Planar Analytic Vector Fields with Generalized Rational First Integrals

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Abstract

The main purpose of this paper is to characterize a germ of planar holomorphic vector field at an elementary singular point having a generalized rational first integral. Our results generalize a result due to Poincaré on a necessary condition of the existence of a rational first integral for planar polynomial systems. As two applications of our main result, we give the necessary and sufficient conditions on the existence of rational first integral for planar quadratic systems having either a weak nondegenerate singular point, or a degenerate elementary singular point.

1. Introduction and statement of the main results

A vector field defined in a domain of $\mathbf{C}^n$ is said to be *analytic* or *holomorphic* if its components are holomorphic functions. A holomorphic mapping $U \to W$, $U \subset \mathbf{C}^n$, $W \subset \mathbf{C}^m$, is defined similarly. An equation of the form

$$\frac{dw}{dt} = P(w,z), \quad \frac{dz}{dt} = Q(w,z), \quad (w,z) \in U \subset \mathbf{C}^2,\ t \in \mathbf{C} \tag{1}$$

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Key words and phrases: first integral, monodromy, normal form.
(2000) AMS Mathematics Subject Classification. 58F14, 34C05.

where P, Q are holomorphic functions, is called a *planar differential system with complex time*, a *planar complex autonomous equation*, or *a planar analytical differential system.* When P, Q are polynomials with real or complex coefficients, system (1) is called a *polynomial system.*

System (1) can be thought as a vector field $X = P\frac{\partial}{\partial w} + Q\frac{\partial}{\partial z}$. A point $q \in U$ is a *singular point* of system (1) if $P(q) = Q(q) = 0$. If $P_w(q)Q_z(q) - P_z(q)Q_w(q) \neq 0$, q is *nondegenerate*, otherwise, q is called *degenerate.* Obviously, a nondegenerate singular point is isolated in the set of singular points. A singular point is called *elementary*, if at least one eigenvalue of the linear part of the field at that point is nonzero.

An *integral curve* of system (1) is a holomorphic curve whose tangent line at each point coincides with the direction field associated with system (1), and which is connected and maximal with respect to this property. We remark that the integral curves are now either singular points or complex curves tangent to the vector field which are holomorphically immersed in $\mathbf{C}^2$. This gives rise to a holomorphic foliation by complex curves.

Two analytical differential systems are called *analytically equivalent* if the first system can be transformed into the second by a biholomorphic change of variables. Two systems are *analytically orbitally equivalent* if and only if there exists a biholomorphic transformation of the phase space which takes complex phase curves of the first to those of the second one.

A continuous function $H : U \to \mathbf{C}$ is a *strong first integral* of the differential system (1) defined in U if H is constant on each integral curve of this system, and H is non-constant on any open subset of U. If H is analytic, then the previous definition of integrability implies that the derivative of H following the direction of the vector field X is zero, i.e. if $XH = 0$ on U. This definition of strong first integral is the usual one which appears in the major part of books on differential equation (see, for instance, [34]).

With this definition the real linear differential system

$$\dot{x} = -x, \quad \dot{y} = -y \tag{2}$$

defined on $\mathbf{R}^2$ has no strong first integrals because the origin is a global attractor. Since we do not like that differential systems so easy as system (2) have no first integrals, we will introduce the notion of weak first integral.

Let Σ be a set of integral curves of system (1) such that $U \setminus \Sigma$ is open. We say that a continuous function $H : U \setminus \Sigma \to \mathbf{C}$ is a *weak first integral* of system (1) defined in U if H is constant on each solution of system (1) contained in $U \setminus \Sigma$, and H is non-constant on any open subset of $U \setminus \Sigma$. Obviously, system (2) has a weak first integral $H : \mathbf{R}^2 \setminus \{\mathbf{0}\} \to \mathbf{R}$, defined by $H(x, y) = xy/(x^2 + y^2)$. We remark that the unique difference between the notions of strong and weak first integral is that a weak first integral does not need to be defined in the whole domain of definition U of system (1). This difference has been noted by many authors (see for instance, [9]). Thus, the first integrals computed by Darboux [13] in 1887 for polynomial systems possessing sufficient algebraic solutions are, in general, weak first integrals.

The theory of integrability is a classical one. It received contributions from the work of Poincaré [30], who mainly was interested in the rational first integrals. Prelle and Singer [31], using methods of differential algebra, showed that if a polynomial vector field has an elementary first integral, then it can be computed using Darboux theory of integrability. Singer [33] proved that if a polynomial vector field has Liouvillian first integrals, then it has integrating factors given by Darbouxian functions. These two results can also be summarized in Corollaries 8.1 and 8.2 of [11]. For planar quadratic systems having a center, Schlomiuk [32] investigated its particular invariant algebraic curves, and provided a generic characterization of those systems in terms of possessing two particular invariant algebraic curves of degrees not exceeding 3. Forsyth [15], Kummer, Churchill and Rod [23], Goriely [19] proved that a system of dimension n has k $(k < n)$ independent algebraically first integrals if and only if it has k independent rational first integrals. This result shows that in order to investigate the algebraic integrability, we only need to study its rational first integrals. For some special three dimensional systems, Labrunie [25] characterized all polynomial first integrals of the (a, b, c) Lotka–Volterra system, Moulin Ollagnier [28] obtained necessary and sufficient conditions for this system to have polynomial first integrals. In [29] Moulin Ollagnier studied the homogenous rational first integrals of the (a, b, c) Lotka–Volterra system. Giacomini, Repetto and Zandron [18] investigated the integrals of motion of three–dimensional non–Hamiltonian dynamical systems. Llibre and Zhang [27] characterized all the invariant algebraic surfaces, the polynomial first integrals, the rational first integrals, the invariant, and the algebraically integrable cases of the Rikitake system.

For the local first integrals of a system of differential equations, Furta [17] obtained that if a system has a singular point with the eigenvalues not satisfying any resonant conditions, then the system does not have any analytic first integrals in a neighbourhood of the singular point. Li, Llibre and Zhang [24] gave necessary and sufficient conditions in order that a system has a formal first integral in a neighbourhood of a singular point when one eigenvalue is zero, and the others are not resonant. These local first integrals are weak ones.

For polynomial systems, probably, the most important and natural weak first integrals are the rational ones. We recall that a rational function f/g is called a first integral of a polynomial system (1) if f and g are coprime and f/g is constant on each integral curve in $\mathbf{C}^2 \setminus \Sigma$, where $\Sigma = \{(w,z) \in \mathbf{C}^2 |\ g(w,z) = 0\}$. Obviously, if a planar polynomial system has a rational first integral, then all integral curves of this system are algebraic. We remark that by a result due to Jouanolou [21] (see [10] for an easier proof), a planar polynomial system of degree m has $2 + m(m+1)/2$ invariant algebraic curves, if and only if it has a rational first integral.

The following result is due to Poincaré [30], which give a necessary condition for a planar polynomial system having a rational first integral.

Poincaré Theorem. *If polynomial system (1) has a rational first integral, then the eigenvalues λ_1, λ_2 associated to any singular point of the system must be resonant in the following sense: there exist integers m_1, m_2 with $|m_1| + |m_2| > 0$ such that $m_1\lambda_1 + m_2\lambda_2 = 0$.*

Our main purpose in this paper is to generalize Poincaré Theorem to the planar analytical systems with elementary singular points, and to obtain the necessary and sufficient conditions in order that the system has generalized rational first integrals. In order to present it we need some preliminary definitions.

Two analytic functions $f(w,z)$, $g(w,z) : U \to \mathbf{C}$ are said to be *coprime* if the set of points: $\{(w,z) \in U |\ f(w,z) = g(w,z) = 0\}$ is isolated. We call the ratio of two coprime analytic functions a *generalized rational function*. A generalized rational function $H = f/g$ defined on U is a first integral of system (1) if $\Sigma := \{(w,z) \in U |\ g(w,z) = 0\}$ is a set of integral curves of system (1) and H is a weak first integral on $U \setminus \Sigma$. Obviously, $H = f/g$ is a first integral of (1) if and only if $(Xf)g - (Xg)f \equiv 0$, $(w,z) \in U$.

Consider now a planar real or complex analytic system

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix} = A \begin{pmatrix} w \\ z \end{pmatrix} + \cdots, \quad t \in \mathbf{F},\ (w,z) \in U \subset \mathbf{F}^2, \tag{3}$$

where the dots denote the terms of order ≥ 2, $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$ and A is a square matrix of order 2 with elements in real or complex field according to $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, respectively. We denote by $\mathbf{N}, \mathbf{Q}$, $\mathbf{Q}^+$ and $\mathbf{Q}^-$ the set of positive integers, rational numbers, positive rational numbers and negative rational numbers respectively. Our main result is the following one.

Theorem 1. *Assume that the origin is an elementary singular point for the analytic differential system (3) with eigenvalues $\lambda_1 \neq 0, \lambda_2$. Then system (3) has a (real when $\mathbf{F} = \mathbf{R}$) generalized rational first integral in some neighbourhood of the origin if and only if one of the following conditions holds.*

(a) $\lambda_1 \neq 0 = \lambda_2$ *and the origin is not an isolated singular point,*

(b) $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \in \mathbf{Q}^+ \setminus \mathbf{N}$,

(c) $\lambda_1 = \lambda_2 \neq 0$, $A = \mathrm{diag}(\lambda_1, \lambda_2)$,

(d) λ_1/λ_2 or $\lambda_2/\lambda_1 \in \mathbf{N} \setminus \{1\}$ *and the germ (3) is analytically equivalent to its linear part,*

(e) $\lambda_1/\lambda_2 \in \mathbf{Q}^-$ *and the germ (3) is analytically orbitally equivalent to its linear part.*

Moreover, if (a) or (e) holds, system (3) has a complex (real) analytic first integral in some neighbourhood of the origin when $\mathbf{F} = \mathbf{C}$ *(respectively, when* $\mathbf{F} = \mathbf{R}$*).*

We note that Poincaré Theorem follows immediately as a corollary of Theorem 1.

As two examples of application of Theorem 1, we consider the existence of rational first integrals of planar quadratic systems with either a weak nondegenerate singular point, or a degenerate elementary singular point. We recall that a singular point of a vector field is said to be *weak*, if the divergence of the field at this singular point vanishes.

Theorem 2. *After a complex affine change of variables and a rescaling of the time (if necessary) any complex quadratic system having a nondegenerate weak singular point can be written in the form*

$$\dot{x} = -y - bx^2 - Cxy - dy^2, \quad \dot{y} = x + ax^2 + Axy - ay^2, \tag{4}$$

where a, b, d, A, C *are complex constants and* $a = 0$ *when* $b+d = 0$. *This system has a rational first integral if and only if one of the following conditions holds.*

(1) $A - 2b = C + 2a = 0$,

(2) $C = a = 0, A \neq 0$, *either* $b = 0$ *and* $A = d$, *or* $b/A = -\frac{1}{2}$ *and* $A = 2d$, *or* $b/A = -1$ *and* $d = 0$, *or* $b/A \in \mathbf{Q} \setminus \{0, -\frac{1}{2}, -1\}$,

(3) $b+d = 0, ACb\Delta \neq 0$ *and* $A/b, C/\sqrt{\Delta} \in \mathbf{Q}$, *where* $\Delta = b(A+b)+C^2/4$,

(4) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$ *and* $ad(d + b) \neq 0$.

Theorem 3. *After a complex (real) affine change of variables and a rescaling of the time (if necessary) any complex (respectively, real) quadratic system having a degenerate elementary singular point can be written in the form*

$$\dot{x} = a_1x^2 + b_1xy + c_1y^2, \quad \dot{y} = y + a_2x^2 + b_2xy + c_2y^2, \tag{5}$$

where a_1, b_1, c_1, a_2, b_2 and c_2 are complex (respectively, real) constants. This system has a complex (respectively, real) rational first integral if and only if either $a_1 = b_1 = c_1 = 0$, or $a_1 = a_2 = 0$ and one of the following conditions holds.

(1) $b_1 + c_2 = 0$,

(2) $b_1 - c_2 = c_1 = b_2 = 0$,

(3) $b_1 c_2 \neq c_1 b_2$ *and* $\lambda_1/\lambda_2 \in \mathbf{Q} \setminus \{1\}$, *where* λ_1 *and* λ_2 *are the eigenvalues of the matrix* $\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}$.

This paper is organized as follows. In Section 2 we prove Theorem 1. We prove Theorem 2 in Section 3. Theorem 3 is proved in Section 4.

2. Proof of Theorem 1

Assume that there exists a phase curve of system (3) which can be extended holomorphically through the origin and the extended curve is holomorphic to the disc $\{z \in \mathbf{C} | \, |z| < 1\}$. By analogy with the real case, we call it *separatrix* and denote it by S.

Consider an oriented loop γ encircling the origin on the separatrix S. We will associate to this loop a germ of a mapping, called *monodromy.* Fix a point $p \in \gamma$ and let Γ be a local transversal section of system (3) at p. The monodromy will be the germ of a conformal mapping $\Delta : (\Gamma, p) \to (\Gamma, p)$. The precise definition is as follows. A rectification of the foliation in a neighbourhood of a point of the loop defines diffeomorphisms between the local transversals at nearby points of the loop by carring a point of a leaf into a point of the same leaf. Choosing a finite number of neighbourhoods which cover the loop, we obtain a diffeomorphism of the local transversal Γ into itself which carries a point of each leaf into a point of the same leaf and fixes the initial point. The monodromy does not depend on the choice of transversal: mappings defined on different transversals are conjugate. The monodromy is a germ of a holomorphic mapping with respect to the natural complex structure on the local transversal (for more details on monodromy, see [2], pages 97-98).

The following lemma is the key to the proof of Theorem 1.

Lemma 4. *If system (3) has a generalized rational first integral in some neighbourhood of its singular point O which has a separatrix S, then the monodromy Δ associated to S is periodic, i.e. there exists a positive integer n such that $\Delta^{[n]} = id$, where $\Delta^{[n]}$ denotes the iteration of Δ with itself n times.*

Proof. First we claim that there exists a neighbourhood W of the loop γ and a positive integer N such that any integral curve of system (3) in W intersects with Γ at most in N points. Indeed, without loss of generality, we assume that the separatrix is the Ow axis: $\{(w,z) \in (\mathbf{C}^2, 0) |\ z = 0\}$. Let $H = f/g$ be a generalized rational first integral of system (3). Since f and g are coprime, one of the two functions $g(w,0)$ and $f(w,0)$, say $g(w,0)$, is not identical zero. We assume, without loss of generality, that $H = 0$ corresponds to the separatrix, i.e. $f(w,0) \equiv 0$. Now we fix w_0 with $0 < |w_0| \ll 1$ such that $g(w_0, 0) \neq 0$ and $f(w_0, z)$ is not identically zero. Let $\Gamma = \{(w,z) \in (\mathbf{C}^2, 0) |\ w = w_0\}$ be the local transversal. Then

$$\#\{\{H(w,z) = c\} \cap \Gamma\} = \#\{z \in (\mathbf{C}, 0) |\, H(w_0, z) - c = 0\}.$$

Let $H(w_0, z) = z^N \varphi(z),\ \varphi(0) \neq 0$. Then the equation $H(w_0, z) - c = 0$ has at most N zeros in $(\mathbf{C}, 0)$ for $|c| \ll 1$. Our claim is proved.

By our claim above, any point is a periodic point of monodromy map Δ with period $\leq N$. Let $n = N!$, then $\Delta^{[n]} = id$. ∎

Lemma 5. *If the germ of an analytic map* $\Delta : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ *is periodic, then there exist positive integers* n, m *such that* $\Delta'(0) = e^{2m\pi i/n}$. *Moreover, if* $\Delta'(0) = 1$, *then* $\Delta = id$.

Proof. Assume that $\Delta^{[n]} = id$, then $(\Delta'(0))^n = 1$, which implies the first statement. Now we assume that $\Delta'(0) = 1$. If $\Delta \neq id$, let $\Delta(z) = z + az^k + \cdots,\ a \neq 0$. Then we obtain

$$id(z) = \Delta^{[n]}(z) = z + naz^k + \cdots \neq z,$$

a contradiction. ∎

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a matrix A. We say the n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ is *resonant* if there exists a n-tuple $m = (m_1, \ldots, m_n)$, where m_j are nonnegative integers with $|m| = \sum_1^n m_j \geq 2$, such that

$$\lambda_s = (m, \lambda) := \sum_{j=1}^{n} m_j \lambda_j$$

for some $s \in \{1, \ldots, n\}$. The vector–valued monomial $x_1^{m_1} \cdots x_n^{m_n} \dfrac{\partial}{\partial x_s}$ is said to be *resonant* if $\lambda_s = (m, \lambda)$. If the convex hull of $\{\lambda_1, \ldots, \lambda_n\}$ in the complex plane does not contain the origin of $\mathbf{C}$, then $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is said to be in the *Poincaré domain*.

Theorem of Poincaré–Dulac. *If the eigenvalues of the linear part of a holomorphic vector field at a singular point belong to the Poincaré domain,* *then in the neighbourhood of the singular point, the field is analytically equivalent to a polynomial field in which the linear part matrix has the Jordan normal form and all vector–valued monomials with coefficients of degree greater than* 1 *are resonant.*

For a proof of Theorem of Poincaré–Dulac, see [12].

Proof of Theorem 1. Sufficiency.

Case (a): $\lambda_1 \neq 0 = \lambda_2$. We will prove that if the singular point is not isolated, system (3) has an analytic first integral in a neighbourhood of the origin. We rewrite system (3) in the form

$$\dot{w} = \lambda_1 w + f(w,z), \quad \dot{z} = g(w,z), \tag{6}$$

where $f, g = O(2)$. By using the Implicit Function Theorem, there exists an analytic function $w = w(z)$ with $w(0) = 0$ such that $\lambda_1 w(z) + f(w(z), z) \equiv 0$. Since the singular point is not isolated, we have $g(w(z), z) \equiv 0$. Therefore, there exist analytic functions $\varphi_1(w,z), \varphi_2(w,z)$ with $\varphi_1(0,0) = \lambda_1$ such that

$$\lambda_1 w + f(w,z) = (w - w(z))\varphi_1(w,z), \quad g(w,z) = (w - w(z))\varphi_2(w,z).$$

Now we consider the system

$$\dot{w} = \varphi_1(w,z), \quad \dot{z} = \varphi_2(w,z). \tag{7}$$

The origin is not a singular point any more of system (7), therefore system (7), and consequently system (6) has an analytic first integral in some neighbourhood of the origin.

Cases (b),(c),(d). By the Poincaré–Dulac Normal Form Theorem (for cases (b) and (c)) or the assumption of sufficiency (for case (d)), system (3) in some neighbourhood of the origin is analytically equivalent to its linear part, i.e. there exists an analytic change of variables $x = x(w,z), y = y(w,z)$ such that system (3) is transformed into

$$\dot{x} = \lambda_1 x, \quad \dot{y} = \lambda_2 y, \quad \lambda_1/\lambda_2 = m/n \in \mathbf{Q}^+. \tag{8}$$

Obviously system (8) has a rational first integral $x^n y^{-m}$. Therefore, system (3) has a generalized rational first integral $H(w,z) = x(w,z)^n y(w,z)^{-m}$.

Case (e). By the assumption of sufficiency, system (3) in some neighbourhood of the origin is analytically orbitally equivalent to its linear part, i.e. there exists an analytic change of variables $x = x(w,z)$, $y = y(w,z)$ such that system (3) is transformed into

$$\dot{x} = \lambda_1 x T(x,y), \quad \dot{y} = \lambda_2 y T(x,y), \quad \lambda_1/\lambda_2 = -m/n \in \mathbf{Q}^-, \tag{9}$$

where $T(0,0) = 1$. System (9) has a polynomial first integral $x^n y^m$, which implies system (3) has an analytic first integral $H(w,z) = x(w,z)^n y(w,z)^m$.

The proof of sufficiency of Theorem 1 is complete.

Necessity.

First we consider the degenerate case: $\lambda_1 \neq 0 = \lambda_2$. We will prove that the singular point is not isolated by contradiction. Assume that system (3) has a generalized rational first integral and the origin is an isolated degenerate elementary singular point. By normal form theory (see [14],[20]) system (3) is analytically orbitally equivalent to the system

$$\dot{w} = -w + zg(w,z), \quad \dot{z} = z^{p+1}, \tag{10}$$

where $p \geq 1$ is an integer, g is holomorphic at the origin with $g(0,0) = 0$. In this normalized chart, the separatrix S is Ow axis: $\{z = 0\}$. Now we calculate its monodromy. Fix a point $(w_0, 0) \in S$ and we choose the loop $\gamma \subset S$ as follows:

$$\gamma = \{(w,z) = (w_0 e^{it}, 0)\ t \in [0, 2\pi]\}.$$

Substituting $w = w_0 e^{it}$ into (10), we obtain

$$\frac{dz}{dt} = \frac{ie^{it}w_0 z^{p+1}}{-e^{it}w_0 + zg(w_0 e^{it}, z)} = -iz^{p+1}(1 + O(z)). \tag{11}$$

Let

$$z(t, z_0) = \sum_{j=1}^{\infty} a_j(t) z_0^j,\ a_1(0) = 1,\ a_j(0) = 0, j \geq 2.$$

Then equation (11) becomes

$$\sum_{j=1}^{\infty} a_j'(t) z_0^j = -i \left(\sum_{j=1}^{\infty} a_j(t) z_0^j \right)^{p+1} (1 + O(z_0)). \tag{12}$$

Comparing the coefficients of z_0^j in equation (12) for $j = 1, 2, \ldots$, we obtain

$$a_1(t) = 1,\ a_j(t) = 0,\ 2 \leq j \leq p,\ a_{p+1}(t) = -it.$$

So the monodromy has the form

$$\Delta(z_0) = z(2\pi, z_0) = z_0 - 2\pi i z_0^{p+1} + O(z_0^{p+2}),$$

which is not periodic by Lemma 5. Thus we have a contradiction by using Lemma 4. The proof of necessity of statement (a) is complete.

Lemma 6. *Assume $\lambda_1 \lambda_2 \neq 0$ and system (3) has a generalized rational first integral in some neighbourhood of the origin, then $\lambda_2/\lambda_1 \in \mathbf{Q}$.*

Proof. First we claim that if $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \mathbf{N}$, there exists a local chart (w, z) such that system (3) has the form

$$\dot{w} = w(\lambda_1 + g(w,z)), \quad \dot{z} = z(\lambda_2 + f(w,z)), \tag{13}$$

where f, g are analytic functions with $f(0,0) = g(0,0) = 0$. Indeed, if $\lambda_1/\lambda_2 \notin \mathbf{R}^+$, where $\mathbf{R}^+$ denotes the set of positive real numbers, by the Invariant Manifold Theorem (see [4]), there exist two separatrices of system (3) which are at the origin tangent to the eigenvectors corresponding to the eigenvalues λ_1 and λ_2, respectively. Choose a local chart (w, z) such that the separatrices are coordinate axes, then system (3) has the form (13). If $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \in \mathbf{R}^+ \setminus \mathbf{N}$, by Poincaré–Dulac Theorem, system (3) is analytically equivalent to its linear part whose Jordan normal form has the form (13) with $f = g \equiv 0$. In short, our claim is proved. We write system (13) in the form

$$\frac{dz}{dw} = \frac{z(\lambda_2 + f(w,z))}{w(\lambda_1 + g(w,z))}. \tag{14}$$

Let

$$\frac{\lambda_2 + f(w,0)}{\lambda_1 + g(w,0)} = \frac{\lambda_2}{\lambda_1} + h(w),$$

then $h(w)$ is an analytic function with $h(0) = 0$. We calculate the monodromy associated to the separatrix $\{z = 0\}$ of system (14) as follows. Substituting $w = w_0 e^{it}$, $t \in [0, 2\pi]$, into (14), we obtain

$$\frac{dz}{dt} = iz\left(\frac{\lambda_2}{\lambda_1} + h(w_0 e^{it}) + O(z)\right). \tag{15}$$

Let

$$z(t, z_0) = \sum_{j=1}^{\infty} a_j(t) z_0^j, \ a_1(0) = 1, \ a_j(0) = 0, \ j \geq 2,$$

then (15) becomes

$$\sum_{j=1}^{\infty} a_j'(t) z_0^j = i\left(\sum_{j=1}^{\infty} a_j(t) z_0^j\right)\left(\frac{\lambda_2}{\lambda_1} + h(w_0 e^{it}) + O(z_0)\right),$$

which implies

$$a_1'(t) = i a_1(t)\left(\frac{\lambda_2}{\lambda_1} + h(w_0 e^{it})\right).$$

Therefore

$$a_1(t) = \exp\left(it\frac{\lambda_2}{\lambda_1} + i\int_0^t h(w_0 e^{is})\, ds\right).$$

Since $h(0) = 0$, $\int_0^{2\pi} h(w_0 e^{is})ds = 0$. So we obtain $\Delta(z_0) = z(2\pi, z_0) = e^{2\pi i\lambda_2/\lambda_1} z_0 + \cdots$. By Lemma 4, Δ is periodic. Now it follows from Lemma 5 that $\lambda_2/\lambda_1 \in \mathbf{Q}$. ∎

It follows from Lemma 6 that if $\lambda_1\lambda_2 \neq 0$ and system (3) has a generalized rational first integral, then either $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \in \mathbf{Q}^+ \setminus \mathbf{N}$, which proves the necessity of statement (b), or $\lambda_1/\lambda_2 = n, 1/n, n \in \mathbf{N}$, or $\lambda_1/\lambda_2 \in \mathbf{Q}^-$.

Assume now λ_2/λ_1 or $\lambda_1/\lambda_2 = n \in \mathbf{N}$. By the Poincaré–Dulac Normal Form Theorem, system (3) is analytically equivalent to its normal form

$$\dot{w} = n\lambda w + az^n, \quad \dot{z} = \lambda z. \tag{16}$$

We claim that if system (16) has a generalized rational first integral, then $a = 0$. By Lemma 4 it is enough to prove that the monodromy is not periodic if $a \neq 0$. Assume that $a \neq 0$. We calculate the monodromy associated to the separatrix $\{z = 0\}$ of system (16) as before. We write system (16) into the form

$$\frac{dz}{dw} = \frac{\lambda z}{n\lambda w + az^n}. \tag{17}$$

Substituting $w = w_0 e^{it}$, $t \in [0, 2\pi]$, into (17), we obtain

$$\frac{dz}{dt} = \frac{iz}{n}\left(1 - \frac{ae^{-it}}{n\lambda} z^n + O(z^{2n})\right). \tag{18}$$

Let

$$z(t, z_0) = \sum_{j=1}^{\infty} a_j(t) z_0^j, \ a_1(0) = 1, \ a_j(0) = 0, \ j \geq 2.$$

Then (18) implies

$$a_1'(t) = \frac{i}{n} a_1(t), \qquad a_j'(t) = 0, \ 2 \leq j \leq n,$$
$$a_{n+1}'(t) = \frac{i}{n} a_{n+1}(t) - \frac{ae^{-it}}{n^2\lambda} a_1(t)^{n+1}.$$

Therefore,

$$a_1(t) = e^{it/n}, \ a_j(t) = 0, \ 2 \leq j \leq n, \ a_{n+1}(t) = -\frac{at}{n^2\lambda} e^{it/n},$$

which implies the monodromy has the form

$$\Delta(z_0) = z(2\pi, z_0) = e^{2\pi i/n} z_0 - \frac{2a\pi}{n^2\lambda} e^{2\pi i/n} z_0^{n+1} + \cdots.$$

Calculating straightforward we have

$$\Delta^{[n]}(z_0) = z_0 - \frac{2a\pi}{n\lambda} z_0^{n+1} + \cdots,$$

which is not an identical mapping if $a \neq 0$. This, by Lemma 5, implies that Δ is not periodic, a contradiction.

The conclusion $a = 0$ implies that system (3) is analytically equivalent to its linear part. In particular, for the case $n = 1$, i.e. $\lambda_1 = \lambda_2$, the matrix of the linear part is diagonal. The proof of the necessity of statements (c) and (d) is complete.

Next we prove the necessity for the case $\lambda_1/\lambda_2 = -m/n \in \mathbf{Q}^-$. According to normal form theory (see [5],[6],[20]), a germ of the analytic vector field at the singular point with ratio of eigenvalues equal to $-m/n$ is analytically orbitally equivalent to its linear part or to the system

$$\dot{w} = w, \quad \dot{z} = z\left(-\frac{m}{n} + u^k(1+\alpha u^k)^{-1} + z^N w^N f_N(w,z)\right) \tag{19}$$

for some natural number k and any given $N \in \mathbf{N}$, where $u = z^n w^m$ and f_N is a function holomorphic at the origin. By Lemma 4, in order to prove the necessity of statement (e), it is enough to prove the monodromy associated to the separatrix $\{z = 0\}$ of system (19) is not periodic.

We write system (19) into the form

$$\frac{dz}{dw} = \frac{z}{w}\left(-\frac{m}{n} + u^k(1+\alpha u^k)^{-1} + z^N w^N f_N(w,z)\right), \tag{20}$$

taking $N > k(m+n)$. Let $w = w_0 e^{it}$, then system (20) becomes

$$\frac{dz}{dt} = iz\left(-\frac{m}{n} + w_0^{km} e^{ikmt} z^{kn} + o(z^{kn})\right). \tag{21}$$

Let

$$z(t, z_0) = \sum_{j=1}^{\infty} a_j(t) z_0^j, \ a_1(0) = 1, \ a_j(0) = 0, \ j \geq 2.$$

Then, system (21) implies

$$a_j'(t) = -\frac{m}{n} i a_j(t), \ 1 \leq j \leq kn,$$
$$a_{kn+1}'(t) = -\frac{m}{n} i a_{kn+1}(t) + i a_1(t)^{kn+1} w_0^{km} e^{ikmt},$$

from which we obtain

$$a_1(t) = e^{-mit/n}, \ a_j(t) = 0, \ 2 \leq j \leq kn, \ a_{kn+1}(t) = i w_0^{km} t e^{-mit/n}.$$

Therefore, the monodromy takes the form

$$\Delta(z_0) = z(2\pi, z_0) = e^{-2m\pi i/n}[z_0 + 2\pi i w_0^{km} z_0^{kn+1} + o(z_0^{kn+1})].$$

Since

$$\Delta^{[n]}(z_0) = z_0 + 2n\pi i w_0^{km} z_0^{kn+1} + o(z_0^{kn+1})$$

is not the identity, the monodromy Δ is not periodic by Lemma 5.

The proof of Theorem 1 is complete.

3. The proof of Theorem 2

First we can assume, without loss of generality, that the origin is a nondegenerate weak singular point and the matrix of linear part of the quadratic system is diagonal, i.e. the quadratic system under consideration has the form

$$\frac{du}{dt} = \lambda u + a_1 u^2 + a_2 uv + a_3 v^2, \quad \frac{dv}{dt} = -\lambda v + b_1 u^2 + b_2 uv + b_3 v^2. \quad (22)$$

After the time rescaling $t \to \frac{i}{\lambda} t$ and the linear change of variables $u = x + iy, v = x - iy$, system (22) has the form

$$\dot{x} = -y + c_1 x^2 + c_2 xy + c_3 y^2, \quad \dot{y} = x + d_1 x^2 + d_2 xy + d_3 y^2. \quad (23)$$

Now we take a rotation of axes:

$$x \to x\cos\theta + y\sin\theta, \quad y \to -x\sin\theta + y\cos\theta, \quad (24)$$

where $\theta \in \mathbf{C}$. Such θ can be found in order that after the rotation (24) the sum of the coefficients of x^2 and y^2 in the second equation of system (23) is zero, i.e. we obtain system (4). Furthermore, if $b + d = 0$ in system (4), as Frommer observed in [16] (see also [32]), the form of system (4) with $b + d = 0$ is preserved under the rotation (24), and by a suitable rotation the coefficients of x^2 and y^2 of the second equation in system (4) can be changed to zero.

Now we assume that system (4) has a rational first integral. By Theorem 1 (case (e)), system (4) has an analytic first integral in a neighbourhood of the origin. By Kapteyn-Bautin Theorem (see [22],[3],[8]), one of the following conditions holds.

(1) $A - 2b = C + 2a = 0$,

(2) $C = a = 0$,

(3) $b + d = 0$,

(4) $C+2a=A+3b+5d=a^2+bd+2d^2=0$.

Assume that system (4) satisfies condition (1). Then it is easy to check that the system is Hamiltonian, i.e. $\dot{x}=-\partial H/\partial y$, $\dot{y}=\partial H/\partial x$ with $H=\frac{1}{2}(x^2+y^2)+\frac{a}{3}x^3+bx^2y-axy^2+\frac{d}{3}y^3$. Therefore H is a polynomial first integral of system (4).
Suppose that system (4) satisfies condition (2). We can assume that $A-2b\neq 0$, otherwise we would be under the assumptions of condition (1). If $A=0$, system (4) has a first integral:

$$H=e^{2by}\left(x^2+\frac{d}{2b}y^2+\frac{1}{2b^2}(b-d)y+\frac{1}{4b^3}(d-b)\right).$$

Consequently, $H=c$ are not algebraic curves. Therefore system (4) can not have any rational first integrals. Hence, $A\neq 0$.

If $A\neq 0$ and $b=0$, system (4) has a first integral:

$$H=\frac{x^2}{2}+\frac{d}{2A}y^2+\frac{1}{A^3}(A-d)[Ay-\ln(1+Ay)].$$

Therefore, system (4) has a rational first integral if and only if $A=d$.

If $b/A=-1/2$, the first integral is

$$H=(1+Ay)^{-1}\left[\frac{x^2}{2}+\frac{d}{A^3}(1+Ay)^2+\right.$$
$$\left.+\frac{1}{A^3}(A-2d)(1+Ay)\ln(1+Ay)+\frac{1}{A^3}(A-d)\right].$$

$H=c$ are algebraic curves if and only if $A=2d$.

If $b/A=-1$, the first integral is

$$H=(1+Ay)^{-2}\left[\frac{x^2}{2}+dA^{-3}(1+Ay)^2\ln(1+Ay)+\right.$$
$$\left.+\frac{(2d-A)}{A^3}(1+Ay)+A^{-3}(A-d)/2\right].$$

$H=c$ are algebraic curves if and only if $d=0$.

If $b/A\neq 0,-\frac{1}{2},-1$, system (4) has a first integral

$$H=(1+Ay)^{2b/A}\left[\frac{x^2}{2}+\frac{d}{2A^2(b+A)}(1+Ay)^2+\right.$$
$$\left.+\frac{A-2d}{A^2(2b+A)}(1+Ay)+\frac{d-A}{2bA^2}\right].$$

Obviously, the curves $H=c$ are algebraic if and only if $b/A \in \mathbf{Q}$.
Assume that system (4) satisfies condition (3). First we recall that $a=0$ when $b+d=0$ and furthermore we can assume that $C \neq 0$, otherwise we would be under the assumptions of condition (2).

If $b=0$ and $A=0$, system (4) has a first integral

$$H=\frac{y^2}{2}+\frac{x}{C}-\frac{1}{C^2}\ln(1+Cx).$$

If $b=0$ and $A\neq 0$, the first integral is

$$H=A^{-1}y-A^{-2}\ln(1+Ay)+C^{-1}x-C^{-2}\ln(1+Cx).$$

If $A=0$, $b\neq 0$ and $\Delta=\frac{C^2}{4}+b(b+A)=0$, the first integral is

$$H=\frac{4}{C^2}\ln\left(1-by+\frac{C}{2}x\right)-\frac{2x}{C}\left(1-by+\frac{C}{2}x\right)^{-1}-\frac{y}{b}.$$

If $A=0$ and $b\Delta\neq 0$, the first integral is

$$\begin{aligned} H &= \frac{1}{2\sqrt{\Delta}}\left[\left(\frac{C}{2}-\sqrt{\Delta}\right)^{-1}\ln(2-2by+Cx-2\sqrt{\Delta}x)\right. \\ &\quad \left. -\left(\frac{C}{2}+\sqrt{\Delta}\right)^{-1}\ln(2-2by+Cx+2\sqrt{\Delta}x)\right]-\frac{y}{b}. \end{aligned}$$

If $bA\neq 0$ and $\Delta=0$, the first integral is

$$H=\frac{4}{C^2}\ln\left(1-by+\frac{C}{2}x\right)-\frac{4}{C^2}(by-1)\left(1-by+\frac{C}{2}x\right)^{-1}+\frac{4b}{AC^2}\ln(1+Ay).$$

The above discussion show that if $bA\Delta=0$, system (4) has no rational first integrals. If $bA\Delta\neq 0$, system (4) has a first integral

$$H=(1+Ay)\left(1-by+\frac{C}{2}x+\sqrt{\Delta}x\right)^{\lambda_-}\left(1-by+\frac{C}{2}x-\sqrt{\Delta}x\right)^{\lambda_+},$$

where $\lambda_{\pm}=\frac{A}{2b\sqrt{\Delta}}(\sqrt{\Delta}\pm\frac{C}{2})$. Therefore, system (4) has a rational first integral if and only if $\lambda_\pm\in\mathbf{Q}$, or equivalently, if and only if $A/b, C/\sqrt{\Delta}\in\mathbf{Q}$.
Suppose that system (4) satisfies condition (4). We can assume that $ad(b+d)\neq 0$, otherwise we would be under the assumptions of condition (2) or (3). Then system (4) becomes

$$\dot{x}=-y+\frac{a^2+2d^2}{d}x^2+2axy-dy^2,\quad \dot{y}=x+ax^2+\frac{3a^2+d^2}{d}xy-ay^2. \tag{25}$$

System (25) has a rational first integral

$$H=\frac{\left(\frac{(dy-ax)^3}{3a^2d^2}+\frac{y(dy-ax)}{a^2d}+\frac{1-3ad^4}{3da^2+3d^3}y+\frac{1}{3a^4+3a^2d^2}\right)^2}{\left[(a^2+d^2)(dy-ax)^2+2d(a^2+d^2)y+d^2\right]^3}.$$

The proof of Theorem 2 is complete.

4. Proof of Theorem 3

First we can assume, without loss of generality, that the origin is a degenerate elementary singular point and the matrix of linear part of the quadratic system is diagonal, i.e. the quadratic system under consideration has the form

$$\frac{dx}{dt}=a_{20}x^2+a_{11}xy+a_{02}y^2,\quad \frac{dy}{dt}=\lambda y+b_{20}x^2+b_{11}xy+b_{02}y^2. \tag{26}$$

After the time rescaling $t\to\frac{t}{\lambda}$, system (26) has the form (5). From statement (a) of Theorem 1, if system (5) has a rational first integral, then the singular point $(0,0)$ is not isolated. This implies that either $a_1=b_1=c_1=0$, or the polynomials $P=: a_1x^2+b_1xy+c_1y^2$ and $Q=: y+a_2x^2+b_2xy+c_2y^2$ have a common linear factor of the form $ax+by, |a|+|b|\neq 0$. If $a_1=b_1=c_1=0$, system (5) has a rational first integral $H=x$. Assume now that $|a_1|+|b_1|+|c_1|\neq 0$. It is easy to see that $ax+by\,|\,Q$ if and only if $a=a_2=0$, which means that the common factor of P and Q is y. Hence, $a_1=0$. After dividing by the common factor y, system (5) becomes

$$\dot{x}=b_1x+c_1y,\quad \dot{y}=1+b_2x+c_2y. \tag{27}$$

Obviously, system (27) and system (5) have the same first integral. Calculating straightforward we have that system (27) has a rational first integral if and only if one of statements (1), (2) and (3) in Theorem 3 holds.

The proof of Theorem 3 is complete.

Acknowledgement

Weigu Li and Xiang Zhang want to thank to the CRM and to the Department of Mathematics of the Universitat Autònoma de Barcelona for their support and hospitality during the period in which this paper was written. They are partially supported by NSFC of China and Weigu Li is supported by the 973 Project of the Ministry of Science and Technology of China. Jaume Llibre is partially supported by DGES grant number PB96–1153 and by CIRIT grant number 1999SER–00349.

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