Extension of Floquet's Theory to Nonlinear Periodic Differential Systems and Embedding Diffeomorphisms in Differential Flows

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Abstract

This paper has two parts. In the first one we generalize the Floquet theory to nonlinear periodic differential systems. In the second part we apply the first one to obtain a sufficient condition in order that a germ of diffeomorphism can be embedded in a differential flow.

1. Introduction and statement of the main results

Consider the differential system with a periodic right hand side in a neighbourhood of a constant solution, i.e. equations of the form

$$\frac{dx}{dt} = f(x,t), \quad \text{where } f \in C^\infty,\ f(0,t) \equiv 0, \text{ and } f(x,t+\omega) \equiv f(x,t), \tag{1}$$

and x belongs to a neighbourhood of $\mathbf{0}$ in the space $\mathbf{R}^n$ and t is real. Floquet's theory (see, for example, [1]) allows us to change variables so

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that the linear part of equation (1) with respect to x when $x = \mathbf{0}$ becomes autonomous. Therefore, we can assume, without loss of generality, that system (1) has the following form

$$\dot{x} = Ax + v(x,t),\ x \in (\mathbf{R}^n, 0),\ t \in \mathbf{R}, \tag{2}$$

where A is a real square matrix of order n, v is a C^∞ vector valued function with $v(x, t+1) \equiv v(x,t)$, $v(x,t) = O(\|x\|^2)$.

A n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ of eigenvalues of A is said to be *weakly nonresonant* for the 1-periodic equation (2) if for all $m = (m_1, \ldots, m_n) \in \mathbf{Z}_+^n$ with $\sum_{i=1}^n m_i \geq 2$, we have that

$$\sum_{i=1}^n m_i\lambda_i - \lambda_j \neq 2k\pi\sqrt{-1}, \quad k \in \mathbf{Z} \setminus \{0\}, \quad 1 \leq j \leq n. \tag{3}$$

The n-tuple λ is said to be *nonresonant* if for all $m = (m_1, \ldots, m_n) \in \mathbf{Z}_+^n$ with $\sum_{i=1}^n m_i \geq 2$, we have that

$$\sum_{i=1}^n m_i\lambda_i - \lambda_j \neq 2k\pi\sqrt{-1}, \quad k \in \mathbf{Z}, \quad 1 \leq j \leq n. \tag{4}$$

We remark that if all eigenvalues of A are real, then they are weakly nonresonant. The n-tuple λ is said to be *hyperbolic* if $\operatorname{Re}\lambda_i \neq 0$, for $1 \leq i \leq n$. Two periodic differential systems with period 1 are said to be C^k *equivalent* if there exists a C^k time dependent diffeomorphism which is 1-periodic in t and transforms one system into the other.

Theorem 1. *If the eigenvalues of A are weakly nonresonant and hyperbolic, then system (2) in a neighbourhood of $\mathbf{0}$ in $\mathbf{R}^n$ is C^∞ equivalent to an autonomous differential system. Moreover, if the eigenvalues are nonresonant, the reduced autonomous system is the linear part of system (2).*

The second part of this paper is concerned the problem of embedding vector fields or flows for diffeomorphisms.

Let f be a diffeomorphism on a smooth manifold M. A smooth flow of M, $\{f^t\}$ ($t \in \mathbf{R}$), is said to be an *embedding flow* of f if $f^1 = f$. The corresponding vector field, $v(x) = \frac{\partial}{\partial t} f^t(x)\big|_{t=0}$, is called an *embedding vector field* of f. We also say that f can be embedded in the vector field $v(x)$. A diffeomorphism which admits an embedding flow or vector field is called *embeddable*. The embedding flow problem appears in a natural way when we try to find the relation between flows with discrete time and flows with continuous time. There are many results on the embedding problem of

1-dimensional diffeomorphisms, see [2],[7],[8],[9],[11],[12]. For higher dimensional case, Palis in [10] pointed out that diffeomorphisms which admit embedding flows with some smoothness are "few" in the Baire sense.

If v is an embedding vector field for a diffeomorphism f of M, then v satisfies the following functional equation (see [2],[11]):

$$v(f^t(x)) = D(f^t(x))v(x), \ x \in M, \, t \in \mathbf{R}.$$

Taking $t = 1$, we know that v satisfies the following *embedding equation*

$$v(f(x)) = Df(x)v(x), \, x \in M. \tag{5}$$

In dimension 1 the embedding vector field v for a given diffeomorphism f can be obtained by solving equation (5), see [11] and [12]. It seems to us that the embedding equation is the most powerful tool for the embedding problem in the 1-dimensional case. For diffeomorphisms in dimension≥ 2, the embedding equation is difficult to solve. Our new approach in this paper for the embedding problem is based on the following fact. Let f, g be two smooth conjugate diffeomorphisms of M, i.e. there exists a diffeomorphism H of M such that $H \circ g = f \circ H$, if g admits an embedding vector field v, then f admits the embedding vector field $H_* v$, where H_* denotes the tangent mapping of H. In other words, if for a given diffeomorphism f there exists an embeddable diffeomorphism g which is smooth conjugate to f, then f is also embeddable. By using this idea, we obtain a sufficient condition in order that a germ of diffeomorphisms at a fixed point be embeddable, see Theorem 4 below.

We begin with the linear case. Let $\mathbf{F}$ be the real field $\mathbf{R}$ or the complex field $\mathbf{C}$, and $M_n(\mathbf{F})$ the set of square matrices of order n with elements of $\mathbf{F}$. Choose a basis in the linear space $\mathbf{F}^n$, then each linear isomorphism of $\mathbf{F}^n$ corresponds to a nonsingular matrix $A \in M_n(\mathbf{F})$. We say that a *linear isomorphism A of $\mathbf{F}^n$ can be embedded into a linear vector field*

$$\dot{x} = Bx, \quad B \in M_n(\mathbf{F}), \, x \in \mathbf{F}^n, \tag{6}$$

if A is equal to the time one map of system (6). Since system (6) has the solutions $x = e^{Bt}x_0$, the linear embedding problem is equivalent to the following. For a given matrix $A \in M_n(\mathbf{F})$, we look for a matrix $B \in M_n(\mathbf{F})$ such that

$$e^B = A, \quad \text{where } e^B := \sum_{k=0}^{\infty} \frac{B^k}{k\,!}. \tag{7}$$

For $A \in M_n(\mathbf{F})$, we call the matrix $B \in M_n(\mathbf{F})$ satisfying (7) the *logarithm* of A, and denote by $\ln A$. We remark that as the logarithm function, the logarithm of a matrix in the complex field $\mathbf{C}$ is not single valued. In fact, for any integer k, $\ln A + 2k\pi\sqrt{-1}E_n$ is a logarithm of A, where E_n is the unit matrix of order n. The real matrix B satisfying (7) is said to be a *real logarithm* of A.

Theorem 2. *The following statements hold.*

(1) Let $A \in M_n(\mathbf{C})$. Then A has a logarithm in $M_n(\mathbf{C})$ if and only if A is nonsingular.

(2) Assume that $A \in M_n(\mathbf{R})$ is nonsingular. Then the following hold.

  (2a) A has a real logarithm if and only if A has no negative real eigenvalues or the Jordan blocks in Jordan Normal Form (JNF, for short) of A corresponding to the negative real eigenvalues appear pairwise, i.e. there are an even number of such blocks: $J_1, \dots, J_{2m}$ with $J_{2i-1} = J_{2i}$, for $i = 1, \dots, m$.

  (2b) A has a unique real logarithm if and only if all eigenvalues of A are positive and the Jordan blocks in JNF of A are pairwise different.

As a corollary of Theorem 2, we have the following known result.

Corollary 3. *Let $A \in M_n(\mathbf{R})$ be nonsingular, then A^2 has a real logarithm.*

Now we turn to the embedding problem for the germs of diffeomorphisms. Consider a germ of diffeomorphism

$$f: x \mapsto Ax + O(\|x\|^2), \ x \in (\mathbf{R}^n, \mathbf{0}). \tag{8}$$

Obviously, if germ (8) admits a smooth vector field

$$\dot{x} = Bx + O(\|x\|^2), \ x \in (\mathbf{R}^n, \mathbf{0}), \tag{9}$$

then $A = e^B$, i.e. the linear part of the germ of the diffeomorphism admits a linear embedding vector field.

Theorem 4. *Let f given in (8) be a germ of C^∞ diffeomorphism. If all eigenvalues of A are not on the unit circle and A has a real logarithm B whose eigenvalues are weakly nonresonant, then the germ f can be embedded in a germ of C^∞ vector field with linear part matrix B.*

As a corollary of Theorem 4, we have the following result.

Corollary 5. *If all eigenvalues of A are real, positive and not equal to* 1*, then the germ of a C^∞ diffeomorphism (8) can be embedded in a germ of C^∞ vector field.*

This paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2 and Corollary 3. Finally, Theorem 4 and Corollary 5 are proved in Section 4.

2. Extension of Floquet's theory to nonlinear periodic differential systems

In this section we prove Theorem 1.

Denote by $\mathcal{F}(\mathbf{F})$ the set of n-dimensional vector valued formal power series of n variables whose coefficients are smooth real ($\mathbf{F}$=$\mathbf{R}$) or complex ($\mathbf{F}$=$\mathbf{C}$) periodic functions of $t \in \mathbf{R}$ of period 1. We write for a formal power series $Y(y,t) \in \mathcal{F}(\mathbf{F})$ that $Y = O(\|y\|^2)$, if Y has no the terms of degree 0 and 1. We say that two formal periodic systems

$$\dot{y} = Ay + Y(y,t) \tag{10}$$

and

$$\dot{z} = Bz + Z(z,t), \tag{11}$$

where $A, B \in M_n(\mathbf{F})$, $Y = O(\|y\|^2), Z = O(\|z\|^2) \in \mathcal{F}(\mathbf{F})$, are $\mathbf{F}$ *formally equivalent* if there exists a formal change of variables $y = Cz + h(z,t)$, $C \in M_n(\mathbf{F}), \det C \neq 0$, $h(z,t) = O(\|z\|^2) \in \mathcal{F}(\mathbf{F})$, which transforms one system into the other.

Lemma 6. *Assume that the eigenvalues of A are weakly nonresonant, then system (10) is $\mathbf{F}$ formally equivalent to a formal autonomous system. Moreover, if the eigenvalues of A are nonresonant, system (10) is $\mathbf{F}$ formally equivalent to its linear part.*

First we recall the next lemma due to Bibikov [3] which plays a key role in the proof of Lemma 6.

Lemma 7. *Denote by $H_n^k(\mathbf{F})$ the linear space of n-dimensional vector valued homogenous polynomials of n variables of degree k with real (if $\mathbf{F} = \mathbf{R}$) or complex (if $\mathbf{F} = \mathbf{C}$) coefficients. Let $A \in M_n(\mathbf{F})$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Define a linear operator L on $H_n^k(\mathbf{F})$ as follows,*

$$Lh = \frac{\partial h}{\partial x} Ax - Ah, \tag{12}$$

for $h(x) \in H_n^k(\mathbf{F})$. Then the set of eigenvalues of L is

$$\left\{ \sum_{i=1}^{n} m_i \lambda_i - \lambda_j \, : \, m_i \in \mathbf{N} \cup \{0\}, \sum_{i=1}^{n} m_i = k, \, 1 \leq j \leq n \right\}. \tag{13}$$

Proof of Lemma 6. Assume that the change of variables $y = z + h(z,t)$, $h = O(\|z\|^2) \in \mathcal{F}(\mathbf{F})$ transforms system (10) into the system $\dot{z} = Az + Z(z,t)$. Then

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial z} Az - Ah = Y(z+h,t) - \frac{\partial h}{\partial z} Z - Z. \tag{14}$$

We will prove Lemma 6 by showing first, if the eigenvalues of A are nonresonant, then (14) has a solution $h = O(\|z\|^2) \in \mathcal{F}(\mathbf{F})$ for $Z = \mathbf{0}$; and second, if the eigenvalues of A are weakly nonresonant, then (14) has a solution $h = O(\|z\|^2) \in \mathcal{F}(\mathbf{F})$ for some $Z = O(\|z\|^2) \in \mathcal{F}(\mathbf{F})$ whose coefficients are independent on t. Let

$$h(z,t) = \sum_{k=2}^{\infty} h_k(z,t), \quad Z(z,t) = \sum_{k=2}^{\infty} Z_k(z,t), \quad Y(y,t) = \sum_{k=2}^{\infty} Y_k(y,t), \tag{15}$$

where h_k, Z_k and Y_k are n-dimensional vector valued homogeneous polynomials in the variables z, z, y respectively of degree k whose coefficients are smooth 1-periodic functions of t. We note that h_k, Z_k and Y_k can also be considered as smooth maps from $\mathbf{R}/\mathbf{Z}$ to $H_n^k(\mathbf{F})$ and denoted by $h_k(t), Z_k(t)$ and $Y_k(t)$, respectively. Substituting (15) into (14), we solve (14) for h_k inductively by comparing the terms of degree k for $k = 2, 3, \ldots$. We set $h_1 = Z_1 = \mathbf{0}$ and assume that we have already determined the terms of degree $\leq k-1$ for some $k \geq 2$. By comparing the terms of degree k with respect to z in (14), we obtain

$$\frac{dh_k(t)}{dt} + Lh_k(t) = F_k(t) - Z_k(t), \tag{16}$$

where L is the linear operator on $H_n^k(\mathbf{F})$ defined in (12), $F_k(t)$ is the coefficient vector of the term of degree k of the expression

$$Y\left(z + \sum_{m=1}^{k-1} h_m(z,t), t\right) - \left(\sum_{m=1}^{k-1} \frac{\partial h_m(z,t)}{\partial z}\right)\left(\sum_{m=1}^{k-1} Z_m(z,t)\right),$$

which is 1-periodic and known already by induction assumption. System (16) has the solution

$$h_k(t) = e^{-Lt} \int_0^t e^{Ls}(F_k(s) - Z_k(s))ds + e^{-Lt} h_k(0). \tag{17}$$

If $Z_k(t)$ is 1-periodic, then the function $h_k(t)$ has periodic 1 if and only if $h_k(0) = h_k(1)$, or equivalently

$$(e^L - E)h_k(0) = \int_0^1 e^{Ls}(F_k(s) - Z_k(s))\, ds, \tag{18}$$

where E is the identical operator of $H_n^k(\mathbf{F})$. From Lemma 7, L has the set of eigenvalues (13). So, if the eigenvalues of A are nonresonant, $e^L - E$ is invertible. Therefore, system (18) has a unique solution for $Z_k = \mathbf{0}$:

$$h_k(0) = (e^L - E)^{-1} \int_0^1 e^{Ls} F_k(s)\, ds.$$

If the eigenvalues of A are weakly nonresonant, we claim that (18) has a solution $h_k(0)$ if $Z_k(s) = C_k$ for some suitable $C_k \in H_n^k(\mathbf{F})$. Indeed, let $\int_0^1 e^{Ls} ds = B$, then $e^L - E = LB = BL$. Now (18) can be written as

$$B(Lh_k(0) + C_k) = \int_0^1 e^{Ls} F_k(s)\, ds. \tag{19}$$

Let $\{\mu_i\}$ be the set of eigenvalues of L, then the set $\{\tilde{\mu}_i\}$ given by

$$\tilde{\mu}_i = \int_0^1 e^{\mu_i s} ds = \begin{cases} 1 & \text{if } \mu_i = 0, \\ \frac{e^{\mu_i} - 1}{\mu_i} & \text{if } \mu_i \neq 0, \end{cases}$$

is the set of eigenvalues of B. Hence, if the eigenvalues of A are weakly nonresonant, B is invertible. Denote by LH_n^k the image of $H_n^k(\mathbf{F})$ under L and R_n^k its complementary subspace: $H_n^k(\mathbf{F}) = LH_n^k \oplus R_n^k$. Let

$$B^{-1} \int_0^1 e^{Ls} F_k(s) ds = B_1 + B_2,$$

with $B_1 \in LH_n^k$, $B_2 \in R_n^k$. Taking $C_k = B_2$, then (19) is equivalent to

$$Lh_k(0) = B_1. \tag{20}$$

Since $B_1 \in LH_n^k$, there exists $h_k(0) \in H_n^k(\mathbf{F})$ satisfying (20). ∎

Let $x(t, x_0)$ be the solution of (2) with initial condition $x(0, x_0) = x_0$. Then the *Poincaré map* of (2) is defined as $P : x_0 \mapsto x(1, x_0)$.

Lemma 8. *Two real $C^k, 1 \le k \le \omega$, periodic differential systems of period 1 are C^k equivalent if and only if their Poincaré maps are C^k conjugate.*

Proof. Necessity. Denote by $x(t, x_0)$, $y(t, y_0)$ the solutions of these two systems under consideration with initial condition $x(0, x_0) = x_0$, $y(0, y_0) = y_0$, respectively. By the assumption of necessity, there exists a C^k time dependent diffeomorphism $y = H_t(x) (\equiv H_{t+1}(x))$ which carries the solutions $x(t, x_0)$ into the solutions $y(t, y_0)$, i.e. $y(t, H_0(x_0)) = H_t(x(t, x_0))$. Let

$\tilde{P}(y_0) := y(1, y_0)$ and $P(x_0) := x(1, x_0)$ be their Poincaré maps respectively, then

$$\tilde{P} \circ H_0(x_0) = y(1, H_0(x_0)) = H_1(x(1, x_0)) = H_0(x(1, x_0)) = H_0 \circ P(x_0).$$

Sufficiency. Denote by X_t and Y_t the flow maps of these two systems under consideration respectively:

$$X_t : x_0 \mapsto x(t, x_0), \quad Y_t : y_0 \mapsto y(t, y_0).$$

Then $X_{t+1} = X_t \circ P$, $Y_{t+1} = Y_t \circ \tilde{P}$. By the assumption of sufficiency, there exists a C^k diffeomorphism H such that $\tilde{P} \circ H = H \circ P$. Let $H_t = Y_t \circ H \circ (X_t)^{-1}$. Then H_t is a time dependent C^k diffeomorphism with $H_0 = H$. Obviously, H_t carries the solutions $x(t, x_0)$ into the solution $y(t, H(x_0))$ and

$$H_{t+1} = Y_{t+1} \circ H \circ (X_{t+1})^{-1} = Y_t \circ \tilde{P} \circ H \circ P^{-1} \circ (X_t)^{-1} = Y_t \circ H \circ (X_t)^{-1} = H_t.$$

∎

Lemma 9. *Let $a_m(t)$, $m = (m_1, m_2, \ldots, m_n) \in \mathbf{Z}_+^n$, be a sequence of C^∞ periodic functions of period 1, then there exists a C^∞ function $f(x, t)$ defined on $\mathbf{R}^n \times \mathbf{R}$ such that $f(x, t+1) \equiv f(x, t)$ and*

$$\left. \frac{\partial^m f(x, t)}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}} \right|_{x=\mathbf{0}} = a_m(t), \quad \textit{for all } m \in \mathbf{Z}_+^n. \tag{21}$$

Proof. Let $\varphi(r) : \mathbf{R}^+ \to [0, 1]$ be a C^∞ function such that

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}; \\ 0 & \text{if } r \ge 1. \end{cases}$$

For $x \in \mathbf{R}^n, m = (m_1, \ldots, m_n) \in Z_+^n$ we introduce the notations:

$$m! := m_1! \, m_2! \cdots m_n!, \; x^m := x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}, \; \|x\| := \sqrt{\sum_{i=1}^n x_i^2}.$$

Let $M_m := \|a_m(t)\|_{C^k} + m!$, where $k = \sum_{i=1}^n m_i$. We define some functions as follows:

$$\varphi_m(x, t) := \frac{a_m(t)}{m!} x^m \varphi(M_m \|x\|), \quad f(x, t) := \sum_{m \in \mathbf{Z}_+^n} \varphi_m(x, t). \tag{22}$$

Note that when $\|x\| \ge \frac{1}{M_m}$, $\varphi_m = 0$. Hence, series (22) and its derivatives of any order are uniformly convergent. This implies $f \in C^\infty$. Since $\varphi(M_m \|x\|) = 1$ for $\|x\| < 1/(2M_m)$, (21) holds. ∎

Remark. If functions $a_m(t)$ are constant, i.e. they are independent on t, then $f(x,t) = f(x)$ is defined in $\mathbf{R}^n$.

A germ of diffeomorphism

$$x \mapsto g(x) = Ax + \cdots, \ x \in (\mathbf{R}^n, \mathbf{0})$$

is called *hyperbolic*, if no eigenvalues of A are on unit circle.

Lemma 10. ([4],[5],[6]) *If two hyperbolic germs of C^∞ diffeomorphisms are formally conjugate, then they are C^∞ conjugate.*

Proof of Theorem 1. From Lemma 6, if the eigenvalues of A are weakly nonresonant or nonresonant, there exists a real formal time dependent variable change $x = z + \tilde{h}(z,t)$ which transforms system (2) to a real formal autonomous system $\dot{z} = Az + \tilde{f}(z)$ or to its linear part $\dot{z} = Az$, respectively. From Lemma 9, there exists a C^∞ vector valued function $h(z,t)$ such that

$$jet^{\infty}_{z=\mathbf{0}} h(z,t) = \tilde{h}(z,t,), \quad h(z,t+1) \equiv h(z,t), \tag{23}$$

where $jet^{\infty}_{z=\mathbf{0}} h(z,t)$ denotes the Taylor series of h at $z = \mathbf{0}$. Assume that under the variable change $x = z + h(z,t)$, system (2) is changed to

$$\dot{z} = Az + F(z,t), \tag{24}$$

then $jet^{\infty}_{z=\mathbf{0}} F(z,t) = \tilde{f}(z)$ ($\mathbf{0}$, respectively) which is time independent. Again by using Lemma 9, there exists a C^∞ vector valued function $f(z)$ such that $jet^{\infty}_{z=\mathbf{0}} f(z) = \tilde{f}(z)$ (if $\tilde{f} = \mathbf{0}$, we set $f = \mathbf{0}$). Let $r(z,t) = F(z,t) - f(z)$, then r is ∞-flat at $z = \mathbf{0}$ and system (24) can be written as

$$\dot{z} = Az + f(z) + r(z,t). \tag{25}$$

Now consider the C^∞ autonomous system

$$\dot{z} = Az + f(z). \tag{26}$$

Denote by $P(z)$ the Poincaré map of (25) and $\tilde{P}(z)$ the time one map of (26), then

$$jet^{\infty}_{z=\mathbf{0}} P(z) = jet^{\infty}_{z=\mathbf{0}} \tilde{P}(z). \tag{27}$$

Since A has no eigenvalues with zero real part by the assumption of Theorem 1, P and $\tilde{P}$ are hyperbolic, which, together with (27) and Lemma 10, implies they are C^∞ conjugate. Now Theorem 1 follows from Lemma 8. ∎

3. Linear isomorphisms which can be embedded in linear flows

In this section we prove Theorem 2 and Corollary 3.
Proof of statement (1) of Theorem 2. Necessity. From (7) it follows that

$$\det A = \det e^B = e^{\mathrm{tr} B} \neq 0.$$

Sufficiency. By JNF theory, there exists a nonsingular matrix $T \in M_n(\mathbf{C})$ such that

$$A = TJT^{-1}, \quad J = \mathrm{diag}(J_1, J_2, \ldots, J_m),$$

where

$$J_s = \lambda_s E_{n_s} + Z_{n_s}, \ \lambda_s \neq 0, \ Z_{n_s} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}_{n_s \times n_s}, \qquad 1 \leq s \leq m.$$

By using Taylor formula

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \tag{28}$$

we obtain a logarithm of J_s as follows:

$$\ln J_s = (\ln \lambda_s) E_{n_s} + \sum_{k=1}^{\infty} \frac{1}{k} (\lambda_s^{-1} Z_{n_s})^k. \tag{29}$$

Note that $Z_{n_s}^k = \mathbf{0}$ for $k > n_s$, series (29) is convergent. Let

$$B = T \,\mathrm{diag}(\ln J_1, \ln J_2, \ldots, \ln J_s)\, T^{-1},$$

then

$$e^B = T \,\mathrm{diag}(J_1, J_2, \ldots, J_s)\, T^{-1} = A.$$

The proof of statement (1) of Theorem 2 is complete. ∎

Proof of statement (2a) of Theorem 2. Sufficiency. By JNF theory, there exists a matrix $T \in M_n(\mathbf{R})$ such that

$$A = TJT^{-1}, \quad J = \mathrm{diag}(A_1, \ldots, A_r, B_1, \ldots, B_s, C_1, \ldots, C_s),$$

where

$$
\begin{aligned}
A_m &= \begin{pmatrix} \lambda_m & & & \\ 1 & \lambda_m & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_m \end{pmatrix}_{n_m \times n_m}, \quad \lambda_m > 0, \quad 1 \le m \le r; \\
B_j &= \begin{pmatrix} \mu_j & & & \\ 1 & \mu_j & & \\ & \ddots & \ddots & \\ & & 1 & \mu_j \end{pmatrix}_{p_j \times p_j}, \quad \mu_j < 0, \quad 1 \le j \le s; \qquad (30) \\
C_k &= \begin{pmatrix} D_k & & & \\ E_2 & D_k & & \\ & \ddots & \ddots & \\ & & E_2 & D_k \end{pmatrix}_{2q_k \times 2q_k}, \quad 1 \le k \le t; \\
D_k &= \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha_k, \beta_k \in \mathbf{R}, \ \beta_k \neq 0.
\end{aligned}
$$

Equation (7) can be written as $e^{T^{-1}BT} = J$. Hence, it is enough to prove that the JNF J of A has a real logarithm. First, by using Taylor formula (29), we have

Lemma 11. *The matrices A_m given in (30) have real logarithms with real eigenvalues* $\ln \lambda_m$.

Lemma 12. *The matrices C_k given in (30) have infinitely many real logarithms.*

Proof. Assume that

$$D_k = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix} = e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}} = e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix},$$

then we obtain

$$a = \frac{1}{2} \ln(\alpha_k^2 + \beta_k^2), \quad b = \arccos\left(\frac{\alpha_k}{\sqrt{\alpha_k^2 + \beta_k^2}} \right) + 2l\pi, \ l \in \mathbf{Z},$$

which implies that D_k has infinitely many real logarithms. Note that

$$C_k = \Lambda_k + Z_k, \ \Lambda_k = \text{diag}(D_k, \ldots, D_k), \quad Z_k = \begin{pmatrix} \mathbf{0} & & & \\ E_2 & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & E_2 & \mathbf{0} \end{pmatrix}.$$

We get infinitely many real logarithms of C_k by using the Taylor formula as follows:

$$\ln C_k = \mathrm{diag}(\ln D_k, \ldots, \ln D_k) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (\Lambda_k^{-1} Z_k)^i.$$

∎

Lemma 13. *Let*

$$\tilde{B} = \begin{pmatrix} \mu & & & \\ 1 & \mu & & \\ & \ddots & \ddots & \\ & & 1 & \mu \end{pmatrix}_{l \times l}, \ \mu \in \mathbf{R} \setminus \{0\},$$

then the matrix $M = \mathrm{diag}(\tilde{B}, \tilde{B})$ *has infinitely many real logarithms.*

Proof. It is easy to see that the matrix M is similar to the matrix

$$\tilde{M} = \Lambda + Z, \text{ where } \Lambda = \begin{pmatrix} \mu E_2 & & & \\ & \mu E_2 & & \\ & & \ddots & \\ & & & \mu E_2 \end{pmatrix}_{2l \times 2l},$$

$$Z = \begin{pmatrix} \mathbf{0} & & & \\ E_2 & \mathbf{0} & & \\ & \ddots & \ddots & \\ & & E_2 & \mathbf{0} \end{pmatrix}_{2l \times 2l}.$$

Note that μE_2 has infinitely many real logarithms:

$$\ln(\mu E_2) = \begin{cases} \begin{pmatrix} \ln \mu & 2k\pi \\ -2k\pi & \ln \mu \end{pmatrix}, & \text{if } \mu > 0; \\ \begin{pmatrix} \ln |\mu| & (2k+1)\pi \\ -(2k+1)\pi & \ln |\mu| \end{pmatrix}, & \text{if } \mu < 0; \end{cases}$$

where $k \in \mathbf{Z}$. So the formula $\ln \Lambda = \mathrm{diag}(\ln(\mu E_2), \ldots, \ln(\mu E_2))$ gives infinitely many real logarithms of Λ. By using Taylor formula (28), we have

$$\ln \tilde{M} = \ln \Lambda + \ln(E_{2l \times 2l} + \Lambda^{-1} Z) = \ln \Lambda + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (\Lambda^{-1} Z)^k,$$

which implies that the matrix $\tilde{M}$, and consequently M has infinitely many real logarithms. ∎

Now the sufficiency of statement (2a) follows easily from Lemma 11, Lemma 12 and Lemma 13.

Necessity. By the assumption of necessity, there exists a matrix $B \in M_n(\mathbf{R})$ such that $A = e^B$. Assume that the JNF of B has the following form:

$$\mathrm{diag}(A_1, \ldots, A_r, B_1, \ldots, B_s, C_1, \ldots, C_t),$$

where A_m, B_j and C_k are the real Jordan blocks given in (30) which correspond to positive, negative and complex eigenvalues respectively. Obviously, $A = e^B$ is similar to the matrix

$$\mathrm{diag}\left(e^{A_1}, \ldots, e^{A_r}, e^{B_1}, \ldots, e^{B_s}, e^{C_1}, \ldots, e^{C_t}\right).$$

Since e^{A_m} and e^{B_j} have positive eigenvalues only, the blocks in the JNF of A corresponding to the negative eigenvalues (if exist) must come from e^{C_k} for some $1 \le k \le t$. Let $P_k = \begin{pmatrix} \cos\beta_k & \sin\beta_k \\ -\sin\beta_k & \cos\beta_k \end{pmatrix}$, then we have

$$e^{C_k} = e^{\alpha_k} \begin{pmatrix} P_k & & \\ & \ddots & \\ & & P_k \end{pmatrix}_{2q_k \times 2q_k} \times$$
$$\times \begin{pmatrix} E_2 & & & & \\ E_2 & E_2 & & & \\ \frac{E_2}{2!} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{E_2}{(q_k-1)!} & \cdots & \frac{E_2}{2!} & E_2 & E_2 \end{pmatrix}_{2q_k \times 2q_k}.$$

Since the eigenvalues of e^{C_k} are negative, $\sin\beta_k = 0, \cos\beta_k = -1$. Therefore,

$$e^{C_k} = -e^{\alpha_k} \begin{pmatrix} E_2 & & & & \\ E_2 & E_2 & & & \\ \frac{E_2}{2!} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{E_2}{(q_k-1)!} & \cdots & \frac{E_2}{2!} & E_2 & E_2 \end{pmatrix}_{2q_k \times 2q_k},$$

whose JNF is the matrix $\mathrm{diag}(\tilde{A}, \tilde{A})$, where

$$\tilde{A} = \begin{pmatrix} -e^{\alpha_k} & & & \\ 1 & -e^{\alpha_k} & & \\ & \ddots & \ddots & \\ & & 1 & -e^{\alpha_k} \end{pmatrix}_{q_k \times q_k}.$$

So the blocks in the JNF of A corresponding to negative eigenvalues appear pairwise. Hence, the proof of statement (2a) of Theorem 2 is complete. ∎

Proof of statement (2b) of Theorem 2. Necessity. First, we claim that if A has a unique real logarithm, then A has only positive eigenvalues. Indeed, from Lemma 12, all eigenvalues of A are real. If A has negative eigenvalues, then by statement (2a), the Jordan blocks in JNF of A corresponding to negative eigenvalues appear pairwise, which is a contradiction with the uniqueness of the logarithm of A by using Lemma 13. Therefore, our claim is proved. Now the necessity follows from Lemma 13.

Sufficiency.

Lemma 14. *Let*

$$\Lambda = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}_{m\times m},$$

then the JNF of e^{Λ} *is*

$$\begin{pmatrix} e^{\lambda} & & & \\ 1 & e^{\lambda} & & \\ & \ddots & \ddots & \\ & & 1 & e^{\lambda} \end{pmatrix}_{m\times m}. \tag{31}$$

Proof. Straightforward calculating, we have

$$e^{\Lambda} = e^{\lambda} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ \frac{1}{2!} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{1}{(m-1)!} & \cdots & \frac{1}{2!} & 1 & 1 \end{pmatrix}_{m\times m} = e^{\lambda} E_m + N,$$

where E_m is the identical matrix of order m and

$$N = e^{\lambda} \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ \frac{1}{2!} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{1}{(m-1)!} & \cdots & \frac{1}{2!} & 1 & 0 \end{pmatrix}_{m\times m}.$$

Since $N^m = \mathbf{0}, N^{m-1} \neq \mathbf{0}$, N is similar to the matrix

$$\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}_{m\times m},$$

which implies $e^{\Lambda} = e^{\lambda}E_m + N$ is similar to the matrix (31). ∎

Assume that

$$J = \mathrm{diag}(J_1, \ldots, J_s),\ J_m = \begin{pmatrix} \lambda_m & & & \\ 1 & \lambda_m & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_m \end{pmatrix}_{n_m\times n_m}, \tag{32}$$

$$\lambda_m > 0,\ 1 \leq m \leq s,$$

is the JNF of A, where J_m, $m = 1, 2, \ldots, s$, are mutually different.

Lemma 15. *The matrix J given in (32) has a unique real logarithm.*

Proof. Assume that there exist $B, C \in M_n(\mathbf{R})$ such that

$$e^B = e^C = J. \tag{33}$$

First, we point out that all eigenvalues of B and C must be real, otherwise, J has two repeated Jordan blocks. So $\ln\lambda_m \in \mathbf{R}$, $m = 1, 2, \ldots, s$, are the eigenvalues of B and C. Let

$$\tilde{J} := \mathrm{diag}(\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_s), \quad \tilde{J}_m = \begin{pmatrix} \ln\lambda_m & & & \\ 1 & \ln\lambda_m & & \\ & \ddots & \ddots & \\ & & 1 & \ln\lambda_m \end{pmatrix}_{n_m\times n_m}.$$

From Lemma 14, $\tilde{J}$ is the JNF of B and C, i.e., there exist nonsingular matrices $T_1, T_2 \in M_n(\mathbf{R})$ such that

$$B = T_1\tilde{J}T_1^{-1}, \quad C = T_2\tilde{J}T_2^{-1}.$$

Substituting them into (33), we obtain that $L := T_2^{-1}T_1$ is commutative with $e^{\tilde{J}}$, i.e.

$$Le^{\tilde{J}} = e^{\tilde{J}}L. \tag{34}$$

Let $M \in M_n(\mathbf{R})$, we denote by $C(M)$ the centralizer of M :

$$C(M) := \{T \in M_n(\mathbf{R}) \,|\, TM = MT\}.$$

Then $C(M)$ is a linear subspace of $M_n(\mathbf{R})$. We claim that

$$C(J) \subset C(\tilde{J}) \subset C(e^{\tilde{J}}). \tag{35}$$

The claim $C(\tilde{J}) \subset C(e^{\tilde{J}})$ is obvious. Next we prove $C(J) \subset C(\tilde{J})$. Let $T \in C(J)$, $T = (t_{ij})_{1\le i,\, j\le s}$, where t_{ij} has the same partition than the matrix J. By multiplication rule of matrices we have

$$TJ = (t_{ij}J_j)_{1\le i,\, j\le s}, \quad JT = (J_i t_{ij})_{1\le i,\, j\le s}.$$

Since $T \in C(J)$,

$$t_{ij}J_j = J_i t_{ij}. \tag{36}$$

Let

$$J_i = \lambda_i E_{n_i} + N_i, \quad N_i = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}_{n_i \times n_i}$$

Then (36) becomes

$$\lambda_j t_{ij} + t_{ij}N_j = \lambda_i t_{ij} + N_i t_{ij}. \tag{37}$$

We claim that

$$(\ln \lambda_j)t_{ij} + t_{ij}N_j = (\ln \lambda_i)t_{ij} + N_i t_{ij}. \tag{38}$$

Indeed, if $\lambda_i = \lambda_j$, from (37), $t_{ij}N_j = N_i t_{ij}$, which implies (38). If $\lambda_i \ne \lambda_j$, from (37), $t_{ij} = \mathbf{0}$. Therefore, (38) also holds. From (38), we get $T\tilde{J} = \tilde{J}T$, i.e. $T \in C(\tilde{J})$. Our claim (35) is proved. From Lemma 14, there exists a nonsingular matrix $H \in M_n(\mathbf{R})$ such that $e^{\tilde{J}} = H^{-1}JH$. For any $T \in C(e^{\tilde{J}})$, i.e. $Te^{\tilde{J}} = e^{\tilde{J}}T$, we have $TH^{-1}JH = H^{-1}JHT$, which implies

$$HTH^{-1} \subset C(J). \tag{39}$$

Now we define an invertible linear operator of $M_n(\mathbf{R})$ as follows:

$$H_* : T \mapsto HTH^{-1}.$$

Then by (39), $H_* C(e^{\tilde{J}}) \subset C(J)$. Thus, from (35),

$$\dim C(e^{\tilde{J}}) = \dim(H_* C(e^{\tilde{J}})) \le \dim C(J) \le \dim C(\tilde{J}) \le \dim C(e^{\tilde{J}}).$$

Therefore,

$$\dim C(J) = \dim C(\tilde{J}) = \dim C(e^{\tilde{J}}),$$

which, together with (35), implies

$$C(J) = C(\tilde{J}) = C(e^{\tilde{J}}). \tag{40}$$

From (34) and (40), $L = T_2^{-1}T_1 \in C(\tilde{J})$. Therefore,

$$B = T_1\tilde{J}T_1^{-1} = T_2\tilde{J}T_2^{-1} = C.$$

∎

Now we prove A has a unique real logarithm. Assume that B and C are two real logarithms of A, i.e. $e^B = e^C = A$. Let $A = TJT^{-1}$, where J given in (32) is the JNF of A and $T \in M_n(\mathbf{R})$. From $e^B = e^C = A$, we obtain $e^{T^{-1}BT} = e^{T^{-1}CT} = J$. By Lemma 15, $T^{-1}BT = T^{-1}CT$, which implies $B = C$.

The proof of Theorem 2 is complete. ∎

Proof of Corollary 3. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, then A^2 has the eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$. Therefore, the blocks in the JNF of A^2 corresponding to negative eigenvalues (if exist) come from the blocks in the JNF of A corresponding to pure imaginary eigenvalues. Since A is a real matrix, all blocks in the JNF corresponding to pure imaginary eigenvalues appear pairwise. In other words, the blocks

$$J = \begin{pmatrix} \beta i & & & \\ 1 & \beta i & & \\ & \ddots & \ddots & \\ & & 1 & \beta i \end{pmatrix}$$

and

$$\tilde{J} = \begin{pmatrix} -\beta i & & & \\ 1 & -\beta i & & \\ & \ddots & \ddots & \\ & & 1 & -\beta i \end{pmatrix}, \quad \beta \in \mathbf{R} \setminus \{0\}$$

appear pairwise. Note that J^2 and $\tilde{J}^2$ are similar to the matrix

$$\begin{pmatrix} -\beta^2 & & & \\ 1 & -\beta^2 & & \\ & \ddots & \ddots & \\ & & 1 & -\beta^2 \end{pmatrix}.$$

Therefore, the blocks in the JNF of A^2 corresponding to negative eigenvalues (if exist) appear pairwise. Now Corollary 3 follows from statement (2a) of Theorem 2. ∎

4. Diffeomorphisms which can be embedded in differential flows

The following result is a key to the proof of Theorem 4 which is mentioned in [1] (page 200).

Lemma 16. *For $B \in M_n(\mathbf{R})$, let $A = e^B$. If*

$$f : x \mapsto Ax + O(\|x\|^2),\ x \in (\mathbf{R}^n, \mathbf{0})$$

is a C^∞ diffeomorphism, then there exists a C^∞ periodic differential system

$$\dot{x} = Bx + r(x,t),\ x \in (\mathbf{R}^n, \mathbf{0}), \tag{41}$$

where $r(x,t) = O(\|x\|^2)$, $r(x,t+1) \equiv r(x,t)$, such that f is the Poincaré map of system (41) in some neighbourhood of $x = \mathbf{0}$.

Proof. Fix $0 < \delta < \frac{1}{4}$ small enough such that

$$\|e^{Bt} - E_n\| < \frac{1}{2},\ t \in [-2\delta, 2\delta].$$

Let $\varphi(t) : [-\frac{1}{2}, \frac{1}{2}] \to [0,1]$ be such a C^∞ function that

$$\varphi(t) = \begin{cases} 1 & \text{if} \quad t \in [-\delta, \delta]; \\ 0 & \text{if} \quad 2\delta < |t| \le \frac{1}{2}. \end{cases}$$

Let $T(t) = \{t + \frac{1}{2}\} - \frac{1}{2}$, where $\{\cdot\}$ denotes the decimal part. Then $T(t)$ is a periodic function of period 1 with $T(t) = t$ for $|t| < \frac{1}{2}$. Let $\psi(t) = \varphi(T(t))$, then $\psi(t)$ is a C^∞ periodic function of period 1 with $\psi(t) = 1$ when $|t| < \delta$. Define a C^∞ map $G : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ as follows:

$$G(x,t) = (g(x,t), t),\ g(x,t) := (1 - \psi(t))x + \psi(t) e^{BT(t)} x.$$

Then $g(x, t+1) \equiv g(x,t)$, $g(x,t) = e^{Bt}x$ for $|t| < \delta$, and

$$\begin{aligned} \|D_x g - E_n\| &= \|(1 - \psi(t))E_n + \psi(t)e^{BT(t)} - E_n\| \\ &\le \psi(t)\|E_n - e^{BT(t)}\| \\ &\le \sup_{|t| \le 2\delta} \|E_n - e^{Bt}\| < \frac{1}{2}. \end{aligned}$$

This implies G is a C^∞ diffeomorphism and takes line segments $\{x = \text{const.}, t \in [-\delta, \delta]\}$ to the solution of the system

$$\dot{x} = Bx, \;\; \dot{t} = 1. \tag{42}$$

Consider the image of the vector field (42) under the action of the tangent mappings of G^{-1} in $\mathbf{R}^n \times \mathbf{R}$:

$$W(x,t) = (G^{-1})_*(Bx, 1) = (w(x,t), 1).$$

Then

$$w(x,t) \equiv w(x, t+1), \text{ and } w(x,t) \equiv \mathbf{0} \text{ for } |t| \leq \delta. \tag{43}$$

Since $G|_{t=0}$ is the identity, the Poincaré map of the system

$$\dot{x} = w(x,t) \tag{44}$$

is the same as the time one map of the linear vector field $\dot{x} = Bx$, i.e. Ax. From (43), the flow map of system (44) from $t = \delta$ to $t = 1$ is also Ax. Let

$$A^{-1} f(x) = x + h(x), \, h(x) = O(\|x\|^2),$$

and $\alpha(t) : \mathbf{R} \to [0,1]$ be a C^∞ periodic function of period 1 such that

$$\alpha(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \frac{\delta}{3}]; \\ 1 & \text{if} \quad t \in [\frac{2\delta}{3}, \delta]. \end{cases}$$

Consider a family of smooth curves on $(\mathbf{R}^n, \mathbf{0}) \times [0, \delta]$ as follows:

$$\gamma_x(t) : \{(y,t) |\, y = x + \alpha(t) h(x), \, t \in [0, \delta]\}, \;\; x \in (\mathbf{R}^n, \mathbf{0}).$$

Then γ_x connects the points $(x, 0)$ and $(A^{-1} f(x), \delta)$. Obviously, $\gamma_x \cap \gamma_y = \emptyset$ for $x \neq y$ and

$$F_t : \, x \mapsto x + \alpha(t) h(x)$$

is a smooth diffeomorphism. For a point $(y,t) \in \gamma_x(t)$, the tangent vector of γ_x at (y,t) is $(\alpha'(t) h(x), 1)$. Now define a C^∞ vector field $(Q(x,t), 1)$ in $(\mathbf{R}^n, \mathbf{0}) \times \mathbf{R}$ as follows:

$$Q(x,t) = \begin{cases} w(x,t) & \text{if} \quad t \notin [0, \delta] \, (\text{mod } \mathbf{Z}); \\ \alpha'(t) h(F_t^{-1}(x)) & \text{if} \quad t \in [0, \delta] \, (\text{mod } \mathbf{Z}). \end{cases} \tag{45}$$

Then $Q(x, t+1) \equiv Q(x,t)$, $Q(\mathbf{0}, t) \equiv \mathbf{0}$. Moreover, $\gamma_{x_0}(t)|_{t \in [0,\delta]}$ is the segment of the solution passing through the point x_0 when $t = 0$ of the system

$$\dot{x} = Q(x,t). \tag{46}$$

This implies that the Poincaré map of system (46) is $A(A^{-1}f(x)) = f(x)$. Now consider the vector field $(v(x,t),1) := G_*(Q(x,t),1)$. Since $G|_{t=0}$ is the identity, the Poincaré map of system (46) and the Poincaré map of the system

$$\dot{x} = v(x,t) \tag{47}$$

are the same, i.e. $f(x)$. Finally, we prove that

$$\left.\frac{\partial v}{\partial x}\right|_{x=\mathbf{0}} = B. \tag{48}$$

We claim that

$$Q(\mathbf{0},t) = w(\mathbf{0},t) = \mathbf{0}, \quad \left.\frac{\partial Q}{\partial x}\right|_{x=\mathbf{0}} = \left.\frac{\partial w}{\partial x}\right|_{x=\mathbf{0}}. \tag{49}$$

The first equality in (49) is obvious by the definition of the functions Q and w. Next we prove the second one. Indeed, by (45), $Q \equiv w$ for $t \notin [0,\delta]\,(\mathrm{mod}\ \mathbf{Z})$. So it is enough to show (49) holds for $t \in [0,\delta]$. It follows easily from (43) and (45). From (49), $(Bx,1) = G_*(w,1)$ and $(v(x,t),1) = G_*(Q(x,t),1)$ must have the same linear part at $x = \mathbf{0}$. ∎

Proof of Theorem 4. From Lemma 16, there exists a periodic differential system (41) such that the diffeomorphism f given in (8) is the Poincaré map of (41). Since all eigenvalues of A do not lie on the unit circle, no eigenvalues of B has zero real part, which, together with the assumption of Theorem 4 that the eigenvalues of B are weakly nonresonant, implies by Theorem 1 that system (41) is C^∞ equivalent to an autonomous system

$$\dot{x} = v(x),\ x \in (\mathbf{R}^n, \mathbf{0}). \tag{50}$$

From Lemma 8, f is C^∞ conjugate to the time one map of autonomous system (50), which implies that f admits a C^∞ embedding vector field in some neighbourhood of $\mathbf{0}$ in $\mathbf{R}^n$. ∎

Proof of Corollary 5. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of A. By assumption of Corollary 5, $\lambda_i \in (0,+\infty) \setminus \{1\}$. By Lemma 11, A has a real logarithm B with eigenvalues $\ln \lambda_i \in \mathbf{R} \setminus \{0\}$. Since $\ln \lambda_i$, $i = 1, \ldots, n$, are real, they are weakly nonresonant. Now Corollary 5 follows from Theorem 4. ∎

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