BOUNDING THE ORDERS OF FINITE SUBGROUPS

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Abstract. We give homological conditions on groups such that whenever the conditions hold for a group G, there is a bound on the orders of finite subgroups of G. This extends a result of P. H. Kropholler. We also suggest other weaker conditions under which the same conclusion should hold.

1. Introduction

Let R be a non-trivial unital ring. An R-module M is said to be of type FP_n if there is a projective resolution

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0$$

of M over R in which $P_0, \ldots, P_n$ are finitely generated. M is said to be of type FP_∞ if M is FP_n for each n. Similarly, M is said to be of type FP (resp. FL) over R if there is a resolution of M of finite length in which each term is a finitely generated projective (resp. free) module. For any discrete group G and commutative ring R, the augmentation homomorphism $RG \to R$ gives R the structure of a module for the group algebra RG. The group G is said to be FP_n (resp. FP_∞, FP, FL) over R if the RG-module R is FP_n (resp. FP_∞, FP, FL) in the above sense. The cohomological dimension of G over R, denoted by $\mathrm{cd}_R(G)$, is the projective dimension of R as an RG-module. For further information concerning these definitions, see [2] or Chapter VIII of [3]. As usual, let $\mathbb{Q}$ and $\mathbb{Z}$ denote the rational numbers and the integers respectively. We prove the following.

Proposition 1. *Let G be a group with $\mathrm{cd}_{\mathbb{Q}}(G) = n < \infty$ and suppose that G is of type FP_{n+1} over $\mathbb{Z}$. Then there is a bound on the orders of finite subgroups of G.*

A similar result was proved by P. H. Kropholler in section 5 of [6], under the extra hypothesis that G should be FP_∞ over $\mathbb{Z}$. His proof made use of

Date: November 9, 2000.
1991 *Mathematics Subject Classification.* Primary 20J05; Secondary 19J05.
Key words and phrases. finiteness conditions, finite subgroups.
The author acknowledges support from SFB 478 Münster and the CRM Barcelona.

the complete cohomology introduced by D. Benson, J. Carlson, G. Mislin and F. Vogel [1, 8]. In contrast, our proof uses Tate cohomology only for finite subgroups of G. (Complete cohomology can be viewed as a generalization of Tate cohomology.)

The author believes that the same result should hold if the finiteness condition on G is weakened to 'type FP_n over $\mathbb{Z}$', but has been unable to prove this. The conclusion does not hold for all groups of type FP_{n-1} over $\mathbb{Z}$. K. S. Brown has shown [4] that for each $n > 0$, the Houghton groups [5] afford an example of a group $G = G(n)$ such that:

(a) G contains the infinite, finitary symmetric group;
(b) $\mathrm{cd}_{\mathbb{Q}}(G) = n$;
(c) G is FP_{n-1} over $\mathbb{Z}$.

The author and B. E. A. Nucinkis have recently constructed groups G of type FP_∞ over $\mathbb{Z}$ with $\mathrm{cd}_{\mathbb{Q}}G$ finite that contain infinitely many conjugacy classes of finite subgroups [7], and it was these examples that led to the author's interest in Proposition 1. It is not known whether there is a bound on the orders of finite subgroups for every G of type FP over $\mathbb{Q}$. Some remarks concerning this question will be made at the end of the paper.

2. Proofs

Before starting, we recall some properties of finiteness conditions. All of the assertions contained in this paragraph are proved in sections VIII.4–VIII.6 of [3]. Suppose that M is an R-module of type FP_n, and that

$$P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is a partial projective resolution of M in which each P_i is finitely generated. Then K, defined as the kernel of the map from P_{n-1} to P_{n-2}, is finitely generated. There is such a partial resolution in which each P_i is finitely generated and free. If also M has projective dimension n, then M is FP. If M has projective dimension n and the P_i are finitely generated free modules, then M is FL if and only if K is stably free. The following lemma is well-known, but doesn't seem to appear in [3], so we briefly sketch a proof.

Lemma 2. *Let C denote an infinite cyclic group. For any R, if G is a group of type FP over R, then $G \times C$ is of type FL over R.*

Proof. There is a free resolution Q_* of R over RC of length one, with $Q_1 \cong Q_0 \cong RC$. Now suppose that

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

is a projective resolution of R over RG in which each P_i is finitely generated, and P_i is free for $i < n$. Let P' be such that $P_n \oplus P'$ is a finitely-generated free RG-module. Writing $\otimes$ for tensor products over R, the total complex

T_* for the double complex $P_* \otimes Q_*$ is a projective resolution of $R \otimes R = R$ over $RG \otimes RC \cong R(G \times C)$, of length $n+1$. Each T_i is finitely generated and T_i is free for $i < n$. Let S_* be the exact chain complex consisting of one copy of $P' \otimes RC$ in degree $n+1$ and one copy in degree n, with the identity map as the boundary. Then $S_* \oplus T_*$ is a finite free resolution of R over $R(G \times C)$. $\square$

In our proof of Proposition 1 we shall also use the following.

Lemma 3. *Let $F_n \to \cdots \to F_0$ be a finite-length chain complex of free $\mathbb{Z}G$-modules, suppose that $H_0(F_*)$ is isomorphic to the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, and that for each $j > 0$, there exists an integer $m_j > 0$ such that multiplication by m_j annihilates $H_j(F_*)$. Then any finite subgroup of G has order dividing $\prod_{j=1}^{n} m_j$.*

Proof. Let H be a finite subgroup of G, and let P_* be a complete resolution for H, i.e., a $\mathbb{Z}$-graded exact sequence of projective $\mathbb{Z}H$-modules which agrees with a projective resolution of $\mathbb{Z}$ in positive degrees. For any H-module M, the Tate cohomology $\widehat{H}^*(H;M)$ may be defined to be the cohomology of the cochain complex $\mathrm{Hom}_H(P_*, M)$ (see chapter VI of [3]). Now let $E_*^{*,*}$ and $E_*'^{*,*}$ be the two spectral sequences arising from the double complex

$$E_0^{i,j} = \mathrm{Hom}_H(P_i, F_j),$$

where E'_* (resp. E_*) is the spectral sequence for which the differential d_0 is induced by the boundary map of P_* (resp. F_*). For each i and j one sees that

$$E_1'^{i,j} \cong \widehat{H}^i(H; F_j) = \{0\},$$

since Tate cohomology with free coefficients vanishes. It follows that both spectral sequences must converge to zero. On the other hand,

$$E_1^{i,j} \cong \mathrm{Hom}_H(P_i, H_j(F_*)), \qquad \text{and} \qquad E_2^{i,j} \cong \widehat{H}^i(H; H_j(F_j)).$$

In particular, $E_2^{0,0} \cong \widehat{H}^0(H;\mathbb{Z})$ is a cyclic group of order $|H|$, and for each i and each $j > 0$, m_j annihilates $E_2^{i,j}$. By induction, for each r with $3 \leq r \leq n+2$, $E_r^{0,0}$ is a cyclic group such that $|H|$ divides $m_1 \ldots m_{r-2} . |E_r^{0,0}|$. But $E_\infty^{0,0} = E_{n+2}^{0,0} = \{0\}$, and hence the claim. $\square$

We shall use Lemma 3 only in the case where F_* has only two non-zero homology groups (including H_0). A proof of this special case without using spectral sequences could be given.

To prove Proposition 1, we reduce to a slightly stronger statement for groups of type FL over $\mathbb{Q}$.

Proposition 4. *Let G be of type FL over $\mathbb{Q}$, with $\mathrm{cd}_{\mathbb{Q}}(G) = n < \infty$ and suppose that G is FP_n over $\mathbb{Z}$. Then there is a bound on the order of finite subgroups of G.*

Proof of Proposition 1. The hypotheses imply that G is of type FP over $\mathbb{Q}$, and FP_{n+1} over $\mathbb{Z}$, where $n = \mathrm{cd}_{\mathbb{Q}}(G)$. Let C denote an infinite cyclic group, and let $G' = G \times C$. Then G' is FL over $\mathbb{Q}$, G' is FP_{n+1} over $\mathbb{Z}$, $\mathrm{cd}_{\mathbb{Q}}(G') = n + 1$, and G' contains a subgroup isomorphic to G. The result follows from Proposition 4. □

Proof of Proposition 4. Let F_* be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ in which the first n terms are finitely generated, and write $\mathbb{Q} \otimes F_*$ for $\mathbb{Q}G \otimes_{\mathbb{Z}G} F_*$, a free resolution for $\mathbb{Q}$ over $\mathbb{Q}G$ in which the first n terms are finitely generated. Now define K by $K = \ker(F_{n-1} \to F_{n-2})$. Thus K is a finitely-generated $\mathbb{Z}G$-module such that $\mathbb{Q} \otimes K$ is a stably free $\mathbb{Q}G$-module. Without loss of generality we may assume that $\mathbb{Q} \otimes K$ is free, since if not we may add to F_* a chain complex of the form

$$0 \to \mathbb{Z}G^{\oplus r} \to \mathbb{Z}G^{\oplus r} \to 0 \to \cdots ,$$

with the non-zero terms in degrees n and $n-1$, and the identity map between them. This has the effect of replacing K by $K \oplus \mathbb{Z}G^{\oplus r}$.

Let L be a free $\mathbb{Z}G$-module with a map $\phi : L \to \mathbb{Q} \otimes K$ inducing an isomorphism from $\mathbb{Q} \otimes L$ to $\mathbb{Q} \otimes K$. Since L is finitely generated, we may assume that the image of ϕ lies inside K. Now the sequence

$$L \to F_{n-1} \to \cdots \to F_1 \to F_0$$

is a chain complex of free $\mathbb{Z}G$-modules with $H_0 \cong \mathbb{Z}$, $H_{n-1} \cong K/L$, and $H_i = \{0\}$ for all other values of i. Now K/L is a finitely generated $\mathbb{Z}G$-module such that $\mathbb{Q} \otimes K/L = \{0\}$, and so K/L consists of torsion of bounded exponent. Thus Lemma 3 applies, and we deduce that the order of any finite subgroup of G is bounded by the exponent of K/L. □

Finally, we consider the problem of bounding the orders of finite subgroups of an arbitrary group of type FP over $\mathbb{Q}$. Such a G is finitely generated, and by Lemma 2, we may assume without loss of generality that G is FL over $\mathbb{Q}$. Let P_0 be a free $\mathbb{Q}G$-module of rank one with generator v, and let P_1 be $\mathbb{Q}G$-free on a set $e_1, \ldots e_m$ bijective with a set $g_1, \ldots, g_m$ of generators for G. Define a map from P_0 to $\mathbb{Q}$ by $v \mapsto 1$ and a map from P_1 to P_0 by $e_i \mapsto (1 - g_i)$. Finally, let

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Q} \to 0$$

be a finite free resolution of $\mathbb{Q}$ over $\mathbb{Q}G$ extending this partial resolution.

Now let F_0 (resp. F_1) be the $\mathbb{Z}G$-submodule of P_0 (resp. P_1) generated by v (resp. $e_1, \ldots, e_m$). For $i \geq 2$, if F_{i-1} has already been chosen, let F_i be a

$\mathbb{Z}G$-lattice in P_i (i.e., a $\mathbb{Z}G$-free $\mathbb{Z}G$-submodule such that $\mathbb{Q}\otimes F_i = P_i$), such that the image of F_i in P_{i-1} is contained in F_{i-1}. This defines a finite chain complex F_* of finitely-generated free $\mathbb{Z}G$-modules such that $H_0(F_*) \cong \mathbb{Z}$ and $H_i(F_*)$ is torsion for $i > 0$. If one could bound the exponent of the torsion in $H_i(F_*)$, Lemma 3 could be applied to bound the orders of finite subgroups of G. Note that in general $H_i(F_*)$ will not be finitely generated as $\mathbb{Z}G$-module. For example, if G is not FP_2 over $\mathbb{Z}$, then $H_1(F_*)$ will not be finitely generated.

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