tag{2.2}$$

such that

$$\langle \nabla f_n, (P,Q) \rangle + sf_n \operatorname{div}(P,Q) = M_{s,n}(x).$$

Proof. Direct computations give

$$\begin{aligned} &\langle \nabla f_n, (P,Q) \rangle + sf_n \operatorname{div}(P,Q) = \\ &\qquad = \left[\{ (sp_1' + 2sq_2 + nq_2)\, g_n + p_1 g_n' \} \, y^{n+1} + \mathcal{F}_n(g_n, g_{n-1}) y^n + \right. \\ &\qquad + \{ \mathcal{F}_{n-1}(g_{n-1}, g_{n-2}) + nq_0(x) g_n(x) \} \, y^{n-1} + \cdots + \\ &\qquad \left. + \{ \mathcal{F}_1(g_1, g_0) + 2q_0(x) g_2(x) \} \, y + \{ (sp_0' + sq_1) g_0 + p_0 g_0' + q_0 g_1 \} \right], \end{aligned}$$

where for each $j = 1, 2 \dots, n$,

$$\mathcal{F}_j(g_j, g_{j-1}, g_j', g_{j-1}') = (sp_0' + sq_1 + jq_1)g_j(x) + p_0 g_j'(x) + (sp_1' + 2sq_2 + (j-1)q_2))g_{j-1}(x) + p_1 g_{j-1}'(x).$$

From the above expressions we can obtain a 1-parameter family of functions $g_n^*(x; c_n) := g_n^*(x)$ such that the coefficient of y^{n+1} vanishes, by solving a linear first order ordinary differential equation. Once we have g_n^*, from $\mathcal{F}_n(g_n^*, g_{n-1}) = 0$ we get $g_{n-1}^*(x; c_n, c_{n-1}) := g_{n-1}^*(x)$ and so on until we have fixed $g_n^*, g_{n-1}^*, \dots, g_0^*$. Finally, we obtain

$$\langle \nabla f_n, (P, Q) \rangle + s f_n \operatorname{div}(P, Q) = [(sp_0' + sq_1)g_0^* + p_0(g_0^*)' + q_0 g_1^*] := M_{s,n}(x),$$

as we wanted to proof. □

Remark 2.6. *Note that in the previous result the functions $g_i^*(x)$, with $i = 0, ..., n$, may not be defined in the whole $\mathbb{R}^2$.*

The proof of the above proposition is reminiscent of that of [C1, GN] where the authors prove that a $(n+1)$-parameter family of functions of the form (2.2) can be constructed for Liénard differential equations in such a way that $df_n/dt = \langle f_n, X \rangle$ is a function in the variable x.

Proof of Theorem A. Theorem A follows directly from Propositions 2.3 and 2.5. □

In next corollary we explicitly state the results of Proposition 2.5 for the generalized Liénard equation with $n = 2$ and $s = -1$. We will use it in next section.

Corollary 2.7. *Consider the generalized Liénard system*

$$\dot{x} = y - F(x) := P(x, y), \quad \dot{y} = -g(x) := Q(x, y).$$

If we take

$$f_2(x, y) = \left(\frac{s(s+1)}{2}(F(x))^2 + c_1 s F(x) + 2G(x) + c_0 \right) + (sF(x) + c_1)y + y^2,$$

where $G(x) = \int_0^x g(z)\, dz$, then

$$\begin{aligned} M_{s,2}(x) &= \langle \nabla f_2, (P, Q) \rangle + s f_2 \operatorname{div}(P, Q) \\ &= -\frac{s(s+1)(s+2)}{2}(F(x))^2 F'(x) - s(s+1)c_1 F(x) F'(x) \\ &\quad - (s+2)g(x)F(x) - 2sF'(x)G(x) - sc_0 F'(x) - c_1 g(x). \end{aligned}$$

In particular, for $s = -1$ we have

$$f_2(x, y) = (-c_1 F(x) + 2G(x) + c_0) + (-F(x) + c_1)y + y^2,$$

and

$$M_{-1,2}(x)=2F'(x)G(x)+c_0F'(x)-g(x)F(x)-c_1g(x).$$

3. Applications

In this section we apply Theorem A to several families of Liénard differential equations. Each family is studied in a different subsection.

3.1. A polynomial Liénard system with two odd monomial terms of arbitrary degree.

We prove the following proposition, which gives a new support to the well-known conjecture of Lins, Melo and Pugh on the maximum number of limit cycles of polynomial Liénard differential equations, see [LMP].

Note also that the celebrated van der Pol system is a particular case of the Liénard system that we will study. It corresponds to the case $n=0$, $k=1$, and $a<0$.

Proposition 3.1. *Consider the Liénard system*

$$\dot{x}=y-F(x)\,,\qquad \dot{y}=-x\,, \tag{3.1}$$

where

$$F(x)=ax^{2n+1}+x^{2k+1}\,, \tag{3.2}$$

with $k,n\in\mathbb{N}$ and $k>n$. Then it has a limit cycle if and only if a is negative. Furthermore this limit cycle is unique, hyperbolic and stable.

Proof. If $a\geq 0$ we have $\operatorname{div}(P,Q)\leq 0$. Hence in this case the system has no limit cycles. Therefore, we will consider in the following just the case $a<0$.

This system has a unique critical point located at the origin. By using the Liapunov function x^2+y^2 it is easy to see that the origin is a repelling point. The infinity of this polynomial Liénard differential equation is a repeller, see for instance [DH, LMP]. Therefore, for $a<0$ the system has at least one limit cycle. To prove the uniqueness and hyperbolicity of the limit cycle we apply Theorem A with $s=-1$ and $n=2$, see also Corollary 2.7. More concretely, we consider the following polynomial function

$$f(x,y)=y^2-F(x)y+x^2+c_0\,, \tag{3.3}$$

with $c_0=\frac{-2k}{2k+1}\left(\frac{-na}{k}\right)^{1/(k-n)}<0$. Corollary 2.7 gives that the function $M_{-1,2}(x):=M(x)$ is the polynomial:

$$M(x)=x^{2n}(2kx^{2(k-n)+2}+2kd_1x^{2(k-n)}+2kd_2x^2+2kd_1d_2+d_3)\,, \tag{3.4}$$

where

$$d_1 = -\left(\frac{-na}{k}\right)^{\frac{1}{k-n}} < 0\,, \quad d_2 = \frac{na}{k} < 0\,,$$
$$d_3 = -\frac{2(k-n)a}{2k+1}\left(\frac{-na}{k}\right)^{\frac{1}{k-n}} > 0\,. \tag{3.5}$$

The function $M(x)$ given in (3.4) can be rewritten as follows:

$$M(x) = x^{2n}(2k(x^2+d_1)(x^{2(k-n)}+d_2)+d_3)\,.$$

From the explicit expressions of d_1 and d_2 given in (3.5) we see that the real roots of the polynomials x^2+d_1 and $x^{2(k-n)}+d_2$ are the same. Therefore, as $d_3 > 0$ we have

$$M(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R} \quad (\text{zero just at } x=0). \tag{3.6}$$

We analyse now the set $\{f=0\}$. As a consequence of (3.6), the curves contained in this set are simple (they have no singular points) and transversal (except at $x=0$) to the flow defined by (P,Q) (they are crossed by the flow just in one direction). Then, any closed component of $\{f=0\}$ does not contain limit cycles and must surround the unique critical point of (3.1), the origin.

The fact that (3.1) has at least one limit cycle forces that $\{f=0\}$ has at least one closed component. Taking into account that $f(x,y)$ is a second degree polynomial in the variable y, we get that $\{f=0\}$ contains exactly one closed component. Then, from Theorem A, we conclude that system (3.1) has exactly one hyperbolic and stable limit cycle. This limit cycle is contained in the 1-connected component of $\mathbb{R}^2 \setminus \{f=0\}$. □

The existence, uniqueness and stability of the limit cycle for system (3.1) in Proposition 3.1 could also be proved by the standard results on Liénard systems given in [H, Theorems 1.5 and 1.6, pp. 57-61]. See also [SC, Z]. The proof that we give here is different and algebraic.

The following facts are well known:

- Polynomial Liénard differential equations with $F(x)=F_e(x)$, where $F_e(x)=F_e(-x)$ is an even polynomial, have a center at the origin and have no limit cycles.
- Polynomial Liénard differential equations with $F(x) = F_e(x) + ax^{2k+1}$ have no limit cycles.

These two results plus Proposition 3.1 imply that the problem of the number of limit cycles for the Liénard differential equations with F containing just two monomials is completely solved. Some cases with 3 monomials are also studied in the literature:

- The case $F(x) = ax + bx^2 + cx^3$. In [LMP] it is proved that this system has exactly one limit cycle when $ac < 0$ and it has no limit cycles when $ac \geq 0$.
- The case $F(x) = ax + bx^3 + cx^5$. In [R] the existence of at most two limit cycles is proved. Moreover, it is also shown that this system has exactly two limit cycles when certain inequalities between the parameters a, b and c are satisfied. This is the only example, at our knowledge, of a polynomial Liénard system with more than a limit cycle, for which the exact number of limit cycles is determined.

In next section we study a case with 4 monomials.

3.2. **A polynomial Liénard system with exactly three limit cycles.** In this section we prove Theorem 1.1 stated in the introduction. This result presents an example of a Liénard system for which we are able to prove that it has exactly 3 limit cycles. We want to remark that this example is not a perturbation of an integrable situation (it has no parameters). Furthermore none of the limit cycles is algebraic. This last assertion follows from results given in [O].

For commodity we state again Theorem 1.1:

Theorem 1.1. *The Liénard system*

$$\dot{x} = y - F(x)\,, \qquad \dot{y} = -x\,, \tag{3.7}$$

with

$$F(x) = x(x^2-2)(x^2-1)(x^2-\frac{1}{4})\,,$$

has exactly three limit cycles. Furthermore they are concentric and hyperbolic.

Proof. We apply Theorem A with $s=-1$, $n=6$ and $f(x,y)$ given by

$$f(x,y) = g_0(x) + g_1(x)y + g_2(x)y^2 + g_3(x)y^3 + g_4(x)y^4 + g_5(x)y^5 + y^6\,,$$

where

$$\begin{aligned}
g_5(x) &= -x^7 + \frac{13}{4}x^5 - \frac{11}{4}x^3 + \frac{x}{2},\\
g_4(x) &= 3x^2 - 3,\\
g_3(x) &= -\frac{20}{9}x^9 + \frac{19}{2}x^7 - \frac{283}{20}x^5 + + \frac{103}{12}x^3 - \frac{3}{2}x,
\end{aligned}$$

$$\begin{aligned}
g_2(x) = \frac{-5}{18}x^{16} + \frac{130}{63}x^{14} - \frac{4207}{720}x^{12} + \frac{1423}{180}x^{10} - \frac{623}{120}x^8\\
+ \frac{22}{15}x^6 + \frac{35}{12}x^4 - 6x^2 + 2,
\end{aligned}$$

$$
\begin{aligned}
g_1(x) = &-\frac{5}{46}x^{23}+\frac{65}{56}x^{21}-\frac{33433}{6384}x^{19}+\frac{1486129}{114240}x^{17}-\frac{26135}{1344}x^{15}\\
&+\frac{56033}{3120}x^{13}-\frac{20343}{1760}x^{11}+\frac{131}{12}x^{9}-\frac{26543}{1680}x^{7}+\frac{563}{40}x^{5}-6x^3+x,\\
g_0(x) = &-\frac{4}{69}x^{30}+\frac{130}{161}x^{28}-\frac{1580639}{318136}x^{26}+\frac{59217241}{3328192}x^{24}-\frac{654745823}{15917440}x^{22}\\
&+\frac{86513747}{1343680}x^{20}-\frac{678321469}{9694080}x^{18}+\frac{1270415899}{23063040}x^{16}\\
&-\frac{4826447}{131040}x^{14}+\frac{9099779}{332640}x^{12}-\frac{342457}{16800}x^{10}+\frac{13901}{1344}x^{8}\\
&-\frac{559}{360}x^{6}-\frac{23}{8}x^{4}+2x^2-\frac{1}{10}\,.
\end{aligned}
$$

The above function has been obtained by using Proposition 2.5. The function $M_{-1,6} := M$ is for this case an even polynomial of degree 36 in the variable x:

$$
\begin{aligned}
M(x) = &\frac{4}{3}x^{36}-\frac{65}{3}x^{34}+\frac{115797}{728}x^{32}-\frac{938009}{1344}x^{30}+\frac{522981675}{256256}x^{28}\\
&-\frac{307455133}{73216}x^{26}+\frac{5973494837}{953856}x^{24}-\frac{11009382341}{1596672}x^{22}\\
&+\frac{210716274919}{36771840}x^{20}-\frac{74840066239}{19740672}x^{18}+\frac{10790844613}{4838400}x^{16}\\
&-\frac{167209909}{131040}x^{14}+\frac{1242568181}{1900800}x^{12}-\frac{14891563}{57600}x^{10}+\frac{1131031}{13440}x^{8}\\
&-\frac{43561}{1440}x^{6}+\frac{141}{16}x^{4}-\frac{33}{40}x^{2}+\frac{1}{20}\,.
\end{aligned}
$$

This polynomial has no real roots, as can be easily verified by applying the Sturm algorithm by means of an algebraic manipulator.

Then we have $M(x) > 0$ for all $x \in \mathbb{R}$. We must analyze now the set $\{f = 0\}$. From the above result, since $\langle \nabla f, X\rangle$ is positive over $\{f = 0\}$, we obtain that the curves contained in this set are simple and transversal to the flow associated to (3.7). We conclude that no limit cycle is contained in $\{f = 0\}$ and that the closed components of this set, if they exist, must be nested closed curves that contain the origin. The problem now is to determine the number of closed curves of this type. In order to solve this problem we consider three circles centered at the origin, with radius $r_1 = \frac{1}{2}$, $r_2 = 1$ and $r_3 = \frac{3}{2}$, respectively. Each one of these circles has no intersection in the (x, y)-affine plane with the curves of the set $\{f = 0\}$. This result can be proved taking the resultant of the polynomials $f(x, y)$ and $x^2 + y^2 - r_i^2$ with respect to the variable y, for a given value of $i = 1, 2, 3$. This resultant is a polynomial of degree 60 in the variable x, for each value of r_i. By applying again the Sturm algorithm it is possible to show that each one of these three polynomials has no real roots.

We evaluate now the polynomial $f(x,y)$ at $x=0$. We obtain an even polynomial of degree six in the variable y.

$$f(0,y)=y^6-3y^4+2y^2-\frac{1}{10}\,.$$

The roots of this polynomial can be calculated in closed form and are given by $\pm y_1$, $\pm y_2$ and $\pm y_3$, with

$$\begin{aligned}
y_3 &= \left(1+\frac{2}{\sqrt{3}}\cos\left(\frac{\varphi}{3}\right)\right)^{1/2}>0,\\
y_2 &= \left(1+\frac{2}{\sqrt{3}}\cos\left(\frac{\varphi}{3}+\frac{4\pi}{3}\right)\right)^{1/2}>0,\\
y_1 &= \left(1+\frac{2}{\sqrt{3}}\cos\left(\frac{\varphi}{3}+\frac{2\pi}{3}\right)\right)^{1/2}>0,
\end{aligned}$$

where φ is defined by the relation

$$\cos\varphi=\frac{3\sqrt{3}}{20}\,. \tag{3.8}$$

From the inequalities

$$\begin{aligned}
&y_1<y_2<y_3, \quad \text{and}\\
&f(0,r_1)>0, \quad f(0,r_2)<0, \quad f(0,r_3)>0,
\end{aligned} \tag{3.9}$$

we conclude that

$$y_1<r_1<y_2<r_2<y_3<r_3\,. \tag{3.10}$$

The six points $(0,\pm y_1)$, $(0,\pm y_2)$ and $(0,\pm y_3)$ of the (x,y)-affine plane belong to the set $\{f=0\}$. As the curves of this set have no singular points and they have no intersection with the three circles of radius r_1, r_2 and r_3, we conclude that the set $\{f=0\}$ contains at least three closed components. Each pair of roots $\pm y_i$ belongs to a different closed component. As $f(x,y)$ is a polynomial of degree six in the variable y, we conclude that the set $\{f=0\}$ contains exactly three closed components. From Theorem A we deduce that system (3.7) has at most three limit cycles.

The origin of this system is an unstable critical point. From inequalities (3.9) and (3.10) we obtain that the first and second closed components of $\{f=0\}$ are traversed by the flow associated to system (3.7) in opposite direction. From the Poincaré-Bendixson Theorem we conclude that in the annular region determined by these two closed components there is at least one limit cycle. Again by theorem A we deduce that it is unique and hyperbolic. The same result is obtained for the annular region determined by the third and second closed components of $\{f=0\}$, but changing the stability of the limit cycle.

In order to prove the existence of a third limit cycle we study the nature of critical points at infinity of system (3.7). For example, if we known that all critical points at infinity are repellor, the equator can serve as an outer boundary of a Poincaré-Bendixson annular region. The inner boundary will be the third closed component of $\{f = 0\}$, which is crossed by the flow in upward direction. The number and the nature of critical points at infinity of Liénard systems of this type is well-known, see again [DH, LMP]. There are two critical points at infinity: a saddle point and a unstable node. From this we see that critical points at infinity are all repeller. We deduce that there exists a unique limit cycle in the annular region determined by the third closed component of $\{f = 0\}$ and the equator.

In conclusion, we have proved that system (3.7) has exactly three limit cycles, all of them being hyperbolic. □

Remark 3.2. *If instead of Liénard system* (3.7) *we consider the perturbation of the linear center*

$$\dot{x} = y - \varepsilon F(x)\,, \qquad \dot{y} = -x\,, \tag{3.11}$$

where F is the same function that in (3.7)*, but ε is small enough, it is easy to show that system* (3.11) *has at least 3 limit cycles. The main achievement of Theorem 1.1 is that in its statement there are no small parameters and the exact number of limit cycles is given.*

3.3. **A polynomial Liénard system without limit cycles.** In this section we just show how our methods can be used to test the non existence of limit cycles for some polynomial Liénard systems. Consider system

$$\dot{x} = y - F(x)\,, \qquad \dot{y} = -g(x)\,, \tag{3.12}$$

with

$$F(x) = ax + bx^2 + cx^3 + dx^4 + ex^5, \qquad g(x) = x + lx^2 + mx^3 + nx^4.$$

Note that the divergence of (3.12) is $-F'(x)$ which is a polynomial of degree 4. The condition that it does not change sign gives sufficient conditions for the non existence of limit cycles. This fact can be tested, for instance, by using the methods described in the Appendix. The use of Theorem A gives other cases for which (3.12) has no limit cycles. If we apply this theorem with $n = 1, s = -1$ and $f(x, y) = y + (c_0 - F(x))$, we get that $M_{-1,1}(x, c_0) = c_0 F'(x) - g(x)$ which is an 1-parameter family of polynomials of degree 4. If we assume that $d \neq 0$, by taking $c_0 = m/(4d)$ we obtain a polynomial that has no the x^3 term. Proposition 4.2 can be applied to this function $M_{-1,1}$ and it gives an algebraic criterium of non existence of limit cycles.

Observe also that the function $M_{-1,1}$ obtained by using Theorem A, can be also obtained by applying Remark 2.4 with $f(x,y) = -1, s = 1$ and $\psi(x,y) = -y/c_0$.

3.4. **A rational Liénard differential equation.** In the next result we prove the uniqueness and hyperbolicity of the limit cycle of a Liénard system with a rational F. The uniqueness (without proving the hyperbolicity) for this system was already proved in [Co]. See also [GG].

Proposition 3.3. *The Liénard system*

$$\begin{aligned} \dot{x} &= P(x,y) = y - F(x), \\ \dot{y} &= Q(x,y) = -x\,, \end{aligned} \tag{3.13}$$

with

$$F(x) = \frac{x(1-cx^2)}{(1+cx^2)}\,,$$

and c a real positive constant, has at most one limit cycle. Furthermore, when it exists it is hyperbolic and unstable.

Proof. We apply Theorem A and Corollary 2.7 with $s = -1$, $n = 2$ and $f(x,y)$ given by the following polynomial function:

$$f(x,y) = y^2 - F(x)y + x^2\,. \tag{3.14}$$

The function $M_{-1,2}(x) := M(x)$ is for this case:

$$M(x) = \frac{-4cx^4}{(1+cx^2)^2} < 0 \quad \text{for all } x \neq 0\,.$$

As $f(x,y)$ given by (3.14) is a second degree polynomial in the variable y, the properties of the set $\{f = 0\}$ are obtained from the discriminant Δ, given by:

$$\Delta = x^2\left(\left(\frac{1-cx^2}{1+cx^2}\right)^2 - 4\right) < 0 \qquad \text{for all } x \neq 0\,. \tag{3.15}$$

Then, the real set $\{f = 0\}$ is only composed by the origin.

Therefore for this case $c(\mathbb{R}^2, f) = 0$ and $\ell(\mathbb{R}^2, f) = 1$. From Theorem A we conclude that system (3.13) has at most one limit cycle. The origin is the only critical point of this system and it is stable. Then, when the limit cycle exists it is hyperbolic and unstable.

□

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4. Appendix: Polynomials of constant sign

Several methods to guarantee that a polynomial $P(x)$ has no real roots are well known. Among them we can quote a method based on the study of quadratic forms [C2], or the construction of the Sturm sequence, see [SB, pp. 281–283].

In this appendix first we recall how to use the Sturm sequence to test if a polynomial of arbitrary degree does not change sign and secondly we deduce the generic conditions for a polynomial of degree 4 to have no real roots. We have decided to include this last result here, because in the classical reference [W] there are several misprints in the results about this subject.

4.1. Conditions to ensure that a polynomial does not change sign.

The Sturm sequence associated to a polynomial $p(x)$ is defined as follows

$$f_0(x) = p(x), \qquad f_1(x) = -p'(x),$$
$$f_{i-1}(x) = f_i(x)q_i(x) - f_{i+1}(x)\,, \qquad i = 1, 2, 3, \ldots$$

where $q_i(x)$ and $f_i(x)$ are also polynomials with $\deg f_{i+1}(x) < \deg f_i(x)$. This sequence ends when some f_i is zero. From the above sequence, when the last non zero f_i is a constant polynomial, or from a new one obtained starting the same procedure from $p(x)/f_i(x)$ it is possible to obtain the number of real roots of p, $\mathrm{NRR}\,(p)$ (without taking into account their multiplicity), see again [SB].

It is not difficult to prove the following result, see [GMM].

Lemma 4.1. *Consider a polynomial p of degree k and the following finite sequence of polynomials*

$$\begin{aligned} R_0(x) &:= p(x), \\ R_l(x) &:= \text{g.c.d.}\,\big(R_{l-1}(x), R'_{l-1}(x)\big)\,, \quad 1 \le l \le m, \qquad m \le k. \end{aligned}$$

We stop the sequence when some R_l is a non zero real constant. Then the following holds: the polynomial p does not change sign if and only if

$$\mathrm{NRR}\,(R_{2i}) = \mathrm{NRR}\,(R_{2i+1}), \ \ \textit{for all } i = 0, 1, \cdots, [(m-1)/2].$$

As a corollary of the above lemma we get that to know if a polynomial does not change sign is decidable by means of inequalities among their coefficients.

4.2. Study of the quartic polynomial.

We start with a polynomial in the form

$$p(x) = x^4 + qx^2 + rx + s\,.$$

Remember that any polynomial of degree 4 can be reduced to this form just by a translation.

Direct computations give

$$
\begin{aligned}
f_0(x) &= x^4 + qx^2 + rx + s\,,\\
f_1(x) &= -f_0'(x) = -(4x^3 + 2qx + r)\,,\\
f_2(x) &= -\frac{1}{2}qx^2 - \frac{3}{4}rx - s\,,\\
f_3(x) &= -\frac{8qs - 9r^2 - 2q^3}{q^2}x + \frac{r(12s + q^2)}{q^2}\,, \quad \text{when} \quad q \neq 0\,,\\
f_4(x) &= \frac{1}{4}\frac{q^2(256s^3 - 128q^2s^2 + 144qsr^2 + 16q^4s - 27r^4 - 4r^2q^3)}{(8qs - 9r^2 - 2q^3)^2}
\end{aligned}
$$

when $8qs - 9r^2 - 2q^3 \neq 0$, $f_5(x) \equiv 0$.

By introducing the notation

$$
\begin{aligned}
\Delta_2 &= q\\
\Delta_3 &= 8qs - 9r^2 - 2q^3 \qquad (*)\\
\Delta_4 &= 256s^3 - 128q^2s^2 + 144qsr^2 + 16q^4s - 27r^4 - 4r^2q^3
\end{aligned}
$$

we have the following table of signs of $f_i(x)$ near infinity if we assume that $\Delta_2\Delta_3\Delta_4 \neq 0$.

	f_0	f_1	f_2	f_3	f_4
sign near $-\infty$	$+$	$+$	$-\,\mathrm{sgn}(\Delta_2)$	$\mathrm{sgn}(\Delta_3)$	$\mathrm{sgn}\,(\Delta_4)$
sign near ∞	$+$	$-$	$-\,\mathrm{sgn}(\Delta_2)$	$-\,\mathrm{sgn}(\Delta_3)$	$\mathrm{sgn}\,(\Delta_4)$

Table 1: Signs near $\pm\infty$ of the Sturm sequence when $\Delta_2\Delta_3\Delta_4 \neq 0$.

From the Sturm theory the signs of Table 1 imply the following result

Proposition 4.2. *Consider $f_0(x) = x^4 + qx^2 + rx + s$ and Δ_i, $i = 2, 3, 4$ defined as in $(*)$. Then, if $\Delta_2\Delta_3\Delta_4 \neq 0$ the polynomial $f_0(x)$ is strictly positive if and only if $\Delta_4 > 0$ and*

$$
\begin{aligned}
&\textit{either} \quad \Delta_2 > 0\,;\\
&\textit{or} \quad \Delta_2 < 0 \quad \textit{and} \quad \Delta_3 < 0\,.
\end{aligned}
$$

Remark 4.3. *The case $\Delta_2\Delta_3\Delta_4 = 0$ can be studied in a similar way but the Sturm sequence has to be modified. For instance, when $\Delta_2 = 0$ and*

$\Delta_3\Delta_4 \neq 0$ *we have the following sequence*

$$f_0(x) = p(x)\,, \quad f_1(x) = -p'(x),$$
$$f_2(x) = -\frac{3}{4}rx - s\,,$$
$$f_3(x) = -\frac{1}{27}\frac{256s^3 - 27r^4}{r^3}\,, \quad f_4(x) \equiv 0\,.$$

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