LOCAL FIRST INTEGRALS OF DIFFERENTIAL SYSTEMS AND DIFFEOMORPHISMS

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Abstract

In this paper using theory of linear operators and normal forms we generalize a result of Poincaré [13] about the non-existence of local first integrals for systems of differential equations in a neighbourhood of a singular point. As an application of the generalized result, and under more weak conditions we obtain a result of Furta [10] about local first integrals of semi-quasihomogeneous systems. Moreover, for diffeomorphisms and periodic differential systems we give definitions of their first integrals, and generalize the previous results about systems of differential equations to diffeomorphisms in a neighbourhood of a fixed point and to periodic differential systems in a neighbourhood of a constant solution.

1. Introduction

Investigation of first integrals for systems of differential equations is a classical work. In recent years there are many authors to look for first integrals or to prove non-integrability of autonomous differential systems (see for instantce, [3], [4], [6], [7], [9], [10], [11], [14] and [15]). In the present paper by using theory of normal forms we give a necessary and sufficient conditions

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for a class of autonomous differential systems to have local (formal) first integrals, which generalize a result of Poincaré [13] and one of Furta [10]. Moreover, we extend these results to local first integrals of diffeomophisms, and of periodic differential systems.

We consider the following autonomous differential system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{C}^n, \tag{1}$$

where $\mathbf{f}$ a vector-valued function of dimension n, the dot denotes the derivative with respect to time variable t. As usual, $\mathbf{C}$ denotes the complex field.

A non-constant function $H : \mathbf{C}^n \to \mathbf{C}$ is a *first integral* of system (1) in an open connected set $U \subset \mathbf{C}^n$ if it is constant on all solution curves $\mathbf{x}(t)$ of system (1) in U, i.e. $H(\mathbf{x}(t)) \equiv$ constant for all values of t for which the solution $\mathbf{x}(t)$ is defined on U. If H is differentiable on U, then H is a first integral if and only if along every solution curve

$$\left\langle \frac{\partial H}{\partial \mathbf{x}}, \mathbf{f}(\mathbf{x}) \right\rangle \equiv 0, \text{ i.e. } \sum_{i=1}^{n} f_i(\mathbf{x}) \frac{\partial H}{\partial x_i} \equiv 0, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of vectors of $\mathbf{C}^n$.

Suppose that $\mathbf{x} = \mathbf{0}$ is a singular point of system (1); i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Assume that $H(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$ are formal series in $\mathbf{x}$. We say that $H(\mathbf{x})$ is a *formal first integral* of system (1) in a neighbourhood of the singular point $\mathbf{x} = \mathbf{0}$, if it satisfies equality (2), where the derivative with respect to $\mathbf{x}$ is the derivative of the formal series, the equality means that the sum of the coefficients of the terms with the same order in $\mathbf{x}$ are equal to zero. Obviously, if a formal first integral is convergent, it is an analytic first integral.

We now give the definition of a first integral of a diffeomorphism. Let $\mathbf{f} : M \to M$ be a diffeomorphism from a n-dimensional manifold M into itself, and $H : U \to \mathbf{C}$ be a C^r function defined on an open subset U of M. Here, $r \geq 1$ means that $r = 1, 2, \ldots, \infty, \omega$. Of course, $r = \omega$ means that the diffeomorphism is analytic. We say that H is a *first integral* of the diffeomorphism $\mathbf{f}$ if it is constant on all orbits of the map $\mathbf{f}$ in U, i.e.

$$H(\mathbf{x}) \equiv H(\mathbf{f}^m(\mathbf{x})), \tag{3}$$

for all integer m such that $\mathbf{f}^m(\mathbf{x}) \in U$. This definition is a natural generalization of the notion of first integral for autonomous differential systems. This is due to the fact that if system (1) has a periodic orbit Γ and has a first integral H^* in its neighborhood, we select a transversal section Σ passing through a point $\mathbf{x}^*$ on Γ, and denote by $P(\mathbf{x})$ the Poincaré map induced by the flow $\phi(t, \mathbf{x})$ of system (1) on Σ, then $H(\mathbf{x}) = H^*(\phi(t, \mathbf{x}))$, where $\mathbf{x}$ is defined in a neighbourhood of $\mathbf{x}^*$ in Σ. Thus H is a first integral of the Poincaré map $P(\mathbf{x})$. If $H(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$ are formal series in $\mathbf{x}$ and satisfy (3), then H is called a *formal first integral* of $\mathbf{f}$ in a neighborhood of the fixed point $\mathbf{x} = \mathbf{x}^*$, where the equality of the two formal series means that the coefficients of the terms with the same order in $\mathbf{x}$ on both two sides of the equality are equal.

We now consider a periodic differential system:

$$\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}), \tag{4}$$

where $(t, \mathbf{x}) \in S^1 \times M$ with $S^1 = \mathbf{R}/(2\pi\mathbf{N})$, and $M = \mathbf{C}^n$ or $\mathbf{R}^n$, $\mathbf{g}$ is a periodic function with period 2π in t and C^r in $S^1 \times M$ with $r \geq 1$.

A non-constant function $H(t, \mathbf{x})$ defined on $S^1 \times U$ with U an open subset of M, is a *first integral* of system (4) if it is a 2π periodic function in t which is constant on all solution curves $\mathbf{x}(t)$ of system (4) in U; i.e. $H(t, \mathbf{x}(t)) \equiv$ constant for all values of t for which the solution $\mathbf{x}(t)$ is defined on U. If $H(t, \mathbf{x})$ is differentiable in $S^1 \times U$, then it is a first integral if and only if along every flow

$$\frac{\partial H}{\partial t} + \left\langle \frac{\partial H}{\partial \mathbf{x}}, \mathbf{g}(t, \mathbf{x}) \right\rangle \equiv 0. \tag{5}$$

Suppose that $\mathbf{x} = \mathbf{0}$ is a constant solution of system (4); i.e. $\mathbf{g}(t, \mathbf{0}) = \mathbf{0}$ for all $t \in S^1$. If $H(t, \mathbf{x})$ and $\mathbf{g}(t, \mathbf{x})$ are formal series in $\mathbf{x}$ with coefficients 2π differentiable periodic functions in t satisfying equality (5), then H is called a *formal first integral* of system (4) in a neighborhood of the constant solution $\mathbf{x} = \mathbf{0}$.

In what follows, all mentioned first integrals have no constant term.

This paper is organized as follows. In Section 2 we prove three basic lemmas (see Lemmas 1, 2 and 3), which will be used later on. In Section 3 we generalize the results of Poincaré [13] and Furta [10] about local first integrals of autonomous differential systems, our results allow certain resonant conditions, which are stated in Theorems 6, 8 and 9. In Section 4 we obtain one result on non-existence of local first integrals for periodic differential systems in a neighborhood of a constant solution, which is given in Theorem 10. In Section 5 we obtain the necessary and sufficient conditions in order that a diffeomorphism have local first integrals in a neighborhood of a fixed point, see Theorems 11 and 12. In Section 5 we obtain one result on non-existence of local first integrals for autonomous differential systems in a neighborhood of a periodic orbit, which are given in Corollary 14.

2. Basic Lemmas

This section consists of two parts. First, using methods of linear algebra we prove two basic results in the space of the homogeneous polynomials. Second, we prove a result about the multiplicity of a zero point of an analytic vector-valued function. They will be our basic tools.

Lemma 1. *Let* $\mathbf{A}$ *be a* $n\times n$ *matrix, and let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be its eigenvalues. Let* Υ_k *be the linear space formed by the homogeneous polynomials of degree* k *with* $k \geq 1$ *in* $\mathbf{C}[x_1, \ldots, x_n]$*, the complex polynomial ring in the variables* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*. We define a linear operator from* Υ_k *into itself by*

$$L(h)(\mathbf{x}) = \left\langle \frac{\partial h}{\partial \mathbf{x}}, \mathbf{A}\mathbf{x} \right\rangle,$$

where $h \in \Upsilon_k$*. Then the set of eigenvalues of* L *is*

$$\Theta = \left\{ \sum_{i=1}^{n} k_i \lambda_i,\ k_i \in \mathbf{Z}^+,\ \sum_{i=1}^{n} k_i = k \right\},$$

where $\mathbf{Z}^+ = \mathbf{N} \cup \{0\}$ *and as usual* $\mathbf{N}$ *denotes the set of positive integers.*

Proof: Let Γ_k be the following base of Υ_k,

$$\Gamma_k = \left\{ \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \ldots x_n^{m_n}, \sum_{i=1}^{n} m_i = k \right\}.$$

The proof is divided into three different cases.

Case 1: The matrix $\mathbf{A}$ is diagonal, i.e. $\mathbf{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then for each $\mathbf{x}^{\mathbf{m}} \in \Gamma_k$ satisfying $\mathbf{m} = (m_1, \ldots, m_n)$ and $\sum\limits_{i=1}^{n} m_i = k$, we have

$$\begin{aligned} L\mathbf{x}^{\mathbf{m}} &= \left\langle \frac{\partial \mathbf{x}^{\mathbf{m}}}{\partial \mathbf{x}}, \mathbf{A}\mathbf{x} \right\rangle = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial \mathbf{x}^{\mathbf{m}}}{\partial x_j} \\ &= (m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n)\mathbf{x}^{\mathbf{m}}. \end{aligned}$$

Thus, it follows easily that the set of eigenvalues of L is Θ on Υ_k.

Case 2: The matrix $\mathbf{A}$ is diagonalizable. So there exists an invertible matrix $\mathbf{T}$ such that $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Let $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$ and $g(\mathbf{y}) = h(\mathbf{T}\mathbf{y})$ with $h \in \Upsilon_k$. Then we have

$$\begin{aligned} L(g)(\mathbf{y}) &= L(h)(\mathbf{T}\mathbf{y}) = L(h)(\mathbf{x}) = \left\langle \frac{\partial h}{\partial \mathbf{x}}, \mathbf{A}\mathbf{T}\mathbf{y} \right\rangle \\ &= \left\langle \frac{\partial g}{\partial \mathbf{y}}\mathbf{T}^{-1}, \mathbf{A}\mathbf{T}\mathbf{y} \right\rangle = \left\langle \frac{\partial g}{\partial \mathbf{y}}, \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} \right\rangle. \end{aligned}$$

Working in a similar way to the proof of Case 1 we obtain the result of the lemma in this case.
Case 3: The matrix $\mathbf{A}$ is not diagonalizable. Let $\mathbf{A}_\varepsilon = \mathbf{A} + \varepsilon\mathbf{B}$, where $\varepsilon > 0$ is sufficiently small and $\mathbf{B}$ is a suitable matrix such that $\mathbf{A}_\varepsilon$ is diagonalizable. Assume that the eigenvalues of $\mathbf{A}_\varepsilon$ are $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_n(\varepsilon)$, then $\lim_{\varepsilon \to 0} \lambda_i(\varepsilon) = \lambda_i$ for $i = 1, 2, \ldots, n$. We define the linear operator on Υ_k

$$L_{\mathbf{A}_\varepsilon}(h)(\mathbf{x}) = \left\langle \frac{\partial h}{\partial \mathbf{x}}, \mathbf{A}_\varepsilon \mathbf{x} \right\rangle, \qquad \text{for each } h \in \Upsilon_k.$$

It follows from Case 2 that the set of eigenvalues of $L_{\mathbf{A}_\varepsilon}$ is

$$\left\{ \sum_{i=1}^{n} k_i \lambda_i(\varepsilon),\ k_i \in \mathbf{Z}^+,\ \sum_{i=1}^{n} k_i = k \right\}.$$

Since $L = \lim_{\varepsilon \to 0} L_{\mathbf{A}_\varepsilon}$, we obtain that the set of eigenvalues of L is Θ on Υ_k. This completes the proof of the lemma. ∎

Lemma 2. *Let* $\mathbf{A}$, λ_i *and* Υ_k *be those of Lemma 1. We define a linear operator* L^* *from* Υ_k *into itself by*

$$L^*(h)(\mathbf{x}) = h(\mathbf{A}\mathbf{x}) - h(\mathbf{x}).$$

Then the set of eigenvalues of L^* *is* $\Theta^* = \left\{ \prod_{i=1}^{n} \lambda_i^{k_i} - 1, k_i \in \mathbf{Z}^+, \sum_{i=1}^{n} k_i = k \right\}$.

Proof: We denote by Γ_k the base of Υ_k used in the proof of Lemma 1. If $\mathbf{A}$ is diagonal, i.e., $\mathbf{A} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then for every $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \ldots x_n^{k_n} \in \Gamma_k$ we have

$$L^*(\mathbf{x}^{\mathbf{k}}) = \prod_{i=1}^{n} (\lambda_i x_i)^{k_i} - \prod_{i=1}^{n} x_i^{k_i} = \left(\prod_{i=1}^{n} \lambda_i^{k_i} - 1 \right) \mathbf{x}^{\mathbf{k}}.$$

Then, it follows easily that the set of eigenvalues of L^* is Θ^*.

If A is not diagonal, then working in a similar way to the proof of Lemma 1 we can complete the proof of this lemma. ∎

Let $\mathbf{f} : \mathbf{C}^n \to \mathbf{C}^n$ be an analytic vector-valued function. A point $\mathbf{x}_0 \in \mathbf{C}^n$ is said to be a *zero point* of $\mathbf{f}$ if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. The zero point $\mathbf{x}_0$ is *m-multiplicity* if m is the maximum number of the zero points, i.e. the preimage of $\mathbf{0}$ in the neighborhood of $\mathbf{x}_0$, of $\mathbf{f}$ under small pertubations in the set of analytic functions.

Let $\mathbf{f}(\mathbf{x}) = (f_1(x_1, \mathbf{x}_2), \mathbf{f}_2(x_1, \mathbf{x}_2))$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, where $\mathbf{f}_2, \mathbf{x}_2$ are vectors of dimensional $n-1$. Assume that the determinant of the Jacobian matrix $\partial \mathbf{f}_2/\partial \mathbf{x}_2(\mathbf{0})$ is not equal to zero. By the Implicit Function Theorem, $\mathbf{f}_2 = \mathbf{0}$ has a unique solution $\mathbf{x}_2 = \tilde{\mathbf{x}}_2(x_1)$ satisfying $\mathbf{f}_2(x_1, \tilde{\mathbf{x}}_2(x_1)) \equiv 0$ in a sufficiently small neighborhood of the origin.

Lemma 3. *If $f_1(x_1, \tilde{\mathbf{x}}_2(x_1)) = a x_1^m + O(x_1^{m+1})$ with $a \neq 0$, then the multiplicity of the zero point $\mathbf{0}$ is m.*

Proof: We first take a special pertubation of $\mathbf{f} : \overline{\mathbf{f}} = (f_1 + p_1(x_1), \mathbf{f}_2)$, where $p_1(x_1) = g_1(x_1)G_1(x_1)$ with g_1 a polynomial of degree $m-1$ and G_1 satisfying $f_1(x_1, \tilde{\mathbf{x}}_2(x_1)) = x_1^m G_1$. By the assumption, $\mathbf{f}_2 = \mathbf{0}$ has a unique solution $\mathbf{x}_2 = \tilde{\mathbf{x}}_2(x_1)$ in a sufficiently small neighborhood of the origin. So, $f_1 + p_1 = (x_1^m + g_1(x_1))G_1(x_1)$. Selecting suitable g_1 with sufficiently small coefficients, we can get that $f_1 + p_1 = 0$ has m-roots in a neighborhood of the origin. This proves that the multiplicity of the origin is at least m.

Now we take arbitrary small perturbations of $\mathbf{f}$: $\tilde{\mathbf{f}} = (f_1 + p_1(x_1, \mathbf{x}_2), \mathbf{f}_2 + \mathbf{p}_2(x_1, \mathbf{x}_2))$. Since $\mathbf{p}_2$ is sufficient small, we also have the determinant of the Jacobian matrix $\partial(\mathbf{f}_2 + \mathbf{p}_2)/\partial \mathbf{x}_2(\mathbf{0})$ is not equal to zero. By the Implicit Function Theorem, $\mathbf{f}_2 + \mathbf{p}_2 = 0$ has a unique solution $\mathbf{x}_2 = \overline{\mathbf{x}}_2(x_1) = \tilde{\mathbf{x}}_2(x_1) + \mathbf{q}(x_1)$, where $\mathbf{q}$ is sufficiently small. Substituting $\mathbf{x}_2$ into $f_1 + p_1$ we get

$$\tilde{f}_1 = f_1(x_1, \overline{\mathbf{x}}_2(x_1)) + p_1(x_1, \overline{\mathbf{x}}_2(x_1)) = a x_1^m Q_1(x_1) + \overline{p}_1,$$

where $Q_1 = 1 + O(x_1)$ is an analytic function, and $\overline{p}_1$ is an analytic function with sufficiently small coefficients. From the Argument Principle we can obtain that $\tilde{f}_1$ has at most m roots in a neighborhood of $x_1 = 0$. This proves that the multiplicity of the origin is at most m. So, the lemma follows. ∎

3. On the non-existence of local first integrals of a system of differential equations

Consider an analytic system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{C}^n, \quad \mathbf{x} = (x_1, \ldots, x_n). \tag{6}$$

Let $\mathbf{x} = \mathbf{0}$ be a singular point of (6), i.e., $\mathbf{f}(\mathbf{0})=\mathbf{0}$. We denote by $\mathbf{A}$ the Jacobian matrix $\partial \mathbf{f}/\partial \mathbf{x}(\mathbf{0})$ of the vector field $\mathbf{f}(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$. The following result known by Poincaré [13] gives the relationship between the eigenvalues of $\mathbf{A}$ and the non-existence of local first integrals of system (6) (for a proof, see for instance [10]).

Proposition 4. *If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ do not satisfy any resonant conditions of the form*

$$\sum_{i=1}^{n} k_i \lambda_i = 0, \qquad k_i \in \mathbf{Z}^+, \quad \sum_{i=1}^{n} k_i \neq 0,$$

then system (6) does not have any analytic first integrals in a neighborhood of $\mathbf{x} = \mathbf{0}$.

3.1 The generalization of Proposition 4

We first recall some notations and a preliminary result about normal forms.

We say that the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ of a matrix $\mathbf{A}$ are *resonant* if there exist $s \in \{1, \ldots, n\}, k_i \in \mathbf{Z}^+$ satisfying $\sum\limits_{i=1}^{n} k_i \geq 2$ such that $\lambda_s = \sum\limits_{i=1}^{n} k_i \lambda_i$. The following result due to Poincaré and Dulac gives the norm form of system (6) (for a proof, see [2]).

Theorem 5. *If* $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \ldots$ *is a formal series and* $\mathbf{A}$ *is a Jordan normal matrix, then system (6) can be reduced to the canonical form*

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{w}(\mathbf{y}), \tag{7}$$

by means of a formal change of variables $\mathbf{x} = \mathbf{y} + \ldots$, *where* $\mathbf{w} = (w_1, \ldots, w_n)$ *and all monomials* $\mathbf{y}^m = y_1^{m_1} \ldots y_n^{m_n}$ *in the series* $w_i(\mathbf{y})$ *are resonant verifying* $\lambda_i = \langle m, \lambda \rangle$ *with* $|m| \geq 2$, *where* $m = (m_1, \ldots, m_n)$, *and* $|m| = \sum\limits_{i=1}^{n} m_i$.

Our following result generalize Proposition 4, it allows that the eigenvalues satisfy a strong one-resonance.

Theorem 6. *Assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ satisfy the following conditions:* $\lambda_1 = 0$ *and* $\sum\limits_{i=2}^{n} k_i \lambda_i \neq 0$ *for any* $k_i \in \mathbf{Z}^+$ *and* $\sum\limits_{i=2}^{n} k_i x \geq 1$. *Then the following statements hold.*

- *(a) For $n > 2$, system (6) has a formal series first integral in a neighborhood of* $\mathbf{x} = \mathbf{0}$ *if and only if the singular point* $\mathbf{x} = \mathbf{0}$ *is not isolated. In particular, if the singular point* $\mathbf{x} = \mathbf{0}$ *is isolated, then system (6) has no analytic first integrals in a neighborhood of* $\mathbf{x} = \mathbf{0}$.
- *(b) For $n = 1, 2$, system (6) has an analytic first integral in a neighborhood of* $\mathbf{x} = \mathbf{0}$ *if and only if the singular point* $\mathbf{x} = \mathbf{0}$ *is not isolated.*

Proof: We first prove statement (a). We rewrite system (6) into the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}), \tag{8}$$

where $\mathbf{g}$ satisfies $\mathbf{g}(\mathbf{0}) = \mathbf{0}$, and $\mathrm{D}\mathbf{g}(\mathbf{0}) = \mathbf{0}$. Then there exists an invertible matrix $\mathbf{M}$ such that

$$\mathbf{A}^* = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{pmatrix} 0 & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & \mathbf{B} \end{pmatrix},$$

is a Jordan normal form of the matrix $\mathbf{A}$, where $\mathbf{0}_{n-1}$ is a zero vector of dimension $n-1$, T denotes the transpose of a matrix, and $\mathbf{B}$ is a $(n-1) \times (n-1)$ matrix. We make the change of variables $\mathbf{y} = \mathbf{M}^{-1}\mathbf{x}$, system (8) becomes

$$\dot{\mathbf{y}} = \mathbf{A}^*\mathbf{y} + \mathbf{q}(\mathbf{y}). \tag{9}$$

From Theorem 5 there exists a formal series $\mathbf{y} = \mathbf{z} + \ldots$ such that the above system is reduced to canonical form (7). We write the new formal differential system as

$$\begin{aligned} \dot{z}_1 &= h_1(z_1, \mathbf{z}_2), \\ \dot{\mathbf{z}}_2 &= \mathbf{B}\mathbf{z}_2 + \mathbf{h}_2(z_1, \mathbf{z}_2), \end{aligned} \tag{10}$$

where $\mathbf{z} = (z_1, \mathbf{z}_2)$, $\mathbf{z}_2 = (z_2, \ldots, z_n)$ and $\mathbf{h}_2 = (h_2, \ldots, h_n)$ are vectors of dimension $n-1$, all monomials in the series h_1 and $\mathbf{h}_2$ are resonant. From the definition of resonant monomial $\mathbf{z}^m = z_1^{m_1} \ldots z_n^{m_n}$ and Theorem 5 we get that $0 = \lambda_1 = \langle m, \lambda \rangle = \sum_{i=2}^{n} m_i \lambda_i$. This condition is equivalent to $m_2 = m_3 = \cdots = m_n = 0$ by the assumption of the theorem. So $h_1(z_1, \mathbf{z}_2) = h_1(z_1)$ with the lowest order at least 2. Similarly, we can prove that $\mathbf{h}_2(z_1, \mathbf{0}_{n-1}) \equiv 0$.

We first prove "only if " part. We assume that the formal first integral of system (10) is

$$H(z_1, \mathbf{z}_2) = \sum_{i=0}^{\infty} a_i(\mathbf{z}_2) z_1^i,$$

where $a_i(\mathbf{z}_2)$ are formal series in $\mathbf{z}_2$. Let

$$\mathbf{h}_2(z_1, \mathbf{z}_2) = \sum_{i=0}^{\infty} \mathbf{b}_i(\mathbf{z}_2) z_1^i,$$

where $\mathbf{b}_i(\mathbf{z}_2)$ are vector-valued formal series in $\mathbf{z}_2$ of dimension $n-1$, and $\mathbf{b}_0(\mathbf{z}_2)$ has degree at least 2. From the definition of first integrals we obtain that

$$\left(\sum_{i=1}^{\infty} i a_i(\mathbf{z}_2) z_1^{i-1}\right) h_1(z_1) + \left\langle \sum_{i=0}^{\infty} \frac{\partial a_i(\mathbf{z}_2)}{\partial \mathbf{z}_2} z_1^i, \mathbf{B}\mathbf{z}_2 + \sum_{i=0}^{\infty} \mathbf{b}_i(\mathbf{z}_2) z_1^i \right\rangle \equiv 0. \tag{11}$$

Equating the constant terms in z_1 yields

$$\left\langle \frac{\partial a_0(\mathbf{z}_2)}{\partial \mathbf{z}_2}, \mathbf{B}\mathbf{z}_2 + \mathbf{b}_0(\mathbf{z}_2) \right\rangle \equiv 0.$$

Let $a_0(\mathbf{z}_2) = c_0 + c_k(\mathbf{z}_2) + O(k+1)$, where c_0 is a constant, $c_k(\mathbf{z}_2)$ is a homogeneous polynomial of degree k with $k \geq 1$, and $O(k+1)$ denotes the sum of terms with degree larger than k. Substituting $a_0(\mathbf{z}_2)$ into the above equality, we get $\langle \partial c_k(\mathbf{z}_2)/\partial \mathbf{z}_2, \mathbf{B}\mathbf{z}_2 \rangle = 0$. Since the eigenvalues of the matrix $\mathbf{B}$ satisfy that $\sum_{i=2}^{n} k_i \lambda_i \neq 0$ for all $k_i \in \mathbf{Z}^+$ with $\sum_{i=2}^{n} k_i = k$, by Lemma 1 the operator L is invertible. So, we obtain that $c_k = 0$. Consequently, we have that $a_0(\mathbf{z}_2)$ is a constant, and equal to $c_0 = a_0$.

We shall prove by induction that $a_i(\mathbf{z}_2)$ is constant for $i = 1, 2, \ldots$. Consider for $N > 1$

$$H(z_1, \mathbf{z}_2) = \sum_{i=0}^{N-1} a_i z_1^i + a_N(\mathbf{z}_2) z_1^N + O_{z_1}(N+1),$$

where a_i for $i = 0, 1, \ldots, N-1$, are constants, and $O_{z_1}(N+1)$ denotes the sum of terms with degree in z_1 at least $N+1$. From equation (11) it follows that

$$\begin{aligned} \left(\sum_{i=1}^{N-1} i a_i z_1^{i-1}\right) h_1(z_1) + N a_N(\mathbf{z}_2) z_1^{N-1} h_1(z_1) + \frac{\partial O_{z_1}(N+1)}{\partial z_1} h_1(z_1) \\ + \left\langle \frac{\partial a_N(\mathbf{z}_2)}{\partial \mathbf{z}_2} z_1^N, \mathbf{B}\mathbf{z}_2 + \sum_{i=0}^{\infty} \mathbf{b}_i(\mathbf{z}_2) z_1^i \right\rangle \\ + \left\langle \frac{\partial O_{z_1}(N+1)}{\partial \mathbf{z}_2}, \mathbf{B}\mathbf{z}_2 + \sum_{i=0}^{\infty} \mathbf{b}_i(\mathbf{z}_2) z_1^i \right\rangle \equiv 0. \end{aligned}$$

Comparing terms with z_1^N, we obtain that

$$\left\langle \frac{\partial a_N(\mathbf{z}_2)}{\partial \mathbf{z}_2}, \mathbf{B}\mathbf{z}_2 + \mathbf{b}_0(\mathbf{z}_2) \right\rangle = 0.$$

Working in a similar way to the proof of $a_0(\mathbf{z}_2) =$ constant, we can prove that $a_N(\mathbf{z}_2) =$ constant. By the method of induction it follows that all

$a_i(\mathbf{z}_2)$ are constant for $i = 0, 1, \ldots$. That is, the formal first integral $H(z_1, \mathbf{z}_2)$ is independent on $\mathbf{z}_2$. We obtain from (11) that either $H(z_1, \mathbf{z}_2) =$ constant or $h_1(z_1) \equiv 0$.

If $h_1(z_1) \not\equiv 0$, then $H(z_1, \mathbf{z}_2) =$ constant, system (10) has no formal series first integrals in a neighborhood of the singular point $\mathbf{0}$. Hence, if the first integral exists, we must have $h_1(z_1) \equiv 0$.

In order to complete the proof of statement (a), we need the following results.

Lemma 7. *For system (9) the following statements hold:*

(1) The singular point $\mathbf{0}$ of system (9) is not isolated if and only if $h_1(z_1) \equiv 0$;

(2) The multiplicity of the singular point $\mathbf{0}$ of system (9) is m if and only if $h_1(z_1) = cz_1^m + \cdots$ with $c \neq 0$.

Proof of the sufficiency of statement (2): Assume that $h_1(z_1) = cz_1^m + \cdots$ with $c \neq 0$. Select a sufficiently higher cut, denoted by $\mathbf{y} = \mathbf{z} + \mathbf{w}(\mathbf{z})$, of the formal series transformation from (9) to (10), where $\mathbf{w}(\mathbf{z})$ is a vector-valued polynomial of degree higher enough. Under the change of variables $\mathbf{y} = \mathbf{z} + \mathbf{w}(\mathbf{z})$, system (9) becomes

$$\dot{z}_1 = \tilde{h}_1(z_1, \mathbf{z}_2), \qquad \dot{\mathbf{z}}_2 = \mathbf{B}\mathbf{z}_2 + \tilde{h}_2(z_1, \mathbf{z}_2),$$

where $\tilde{h}_1$ and h_1, $\tilde{h}_2$ and h_2 are different only on the sufficiently higher terms. Using the Implicit Function Theorem, from $\mathbf{B}\mathbf{z}_2 + \tilde{h}_2(z_1, \mathbf{z}_2) = \mathbf{0}_{n-1}$ we get a unique solution $\mathbf{z}_2 = \mathbf{z}_2(z_1) = O(z_1^N), N \gg m$ in a neighborhood of the singular point $\mathbf{0}$. Substituting it into $\tilde{h}_1(z_1, \mathbf{z}_2)$ we get $\tilde{h}_1(z_1, \mathbf{z}_2(z_1)) = cz_1^m + \cdots$, which implies that the multiplicity of the singular point $\mathbf{0}$ is m by Lemma 3.

The necessity of statement (1) is a corollary of the sufficiency of statement (2).

Proof of the sufficiency of statement (1). If the singular point $\mathbf{0}$ of system (9) is isolated, we can claim that its multiplicity is finite. Indeed, let the right hand side of the last $n-1$ equations in (9) be equal to zero. By the assumptions of the theorem and using the Implicit Function Theorem, we can get a unique solution $(y_2, \ldots, y_n) = (\tilde{y}_2, \ldots, \tilde{y}_n)(y_1)$, of this system of the equations in a neighborhood of the origin. Substituting this solution into the right hand side of first equation in (9) and using the assumption that the singular point $\mathbf{0}$ is isolated, we have

$$q_1(y_1, \tilde{y}_2(y_1), \ldots, \tilde{y}_n(y_1)) = q_0 y_1^m + O(y_1^{m+1}), \tag{12}$$

for some integer $m \geq 2$, where q_0 is a nonzero constant and q_1 is the first component of the vector function $\mathbf{q}$. It follows from Lemma 3 that the multiplicity of $\mathbf{0}$ is m.

On the other hand, we select a sufficiently higher cut $\mathbf{y} = \mathbf{z} + \mathbf{w}(\mathbf{z})$ of the formal series transformation from (9) to (10), where $\mathbf{w}(\mathbf{z})$ is a vector-valued polynomial of degree higher enough. Then system (10) and the system obtained from (9) using the analytic transformation $\mathbf{y} = \mathbf{z} + \mathbf{w}(\mathbf{z})$ are different only on the terms with degree sufficiently higher. Since $h_1 \equiv 0$ in (10), we obtain that the multiplicity of the singular point $\mathbf{0}$ is bigger than any given natural number, which is in contradiction with (12). This proves that the singular point $\mathbf{0}$ of system (9) is isolated.

Proof of the necessity of statement (2). First, by statement (1) we have $h_1 \neq 0$. Assume that $h_1 = c_1 z_1^k + \cdots$ with $c_1 \neq 0$. Then by the sufficiency of statement (2) it follows easily that $m = k$. This completes the proof the lemma. ∎

Continuity of proof of statement (a) of Theorem 6: Since $h_1 \equiv 0$, from Lemma 7 we can easily prove that the "only if" part holds.

We now prove the "if" part. Since the singular point $\mathbf{0}$ is not isolated, it follows from Lemma 7 that in system (10) the function $h_1(z_1, \mathbf{z}_2) = h_1(z_1) \equiv 0$. So, $H(z_1, \mathbf{z}_2) = z_1$ is a formal first integral of system (10) in a neighborhood of the singular point $\mathbf{z} = \mathbf{0}$. Moreover, we get that system (6) has a formal first integral in a neighborhood of the singular point $\mathbf{x} = \mathbf{0}$. This proves statement (a).

Proof of statement (b). For $n = 1$, from the definitions of first integral and isolated singular point we can prove easily that the statement holds.

For $n = 2$, from statement (a) it follows easily that the "only if" part holds. We now prove the "if" part.

Working in a similar way to the proof of statement (a), we only need to consider system (9) with $\mathbf{y} = (y_1, y_2)$. Since the singular point $\mathbf{y} = \mathbf{0}$ of system (9) is not isolated, from the proof of statement (a) we obtain that in a sufficiently small neighborhood U of $\mathbf{0}$ there exists a unique solution $y_2 = G(y_1)$ of the equation $\lambda_2 y_2 + q_2(\mathbf{y}) = 0$ such that $q_1(y_1, G(y_1)) \equiv 0$.

We make the change of variables

$$z_1 = y_1, \qquad z_2 = y_2 - G(y_1). \tag{13}$$

Then, system (9) is transformed into

$$\dot{z}_1 = p_1(z_1, z_2) z_2, \qquad \dot{z}_2 = p_2(z_1, z_2) z_2, \tag{14}$$

with $p_2(0,0) = \lambda_2 \neq 0$. Obviously, p_1 and p_2 are analytic. System (14) and the system

$$\dot{z}_1 = p_1(z_1, z_2), \qquad \dot{z}_2 = p_2(z_1, z_2), \tag{15}$$

have the same first integrals. System (15) is regular at $(0,0)$, by the Box Flow Theorem it has an analytic first integral in a sufficiently small neighborhood of $(0,0)$. So system (14) has also an analytic first integral in the corresponding neighborhood of $(0,0)$. Since transformation (13) is an analytic diffeomorphism, so system (9) has an analytic first integral in a suitable neighborhood of the singular point $(0,0)$. This completes the proof of statement (b). ∎

3.2 First integrals of semi-quasilinear systems

System (6) is said to be *quasi-homogeneous* of degree m with exponents $s_1, \ldots, s_n \in \mathbf{Z}, m \in \mathbf{N}\backslash\{1\}$, if for any $\rho \in \mathbf{R}^+$ and $\mathbf{x} \in \mathbf{C}^n$ the following equality holds

$$f_i(\rho^{s_1}x_1, \rho^{s_2}x_2, \ldots, \rho^{s_n}x_n) = \rho^{s_i+m-1}f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{16}$$

A weak version of this definition is the following. System (6) is called *quasi-homogeneous* with respect to exponents $w_1, \ldots, w_n \in \mathbf{R}, \mu \in \mathbf{R}^+$, if

$$f_i(\mu^{w_1}x_1, \mu^{w_2}x_2, \ldots, \mu^{w_n}x_n) = \mu^{w_i+1}f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{17}$$

Obviously, by selecting $\rho = \mu^{1/(m-1)}$, equality (16) becomes (17).

In order to simplify the notation we denote by $\mathbf{f}_m(\mathbf{x})$ a quasi-homogeneous polynomial system of degree m, by $\mu^{\mathbf{W}}$ the diagonal matrix diag$(\mu^{w_1}, \ldots, \mu^{w_n})$. Under this notation, the last equality can be written as $\mathbf{f}(\mu^{\mathbf{W}}\mathbf{x}) = \mu^{\mathbf{W}+\mathbf{E}}\mathbf{f}(\mathbf{x})$ where $\mathbf{E}$ the unit matrix.

It is easy to see that if system (6) is quasi-homogeneous of degree m with exponents $s_1, \ldots, s_n$, then under the transformation

$$\mathbf{x} \longrightarrow \mu^{\mathbf{W}}\mathbf{x}, \quad t \longrightarrow \mu^{-1}t, \quad \mathbf{W} = \frac{1}{m-1}\mathbf{S}, \quad \mathbf{S} = \text{diag}(s_1, \ldots, s_n),$$

it is invariant.

System (6) is called *semi-quasi-homogeneous* if

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}_m(\mathbf{x}) + \widetilde{\mathbf{f}}(\mathbf{x}), \tag{18}$$

where $\widetilde{\mathbf{f}}(\mathbf{x})$ is the sum of quasi-homogeneous polynomials of degree all larger than m or all less than m. In the former (respectively, later), it is called *positively semi-quasi-homogeneous* (respectively, *negatively semi-quasi-homogeneous*). The system $\dot{\mathbf{x}} = \mathbf{f}_m(\mathbf{x})$ is called the *quasi-homogeneous cut* of system (6) with $\mathbf{f}$ of the form (18).

We assume that $\mathbf{f}(\mathbf{x})$ is semi-quasi-homogeneous of type (18). We call every solution $\mathbf{c} \neq 0$ of the algebraic system

$$\mathbf{f}_m(\mathbf{c}) + \mathbf{W}\mathbf{c} = \mathbf{0}, \tag{19}$$

a *balance* associated to system (6) with $\mathbf{f}(\mathbf{x})$ of the form (18), where $\mathbf{W} = \text{diag}(w_1, \ldots, w_n)$. The matrix

$$\mathbf{K} = D\mathbf{f}_m(\mathbf{c}) + \mathbf{W},$$

is the so-called *Kowalevskaya matrix*, where $D\mathbf{f}_m(\mathbf{c})$ is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x} = \mathbf{c}$. Its eigenvalues are called the *Kowalevskaya exponents*. (Sophia Kowalevskaya was the first to introduce the matrix $\mathbf{K}$ to compute the Laurent series solutions of the rigid body motion [12].)

Our following result is an extention of Theorem 1 of Furta [10].

Theorem 8. *Assume that system (6) is semi-quasi-homogeneous of type (18), and that for a balance* $\mathbf{c}$ *its Kowalevskaya exponents* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *satisfy the conditions:* $\lambda_1 = 0$, *and*

$$\sum_{i=2}^{n} k_i \lambda_i \neq 0, \qquad \text{for arbitrary } k_i \in \mathbf{Z}^+, \text{ and } \sum_{i=2}^{n} k_i \neq 0.$$

Then if the solution $\mathbf{c}$ *of (19) is isolated, system (6) does not have polynomial first integrals. Moreover, if system (6) is positively semi-quasi-homogeneous, it does not have formal first integrals in a neighborhood of* $\mathbf{0}$.

Proof: We try to search for first integrals in the form of formal Maclaurin series

$$H(\mathbf{x}) = \sum_{\substack{i_1+\cdots+i_n=1 \\ i_1\geq 0,\ldots,i_n\geq 0}}^{\infty} h_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}. \tag{20}$$

In the case that system (6) is negatively semi-quasi-homogeneous, we only search for polynomial first integrals, that is, the above sum has finitely many terms.

We do the following transformation:

$$\mathbf{x} \longrightarrow \mu^{\mathbf{W}}\mathbf{x}, \qquad t \longrightarrow \mu^{-1}t, \qquad \mu = \rho^{m-1}.$$

Then system (6) becomes

$$\dot{\mathbf{x}} = \mathbf{f}_m(\mathbf{x}) + \widetilde{\mathbf{f}}(\mathbf{x}, \rho), \tag{21}$$

where $\widetilde{\mathbf{f}}(\mathbf{x},\rho)$ is a formal series with respect to ρ or ρ^{-1} according to the system is positively or negatively semi-quasi-homogeneous. The function $H(\mathbf{x})$ goes over to

$$H(\mathbf{x},\rho)=H_l(\mathbf{x})+\rho H_{l+1}(\mathbf{x})+\rho^2 H_{l+2}(\mathbf{x})+\cdots,$$

or

$$H(\mathbf{x},\rho)=H_l(\mathbf{x})+\rho^{-1} H_{l-1}(\mathbf{x})+\rho^{-2} H_{l-2}(\mathbf{x})+\cdots,$$

depending on system (6) is positively or negatively semi-quasi-homogeneous respectively, where $H_j(\mathbf{x})=0$ for $j\le 0$. and $H_j(\rho^{\mathbf{S}}\mathbf{x})=\rho^j H_j(\mathbf{x})$ with $j>0$ and $\mathbf{S}=(m-1)\mathbf{W}$.

If $H(\mathbf{x})$ is a first integral of system (6), then $H_l(\mathbf{x})$ is a polynomial first integral of the quasi-homogeneous cut of system (6):

$$\dot{\mathbf{x}}=\mathbf{f}_m(\mathbf{x}). \tag{22}$$

Using the change of variable $\mathbf{x}=t^{-\mathbf{W}}(\mathbf{c}+\mathbf{u})$, where $\mathbf{c}$ satisfies $\mathbf{f}_m(\mathbf{c})+\mathbf{W}\mathbf{c}=0$, we get that

$$H_l(\mathbf{x})=t^{-l/(m-1)}H_l(\mathbf{c}+\mathbf{u})=\overline{H}(u_0,\mathbf{u}),$$

where we select $u_0=t^{-1/(m-1)}$ as a new auxiliary variable. System (22) becomes

$$t\dot{\mathbf{u}}=\mathbf{K}\mathbf{u}+\overline{\mathbf{f}}_m(\mathbf{u}),\qquad \overline{\mathbf{f}}_m(\mathbf{u})=\mathbf{W}\mathbf{c}+\mathbf{f}_m(\mathbf{c}+\mathbf{u})-\frac{\partial \mathbf{f}_m(\mathbf{c})}{\partial \mathbf{x}}\mathbf{u},$$

where $\mathbf{K}$ is the Kovalevskaya matrix associated to the balance $\mathbf{c}$.

Let $\tau=\log t$. Then $\overline{H}(u_0,\mathbf{u})$ is a polynomial first integral of the system

$$u_0'=-\frac{1}{m-1}u_0,\qquad \mathbf{u}'=\mathbf{K}\mathbf{u}+\overline{\mathbf{f}}_m(\mathbf{u}), \tag{23}$$

where prime denotes the derivative with respect to τ.

On the other hand, we can prove easily that the eigenvalues of the linear part of system (23) at $u_0=0$, $\mathbf{u}=0$ are $\lambda_0=-1/(m-1), \lambda_1=0,\lambda_2,\ldots,\lambda_n$. Since -1 is always a Kowalevskaya exponent, let $\lambda_2=-1$. From the assumptions of the theorem we have

$$-k_0+(m-1)\sum_{i=2}^{n}k_i\lambda_i=(k_0+(m-1)k_2)\lambda_2+(m-1)\sum_{i=3}^{n}k_i\lambda_i\neq 0,$$

where $k_i\in\mathbf{Z}^+$, $k_0+(m-1)\sum_{i=2}^{n}k_i\neq 0$. Since $\mathbf{0}=\mathbf{K}\mathbf{u}+\overline{\mathbf{f}}_m(\mathbf{u})=\mathbf{f}_m(\mathbf{c}+\mathbf{u})+\mathbf{W}(\mathbf{c}+\mathbf{u})$ has isolated roots. We obtain from Theorem 6 that system (23) has no formal first integral. This is in contradiction with the previous argument. This proves the theorem. ∎

The following examples show some applications of Theorem 8.

Example 1. Consider the following semi-quasi-homogeneous system:

$$\dot{x} = y + x^2 + xy + y^3, \qquad \dot{y} = 6xy - 2x^3 + xy^2. \tag{24}$$

Note that under the change of variables $x \to \rho x$, $y \to \rho^2 y$, $t \to \rho^{-1} t$, the system

$$\dot{x} = y + x^2, \qquad \dot{y} = 6xy - 2x^3, \tag{25}$$

is invariant. So, system (25) is a quasi-homogeneous system of degree 2 with exponents $(s_1, s_2) = (1, 2)$. Moreover, system (24) is positively semi-quasi-homogeneous. We denote by $\mathbf{f}_2(x, y)$ the right hand sides of system (25), that is, $\mathbf{f}_2 = (y + x^2, 6xy - 2x^3)$. The following algebraic system

$$\mathbf{f}_2(\mathbf{c}) + \mathbf{W}\mathbf{c} = \mathbf{0}, \qquad \mathbf{c} = (c_1, c_2), \quad \mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

has a unique balance $\mathbf{c} = (-\frac{1}{2}, \frac{1}{4})$. The Kowalevskaya matrix associated to this balance is

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

and it has the eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$. From Theorem 8 system (24) has no analytic first integrals in a neighborhood of the origin. We note that Theorem 1 of [10] cannot be applied to system (24).

Example 2. Consider the two dimensional Volterra-Lotka system

$$\dot{x} = x(\alpha + ax + by), \qquad \dot{y} = y(\beta + cx + dy), \qquad |a| + |b| + |c| + |d| \neq 0. \tag{26}$$

It is negatively semi-quasi-homogeneous with the exponents $(1, 1)$. Furta [10] proved that if $(b-d)(c-a)/(ad-cb) \notin \mathbf{Q}^+$ with $\mathbf{Q}^+$ the set of positive rational numbers, system (26) has no polynomial first integrals. Using our Theorem 8 we can claim that if $b = d \neq 0$ or $a = c \neq 0$, then system (26) has no polynomial first integrals. Indeed, we consider the quasi-homogeneous cut

$$\dot{x} = x(ax + by), \qquad \dot{y} = y(cx + dy),$$

then $\mathbf{W} = \text{diag}(1, 1)$. Let $\mathbf{f}_2(x, y) = (x(ax + by), y(cx + dy))^T$. Straightforward calculations show that the algebraic equation

$$\mathbf{f}_2(\mathbf{c}) + \mathbf{W}\mathbf{c} = \mathbf{0},$$

has two isolated solutions $(0, -\frac{1}{d})$ (if $d \neq 0$) and $(-\frac{1}{a}, 0)$ (if $a \neq 0$), that is, two isolated balances of (26). The corresponding Kowalevskaya matrix $\mathbf{K}$ has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0$. So, from Theorem 8 the claim follows. Essentially, the above conclusions are also obtained by [5], in which the authors give the necessary and sufficient conditions for system (26) to have polynomial first integrals.

Finally, we consider the case in which system (6) has $0 \leq k < n$ independent formal first integrals. Let $H_i(\mathbf{x})$ for $i = 1, \ldots, k$ be first integrals of system (6). They are called *independent* if the Jacobian matrix $\partial(H_1, \ldots, H_k)/\partial\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ has rank k. We assume that system (6) has the form (8).

Theorem 9. *If system (8) has $0 \leq k < n$ independent formal first integrals $H_1, \ldots, H_k$, and the matrix $\mathbf{A}$ has $n-k$ eigenvalues $\lambda_1, \ldots, \lambda_{n-k}$ satisfying*

$$\sum_{i=1}^{n-k} k_i\lambda_i \neq 0, \qquad k_i \in \mathbf{Z}^+, \quad \sum_{i=1}^{n-k} k_i \neq 0,$$

then all formal first integrals of system (8) are formal series in $H_1, \ldots, H_k$.

Proof: We make the formal change of variables

$$y_i = H_i(x_1, \ldots, x_n) \text{ for } i = 1, \ldots, k, \quad \begin{pmatrix} y_{k+1} \\ \vdots \\ y_n \end{pmatrix} = \mathbf{Bx},$$

where $\mathbf{B}$ is a $(n-k) \times n$ matrix such that the determinant of the Jacobian matrix $\partial(y_1, \ldots, y_n)/\partial\mathbf{x}|_{\mathbf{x}=0}$ is not equal to zero. So, this transformation has a formal inverse. Under these changes of variables system (8) becomes

$$\dot{\mathbf{y}}_k = \mathbf{0}_k, \quad \dot{\mathbf{y}}_{n-k} = \mathbf{M}\mathbf{y}_{n-k} + \mathbf{f}^*_{n-k}(\mathbf{y}), \tag{27}$$

where $\mathbf{y}_k = (y_1, \ldots, y_k)$, $\mathbf{y}_{n-k} = (y_{k+1}, \ldots, y_n)$, $\mathbf{f}^*_{n-k} = O(\|\mathbf{y}\|^2)$ and $\mathbf{M}$ is a constant $(n-k) \times (n-k)$ matrix, whose eigenvalues are $\lambda_1, \ldots, \lambda_{n-k}$.

If $H(\mathbf{y})$ is a formal first integral of system (27), without loss of generality, we can assume that it has the form

$$H(\mathbf{y}) = \sum_{|m|=0}^{\infty} a_m(\mathbf{y}_{n-k})\mathbf{y}_k^m,$$

where $\mathbf{y}_k^m = y_1^{m_1} \ldots y_k^{m_k}$, $m_i \in \mathbf{Z}^+$ and $|m| = m_1 + \cdots + m_n$. By the definition of formal first integral we have

$$\left\langle \frac{\partial H(\mathbf{y})}{\partial \mathbf{y}_{n-k}}, \mathbf{M}\mathbf{y}_{n-k} + \mathbf{f}^*_{n-k}(\mathbf{y}) \right\rangle = 0. \tag{28}$$

Comparing the terms with $\mathbf{y}_k^0$ we get

$$\left\langle \frac{\partial a_0(\mathbf{y}_{n-k})}{\partial \mathbf{y}_{n-k}}, \mathbf{M}\mathbf{y}_{n-k} \right\rangle = 0.$$

From Lemma 1 and the assumptions of the theorem we obtain that $a_0(\mathbf{y}_{n-k}) = \text{constant}$. By induction and working in a similar way to the proof of Theorem 6 we can prove that $a_m(\mathbf{y}_{n-k}) = \text{constant}$ for $|m| = 0, 1, \ldots$. This means that if $H(\mathbf{y})$ is a formal first integral of system (27), then it depends on $\mathbf{y}_k$ only. Hence, every formal first integral of system (8) is a formal series in $H_1, \ldots, H_k$ only. This proves the theorem. ∎

We remark that this theorem is a generalization of Theorem 1 in [10]. Because when $k = 0$, our result is that of Theorem 1 in [10].

4. Local first integrals of periodic differential systems in a neighborfood of a constant solution

Let $\mathbf{x} = \mathbf{0}$ be a constant solution of the periodic differential system (4). We rewrite $\mathbf{g}(t, \mathbf{x})$ as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x} \in \mathbf{C}^n, \tag{29}$$

where $\mathbf{f}(t, \mathbf{x}) = O(\mathbf{x}^2)$, $\mathbf{A}(t)$ and $\mathbf{f}(t, \mathbf{x})$ are 2π periodic function in t.

We consider the linear differential system associated to (29)

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}. \tag{30}$$

The linear operator $F : \mathbf{C}^n \to \mathbf{C}^n$ converting the initial condition $\mathbf{x}_0$ for $t = 0$ into the value $F(\mathbf{x}_0)$ of the solution with this initial condition at $t = 2\pi$ is called *monodromy operator*. We denote by μ_i for $i = 1, \ldots, n$ the eigenvalues of the monodromy operator. These eigenvalues are called *eigen multipliers* of the periodic system (30). Floquet's theorem gives the normal form of the linear periodic differential systems (see for instance, [1]).

Floquet's Theorem. *There exists a change of variables* $\mathbf{x} = \mathbf{B}(t)\mathbf{y}$*, linear in* $\mathbf{x}$ *and* 2π *periodic in* t*, which transforms system (30) into* $\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y}$ *with* $\mathbf{\Lambda}$ *a constant matrix. Moreover, we have* $\mu_i = \exp(2\pi\lambda_i)$*, where* λ_i *for* $i = 1, \ldots, n$*, are the eigenvalues of the matrix* Λ.

Theorem 10. *If the eigen multipliers of system (30) do not satisfy any resonant conditions of the type*

$$\prod_{i=1}^{n} \mu_i^{k_i} = 1, \qquad k_i \in \mathbf{Z}^+, \quad \sum_{i=1}^{n} k_i \neq 0, \tag{31}$$

then system (29) does not have any formal first integrals in a neighborhood of the constant solution $\mathbf{x} = 0$.

Proof: Assume that $H(t, \mathbf{x})$ is a formal first integral with period 2π in t of system (29). We make the change of variables $\mathbf{x} = \mathbf{B}(t)\mathbf{y}$ given in Floquet's thoerem, then system (29) becomes

$$\dot{\mathbf{y}} = \mathbf{\Lambda y} + \mathbf{f}^*(t, \mathbf{y}). \tag{32}$$

Furthermore, $H(t, \mathbf{x}) = H^*(t, \mathbf{y})$ is a first integral of this system. Expanding H^* into Maclaurin series in $\mathbf{y}$

$$H^*(t, \mathbf{y}) = \sum_{i=0}^{\infty} H_i^*(t, \mathbf{y}),$$

where $H_i^*(t, \mathbf{y})$ is a homogeneous polynomial in $\mathbf{y}$ of degree i with 2π periodic functions in t as coefficients.

From the definition of the first integrals we have

$$\sum_{i=0}^{\infty} \frac{\partial H_i^*(t, \mathbf{y})}{\partial t} + \left\langle \sum_{i=0}^{\infty} \frac{\partial H_i^*(t, \mathbf{y})}{\partial \mathbf{y}}, \mathbf{\Lambda y} + \mathbf{f}^*(t, \mathbf{y}) \right\rangle \equiv 0. \tag{33}$$

Equating the terms with degree 0 in $\mathbf{y}$ we obtain easily that $H_0^* =$ constant. Without loss of generality, we set $H_0^* = 0$. Equating the terms with degree 1 in $\mathbf{y}$ we get that

$$\frac{\partial H_1^*(t, \mathbf{y})}{\partial t} + \left\langle \frac{\partial H_1^*(t, \mathbf{y})}{\partial \mathbf{y}}, \mathbf{\Lambda y} \right\rangle = 0. \tag{34}$$

Let Υ_k be the linear space formed by homogeneous polynomials of degree k in $\mathbf{y}$. Let

$$L: \ h \longrightarrow \left\langle \frac{\partial h}{\partial \mathbf{y}}, \mathbf{\Lambda y} \right\rangle$$

be the linear operator from Υ_k into itself. Then, from Lemma 1 the set of eigenvalues of L is $\left\{ \sum_{i=1}^{n} k_i \lambda_i, \ k_i \in \mathbf{Z}^+, \ \sum_{i=1}^{n} k_i = k \right\}$.

Since every polynomial is uniquely determined by its coefficients, we consider equation (34) as a system of equations in the coefficients of H_1^* in $\mathbf{y}$. Then, using the operator L equation (34) can be written as

$$\frac{dH_1^*}{dt} + LH_1^* = 0.$$

Its solution is $H_1^*(t) = \exp(-Lt)H_1^*(0)$. In order that H_1^* is a 2π periodic function, we must have $(\exp(-2\pi L) - \mathbf{E})H_1^*(0) = 0$, where $\mathbf{E}$ is a convenient unit matrix.

Since the eigenvalues of the matrix $\exp(-2\pi L)$ are $\exp\left(-2\pi \sum_{i=1}^{n} k_i \lambda_i\right)$ for $\sum_{i=1}^{n} k_i = 1$, i.e., $\left(\prod_{i=1}^{n} \mu_1^{k_1}\right)^{-1}$, it follows from the assumptions of the theorem that $H_1^*(0) = \mathbf{0}$. Moreover, we have $H_1^*(t, \mathbf{y}) = 0$.

In what follows, we use induction. Assume that $H_i^*(t, y) = 0$ for $i = 1, \ldots, k-1$. Then, from (33) we get

$$\frac{\partial H_k^*(t, \mathbf{y})}{\partial t} + \left\langle \frac{\partial H_k^*(t, \mathbf{y})}{\partial \mathbf{y}}, \mathbf{\Lambda y} \right\rangle = 0.$$

Working in a similar way to solve equation (34), we can prove that $H_k^*(t, \mathbf{y}) = 0$. In short, by induction we obtain that $H^*(t, \mathbf{y}) = 0$. This means that system (32), that is system (29), does not have formal first integrals. This proves the theorem. ∎

5. Local first integrals of diffeomorphisms

Let $\mathbf{f}(\mathbf{x})$ be an analytic diffeomorphism defined in $\mathbf{C}^n$. We assume that $\mathbf{x}_0$ is a fixed point of f. For simplicity, we can assume that $\mathbf{x}_0 = \mathbf{0}$. Since $\mathbf{f}$ is a diffeomorphism, it can be expressed as

$$\mathbf{f} = \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{x}), \tag{35}$$

where $\mathbf{A}$ is a $n \times n$ constant matrix, and $\mathbf{F}(\mathbf{x}) = O(\|\mathbf{x}\|^2)$. The following result gives the necessary conditions for a diffeomorphism to have a first integral.

Theorem 11. *If the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix A do not satisfy any resonant conditions of the type*

$$\prod_{i=1}^{n} \lambda_i^{k_i} = 1, \qquad k_i \in \mathbf{N} \cup \{0\}, \qquad \sum_{i=1}^{n} k_i \geq 1,$$

then the diffeomorphism $\mathbf{f}(\mathbf{x})$ does not have any nontrivial formal first integral in a neighborhood of the fixed point $\mathbf{x} = \mathbf{0}$.

Proof: We will prove this theorem by contradiction. Assume that $H(\mathbf{x})$ is a formal first integral of the diffeomorphism $\mathbf{f}$ in a neighborhood U of the fixed point $\mathbf{0}$. Then

$$H(\mathbf{f}(\mathbf{x})) = H(\mathbf{x}), \tag{36}$$

for all $\mathbf{x} \in U$ such that $\mathbf{f}(\mathbf{x}) \in U$.

We expand $H(\mathbf{x})$ into the Maclaurin series

$$H(\mathbf{x}) = \sum_{j=l}^{\infty} H_j(\mathbf{x}),$$

where $H_j(\mathbf{x}) = \sum_{i_1+\cdots+i_n=j} a_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$ is a homogeneous polynomial of degree j in $\mathbf{x}$ for $j = l, l+1, \ldots, \infty$, and $l \geq 1$. Equating the terms in (36) with degree 1 in $\mathbf{x}$ we have $H_1(\mathbf{Ax}) = H_1(\mathbf{x})$, i.e. $L^*(H_1)(\mathbf{x}) = 0$, where L^* is the operator defined in Lemma 2. From the assumptions of this theorem and Lemma 2 it follows that L^* is an invertible linear operator. Hence, $H_1(\mathbf{x}) = 0$.

By induction and working in a similar way to the proof of Theorem 8, we can prove that $H_i(\mathbf{x}) = 0$ for $i \geq 1$. The details are omitted. Therefore, the diffeomorphism $\mathbf{f}$ does not have any formal first integral in a neighborhood of the fixed point $\mathbf{x} = \mathbf{0}$. This completes the proof of the theorem. ∎

The following examples show that if the the eigenvalues of the coefficient matrix of the linear part of a diffeomorphism satisfy resonant conditions, then whether the diffeomorphism has a first first integral depends on additional assumptions.

Example 3. Consider the following diffeomorphism

$$\overline{x} = x - x^3, \qquad \overline{y} = y - y^3,$$

in a neighborhood of the origin. The coefficient matrix of the corresponding linear part has two eigenvalues $\lambda_1 = \lambda_2 = 1$. Since $(0,0)$ is stable, the above diffeomorphism does not have any analytic first integrals in a neighborhood of $(0,0)$.

Example 4. The following diffeomorphism

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \cos f(x,y) & \sin f(x,y) \\ -\sin f(x,y) & \cos f(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

in a neighborhood of the origin, has the first integral $H(x,y) = x^2 + y^2$, where $f(x,y)$ is an arbitrary analytic function. The coefficient matrix of the linear part of the diffeomorphism has two eigenvalues $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$, where $\theta = f(0,0)$ and $i = \sqrt{-1}$. Obviously, $\lambda_1 \lambda_2 = 1$.

Example 5. We consider the following diffeomorphism

$$\overline{x} = 2x, \qquad \overline{y} = \frac{1}{2}y + xy^2,$$

in a neighborhood of the origin. The two eigenvalues of the linear part $\lambda_1 = 2, \lambda_2 = \frac{1}{2}$ satisfy $\lambda_1\lambda_2 = 1$. We claim that this last diffeomorphism does not have any analytic first integrals in a neighborhood of the fixed point $(0,0)$.

We now prove the claim. By contradiction we assume that $H(x,y)$ is an analytic first integral. Expanding H into Maclaurin series

$$H(x,y) = \sum_{i+j=1}^{\infty} a_{ij} x^i y^j,$$

and from

$$H(x,y) = H(\overline{x},\overline{y}) = H(2x, \frac{1}{2}y + xy^2), \tag{37}$$

we can prove easily that $a_{10} = a_{01} = a_{20} = a_{02} = 0$. In order to use induction, we can assume that

$$H(x,y) = h_{k-1}(xy) + \sum_{i+j=k}^{\infty} a_{ij} x^i y^j,$$

where $h_{k-1}(xy)$ is a polynomial of degree at most $k-1$. Using this expansion and $\bar{x}\bar{y} = xy(1+2xy)$, we obtain from (37) that

$$h_{k-1}(xy) + \sum_{i+j=k}^{\infty} a_{ij} x^i y^j = h_{k-1}(\bar{x}\bar{y}) + \sum_{i+j=k}^{\infty} a_{ij} 2^i x^i y^j \left(\frac{1}{2} + xy\right)^j.$$

Equating the terms with degree k and $i \neq j$, we get that $a_{ij}x^iy^j = 2^{i-j}a_{ij}x^iy^j$. Hence, $a_{ij} = 0$ for $i \neq j$. By induction we have

$$H(x,y) = \sum_{i=1}^{\infty} a_{ii} (xy)^i = h(xy).$$

Let $z = xy$, $g(z) = z + 2z^2$. Then equation (37) can be written as $h(z) = h(g(z))$. Selecting $-\frac{1}{2} < z_0 < 0$, then $\{z_k = g^k(z_0)\}$ is a monotonic increasing sequence, and $\lim_{k\to\infty} z_k = 0$. Hence $h(z_k) = 0$. Moreover, we have $h(z) = 0$. This proves the claim.

Now we consider the case in which the eigenvalues of an analytic diffeomorphism at a fixed point satisfy certain resonant.

Theorem 12. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the linear part of an analytic diffeomorphism $\mathbf{f}$ at the fixed point $\mathbf{x} = \mathbf{0}$. Assume that $\lambda_1 = 1$, and that $\lambda_2, \ldots, \lambda_n$ do not satisfy any resonant conditions of the type*

$$\prod_{i=2}^{n} \lambda_i^{k_i} = 1, \qquad k_i \in \mathbf{Z}^+, \quad \sum_{i=2}^{n} k_i \neq 0.$$

Then $\mathbf{f}$ *has a formal first integral in a neighborhood of* $\mathbf{x} = \mathbf{0}$ *if and only if the fixed point* $\mathbf{x} = \mathbf{0}$ *is not isolated.*

In order to prove Theorem 12 we need the following notation and result due to Poincaré and Dulac (for a proof, see for instance [8]). Let $\mathbf{f}$ be that given in (35), $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the linear operator $\mathbf{A}$. We say that these eigenvalues are *resonant*, if they satisfy

$$\lambda_s = \lambda^m, \qquad \lambda^m = \lambda_1^{m_1} \ldots \lambda_n^{m_n}, \quad m_k \in \mathbf{Z}^+, \quad \sum_{k=1}^n m_k \geq 2,$$

for some $s \in \{1, \ldots, n\}$.

Theorem 13. *If the matrix* $\mathbf{A}$ *is in Jordan form, then the diffeomorphism* $\mathbf{f}$ *can be reduced to the normal form* $\mathbf{Ay}+\mathbf{w}(\mathbf{y})$ *by a formal change of variables* $\mathbf{x} = \mathbf{y} + \cdots$, *where* $\mathbf{w} = (w_1, \ldots, w_n)$ *and all monomials* $\mathbf{y}^m = y_1^{m_1} \ldots y_n^{m_n}$ *in the series* $w_s(\mathbf{y})$ *are resonant verifying* $\lambda_s = \lambda^m$ *with* $|m| \geq 2$, *where* $m = (m_1, \ldots, m_n)$, *and* $|m| = \sum\limits_{i=1}^n m_i$.

Proof of Theorem 12: First, we prove the "only if" part. Without loss of generality, we can assume that the diffeomorphism $\mathbf{f}$ is of the form (35) with $\mathbf{F}(\mathbf{x}) = O(|\mathbf{x}|^2)$. Assume that $H(\mathbf{x})$ is a first integral of $\mathbf{f}$ in a neighborhood of $\mathbf{x} = \mathbf{0}$. We make the transformation $\mathbf{x} = \mathbf{My}$, where $\mathbf{M}$ is an invertible matrix such that $\mathbf{M}^{-1}\mathbf{AM}$ is a Jordan form. Then $\mathbf{f}$ becomes

$$\overline{\mathbf{f}} = \mathbf{A}^*\mathbf{y} + \mathbf{G}(\mathbf{y}), \tag{38}$$

where

$$\mathbf{A}^* = \mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 1 & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & \mathbf{B} \end{pmatrix}.$$

Moreover, $H(\mathbf{My})$ is a first integral of $\overline{\mathbf{f}}$ in a neighborhood of $\mathbf{y} = \mathbf{0}$. For simplicity, we denote by $H(\mathbf{y})$ the function $H(\mathbf{My})$.

From Theorem 13 there exists a formal series change $\mathbf{y} = \mathbf{z} + \cdots$ such that (38) becomes

$$g_1(\mathbf{z}) = z_1 + G_1(\mathbf{z}), \qquad \mathbf{g}_2(\mathbf{z}) = \mathbf{B}\mathbf{z}_2 + \mathbf{G}_2(\mathbf{z}), \tag{39}$$

where $\mathbf{z} = (z_1, \mathbf{z}_2)$, $\sigma_2 = (\sigma_2, \ldots, \sigma_n)$ for $\sigma = \mathbf{g}, \mathbf{z}, \mathbf{G}$, and $G_1 = O(|\mathbf{z}|^2)$, $\mathbf{G}_2 = O(|\mathbf{z}|^2)$ contain only resonant terms. For any term $z_1^{m_1} \ldots z_n^{m_n}$ in $G_1(\mathbf{z})$, since $1 = \lambda_1 = \lambda_2^{m_2} \ldots \lambda_n^{m_n}$, we have $m_2 = \cdots = m_n = 0$. So $G_1(\mathbf{z}) = G_1(z_1)$.

By the definition of first integral we have

$$H(z_1 + G_1(z_1), \mathbf{B}\mathbf{z}_2 + \mathbf{G}_2(\mathbf{z})) = H(z_1, \mathbf{z}_2). \tag{40}$$

We write H and $\mathbf{G}_2$ into the form $H = \sum_{i=0}^{\infty} h_i(\mathbf{z}_2) z_1^i$, and $\mathbf{G}_2 = \sum_{i=0}^{\infty} \mathbf{G}_i(\mathbf{z}_2) z_1^i$, respectively. Then (40) has the form

$$\sum_{i=0}^{\infty} h_i \left(\mathbf{B}\mathbf{z}_2 + \sum_{j=0}^{\infty} \mathbf{G}_j(\mathbf{z}_2) z_1^j \right) (z_1 + G_1(z_1))^i = \sum_{i=0}^{\infty} h_i(\mathbf{z}_2) z_1^i. \tag{41}$$

Comparing the coefficients of z_1^0 we get $h_0(\mathbf{B}\mathbf{z}_2 + \mathbf{G}_0(\mathbf{z}_2)) = h_0(\mathbf{z}_2)$. Using Lemma 2 and working in a similar way to the proof of Theorem 6 we have $h_0(\mathbf{z}_2) =$ constant. Again using the method in the proof of Theorem 6 we obtain that $h_i(\mathbf{z}_2) =$ constant $= h_i$ for $i = 1, 2, \ldots$. Hence, $H = \sum_{i=l}^{\infty} h_i z_1^i$ with $l \geq 1$ and $h_l \neq 0$. Equation (41) becomes

$$\sum_{i=l}^{\infty} h_i (z_1 + G_1(z_1))^i = \sum_{i=l}^{\infty} h_i z_1^i.$$

Since $h_l \neq 0$, we can claim that $G_1(\mathbf{z}) = G_1(z_1) = 0$. Indeed, if not, let $G_1(z_1) = c_p z_1^p + \cdots$ with $p > 1$ and $c_p \neq 0$. Then, this last equality gives

$$h_l z_1^l + \cdots + h_{p+l-1} z_1^{p+l-1} + h_l c_p l z_1^{p+l-1} + O(z_1^{p+l}) = \sum_{i=l}^{\infty} h_i z_1^i.$$

It follows easily that $c_p = 0$, and in contradiction. This proves the claim.

Let $G_1^*(\mathbf{z}) = g_1(\mathbf{z}) - z_1, \mathbf{G}_2^*(\mathbf{z}) = \mathbf{g}_2(\mathbf{z}) - \mathbf{z}_2$. Then from (39) and Lemma 3, and working in a similar way to the proof of Theorem 6 in which the singular point $\mathbf{0}$ is not isolated, we can verifies that the fixed point $\mathbf{0}$ of (38) is not isolated if and only if $G_1 \equiv 0$. This completes the proof of the "only if" part.

We prove the "if" part. If the fixed point $\mathbf{0}$ is not isolated, then the analytic diffeomorphism (35) can be transformed to (39) with $G_1(\mathbf{z}) = 0$ by a formal series change. Obviously, (39) with $G_1 = 0$ has the formal first integral $H = x_1$. This completes the proof of the theorem. ∎

Remark. In Theorem 12 if $\lambda_1 = -1$ and $\lambda_2, \ldots, \lambda_n$ do not satisfy the following conditions:

$$\prod_{i=2}^{n} \lambda_i^{m_i} = \pm 1, \qquad k_i \in \mathbf{Z}^+, \quad \sum_{i=2}^{n} k_i \neq 0,$$

then the conclusion in Theorem 12 is replaced as follows: $\mathbf{f}$ has a formal first integral if and only if $\mathbf{x} = \mathbf{0}$ is not an isolated fixed point of $\mathbf{f} \circ \mathbf{f}$.

We remark that for the analytic diffeomorphism of dimension 2 we cannot obtain a similar result as statement (b) of Theorem 6, that is, it has an analytic first integral in a neighborhood of $\mathbf{x} = \mathbf{0}$ if and only if the fixed point $\mathbf{x} = \mathbf{0}$ is not isolated. We propose the following

Open question. *Are there examples of the analytic diffeomorphism* $\mathbf{f}$ *of dimension* 2 *with* $\lambda_1 = 1, |\lambda_2| \neq 1$*, the two eigenvalues of* $\mathbf{f}$ *at the fixed point* $\mathbf{x} = \mathbf{0}$*, such that* $\mathbf{f}$ *has no analytic first integrals if* $\mathbf{x} = \mathbf{0}$ *is not isolated?*

6. Local first integrals of autonomous differential systems in a neighborhood of a periodic orbit

In this section, using Theorem 11 we give a sufficient condition in order that an autonomous differential system has no analytic first integrals in a neighborhood of a periodic orbit.

Suppose that system (6) has a periodic orbit. Choose a point A on the orbit and draw a transversal to the orbit (a smooth transverse hypersurface of dimension $n-1$). All orbits which start at points of the transversal sufficiently close to the point A return to the transversal. Let B be a point of the transversal, which is sufficiently close to the point A. An orbit starting from B returns to the transversal, the first intersection point is denoted by C. Thus, we get a map from the transversal to itself ($B \to C$) in a neighborhood of A. The germ at A of this map is called the *Poincaré map* (see for instance, [1], page 26–28). The point on the periodic orbit is a fixed point of the Poincaré map.

The *Multipliers of a periodic orbit* are defined to be the eigenvalues of the linear part of the Poincaré map at the fixed point corresponding to the periodic orbit. From Theorem 11 we obtain the following result.

Corollary 14. *If system (6) has a periodic orbit and its multipliers do not satisfy any resonant conditions of the type (31), then system (6) does not have any analytic first integrals in a neighborhood of the periodic orbit.*

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