Melnikov Functions for Period Annulus, Nondegenerate Centers, Heteroclinic and Homoclinic Cycles*

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Abstract

We give sufficient conditions in terms of the Melnikov functions in order that an analytic or a polynomial differential system in the real plane has a period annulus.

We study the first nonzero Melnikov function of the analytic differential systems in the real plane obtained by perturbing a Hamiltonian system having either a nondegenerate center, a heteroclinic cycle, a homoclinic cycle, or three cycles obtained connecting the four separatrices of a saddle. All the singular points of these cycles are hyperbolic saddles.

Finally, using the first nonzero Melnikov function we study the number of limit cycles that can bifurcate either from a nondegenerate center, or from a homoclinic cycle of a hyperbolic saddle of a Hamiltonian system when we perturb it inside the class of analytic differential systems.

1. Introduction and statement of the main results

We consider the planar vector fields $\mathcal{X}_\epsilon$ associated to the system:

$$\begin{aligned} \dot{x} &= X(x,y,\lambda,\epsilon) = p(x,y) + \epsilon P(x,y,\lambda,\epsilon), \\ \dot{y} &= Y(x,y,\lambda,\epsilon) = q(x,y) + \epsilon Q(x,y,\lambda,\epsilon), \end{aligned} \tag{1}$$

where X, Y depend analytically on their variables and parameters $\lambda \in \Lambda$, and $\epsilon \in \mathbf{R}$, $\Lambda \subset \mathbf{R}^r$ is an open region. Assume that for $\epsilon = 0$, system (1) has a period annulus; i.e., a continuous family of periodic orbits. As usual,

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the dot denotes derivative with respect to the time variable t. We say that system (1) with $\epsilon = 0$ is the *unperturbed* system, while system (1) with $\epsilon \neq 0$ is the *perturbed* one.

Given any compact subset $\mathbf{D}$ of Λ and $\epsilon_0 > 0$ small, we assume that there is a transversal section J to the vector fields $\mathcal{X}_\epsilon$ in the region covered by the period annulus for $|\epsilon| < \epsilon_0$ and $\lambda \in \mathbf{D}$. Let u be an analytical parameterization of J. Then there is a subsection $\Sigma \subset J$ such that the Poincaré return map $(u, \lambda, \epsilon) \mapsto \Pi(u, \lambda, \epsilon)$ is defined from $\Sigma \times \mathbf{D} \times (-\epsilon_0, \epsilon_0)$ to J. The *displacement function* $d(u, \lambda, \epsilon)$ is defined as $d(u, \lambda, \epsilon) = \Pi(u, \lambda, \epsilon) - u$. Since system (1) has a period annulus for $\epsilon = 0$, we have $d(u, \lambda, 0) \equiv 0$, and thus for $u \in \Sigma, \lambda \in \mathbf{D}$ and $\epsilon_0 > 0$ sufficiently small, we have

$$d(u, \lambda, \epsilon) = \sum_{i=1}^{\infty} M_i(u, \lambda)\epsilon^i. \tag{2}$$

The function M_i is called the i-th *Melnikov function*. In what follows the notation $|\epsilon| \ll 1$ means for all ϵ such that $|\epsilon| < \epsilon_0$ with $\epsilon_0 > 0$ sufficiently small. The first part of this paper is dedicated to period annulus.

Theorem 1. *For any compact set $\mathbf{D} \subset \Lambda$, $\epsilon_0 > 0$ and a transversal section Σ for which the displacement function (2) is defined, there exists a natural number N depending on $\mathbf{D}$ such that for any $\lambda_0 \in \mathbf{D}$, if $M_i(u, \lambda_0) \equiv 0$, for $u \in \Sigma, 1 \leq i \leq N$, then system (1) has a period annulus for $\lambda = \lambda_0$ and $|\epsilon| \ll 1$.*

Theorem 2. *Assume that*

$$P(x, y, \lambda, \epsilon) = P(x, y, \epsilon) = \sum_{i=0}^{l} P_i(x, y)\epsilon^i,$$
$$Q(x, y, \lambda, \epsilon) = Q(x, y, \epsilon) = \sum_{i=0}^{l} Q_i(x, y)\epsilon^i,$$

and $P_i(x, y)$, $Q_i(x, y)$ are polynomials in the variables x and y of degree at most n, then there exists a natural number N depending on the unperturbed system $\mathcal{X}_0$ and on the natural numbers l, n such that if $M_i(u) \equiv 0$ for $1 \leq i \leq N$, then system (1) has a period annulus for $|\epsilon| \ll 1$.

The second part of this paper is concerned with the properties of Melnikov functions near a nondegenerate center and a hyperbolic heteroclinic or homoclinic cycle for the perturbed Hamiltonian systems.

We first recall some definitions. Let $\mathcal{X}$ be a vector field in the plane. A *center* is a singular point of $\mathcal{X}$ for which there is a neighbourhood filled of periodic orbits with the exception of the singular point. A center of $\mathcal{X}$ is called *nondegenerate* if it has a pair of pure imaginary eigenvalues. A *heteroclinic cycle* Γ for $\mathcal{X}$ is a finite collection of separatrices of hyperbolic sectors $\gamma_1, \gamma_2, \ldots, \gamma_h$ and a finite collection of singular points $p_1, p_2, \ldots, p_n$ such that the α-limit set of γ_i is p_i for $i = 1, \ldots, n$, the ω-limit set of γ_i is p_{i+1} for $i = 1, 2, \ldots, n-1$ and the ω-limit set of γ_n is p_1. Moreover, some of the p_i can be repeated. A heteroclinic cycle Γ is called *hyperbolic*, if all its singular points are hyperbolic saddles. A heteroclinic cycle becomes a *homoclinic* one, if it consists of one singular point and one separatrix. Now we consider the following perturbed Hamiltonian system:

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y} + \epsilon P(x,y,\epsilon), \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + \epsilon Q(x,y,\epsilon), \end{aligned} \tag{3}$$

where H, P, Q are analytical functions in the variables $(x, y) \in \mathbf{R}^2$ and in the parameter $\epsilon \in (\mathbf{R}, 0)$. Here $(\mathbf{R}, 0)$ denotes a small neighbourhood of zero, and $C^\omega(\mathbf{R}, 0)$ denotes the set of analytic functions in a small neighbourhood of zero. In this case, as usual, we parameterize the transversal section J by the Hamiltonian constant $h = H$. We assume that the unperturbed Hamiltonian system has a continuous family of periodic orbits $\gamma_h \subset H^{-1}(h)$ for $0 < h \ll 1$.

Theorem 3. *For system (3) assume that when $h \searrow 0, \gamma_h \to (0,0)$, a nondegenerate center of the unperturbed system. Then the following hold.*

(1) $M_1(h)$ can be analytically continued to $h = 0$, and

$$M_1(0) = 0, \qquad M_1'(0) = \frac{2\pi}{\beta} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Big|_{(x,\, y,\, \epsilon) = (0,\, 0,\, 0)},$$

where $\pm i\beta$ with $\beta > 0$ are the eigenvalues of the center.

(2) If $M_i(h) \equiv 0$ for $1 \le i \le k-1$, then $M_k(h)$ can be analytically continued to $h = 0$ and $M_k(0) = 0$.

Theorem 4. *For system (3) assume that when $h \searrow 0, \gamma_h \to \gamma_0$, a heteroclinic cycle of the unperturbed system consisting of n hyperbolic saddles $p_1, p_2, \ldots, p_n$ (eventually they can be repeated) and the corresponding n separatrices. Then the following hold.*

(1) There exist analytical functions $a_1(h), b_1(h) \in C^\omega(\mathbf{R}, 0)$ such that

$$M_1(h) = a_1(h) + b_1(h) \ln h, \quad 0 < h \ll 1,$$

with

$$a_1(0) = \int_{\gamma_0} P(x,\, y,\, 0)\, dy - Q(x,\, y,\, 0)\, dx,$$
$$b_1(0) = 0,$$
$$b_1'(0) = -\sum_{i=1}^{n} \frac{1}{\lambda_i} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Bigg|_{(x,\, y)=p_i,\, \epsilon=0},$$

where $-\lambda_i < 0 < \lambda_i$ *are the eigenvalues of the saddle* p_i*. Moreover, if*

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Bigg|_{(x,\, y)=p_i,\, \epsilon=0} = 0, \text{ for } i = 1, \ldots, n,$$

then

$$a_1'(0) = \int_{\gamma_0} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Bigg|_{\epsilon=0} dt.$$

(2) *If* $n = 1$*, and* γ_0 *is a homoclinic cycle of a hyperbolic saddle, and* $M_i(h) \equiv 0$ *for* $1 \le i \le k-1$*, then there exist analytical functions* $a_k(h), b_k(h) \in C^\omega(\mathbf{R}, 0)$ *with* $b_k(0) = 0$*, such that*

$$M_k(h) = a_k(h) + b_k(h) \ln h, \quad 0 < h \ll 1.$$

In general, statement (2) of Theorem 4 cannot be generalized to heteroclinic cycles with two saddles or more. Now we consider the so called 8-figure heteroclinic cycles, i.e., the cycles consisting of one saddle and its two homoclinic orbits. Assume that for $\epsilon = 0$, system (3) has two homoclinic orbits $\gamma_0^\pm$ of a hyperbolic saddle, called the 8-figure cycle, and three families of periodic orbits: $\gamma_h \subset \{H^{-1}(h), h > 0\}$, $\gamma_h^+ \cup \gamma_h^- \subset \{H^{-1}(h), h < 0\}$, such that $\gamma_h \to \gamma_0^+ \cup \gamma_0^-$ when $h \searrow 0$, and $\gamma_h^\pm \to \gamma_0^\pm$ when $h \nearrow 0$, see Figure 1. Three classes of Melnikov functions are defined corresponding to the three period annuli: $M_k(h)$, $M_k^\pm(h)$ for $k \ge 1$. Then we have

Theorem 5. *If* $M_i = M_i^+ = M_i^- \equiv 0$ *for* $0 \le i \le k-1$ *with* $k \ge 1$*, then the following hold.*

(1) *If one of three functions* M_k, M_k^+, M_k^- *can be analytically continued to* $h = 0$*, then the other two can also be continued.*

(2) *If two of three functions* M_k, M_k^+, M_k^- *are identically zero, then the third one is identically zero.*

(3) *There exist analytical functions* $a_k(h)$*,* $b_k(h) \in C^\omega(\mathbf{R}, 0)$ *with* $b_k(0) = 0$ *such that* $M_k(h) = a_k(h) + b_k(h) \ln h$ *for* $0 < h \ll 1$.

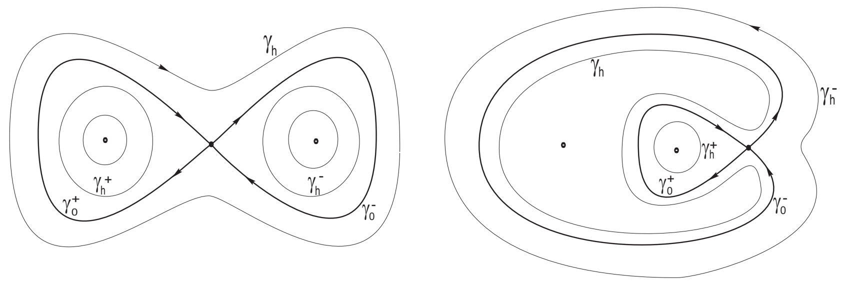

Figure 1: The two 8-figure heteroclinic cycles.

The third part of this paper is concerned with the determination of the cyclicity of a center or of a homoclinic cycle using Melnikov functions. Assume that the origin $(0,0)$ is a nondegenerate center of system (3) for $\epsilon = 0$. Without loss of generality, we can assume that

$$P(0,0,\epsilon) = Q(0,0,\epsilon) \equiv 0, \tag{4}$$

which means that our perturbation preserves the singular point $(0,0)$ fixed. Let $f(h) \in C^{\omega}(\mathbf{R}, 0)$. If $f(h) = ah^n + O(h^{n+1})$ with $a \neq 0$, we define $m(f) = n$. If $f \equiv 0$, we define $m(f) = \infty$.

Theorem 6. *Let the origin $(0,0)$ be a nondegenerate center of system (3) for $\epsilon = 0$. Assume that (4) holds, and there exist integers $k \geq 1$, $m \geq 0$ such that*

$$M_i(h) \equiv 0 \text{ for } i \leq k-1, \text{ and } m(M_k(h)) = n+1 \text{ with } 0 \leq n < \infty.$$

Then system (3) has at most n (taking into account their multiplicity) limit cycles in some neighbourhood of the origin for $|\epsilon| \ll 1$.

Theorem 7. *Assume that when $h \searrow 0$, $\gamma_h \to \gamma_0$, where γ_0 is a homoclinic cycle of a hyperbolic saddle. Let $k \geq 1$ be such an integer that $M_i(h) \equiv 0$ for $0 \leq i \leq k-1$ and $M_k(h) = a_k(h) + b_k(h) \ln h$ is not identically zero, then there exists a neighbourhood U of γ_0 such that for $|\epsilon| \ll 1$, system (3) in U has at most $2m(b_k) - 1$ limit cycles if $m(a_k) \geq m(b_k)$; and $2m(a_k)$ limit cycles if $m(a_k) < m(b_k)$, these estimates hold taken into account the multiplicity of the limit cycles.*

We remark that Roussarie in [10] obtained the result of Theorem 7 for $k = 1$. As an application of Theorem 7, we will prove following Theorem 8 which is a particular case of a result due to Roussarie [11]. First we recall that a homoclinic cycle γ is said to be of *infinite codimension* if there exists a continuous family of periodic orbits tending to the cycle γ.

Theorem 8. *Let $\mathcal{X}_\epsilon$ be an one parameter analytic family of planar vector fields. Assume that $\mathcal{X}_0$ has a homoclinic cycle γ of a hyperbolic saddle of infinite codimension, then there exists a neighbourhood U of γ and a natural number N such that $\mathcal{X}_\epsilon$ has at most N limit cycles (taking into account their multiplicity) in U for $|\epsilon| \ll 1$.*

This paper is organized as follows. In Section 2, we prove Theorems 1 and 2. In Section 3 we first recall three important results, one is about the formula for computing Melnikov functions of arbitrary order, the other two are about the normalization of planar Hamiltonian vector fields near a nondegenerate center or a hyperbolic saddle, which are the main tools in this paper, and then prove Theorem 3. In Section 4 we prove Theorem 4 and 5. In Section 5 we prove Theorem 6. In Section 6 we prove Theorem 7, and in Section 8 (after Proposition 27) we prove Theorem 8.

2. Proofs of Theorems 1 and 2

Lemma 9. *Let $d(u,\lambda,\epsilon)$ be the displacement function as defined in Section 1. Assume that, for $\lambda = \lambda_0 \in \mathbf{D}$, $d(u,\lambda_0,\epsilon) \equiv 0$, then there exists a neighbourhood U of λ_0 and a natural number N such that for any $\lambda \in U$, if $M_i(u,\lambda) \equiv 0, 1 \le i \le N$, then $M_i(u,\lambda) \equiv 0$ for all natural number i, i.e., system (1) has a period annulus for $|\epsilon| \ll 1$.*

Proof. By the assumption, $M_i(u,\lambda_0) \equiv 0$ for all i. For $u_0 \in \Sigma$, let

$$M_i(u,\lambda) = \sum_{j=0}^{\infty} a^i_j(\lambda,u_0)(u-u_0)^j.$$

Then $a^i_j(\lambda_0,u_0) = 0, i \ge 1, j \ge 0$. Denote by $\mathcal{A}$ the ring of germs of analytic functions at λ_0 and $\mathcal{I} = \mathcal{I}\{\hat{a}^i_j(\cdot,u_0)\}_{i\ge 1, j\ge 0}$ the ideal generated by the germs of the analytical functions a^i_j at $\lambda = \lambda_0$. Since the ring $\mathcal{A}$ is Noetherian (see for instance [3], p.161, Theorem 6.3.3), and so $\mathcal{I}$ is generated by a finite number of germs $\hat{a}^i_j$:

$$\mathcal{I} = \mathcal{I}\{\hat{a}^{i_1}_{j_1}, \hat{a}^{i_2}_{j_2}, \ldots, \hat{a}^{i_n}_{j_n}\}.$$

Let $N = \max\{i_1, i_2, \ldots, i_n\}$. Then, obviously, Lemma 9 holds. ∎

Proof of Theorem 1. Let

$$\begin{aligned} D_i &= \{\lambda \in \mathbf{D} \mid \exists k, k \leq i \text{ with } M_k(u,\lambda) \text{ not identically vanishing}\} \\ D &= \{\lambda \in \mathbf{D} \mid M_i(u,\lambda) \equiv 0, \forall i > 0\}. \end{aligned}$$

Then

$$\mathbf{D} = \left(\bigcup_{i=1}^{\infty} D_i \right) \bigcup D.$$

If the conclusion is not true, then there exists a sequence of parameter values $\lambda_n \in \mathbf{D}$ such that

$$M_i(u,\lambda_n) \equiv 0 \text{ for } 1 \leq i \leq n \text{ and } d(u,\lambda_n,\epsilon) \text{ is not identically zero.}$$

By the compactness of $\mathbf{D}$, we can assume that $\lambda_n \to \bar{\lambda} \in \mathbf{D}$. Since D_i are open subsets of $\mathbf{D}$, $\bar{\lambda} \in D$, which is a contradiction with Lemma 9. ∎

Proof of Theorem 2. Let

$$P_i(x,y) = \sum_{0 \leq j+k \leq n} p^i_{j,\,k} x^j y^k, \quad Q_i(x,y) = \sum_{0 \leq j+k \leq n} q^i_{j,\,k} x^j y^k.$$

We consider the coefficients of the polynomials $p^i_{j,\,k}, q^i_{j,\,k}$ and ϵ as the parameters. Note that system (4) preserves unchanged under the parameter change $\epsilon \to \delta^{-1}\epsilon$, $p^i_{j,\,k} \to \delta^i p^i_{j,\,k}$, $q^i_{j,\,k} \to \delta^i q^i_{j,\,k}$. Therefore, we can assume that $|p^i_{j,\,k}| \leq 1$, $|q^i_{j,\,k}| \leq 1$. Hence Theorem 2 becomes a corollary of Theorem 1. ∎

Example 1. For the quadratic perturbations of Bagdanov-Takens system (see [4]):

$$\begin{aligned} \dot{x} &= y + \epsilon P(x,y) \\ \dot{y} &= -x - x^2 + \epsilon Q(x,y) \end{aligned}$$

where P, Q are polynomials of degree at most 2, $N = 4$

Example 2. For the quadratic perturbation of quadratic Hamiltonian system which preserves the center fixed (see [7]):

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y} + \epsilon P(x,y) \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + \epsilon Q(x,y) \end{aligned}$$

where H is a polynomial of degree 3, the origin $(0,0)$ is a center of the unperturbed system and P, Q are polynomails of degree ≤ 2 with $P(0,0) = Q(0,0) = 0$, $N = 6$.

3. Analyticity of Melnikov functions at a center

We first recall three results which are necessary in the proof of our theorems. The first one is about the computation of the Melnikov functions of system (3). We consider now the equivalent form of system (3):

$$\omega_\epsilon = \left(\frac{\partial H}{\partial x} + \epsilon Q\right) dx + \left(\frac{\partial H}{\partial y} - \epsilon P\right) dy = 0.$$

Let

$$\omega_\epsilon = \sum_{i=0}^{\infty} \omega_i \epsilon^i.$$

Then $\omega_0 = dH$, and ω_i's are analytical 1-form. The following result is due to Poggiale[9], its proof can be found in [12].

Proposition 10. *(1)* $M_1(h) = -\int_{\gamma_0} \omega_1$;

(2) If $M_i(h) \equiv 0$ *for* $1 \leq i \leq k$, *then*

$$M_{k+1}(h) = \int_{\gamma_h} \left(\sum_{i=1}^{k} g_i \omega_{k-i+1} - \omega_{k+1} \right),$$

where the analytic functions $g_i, i = 1, 2, \ldots, k$, *are defined inductively by*

$$\omega_i - g_i dH = \sum_{j=1}^{i-1} g_j \omega_{i-j} + dR_i. \tag{5}$$

The next two classical results are about the normalization of planar Hamiltonian system near a nondegenerate center or a hyperbolic saddle respectively (for the proofs, see, for instance, [5] and [8]).

Consider now the following planar Hamiltonian system

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y}, \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x}, \end{aligned} \tag{6}$$

where H is an analytical function defined in some neighbourhood of the origin $(0,0)$.

Proposition 11. *Assume that the origin* $(0,0)$ *is a nondegenerate center of system (6) with eigenvalues* $\pm i\beta$, $\beta > 0$, *then there exist an analytical* *area-preserving transformation of variables:* $(x,y) = F(u,v)$ *in some neighbourhood of the origin and a function* $f \in C^\omega(\mathbf{R}, 0)$ *with* $f(0) = 0$, $f'(0) = \frac{\beta}{2}$ *such that* $f(u^2+v^2) = H \circ F(u,v)$ *and system (6) is changed to the form:*

$$\begin{aligned} \dot{u} &= \quad 2vf'(u^2+v^2), \\ \dot{v} &= -2uf'(u^2+v^2). \end{aligned}$$

Proposition 12. *Assume that the origin* $(0,0)$ *is a hyperbolic saddle of system (6) with eigenvalues* $\pm\lambda$, $\lambda > 0$, *then there exist an analytical area-preserving transformation of variables:* $(x,y) = F(u,v)$ *in some neighbourhood of the origin and a function* $f \in C^\omega(\mathbf{R}, 0)$ *with* $f(0) = 0$, $f'(0) = \lambda$ *such that* $f(uv) = H \circ F(u,v)$ *and system (6) is changed to the form:*

$$\begin{aligned} \dot{u} &= \quad uf'(uv), \\ \dot{v} &= -vf'(uv). \end{aligned}$$

Lemma 13. *Assume that* $f \in C^\omega(\mathbf{R}, 0), f(0) = 0, f'(0) > 0$, *and* $F \in C^\omega(\mathbf{R}^2, \mathbf{0})$. *Define function*

$$M(h) := \iint_{f(x^2+y^2)\le h} F(x,y)\, dx\, dy, \quad 0 < h \ll 1.$$

Then the following statements hold.

(1) $M(h)$ *can analytically be continued to* $h = 0$, *and*

$$M(0) = 0,\ M'(0) = \frac{\pi}{f'(0)} F(0,0).$$

(2) If

$$F(x,y) = \sum_{n=0}^{\infty} F_n(x,y),\ F_n(x,y) = \sum_{i=0}^{n} b_{i,\,n} x^{n-i} y^i, \tag{7}$$

then

$$M(h) \equiv 0 \Longleftrightarrow C_m := \sum_{k=0}^{m} (2m-2k-1)!!\,(2k-1)!!\, b_{2k,\,2m} = 0, \forall m \ge 0, \tag{8}$$

where $(-1)!! := 1$.

Proof. Assume that series (7) is convergence in the square $D = \{(x,y) \in \mathbf{C}^2 |\, |x| \le R,\ |y| \le R\}$. Let $K = \sup_D |F|$. By Cauchy inequality, $|b_{2k,\,2m}| \le KR^{-2m}$. Let $d_m = \frac{C_m}{(2m+2)!!}$, then

$$|d_m| \le (m+1)KR^{-2m}, \tag{9}$$

which implies the function $g(r) := \sum_{m=0}^{\infty} d_m r^{m+1}$ is analytic in the region $|r| \le R^2$. Now we calculate the function $M(h)$. By introducing the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $s = \sqrt{f^{-1}(h)}$, we have

$$\begin{aligned} M(h) &= \int_0^s dr \int_0^{2\pi} rF(r\cos\theta, r\sin\theta)\, d\theta \\ &= \sum_{n=0}^{\infty} \int_0^s r^{n+1} dr \int_0^{2\pi} \sum_{i=0}^{n} b_{i,\,n} \cos^{n-i}\theta \sin^i\theta\, d\theta \\ &= \sum_{m=0}^{\infty} \int_0^s r^{2m+1} dr \int_0^{2\pi} \sum_{i=0}^{2m} b_{i,\,2m} \cos^{2m-i}\theta \sin^i\theta\, d\theta \\ &= \sum_{m=0}^{\infty} \int_0^s r^{2m+1} dr \int_0^{2\pi} \sum_{k=0}^{m} b_{2k,\,2m} \cos^{2m-2k}\theta \sin^{2k}\theta\, d\theta \\ &= 2\pi \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(2m-2k-1)!!\,(2k-1)!!\, b_{2k,\,2m}}{(2m+2)!!} s^{2m+2} \\ &= 2\pi \sum_{m=0}^{\infty} d_m (f^{-1}(h))^{m+1} = 2\pi g(f^{-1}(h)), \end{aligned}$$

which is analytical at $h = 0$, and satisfies the following

$$M(0) = 2\pi g(0) = 0,\ M'(0) = 2\pi g'(0)(f'(0))^{-1} = \frac{\pi}{f'(0)} F(0,0).$$

In the computation above, we have used that

$$\int_0^{2\pi} \cos^{2m-2k}\theta \sin^{2k}\theta\, d\theta = \frac{(2m-2k-1)!!\,(2k-1)!!}{(2m)!!},$$

for a proof, see [2]. Statement (2) is obvious by noting that

$$M(h) \equiv 0 \Longleftrightarrow g(r) \equiv 0 \Longleftrightarrow d_m = 0, \forall\, m.$$

∎

Remark 14. *Condition (8) is equivalent to*

$$\int_0^{2\pi} F(r\cos\theta, r\sin\theta)\, d\theta \equiv 0,\ \ 0 \le r \ll 1. \tag{10}$$

Now we consider an analytical system

$$\begin{aligned} \dot{u} &= \frac{\partial H}{\partial v}, \\ \dot{v} &= -\frac{\partial H}{\partial u}. \end{aligned} \tag{11}$$

Lemma 15. *Assume that system (11) has a family of periodic orbits* γ_h : $H(u,v) = h$, $0 < h < \bar{h}$. *The origin* $(0,0) = H^{-1}(0)$ *is a nondegenerate center with eigenvalues* $\pm i\beta$ *with* $\beta > 0$. *Let* $\omega = -P(u,v)dv + Q(u,v)\,du$ *be an analytical 1-form defined in some neighbourhood of the origin, then the function* $M(h) := \int_{\gamma_h} \omega$ *can be analytically continued to* $h = 0$, *and*

$$M(0) = 0, \quad M'(0) = \frac{2\pi}{\beta}\left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right)\bigg|_{(u,\,v)=(0,0)}.$$

Proof. By Proposition 11, there exist an area-preserving transformation

$$u = u(x,y),\ v = v(x,y),\ u(0,0) = 0,\ v(0,0) = 0, \tag{12}$$

and a function $f \in C^\omega(\mathbf{R}, 0)$ with $f(0) = 0$, $f'(0) = \frac{\beta}{2}$, such that

$$H(u(x,y), v(x,y)) = f(x^2 + y^2).$$

Thus, by Green's formula, we obtain

$$\begin{aligned} M(h) &= \iint_{H(u,v)\le h}\left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right) du\, dv \\ &= \iint_{f(x^2+y^2)\le h} F(x,y)\, dx\, dy, \\ F(x,y) &= \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right)\bigg|_{\substack{u=u(x,\,y)\\ v=v(x,\,y)}}. \end{aligned}$$

By Lemma 13, $M(h)$ can be analytically continued to $h = 0$, with

$$M(0) = 0, \quad M'(0) = \frac{2\pi}{\beta}\left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right)\bigg|_{(u,\,v)=(0,0)}.$$

∎

Statement (1) of Theorem 3 follows from Proposition 10 and Lemma 15. Next, we prove statement (2) of this theorem.

Lemma 16. *Let* H, γ_h, ω *be defined as in Lemma 15, then* $M(h) = \int_{\gamma_h} \omega \equiv 0$ *if and only if there exists a real analytical function* $z = z(x,y)$ *defined in some neighbourhood of the origin* $(0,0)$ *satisfying the following linear partial differential equation*

$$\frac{\partial H}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial H}{\partial v}\frac{\partial z}{\partial u} = \frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}. \tag{13}$$

Proof. Sufficiency. Assume $z(x,y)$ satisfies equation (13), then $\omega - zdH$ is a total differential of some function, i.e., there exists an analytical function R defined in some neighbourhood of the origin such that $\omega - zdH = dR$. Therefore, $\int_{\gamma_h} \omega = \int_{\gamma_h} (zdH + dR) \equiv 0$.

Necessity. Let $u = u(x,y)$, $v = v(x,y)$ be the area-preserving normalization transformation (12). We denote by

$$\begin{aligned}\bar{z}(x,y) =& z(u(x,y), v(x,y)),\\ \bar{H}(x,y) =& f(x^2+y^2) = H(u(x,y), v(x,y)),\\ F(x,y) =& \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right)\Bigg|_{\substack{u=u(x,y)\\ v=v(x,y)}}.\end{aligned}$$

Then

$$\frac{\partial \bar{z}}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial \bar{z}}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y},$$

consequently

$$\begin{pmatrix} \dfrac{\partial z}{\partial u} \\ \dfrac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial v}{\partial y} & -\dfrac{\partial v}{\partial x} \\ -\dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \bar{z}}{\partial x} \\ \dfrac{\partial \bar{z}}{\partial y} \end{pmatrix}.$$

Substituting it into (13), we get

$$\left(\frac{\partial u}{\partial x}\frac{\partial \bar{z}}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial \bar{z}}{\partial x}\right)\frac{\partial H}{\partial u} + \left(\frac{\partial v}{\partial x}\frac{\partial \bar{z}}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial \bar{z}}{\partial x}\right)\frac{\partial H}{\partial v} = F(x,y),$$

or equivalently

$$\frac{\partial \bar{H}}{\partial x}\frac{\partial \bar{z}}{\partial y} - \frac{\partial \bar{H}}{\partial y}\frac{\partial \bar{z}}{\partial x} = F(x,y) \tag{14}$$

By using $\bar{H}(x,y) = f(x^2+y^2)$, (14) can be written in the form

$$\frac{\partial \bar{z}}{\partial y}x - \frac{\partial \bar{z}}{\partial x}y = R(x,y), \quad R(x,y) := \frac{F(x,y)}{2f'(x^2+y^2)}. \tag{15}$$

If

$$R(x,y) = \sum_{n=0}^{\infty} R_n(x,y),\ R_n(x,y) = \sum_{i=0}^{n} b_{i,n} x^{n-i} y^i,$$

then, by Lemma 13 and Remark 14,

$$\int_0^{2\pi} R(r\cos\theta, r\sin\theta)\, d\theta = \frac{1}{2f'(r^2)} \int_0^{2\pi} F(r\cos\theta, r\sin\theta)\, d\theta \equiv 0.$$

This implies that the coefficients $b_{i,\,n}$ must satisfy (8). Let

$$\bar{z} = \sum_{n=0}^{\infty} z_n, \quad z_n = \sum_{k=0}^{n} a_{k,\,n} x^{n-k} y^k. \tag{16}$$

Substituting (16) into (15), we get

$$\frac{\partial z_n}{\partial y} x - \frac{\partial z_n}{\partial x} y = R_n, \quad n = 0, 1, 2, \ldots \tag{17}$$

Setting $a_{-1,\,n} = a_{n+1,\,n} = 0$, from (17), we obtain

$$\sum_{k=0}^{n} [(k+1)a_{k+1,\,n} - (n-k+1)a_{k-1,\,n}] x^{n-k} y^k = \sum_{k=0}^{n} b_{k,\,n} x^{n-k} y^k,$$

or

$$(k+1)a_{k+1,\,n} - (n-k+1)a_{k-1,\,n} = b_{k,\,n}, \quad k = 0, 1, \ldots, n. \tag{18}$$

The determinant of system (18) is

$$\begin{aligned} \Delta &= \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -n & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 1-n & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-1 & 0 \\ 0 & 0 & 0 & \cdots & -2 & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{vmatrix} \\ &= \begin{cases} 0 & \text{if } n = 2m; \\ [(2m+1)!!]^2 & \text{if } n = 2m+1. \end{cases} \end{aligned}$$

Therefore, system (18) has a unique solution for n odd. For $n = 2m$ even, system (18) can be divided into two independent systems:

$$2ka_{2k,\,2m} - 2(m-k+1)a_{2k-2,\,2m} = b_{2k-1,\,2m}, \quad k = 1, 2, \ldots, m, \tag{19}$$

$$(2k+1)a_{2k+1,\,2m} - (2m-2k+1)a_{2k-1,\,2m} = b_{2k,\,2m}, \quad k = 0, 1, \ldots, m. \tag{20}$$

System (19) contains m equations and $m+1$ unknown numbers and its matrix of coefficients has rank m. This implies that it has a solution of one dimension. System (20) contains $m+1$ equations and m unknown numbers, and its matrix of coefficients has rank m. Note that the determinant of the augmented matrix of system (20) is

$$\begin{aligned}
&\left|\begin{array}{ccccccc}
1 & 0 & 0 & \cdots & 0 & 0 & b_{0,\,2m} \\
1-2m & 3 & 0 & \cdots & 0 & 0 & b_{2,\,2m} \\
0 & 3-2m & 5 & \cdots & 0 & 0 & b_{4,\,2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2m-3 & 0 & b_{2m-4,\,2m} \\
0 & 0 & 0 & \cdots & -3 & 2m-1 & b_{2m-2,\,2m} \\
0 & 0 & 0 & \cdots & 0 & -1 & b_{2m,\,2m}
\end{array}\right| \\
= &\ \sum_{k=0}^{m}(2m-2k-1)!!\,(2k-1)!!\,b_{2k,\,2m} \\
= &\ 0,
\end{aligned}$$

the last equality follows from (8). Therefore, system (20) has a unique solution. The argument above shows that system (18) has always solutions, and if we set $a_{0,\,2m}=0$, the solution is unique. Next we prove that the series (16) defined by the unique solution is convergence in some neighbourhood of the origin $(0,0)$. Assume that $R(x,y)$ is convergence in the square $D=\{(x,y)\in \mathbf{C}^2 \mid |x|\le \bar r,\, |y|\le \bar r\}$. Let $C=\sup_D |R|$, then by Cauchy inequality,

$$|b_{i,\,n}|\le \bar r^{-n}C. \tag{21}$$

We claim that

$$|a_{i,\,n}|\le 2^n\bar r^{-n}C. \tag{22}$$

We will prove (22) only for $n=2m+1$, $i=2k+1$. All other cases can be proved in a similar way. Indeed, by (18),

$$a_{-1,\,n}=0,\quad a_{1,\,n}=b_{0,\,n},\quad a_{3,\,n}=\frac{1}{3}b_{2,\,n}+\frac{n-1}{3}b_{0,\,n},$$
$$a_{5,\,n}=\frac{1}{5}b_{4,\,n}+\frac{n-3}{3\cdot 5}b_{2,\,n}+\frac{(n-1)(n-3)}{3\cdot 5}b_{0,\,n},$$

and in general,

$$a_{2k+1,\,n}=\frac{1}{2k+1}b_{2k,\,n}+\frac{n-2k+1}{2k+1}a_{2k-1,\,n}.$$

Let

$$e_n=\max_k\left\{\frac{(n-1)(n-3)\cdots(n-2k+1)}{(2k+1)!!}\right\}.$$

By induction,

$$a_{2k+1,\,n} = \sum_{j=0}^{k} l_{j,\,n} b_{2j,\,n}, \tag{23}$$

where $l_{j,\,n}$ are some constants with $|l_{j,\,n}| \le e_n$. Now we calculate the value of e_n. For $m = 2p$ even,

$$\begin{aligned} e_n &= \max_k \left\{ \frac{4p(4p-2)\cdots(4p-2k+2)}{(2k+1)!!} \right\} = \frac{4p(4p-2)\cdots(2p+2)}{(2p+1)!!} \\ &= \frac{(4p)!!}{(2p+1)!} = \frac{2^m}{m+1}. \end{aligned}$$

Similarly, for $m = 2p+1$ odd, we also have $e_n = \frac{2^m}{m+1}$. Now from (23),

$$\begin{aligned} |a_{2k+1,\,n}| &\le \sum_{j=0}^{k} e_n |b_{2j,\,n}| \le 2^m \max_j |b_{2j,\,n}| \\ &\le 2^m \bar{r}^{-n} C \le \left(\frac{2}{\bar{r}}\right)^n C, \end{aligned}$$

which implies that (16) is convergence in the square $\left\{|x| < \frac{\bar{r}}{2},\, |y| < \frac{\bar{r}}{2}\right\}$. ∎

Next we prove statement (2) of Theorem 3 by induction with respect to k.

Suppose $k = 1$. The 1-form $\omega_1 - g_1 dH$ with $g_1 = z(x,y)$ is a total differential of some function if and only if $z(x,y)$ is a solution of (13). By Lemma 16, if $\int_{\gamma_h} \omega_1 \equiv 0$, then there exists an analytical function $g_1 = z(x,y)$ defined in some neighbourhood of the origin satisfying (13). This implies that $\omega_1 - g_1 dH = dR_1$ for some analytical function R_1 defined in some neighbourhood of the origin. Therefore, $g_1\omega_1 - \omega_2$ is an analytical 1-form defined in some neighbourhood of the origin. By Lemma 15 and Proposition 10, $M_2(h) = \int_{\gamma_h} g_1\omega_1 - \omega_2$ can be analytically continued to $h = 0$ and $M_2(0) = 0$. Now we assume that

$$M_j(h) = \int_{\gamma_h} \left(\sum_{i=1}^{j-1} g_i \omega_{j-i} - \omega_j \right) \equiv 0 \text{ for } 1 \le j \le k-1.$$

Applying Lemma 16 to the function $M_{k-1}(h)$, we get an analytical function g_{k-1} defined in some neighbourhood of the origin $(0,0)$ such that (5) holds for $i = k-1$. By Lemma 15 and Proposition 10, $M_k(h) = \int_{\gamma_h} (\sum_{i=1}^{k-1} g_i \omega_{k-i} - \omega_k)$ can be analytically continued to $h = 0$ and $M_k(0) = 0$. Therefore, the proof of Theorem 3 is now completed.

4. Melnikov functions near homoclinic and heteroclinic cycles

In this section we shall prove Theorem 4.

Lemma 17. *Assume that* $f \in C^\omega(\mathbf{R}, 0)$ *with* $f(0) = 0$, $f'(0) > 0$; $P(u,v)$ *and* $Q(u,v)$ *are analytical functions in the square* $\{(u,v) \in \mathbf{C}^2 \mid |u| \leq \delta_1,\ |v| \leq \delta_1\}$*:*

$$P(u,v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} p_{i,\,n} u^{n-i} v^i, \quad Q(u,v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} q_{i,\,n} u^{n-i} v^i.$$

Let

$$\begin{aligned} D &= [-\delta, \delta] \times [-\delta, \delta] \subset \mathbf{R}^2, \quad 0 < \delta < \delta_1, \\ \gamma_h &= \{(u,v) \in D \mid f(uv) = h,\ u \geq 0, v \geq 0\}. \end{aligned}$$

Define the function

$$M(h) = \int_{\gamma_h} \omega, \qquad \omega = -P\, dv + Q\, du.$$

Let $s = f^{-1}(h)$*. Then there exist functions* $a(s), b(s) \in C^\omega(\mathbf{R}, 0)$ *such that*

$$M(h) = a(s) + b(s) \ln s,\ 0 < h \ll 1,$$

where

$$a(0) = \int_{\gamma_0} \omega,\ b(s) = -\sum_{m=0}^{\infty} (p_{m,\,2m+1} + q_{m+1,\,2m+1}) s^{m+1}.$$

Proof. Calculating straightforward, we have

$$\begin{aligned} \int_{\gamma_h} Q\, du &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} q_{i,\,n} \int_{s\delta^{-1}}^{\delta} u^{n-2i} s^i\, du \\ &= \sum_{m=0}^{\infty} \sum_{i=0}^{2m} q_{i,\,2m} \int_{s\delta^{-1}}^{\delta} u^{2m-2i} s^i\, du \\ &\quad + \sum_{m=0}^{\infty} \sum_{i=0}^{2m+1} q_{i,\,2m+1} \int_{s\delta^{-1}}^{\delta} u^{2m+1-2i} s^i\, du \\ &= \sum_{m=0}^{\infty} \sum_{i=0}^{2m} \frac{q_{i,\,2m}}{2m-2i+1} (\delta^{2m-2i+1} s^i - \delta^{-2m+2i-1} s^{2m-i+1}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=0}^{\infty}\sum_{\substack{i=0\\ i\neq m+1}}^{2m+1}\frac{q_{i,\,2m+1}}{2m-2i+2}(\delta^{2m-2i+2}s^{i}-\delta^{-2m+2i-2}s^{2m-i+2})\\
&+\,2\ln\delta\sum_{m=0}^{\infty}q_{m+1,\,2m+1}s^{m+1}-\sum_{m=0}^{\infty}q_{m+1,\,2m+1}s^{m+1}\ln s\\
=\;& I_1(s)-\left(\sum_{m=0}^{\infty}q_{m+1,\,2m+1}s^{m+1}\right)\ln s. \qquad (24)
\end{aligned}
$$

Similarly, we can get

$$
\int_{\gamma_h} P\,dv = I_2(s)+\left(\sum_{m=0}^{\infty}p_{m,\,2m+1}s^{m+1}\right)\ln s, \qquad (25)
$$

which, together with (24), implies

$$
M(h)=\int_{\gamma_h}-P\,dv+Q\,du=a(s)+b(s)\ln s.
$$

where

$$
a(s)=I_1(s)-I_2(s), \quad b(s)=-\sum_{m=0}^{\infty}(p_{m,\,2m+1}+q_{m+1,\,2m+1})s^{m+1}.
$$

By using the Cauchy inequality, it is easy to prove that the functions $a(s)$ and $b(s)$ are analytical in some neighbourhood of $s=0$. ∎

From Lemma 17 it follows:

Corollary 18. *Under the assumption of Lemma 17, the function $M(h)$ can be analytically continued to $h=0$ if and only if*

$$
p_{m,\,2m+1}+q_{m+1,\,2m+1}=0, \quad \forall\, m\geq 0. \qquad (26)
$$

We remark that condition (26) is equivalent to say that $b(s)\equiv 0$ for $0<s\ll 1$.

Lemma 19. *Let γ_0 be as in Theorem 4 a heteroclinic cycle of system (3) consisting of n hyperbolic saddles $p_1, p_2, \ldots, p_n$ (eventually they can be repeated) and n separatrices, $\omega=-P(x,y)\,dy+Q(x,y)\,dx$ an analytical 1-form defined in some neighbourhood of γ_0, and let $M(h):=\int_{\gamma_h}\omega$, then there exist analytical functions $a(h), b(h)\in C^{\omega}(\mathbf{R},0)$ with*

$$
a(0)=\int_{\gamma_0}\omega,\; b(0)=0,\; b'(0)=-\sum_{i=1}^{n}\frac{1}{\lambda_i}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\Bigg|_{(x,\,y)=p_i}
$$

where $-\lambda_i < 0 < \lambda_i$ *are the eigenvalues of the saddle* p_i*, such that*

$$M(h) = a(h) + b(h)\ln h, \quad 0 < h \ll 1.$$

Moreover, if

$$\left.\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\right|_{(x,\,y)=p_i} = 0 \quad \textit{for } i = 1, 2, \ldots, n, \tag{27}$$

then

$$a'(0) = \int_{\gamma_0} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dt. \tag{28}$$

Proof. According to Proposition 12, for every $1 \le i \le n$, there exist an analytical function $H_i \in C^\omega(\mathbf{R}, 0)$ with $H_i(0) = 0$, $H_i'(0) = \lambda_i$, and an area-preserving normalization coordinate transformation

$$F_i : x = x(u, v, i),\, y = y(u, v, i),$$

from some neighbourhood of the origin $(0, 0)$ to some neighbourhood U_i of p_i such that in the new coordinate (u, v) system (3) for $\epsilon = 0$ takes the form

$$\dot{u} = \frac{\partial G_i}{\partial v}, \quad \dot{v} = -\frac{\partial G_i}{\partial u},$$

where

$$G_i(u, v) = H_i(uv) = H \circ F_i(u, v).$$

Denote by $D = \{|u| \le \delta,\, |v| \le \delta\}$, and fix $\delta > 0$ small enough such that $F_i(D) \subset U_i$, $i = 1, 2, \ldots, n$. We note that $u = 0$ and $v = 0$ are the separatrices of the saddle $(0, 0)$ for the system $(\dot{u}, \dot{v})$. Let

$$\Gamma_i^+ = F_i(\{u = \delta\}), \quad \Gamma_i^- = F_i(\{v = \delta\}).$$

Any closed orbits near γ_0 is separated by $\Gamma_i^\pm$, $i = 1, 2, \ldots, n$ into $2n$ segments: γ_h^i, $i = 1, 2, \ldots, 2n$, in which γ_h^{2i} are close to p_i and γ_h^{2i-1} connects $\gamma_h \cap \Gamma_{i-1}^+$ and $\gamma_h \cap \Gamma_i^-$, see Figure 2. Then

$$M(h) = \int_{\gamma_h} \omega = \sum_{i=1}^{2n} \int_{\gamma_h^i} \omega = \sum_{i=1}^{n} \int_{\gamma_h^{2i}} \omega + \sum_{i=1}^{n} \int_{\gamma_h^{2i-1}} \omega. \tag{29}$$

Since γ_h^{2i-1} depend analytically on h,$\int_{\gamma_h^{2i-1}} \omega$ are analytical at $h = 0$. Next we consider the integrals $\int_{\gamma_h^{2i}} \omega$. Let $\bar{\gamma}_h^i = \{(u, v) \in D \,|\, H_i(uv) = h, \}$. Substituting $x = x(u, v, i)$, $y = y(u, v, i)$ into the integral $\int_{\gamma_h^{2i}} \omega$, we obtain

$$\int_{\gamma_h^{2i}} \omega = \int_{\bar{\gamma}_h^i} -\bar{P}_i(u, v)\, dv + \bar{Q}_i(u, v)\, du,$$

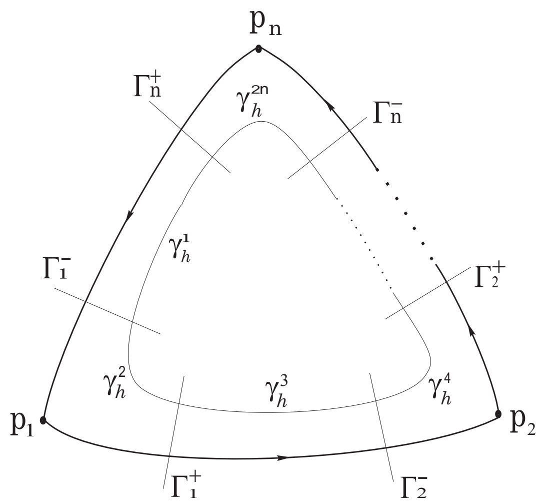

Figure 2.

where

$$\bar{P}_i(u,v) = P \circ F_i \frac{\partial y}{\partial v} - Q \circ F_i \frac{\partial x}{\partial v}, \quad \bar{Q}_i(u,v) = Q \circ F_i \frac{\partial x}{\partial u} - P \circ F_i \frac{\partial y}{\partial u}.$$

Computing straightforward, we get

$$\begin{aligned}\frac{\partial \bar{P}_i}{\partial u} = \left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial v} + P \circ F_i \frac{\partial^2 y}{\partial u \partial v} \\ - \left(\frac{\partial Q}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial v} - Q \circ F_i \frac{\partial^2 x}{\partial u \partial v},\end{aligned}$$

$$\begin{aligned}\frac{\partial \bar{Q}_i}{\partial u} = \left(\frac{\partial Q}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial x}{\partial u} + Q \circ F_i \frac{\partial^2 x}{\partial u \partial v} \\ - \left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial y}{\partial u} - P \circ F_i \frac{\partial^2 y}{\partial u \partial v},\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial \bar{P}_i}{\partial u} + \frac{\partial \bar{Q}_i}{\partial v} &= \frac{\partial P}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) + \frac{\partial Q}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \\ &= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \circ F_i.\end{aligned} \tag{30}$$

The last equality above follows from the fact that F_i is area-preserving. Let $s = H^{-1}(h)$. From Lemma 17, there exist analytical functions $a_i(s)$, $b_i(s) \in C^\omega(\mathbf{R}, 0)$ with

$$a_i(0) = \int_{\bar{\gamma}_0^i} -\bar{P}_i\, dv + \bar{Q}_i\, du,\ b_i(0) = 0$$

and

$$b_i'(0) = \left(\frac{\partial \bar{P}_i}{\partial u} + \frac{\partial \bar{Q}_i}{\partial y}\right)\bigg|_{(u,\,v)=(0,\,0)}$$

such that

$$\int_{\gamma_h^{2i}} \omega = \int_{\bar{\gamma}_h^i} -\bar{P}_i\, dv + \bar{Q}_i\, du = a_i(s) + b_i(s)\ln s, \quad 0 < h \ll 1.$$

Consequently,

$$\int_{\gamma_h^{2i}} \omega = \bar{a}_i(h) + \bar{b}_i(h)\ln h, \tag{31}$$

where

$$\bar{a}_i(h) = a_i \circ H_i^{-1}(h) + b_i \circ H_i^{-1}(h) \ln\left(\frac{H_i^{-1}(h)}{h}\right) \text{ and } \bar{b}_i(h) = b_i \circ H_i^{-1}(h)$$

are analytical at $h = 0$ and satisfy

$$\begin{aligned}
\bar{b}_i(0) &= b_i(0) = 0, \\
\bar{b}_i'(0) &= \frac{b_i'(0)}{\lambda_i} = -\frac{1}{\lambda_i}\left(\frac{\partial \bar{P}}{\partial u} + \frac{\partial \bar{Q}}{\partial v}\right)\bigg|_{(u,\,v)=(0,\,0)} \\
&= -\frac{1}{\lambda_i}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\bigg|_{(x,\,y)=p_i}.
\end{aligned}$$

Substituting (31) into (29), we obtain

$$M(h) = a(h) + b(h)\ln h,$$

where

$$a(h) = \sum_{i=1}^{n} \int_{\gamma_h^{2i-1}} \omega + \sum_{i=1}^{n} \bar{a}_i(h), \quad b(h) = \sum_{i=1}^{n} \bar{b}_i(h)$$

are analytical at $h = 0$ and satisfy

$$a(0) = \sum_{i=1}^{n} \int_{\gamma_0^{2i-1}} \omega + \sum_{i=1}^{n} \bar{a}_i(0) = \sum_{i=1}^{n} \int_{\gamma_0^{2i-1}} + \sum_{i=1}^{n} \int_{\gamma_0^{2i}} \omega = \int_{\gamma_0} \omega,$$
$$b(0) = \sum_{i=1}^{n} \bar{b}_i(0) = 0,$$
$$b'(0) = \sum_{i=1}^{n} \bar{b}_i'(0) = -\sum_{i=1}^{n} \frac{1}{\lambda_i} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Big|_{(x,\,y)=p_i}$$

Now we prove (28). First we point out that if (27) holds, it follows from $b(0) = b'(0) = 0$ that $M(h) \in C^1$. We claim that the integral in(28) is convergence. Indeed, let $p(t) \subset \gamma_0$ be a solution of system (3) for $\epsilon = 0$ and assume that $\lim_{t\to+\infty} p(t) = p_i$, $\lim_{t\to-\infty} p(t) = p_{i-1}$. Note that p_i is a hyperbolic saddle,we have, as $t \to +\infty$, $\|p(t) - p_i\| = O(\exp(-ct))$ for some $c > 0$. Hence,

$$\left\| \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \circ p(t) \right\| = \left\| \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \circ p(t) - \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Big|_{p_i} \right\|$$
$$= O(\|p(t) - p_i\|) = O(\exp(-ct)), \text{ as } t \to +\infty.$$

So, the integral $\int_0^{+\infty} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt$ is convergence. Similarly, the integral $\int_{-\infty}^{0} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt$ is convergence too. Our claim is proved. From $H(x, y) = h$, we get

$$\frac{\partial H}{\partial x}\frac{\partial x}{\partial h} = 1, \quad \frac{\partial H}{\partial y}\frac{\partial y}{\partial h} = 1,$$

which implies

$$\frac{\partial x}{\partial h} dy = -dt, \quad \frac{\partial y}{\partial h} dx = dt.$$

Thus,

$$\begin{aligned} M'(h) &= \frac{\partial}{\partial h} \int_{\gamma_h} (-P(x, y)\, dy + Q(x, y)\, dx) \\ &= \int_{\gamma_h} \left(-\frac{\partial P}{\partial x}\frac{\partial x}{\partial h} dy + \frac{\partial Q}{\partial y}\frac{\partial y}{\partial h} dx \right) = \int_{\gamma_h} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt. \end{aligned} \tag{32}$$

Let $h \to 0$, we get

$$a'(0) = M'(0) = \lim_{h\to 0} M'(h) = \lim_{h\to 0} M'(h) = \int_{\gamma_0} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt.$$

∎

From Lemma 19, we get immediately statement (1) of Theorem 4. Next we prove statement (2).

Lemma 20. *Let $\omega = -P(x,y)\,dy + Q(x,y)\,dx$ be an analytical 1-form defined in some neighbourhood of γ_0, where γ_0 is a homoclinic orbit of a hyperbolic saddle $p = (0,0)$ of system (6), then $M(h) := \int_{\gamma_h} \omega$ can be analytically continued to $h = 0$ if and only if for any area-preserving normalization coordinate transformation $F(u,v)$ near p given by Proposition 12, the Taylor series of the function $\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \circ F(u,v)$ at $(0,0)$ does not contain the terms $u^m v^m$ for any integers $m \geq 0$.*

Proof. According to Proposition 12, there exists a function $f \in C^\omega(\mathbf{R}, 0)$ with $f(0) = 0$, $f'(0) > 0$ such that $f(uv) = H \circ F(u,v)$. Let

$$\Gamma^+ = F(\{u = \delta\}), \quad \Gamma^- = F(\{v = \delta\}).$$

Any closed orbits near γ_0 is separated by $\Gamma^\pm$ into two segments: γ_h^1 and γ_h^2, in which γ_h^2 is close to the saddle p and γ_h^1 connects $\gamma_h \cap \Gamma^+$ and $\gamma_h \cap \Gamma^-$ in the complement of some neighbourhood of p. Since the integral $\int_{\gamma_h^1} \omega$ is analytical at $h = 0$,

$$M(h) = \int_{\gamma_h} \omega = \int_{\gamma_h^1} \omega + \int_{\gamma_h^2} \omega$$

can be analytically continued to $h = 0$ if and only if the integral $\int_{\gamma_h^2} \omega$ can be analytically continued. Let $\bar{\gamma}_h^2 = \{(u,v) | f(uv) = h,\ 0 < u, v \leq \delta\}$, then as in the proof of Lemma 19, we have

$$\int_{\gamma_h^2} \omega = \int_{\bar{\gamma}_h^2} -\bar{P}(u,v)\,dv + \bar{Q}(u,v)\,du,$$

where

$$\bar{P}(u,v) = P \circ F \frac{\partial y}{\partial v} - Q \circ F \frac{\partial x}{\partial v}, \quad \bar{Q}(u,v) = Q \circ F \frac{\partial x}{\partial u} - P \circ F \frac{\partial y}{\partial u}.$$

Let

$$\bar{P}(u,v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} p_{i,\,n} u^{n-i} v^i, \quad \bar{Q}(u,v) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} q_{i,\,n} u^{n-i} v^i.$$

From Corollary 18, the integral $\int_{\gamma_h^2} \omega$ can be analytically continued to $h = 0$ if and only if (26) holds which is equivalent to say that the coefficients of the terms $u^m v^m$ for any $m \geq 0$ in the Taylor series of the function $\frac{\partial \bar{P}}{\partial u} + \frac{\partial \bar{Q}}{\partial v}$ at $(0,0)$ are zero. Now the statement of Lemma 20 follows from (30). ∎

Lemma 21. *Let ω and $M(h)$ be defined as in Lemma 20, then the function $M(h)$ can be analytically continued to $h = 0$ if and only if there exists an analytical function $z = z(x, y)$ defined in some neighbourhood of the saddle p, such that $z(x, y)$ satisfies the following linear partial differential equation:*

$$\frac{\partial z}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial H}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}. \tag{33}$$

Proof. Let $(x, y) = F(u, v)$ be the area-preserving normalization coordinate transformation near p given by Proposition 12 and let $\bar{z} = z \circ F, f(uv) = H \circ F(u, v)$, then (33) can be changed to the form

$$\frac{\partial \bar{z}}{\partial u}u - \frac{\partial \bar{z}}{\partial v}v = R(u, v),\ \ R(u, v) = \frac{1}{f'(uv)}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \circ F(u, v). \tag{34}$$

Obviously, the Taylor series of $f'(uv)R(u, v)$ at $(0, 0)$ does not contain the terms $u^m v^m$ for any integers $m \geq 0$ if and only if $R(u, v)$ has the same property. Let

$$R(u, v) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} b_{i,\,n} u^{n-i} v^i, \tag{35}$$

$$\bar{z}(u, v) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} a_{i,\,n} u^{n-i} v^i. \tag{36}$$

Substituting them into (34), we get

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}[(n - i)a_{i,\,n} - a_{i,\,n}i]u^{n-i}v^i = \sum_{n=0}^{\infty}\sum_{i=0}^{n} b_{i,\,n}u^{n-i}v^i,$$

or equivalently

$$(n - 2i)a_{i,\,n} = b_{i,\,n}, \quad \text{for } n = 0, 1, 2, \ldots, \text{ and } i = 0, 1, \ldots, n. \tag{37}$$

System (37) has solutions if and only if

$$b_{m,\,2m} = 0,\ \forall m \geq 0. \tag{38}$$

Moreover, if (38) holds, we can choose

$$a_{i,\,n} = \begin{cases} \dfrac{b_{i,\,n}}{n - 2i}, & \text{if } n \neq 2i; \\ 0, & \text{if } n = 2i. \end{cases}$$

Since $|a_{i,\,n}| \leq |b_{i,\,n}|$, the convergence radius of (36) is at least equal to the convergence radius of (35). So the function $\bar{z}$ defined in (36) is analytical in some neighbourhood of the origin. Now, the lemma follows using Lemma 20. ∎

Lemma 22. *Let ω and $M(h)$ be defined as in Lemma 20, then the function $M(h)$ is constant for $0 < h \ll 1$ if and only if there exists an analytical function $z = z(x, y)$ defined in some neighbourhood of γ_0 such that (33) holds.*

Proof. Sufficiency. We consider the characteristic equation of (33):

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \quad \dot{z} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}. \tag{39}$$

Assume that there exists an analytical function $z = z(x, y)$ defined in some neighbourhood of γ_0 such that $z(x, y)$ satisfies equation (33). This implies that the surface $S = \{(x, y, z) \in \mathbf{R}^3 \mid z = z(x, y)\}$ is invariant under the flow of (39). Therefore, $S \cap \{(x, y, z) \in \mathbf{R}^3 \mid H(x, y) = h\}$ for $0 < h \ll 1$ is a periodic orbit of (39). So from (32), we have

$$M'(h) = \int_{\gamma_h} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dt = \int_{\gamma_h} \dot{z}\, dt = 0, \tag{40}$$

which implies that $M(h)$ is constant for $0 < h \ll 1$.

Necessity. Assume now $M(h)$ is constant. By Lemma 21, equation (33) has an analytical solution $z(x, y)$ in some neighbourhood of the saddle $p = (0, 0)$. We claim that this solution $z(x, y)$ can be extended continuously to a single valued analytic function in some neighbourhood of γ_0. Indeed, from (33), $\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\Big|_{(x,\,y)=(0,\,0)} = 0$, so the straight line $\{(0, 0, z) \mid z \in \mathbf{R}\}$ consists of singular points of (39). Then the invariant surface S of (39) contains local stable and unstable manifolds of the singular point $p_0 = (0, 0, z(0, 0))$. Let Γ^- and Γ^+ be the planes transversal to the flow of (39) at some point of local stable manifold and local unstable one respectively. Then $l^{\pm} := S \cap \Gamma^{\pm}$ are analytical curves in $\Gamma^{\pm}$, see Figure 3. Let A be the projection from Γ^+ to Γ^- along the orbits of (39). Then $l'_- := Al^+$ is an analytical curve in Γ^-. Introducing the set $U = \{(x, y, z) \in \mathbf{R}^3 \mid H(x, y) > 0\}$, by (40) we have U is filled with periodic orbits. This implies $l'_- \cap U = l^- \cap U$. Therefore, by the analyticity, $l'_- = l^-$. Thus, the union of the orbits passing through l^+ and S constructs an analytical invariant surface of system (39), which is the graph of an analytical function $z(x, y)$ defined in some neighbourhood of γ_0. From the invariance, $z(x, y)$ is a solution of equation (33). ∎

Lemma 23. *Let ω and $M(h)$ be defined as in Lemma 20. Then $M(h) \equiv 0$ if and only if there exist analytical functions $z(x, y)$ and $R(x, y)$ defined in some neighbourhood of γ_0 such that*

$$\omega = z dH + dR. \tag{41}$$

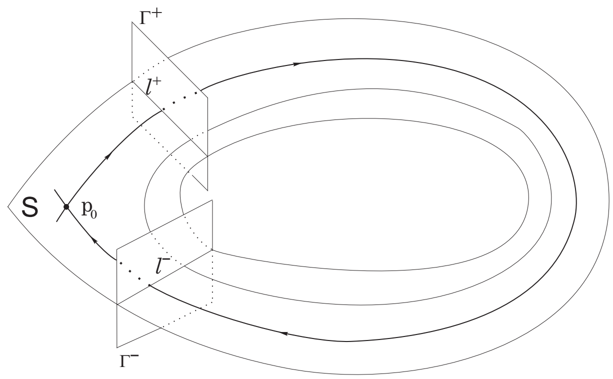

Figure 3.

Proof. Sufficiency. Assume that formula (41) holds. Since the function H is constant along the closed curve γ_h, so $dH = 0$. This implies that $\int_{\gamma_h} \omega = \int_{\gamma_h} zdH + dR = \int_{\gamma_h} dR = 0$.

Necessity. By Lemma 22, there exists an analytical function $z(x, y)$ defined in some neighbourhood of γ_0 such that equation (33) holds. Let

$$Mdx + Ndy = \omega - zdH.$$

From (33),

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{42}$$

Now we define the function $R(x, y)$ in the following way: for any point (x, y) near γ_0, let

$$R(x, y) = \int_{(0,\,0)}^{(x,\,y)} Mdx + Ndy. \tag{43}$$

By (42) and the fact

$$\int_{\gamma_h} Mdx + Ndy = \int_{\gamma_h} (\omega - zdH) = \int_{\gamma_h} \omega = M(h) \equiv 0,$$

the integral in (43) defines a single valued analytical function in some neighbourhood of γ_0 satisfying (41). ∎

The proof of statement (2) of Theorem 4. Suppose $k = 2$. Then $M_1(h) = -\int_{\gamma_h} \omega_1 \equiv 0$. By Lemma 23, there exist analytical functions z_1, R_1 defined in some neighbourhood of γ_0 such that $\omega_1 = z_1 dH + dR_1$. By Proposition

10, $M_2(h) = \int_{\gamma_h}(z_1\omega_1 - \omega_2)$. Now by using Lemma 19, we get that statement (2) holds for $k = 2$. Similarly, assume that $M_i(h) \equiv 0$, $1 \le i \le k-1$. Again from Lemma 23, there exist analytical functions a_{k-1}, R_{k-1} defined in some neighbourhood of γ_0 satisfying (5). Now by Proposition 10 and Lemma 19, statement (2) holds for k. This completes the proof of Theorem 4. ∎

Proof of Theorem 5. Statement (1) is just a corollary of Lemma 20. Statement (2) is a corollary of Lemma 24 below. Finally statement (3) follows easily from Lemma 24 and Lemma 19. ∎

All notations used in Lemma 24 below are the same as in the statement of Theorem 5.

Lemma 24. *Let ω be an analytical 1-form defined in some neighbourhood of the eight figure cycle, then two of the three integrals $\int_{\gamma_h}\omega, \int_{\gamma_h^+}\omega, \int_{\gamma_h^-}\omega$ are identically zero if and only if there exist analytical functions $z(x,y)$, $R(x,y)$ defined in some neighbourhood of $\gamma_0^+ \cup \gamma_0^-$ such that $\omega = zdH + dR$.*

Proof. The sufficiency can be proved by using the same argument as the proof of Lemma 23. We now prove the necessity. First by Lemma 21, equation (33) has an analytical solution $z(x,y)$ in some neighbourhood of the saddle p. By using the same argument of the proof of Lemma 22, the function $z(x,y)$ can be extended continuously to a single valued analytic function in the some neighbourhood of the eight figure. Let $Mdx + Ndy = \omega - zdH$. Then from (33), formula (42) holds. Now we define the function R by the integral in (43). Then by (42) and the assumption that two of the three integrals $\int_{\gamma_h}\omega$, $\int_{\gamma_h^+}\omega$, $\int_{\gamma_h^-}\omega$ are identically zero, the function R is a single-valued analytical one in some neighbourhood of eight figure $\gamma_0^+ \cup \gamma_0^-$ and satisfies (41). ∎

5. Proof of Theorem 6

By Proposition 11, we can assume that there exists a function $f \in C^\omega(\mathbf{R}, 0)$, with $f(0) = 0, f'(0) = \frac{\beta}{2} > 0$ such that $H(x,y) = f(x^2+y^2)$, i.e. system (3) has the form

$$\begin{aligned} \dot{x} &= -yf'(x^2+y^2) + \epsilon P(x,y,\epsilon), \\ \dot{y} &= \quad xf'(x^2+y^2) + \epsilon Q(x,y,\epsilon). \end{aligned} \tag{44}$$

System (44) in polar coordinates can be written in the form

$$\frac{dr}{d\theta} = \epsilon R(r,\theta,\epsilon), \tag{45}$$

where the analytical function R is defined on the cylinder $(r,\theta) \in \mathbf{R} \times \mathbf{R}/(2\pi\mathbf{Z})$ and satisfies

$$R(0,\theta,\epsilon) \equiv 0,\ R(-r,\theta+\pi,\epsilon) \equiv -R(r,\theta,\epsilon). \tag{46}$$

Let L and L' be the straight line $\theta = 0$ and $\theta = 2\pi$. We consider the return map $F : L \to L'$ as follows. If $x \in L$ then $F(x,\epsilon) = r(2\pi,x,\epsilon)$, where $r(\theta,x,\epsilon)$ denotes the solution of (45) such that $r(0,x,\epsilon) = x$. Obviously, $F(x,\epsilon)$ is analytical at $(0,0)$ and $F(0,\epsilon) \equiv 0$, $F(x,0) \equiv x$. The displacement function is defined as $d(x,\epsilon) := F(x,\epsilon) - x$. Denote by $L_+ = \{x \in L,\ x \geq 0\}$, $L'_+ = \{x \in L',\ x \geq 0\}$, then L_+ and L'_+ can be parameterized by the Hamiltonian $h = f(x^2)$. Denote by $g(h) = \sqrt{f^{-1}(h)}$, then $g^{-1} \circ F \circ g(h)$ for $h \geq 0$ is the expression of return map in the coordinate h. By the assumption of Theorem 6,

$$(g^{-1} \circ F \circ g)(h) = h + M_k(h)\epsilon^k + o(\epsilon^k),$$

or

$$h + M_k(h)\epsilon^k + o(\epsilon^k) = g^{-1}(g(h) + d \circ g(h)). \tag{47}$$

Replace $g^{-1}(x)$ by $f(x^2), h$ by $f(x^2), g(h)$ by x, (47) becomes

$$\begin{aligned} f(x^2) + M_k(f(x^2))\epsilon^k + o(\epsilon^k) &= f(x^2 + 2xd(x,\epsilon) + d(x,\epsilon)^2) \\ &= f(x^2) + f'(x^2)(2xd(x,\epsilon) + d(x,\epsilon)^2)(1+o(1)). \end{aligned} \tag{48}$$

Note that $d(0,\epsilon) = d(x,0) \equiv 0$, we have $d^2 = O(x\epsilon)d$. So (48) can be written as

$$M_k(f(x^2)) + o(1) = \frac{2}{\epsilon^k} f'(x^2)xd(x,\epsilon)(1+o(1)).$$

When $\epsilon \to 0$ we get

$$M_k(f(x^2)) = 2xf'(x^2)\lim_{\epsilon\to 0}\frac{d(x,\epsilon)}{\epsilon^k}. \tag{49}$$

Substituting $M_k(h) = ah^{n+1} + O(h^{n+2})$, $f(x^2) = \frac{\beta}{2}x^2(1+O(x^2))$ into (49), we obtain

$$a\left(\frac{\beta}{2}\right)^{n+1} x^{2n+2}(1+O(x^2)) = 2x\left(\frac{\beta}{2} + O(x^2)\right)\lim_{\epsilon\to 0}\frac{d(x,\epsilon)}{\epsilon^k},$$

which implies

$$\lim_{\epsilon\to 0}\epsilon^{-k}d(x,\epsilon) = bx^{2n+1}(1+O(x^2)),\ b = a\beta^n 2^{-n-1} \neq 0.$$

Thus

$$d(x,\epsilon) = bx^{2n+1}(1+O(x^2))\epsilon^k + O(x\epsilon^{k+1}).$$

For $x \neq 0$ and $\epsilon \neq 0$, obviously,

$$d(x,\epsilon) = 0 \Longleftrightarrow x^{2n}(1+O(x^2)) + O(\epsilon) = 0. \tag{50}$$

By Rolle theorem (50) has at most $2n$ nonzero roots in a neighbourhood of $x = 0$. This means that system (45) has at most $2n$ nonzero limit cycles with period 2π. By (46), if $r(\theta)$ is a 2π periodic orbit of (45) then $-r(\theta+\pi)$ is also a 2π periodic orbit of (45). Therefore, system (45) has at most n 2π periodic orbits in the region $r > 0$ for $\epsilon \neq 0$, which implies system (44) has at most n limit cycles in some neighbourhood of the origin.

This completes the proof of Theorem 6.

6. Proof of Theorem 7

Denote by $n = \min\{m(a_k),\, m(b_k)\} < \infty$. According to the normal form theory, see [6] or [12](page 93), system (3) near the hyperbolic saddle is $C^{2M}(M \gg \max\{n,\, k\})$ orbitally equivalent to the polynomial family:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y\left(1 - \sum_{i=1}^{N} \alpha_i(\epsilon)(xy)^{i-1}\right), \end{aligned} \tag{51}$$

where $N > M$ is a natural number depending on M, and $\alpha_i(\epsilon)$ are analytical functions with $\alpha_i(0) = 0$. For ϵ sufficiently small we take the following transversal sections to system (51):

$$I = \left\{y = 1,\, 0 \leq x \leq \frac{1}{2}\right\}, \quad J = \left\{x = 1,\, 0 \leq y \leq \frac{1}{2}\right\}.$$

For any point $p \in I$, the orbit of (51) starting from p will intersect the section J at a point q, this correspondence $p \to q$ define a map $\Delta(\cdot,\epsilon) : I \to J$, called *transition map.*

Lemma 25. ([6], Fundamental Lemma 11) *Let $p = (\xi, 1), q = (1, \Delta(\xi,\epsilon))$, then $\Delta : I \to J$ can be expressed as*

$$\Delta(\xi,\epsilon) = F(\xi, z, \alpha(\epsilon)),$$

where $F = F(\xi, z, \alpha)$ is a solution of the following homogeneous first order partial differential equation:

$$\frac{\partial F}{\partial z}\left(-1 + z\sum_{i=2}^{N} \alpha_i \xi^{i-2}\right) + \frac{\partial F}{\partial \xi}\sum_{i=1}^{N} \alpha_i \xi^{i-1} = 0 \tag{52}$$

with the initial condition

$$F(\xi, z, \alpha)|_{z=0} = \xi \tag{53}$$

and

$$z = L(\xi, \alpha_1) = \begin{cases} \alpha_1^{-1}(\xi^{1-\alpha_1} - \xi), & \alpha_1 \neq 0; \\ -\xi \ln \xi, & \alpha_1 = 0, \end{cases}$$

$$\alpha(\epsilon) = (\alpha_1(\epsilon), \alpha_2(\epsilon), \ldots, \alpha_N(\epsilon)),\ \alpha(0) = \mathbf{0}.$$

Lemma 26. *The function F can be written in the form*

$$F(\xi, z, \alpha) = \xi + \sum_{i=1}^{N} \alpha_i \xi^{i-1} z G(\xi, z, \alpha),$$

where G is analytic at $(0, 0, \mathbf{0})$ and satisfies

$$G(\xi, 0, \alpha) = G(\xi, z, \mathbf{0}) = 1. \tag{54}$$

Proof. First we note that the Cauchy problem (52),(53) can be solved in the class of formal series, which are automatically convergent by the Cauchy-Kowalewski theorem. Now We shall seek the solution of the problem (52),(53) in the form of a convergent series

$$F = \sum_{m=0}^{\infty} z^m f_m(\xi, \alpha). \tag{55}$$

Substituting this series in (52) and equating terms with equal powers of z, we obtain the following recurrence equations:

$$(m+1) f_{m+1} = m Q_\alpha f_m + \frac{\partial f_m}{\partial \xi} P_\alpha, \quad m = 0, 1, 2, \ldots, \tag{56}$$

where $Q_\alpha = \sum_{i=2}^{N} \alpha_{i-2} \xi^{i-2}$, $P_\alpha = \sum_{i=1}^{N} \alpha_i \xi^{i-1}$.

The initial function f_0 is obtained from the initial condition (53): $f_0 = \xi$. The system of recurrence equations indicates the f_m are polynomials in ξ and α. In particular, we have $f_1 = P_\alpha$. Let $h_1 = 1$ and h_m, $m \geq 2$ be defined by the following recurrence equations:

$$(m+1) h_{m+1} = m Q_\alpha h_m + \frac{\partial f_m}{\partial \xi}. \tag{57}$$

Let $\bar{f}_m = h_m P_\alpha$. Then $\bar{f}_m$ satisfy the following recurrence equations

$$(m+1)\bar{f}_{m+1} = mQ_\alpha \bar{f}_m + \frac{\partial f_m}{\partial \xi} P_\alpha, \quad m = 1, 2, \ldots. \tag{58}$$

Note that $f_1 = \bar{f}_1$ and from (56) and (58) we conclude by induction that $f_m = \bar{f}_m$ for $m \geq 1$. Let

$$G(\xi, z, \alpha) := \sum_{m=1}^{\infty} h_m(\xi, \alpha) z^{m-1}. \tag{59}$$

Then

$$F(\xi, z, \alpha) = \sum_{m=0}^{\infty} f_m(\xi, \alpha) z^m = \xi + \sum_{m=1}^{\infty} h_m P_\alpha z^m = \xi + P_\alpha z G(\xi, z, \alpha). \tag{60}$$

We shall show that the series in (59) is convergent for $|z| \ll 1$. Let $C = \sup_{|\xi| \leq 1, \|\alpha\| \leq 1} |Q_\alpha|$. From (57) we obtain for $|\xi| \leq 1$, $\|\alpha\| \leq 1$,

$$m|h_m| \leq C(m-1)|h_{m-1}| + \left| \frac{\partial f_{m-1}}{\partial \xi} \right|,$$

which implies that

$$m|h_m| \leq \sum_{i=1}^{m} \left| \frac{\partial f_{i-1}}{\partial \xi} \right| C^{m-1}. \tag{61}$$

Assume that the series (55) is convergent in the disc $\{z : |z| \leq r < \min\{1, C^{-1}\}\}$. Let $L = \max_{|z| \leq r}\{|F|, |F'_\xi|\}$. By Cauchy inequality,

$$\left| \frac{\partial f_{i-1}}{\partial \xi} \right| = \frac{1}{(i-1)!} \left| \frac{\partial^{i-1} F'_\xi}{\partial z^{i-1}} \right| \Bigg|_{z=0} \leq \frac{L}{r^{i-1}}.$$

Substituting it into (61), we get

$$|h_m| \leq \frac{1}{m} \sum_{i=1}^{m} L r^{1-i} C^{m-i} \leq \frac{LrC^m}{m} \sum_{i=1}^{m} (rC)^{-i} \leq L r^{1-m},$$

which implies that the series (59) is convergent for $|z| < r$.

Now we shall show that the function G satisfies (54). From (52), $\frac{\partial F}{\partial z}\Big|_{z=0} = \frac{\partial F}{\partial \xi}\Big|_{z=0} P_\alpha$, but $F|_{z=0} = \xi$ and hence $\frac{\partial F}{\partial z}\Big|_{z=0} = P_\alpha$, which, together with (60), implies $G(\xi, 0, \alpha) = 1$. From (56), we have $f_m(\xi, \mathbf{0}) = 0$ for $m \geq 1$, which, together with (57), implies $h_m(\xi, \mathbf{0}) = 0$ for $m \geq 2$. Hence,

$$G(\xi, z, \mathbf{0}) = \sum_{m=1}^{\infty} h_m(\xi, \mathbf{0}) z^{m-1} = h_1(\xi, \mathbf{0}) = 1.$$

This completes the proof of the lemma. ∎

Without loss of generality in a neighbourhood of the saddle point of γ_0 we assume that we have the same coordinate system as in Lemma 25 around the origin. Let $f(\xi,\epsilon): J \to I$ denote the Poincaré map along the orbits of system (3), then $f(\xi,\epsilon)$ is a C^{2M} diffeomorphism satisfying $f(\xi,0) \equiv \xi$. The return map is then defined as $P = f \circ \Delta : I \to I$. The transversal section I can also be parameterized by the Hamiltonian h. Denote by $\bar{P} : I \to I$ the return map under the coordinate h, then for $h > 0$, by the assumption of Theorem 7, we have

$$\bar{P}(h,\epsilon) - h = M_k(h)\epsilon^k + o(\epsilon^k).$$

Let $\xi = g(h,\epsilon)$ with $g(0,0) = 0$ and $\frac{\partial g}{\partial h}(0,0) = c > 0$ be the coordinate change from h to ξ, then $g^{-1} \circ P \circ g = \bar{P}$. For $\xi > 0$, using the Mean Value Theorem, we have

$$\begin{aligned} P(\xi,\epsilon) - \xi = g \circ \bar{P}(h,\epsilon) - g(h,\epsilon) &= \frac{\partial g}{\partial h}(\bar{h},\epsilon)(\bar{P}(h,\epsilon) - h) \\ &= \frac{\partial g}{\partial h}(\bar{h},\epsilon)(M_k(h)\epsilon^k + o(\epsilon^k)), \end{aligned}$$

where $\bar{h} \in (h, \bar{P}(h,\epsilon))$. Thus, for $0 < h \ll 1$, we have

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon^{-k}(P(\xi,\epsilon) - \xi) =& g'_h(h,0)M_k(h) \\ =& (c + o(1))M_k(g^{-1}(\xi,0)) \\ :=& \bar{M}(\xi) = \begin{cases} d\xi^n \ln\xi(1 + o(1)), & \text{if } m(a_k) \geq m(b_k) = n, \\ d\xi^n(1 + o(1)) \quad , & \text{if } n = m(a_k) < m(b_k); \end{cases} \end{aligned} \tag{62}$$

where $d \neq 0$ is a constant and $o(1) \to 0$, as $\xi \to 0$. The formula (62) can be written as

$$P(\xi,\epsilon) = \xi + \bar{M}(\xi)\epsilon^k + o(\epsilon^k).$$

Let

$$f^{-1}(\xi,\epsilon) = \xi + \sum_{i=1}^{k} f_i(\xi)\epsilon^i + o(\epsilon^k),$$

then

$$\Delta(\xi,\epsilon) = (f^{-1} \circ P)(\xi,\epsilon) = \xi + \bar{M}(\xi)\epsilon^k + \sum_{i=1}^{k} f_i(\xi)\epsilon^i + o(\epsilon^k), \tag{63}$$

which, together with Lemmas 25 and 26, implies

$$\sum_{i=1}^{N} \alpha_i(\epsilon)\xi^{i-1} zG(\xi,z,\alpha(\epsilon)) = \bar{M}(\xi)\epsilon^k + \sum_{i=1}^{k} f_i(\xi)\epsilon^i + o(\epsilon^k). \tag{64}$$

Let

$$l = \min_{1 \le i \le M} \{k, m(\alpha_i)\}.$$

Obviously, $l \le k$. We claim that

$$l = k. \tag{65}$$

Indeed, if (65) does not hold, i.e. if $l < k$, let $j = \min\{1 \le i \le M \mid m(\alpha_i) = l\}$, and

$$R(\xi, \epsilon) := \sum_{i=M+1}^{N} \alpha_i(\epsilon)\xi^{i-1} zG(\xi, z, \alpha(\epsilon)) = \sum_{i=1}^{k-1} g_i(\xi)\epsilon^i + o(\epsilon^k \xi^M).$$

Then $g_i(\xi) = o(\xi^M)$. Substituting it into (64), we obtain

$$\begin{aligned} \epsilon^l \sum_{i=1}^{M} \tilde{\alpha}_i(\epsilon)\xi^{i-1} zG(\xi, z, \alpha(\epsilon)) + \sum_{i=1}^{k-1} (g_i(\xi) - f_i(\xi))\epsilon^i + o(\epsilon^k \xi^M) \\ = \bar{M}(\xi)\epsilon^k + f_k(\xi)\epsilon^k + o(\epsilon^k), \end{aligned} \tag{66}$$

where $\tilde{\alpha}_i(\epsilon) := \epsilon^{-l}\alpha_i(\epsilon)$ and $\tilde{\alpha}_j(0) \ne 0$. This implies

$$g_i(\xi) = f_i(\xi) \text{ for } 1 \le i \le l - 1. \tag{67}$$

Dividing the both side of equation (66) by ϵ^l and then let $\epsilon \to 0$, we get

$$\tilde{\alpha}_j(0)\xi^j \ln \xi (1 + o(1)) = o(\xi^M) - f_l(\xi),$$

which is a contradiction because the dominant terms of the both sides of the equality are different. Our claim is proved. It follows from (65) and (67) that

$$R(\xi, \epsilon) = \sum_{i=1}^{k-1} f_i(\xi)\epsilon^i + o(\epsilon^k \xi^M).$$

Thus, Δ and f^{-1} can be written as

$$\Delta(\xi, \epsilon) = \xi + \epsilon^k \sum_{i=1}^{M} \bar{\alpha}_i(\epsilon)\xi^{i-1} zG(\xi, z, \alpha(\epsilon)) + \sum_{i=1}^{k-1} f_i(\xi)\epsilon^i + o(\epsilon^k \xi^M),$$

$$f^{-1}(\xi, \epsilon) = \xi + \sum_{i=1}^{k-1} f_i(\xi)\epsilon^i + \epsilon^k \sum_{i=0}^{M} \bar{\beta}(\epsilon)\xi^i + o(\epsilon^k \xi^M),$$

where $\bar{\alpha}_i(\epsilon) := \epsilon^{-k}\alpha_i(\epsilon), \bar{\beta}_i(\epsilon)$ are sufficiently smooth functions. Substituting them into (63), we get

$$\sum_{i=1}^{M} \bar{\alpha}_i(\epsilon)\xi^{i-1} zG(\xi, z, \alpha(\epsilon)) = \bar{M}(\xi) + \sum_{i=0}^{M} \bar{\beta}_i(\epsilon)\xi^i + o(\xi^M) + o(1).$$

If $\epsilon \to 0$, by using (54), the above equality is changed to

$$-\sum_{i=1}^{M} \bar{\alpha}_i(0)\xi^i \ln \xi = \bar{M}(\xi) + \sum_{i=0}^{M} \bar{\beta}_i(0)\xi^i + o(\xi^M)$$

or equivalently

$$\bar{M}(\xi) = -\sum_{i=1}^{M} \bar{\alpha}_i(0)\xi^i \ln \xi - \sum_{i=0}^{M} \bar{\beta}_i(0)\xi^i + o(\xi^M).$$

By (62), we have, if $m(a_k) \geq m(b_k) = n$,

$$\bar{\beta}_i(0) = \bar{\alpha}_i(0) = 0, \text{ for } i \leq n-1, \text{ and } \bar{\alpha}_n(0) = -d \neq 0; \tag{68}$$

and if $m(b_k) > m(a_k) = n$,

$$\bar{\beta}_i(0) = \bar{\alpha}_i(0) = \bar{\alpha}_n(0) = 0, \text{ for } i \leq n-1, \text{ and } \bar{\beta}_n(0) = -d \neq 0. \tag{69}$$

The number of the limit cycles near the homoclinic cycle is equal to the number of the fixed points of return map: $P(\xi, \epsilon) = \xi$ in $0 < \xi \ll 1$, or the number of the zeros of the equation:

$$\Delta(\xi, \epsilon) - f^{-1}(\xi, \epsilon) = 0. \tag{70}$$

For $\epsilon \neq 0$, equation (70) is equivalent to

$$\sum_{i=1}^{M} \bar{\alpha}_i(\epsilon)\xi^{i-1} zG(\xi, z, \alpha(\epsilon)) - \sum_{i=0}^{M} \bar{\beta}_i(\epsilon)\xi^i + o(\xi^M) = 0. \tag{71}$$

It is shown in [6] that equation (71) has at most $2n-1$ zeros if (68) holds; and at most $2n$ zeros if (69) holds. We refer the reader to [6] for more details.

This completes the proof of Theorem 7.

7. Proof of Theorem 8

We shall need the following result.

Proposition 27. *Let X be an analytical vector field defined in some open region of $\mathbf{R}^2$. Assume that X has a continuous family of periodic orbits (the period annulus) $\gamma_s, 0 < s < \bar{s}$, and γ_0 is a nondegenerate center p or a homoclinic orbit of a hyperbolic saddle p such that $\lim_{s \searrow 0} \gamma_s = \gamma_0$, then for any $s \in [0, \bar{s})$, there exists an analytical function $\rho > 0$ defined in some neighbourhood of γ_s such that $\operatorname{div}(\rho X) \equiv 0$, i.e., ρX is a Hamiltonian vector field.*

Proposition 27 will be proved through two lemmas.

Lemma 28. *Under the assumptions of Proposition 27 the following hold. For any $s \in [0, \bar{s})$, the vector field X has an analytical first integral H defined in some neighbourhood of γ_s such that*

$$\det D^2 H(p) \neq 0 \text{ and } DH = 0 \Leftrightarrow X = 0. \tag{72}$$

Proof. If γ_s is a periodic orbit, the lemma is trivial. For γ_0 being a nondegenerate center, by the Poincaré Normal Form Theorem (for a proof, see [1]), there exists an analytic change of coordinates that brings the initial system to the normal form

$$\begin{aligned} \dot{x} &= -yf(x^2 + y^2), \\ \dot{y} &= \quad xf(x^2 + y^2). \end{aligned}$$

Obviously, the system above has a first integral $H = x^2 + y^2$ satisfying (72). Now we assume that γ_0 is a homoclinic orbit of a hyperbolic saddle. Since there exists a family of periodic orbits tending to γ_0, the saddle values of any order must be zero. Therefore under the normalized coordinate the vector field X near the saddle takes the following form (see [1]):

$$\begin{aligned} \dot{x} &= -\lambda x(1 + R(xy)) \\ \dot{y} &= \quad \lambda y(1 + R(xy)), \ R \in C^\omega(\mathbf{R}, 0), \ R(0) = 0. \end{aligned} \tag{73}$$

$H = xy$ is a first integral of system (73). This implies that the vector field X has an analytical first integral in some neighbourhood U of the saddle satisfying (72). We claim that the first integral H can be extended to some neighbourhood of γ_0. Indeed, let $I, J \subset U$ be two transversal sections to X at some point of the local stable manifold and the local unstable one, respectively. Sections I and J can be parameterized by using $h = H$. Without loss of generality, we assume that the intersections of the periodic orbits γ_s with I and J correspond to points with $h > 0$. Let $f : J \to I$ denote the Poincaré map along the orbits of X, then $f \in C^\omega(\mathbf{R}, 0)$. Let $G(h) = H(f(h)) - h$. Since all orbits starting from the points of J with $h > 0$ are periodic, we have $G(h) \equiv 0$ for $0 < h \ll 1$. By the analyticity, $G(h) \equiv 0$ for $|h| \ll 1$. Now we define the function H in some neighbourhood of γ_0 as follows. For any x close to γ_0, denote by $\gamma(x)$ the orbit of X passing through x, then $H(x) := H(\gamma(x) \cap I) = H(\gamma(x) \cap J)$. Obviously, $H(x)$ takes a constant on each orbit, i.e., H is a first integral. ∎

Let H be the first integral in some neighbourhood of γ_s as in Lemma 28. Consider the Hamiltonian vector field

$$Y := JDH, \; J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then X and Y define the same direction field. Let $\rho = \|Y\|/\|X\|$, then ρ is positive and analytical at any regular point. Obviously, $Y = \rho X$ or $Y = -\rho X$. The next lemma shows that the function ρ can be analytically continued to the saddle p.

Lemma 29. *Assume that the analytical vector fields $X_1 = (P_1, Q_1)$ and $X_2 = (P_2, Q_2)$ define the same direction field and the origin $(0,0)$ is an isolated singular point of X_1 and X_2. If*

$$\prod_{i=1}^{2} \left[\left(\frac{\partial P_i}{\partial x} \right)^2 + \left(\frac{\partial P_i}{\partial y} \right)^2 \right] \Bigg|_{(0,0)} \neq 0 \tag{74}$$

or

$$\prod_{i=1}^{2} \left[\left(\frac{\partial Q_i}{\partial x} \right)^2 + \left(\frac{\partial Q_i}{\partial y} \right)^2 \right] \Bigg|_{(0,0)} \neq 0,$$

then there exists a positive analytical function ρ defined in some neighbourhood of the origin such that $\|X_1\| = \rho \|X_2\|$.

Proof. We will prove the lemma under the assumption $\frac{\partial P_1}{\partial x}(0,0) \neq 0$. The proofs for other cases are similar. Since X_1 and X_2 define the same direction field, they have the same vertical isocline:

$$V : \{(x,y) \,|\, P_1(x,y) = 0\} = \{(x,y) \,|\, P_2(x,y) = 0\}.$$

By using Implicit Function Theorem for P_1 at $(0,0)$, there exists a function $g(y) \in C^\omega(\mathbf{R}, 0)$ with $g(0) = 0$, such that

$$V = \{(x,y) \,|\, x = g(y)\}. \tag{75}$$

Thus,

$$P_1(x,y) = (x - g(y))\bar{P}_1(x,y),\ \bar{P}_1 \in C^\omega(\mathbf{R}^2, \mathbf{0}),\ \bar{P}_1(0,0) = \frac{\partial P_1}{\partial x}(0,0) \neq 0.$$

We claim that $\frac{\partial P_2}{\partial x}(0,0) \neq 0$. Indeed, if $\frac{\partial P_2}{\partial x}(0,0) = 0$, then by (74), $\frac{\partial P_2}{\partial y}(0,0) \neq 0$. By the Implicit Function Theorem, the curve V is tangent to the x-axis at the origin, which is a contradiction with (75). Thus, we get

$$P_2(x,y) = (x - g(y))\bar{P}_2(x,y),\ \bar{P}_2 \in C^\omega(\mathbf{R}^2, \mathbf{0}),\ \bar{P}_2(0,0) = \frac{\partial P_2}{\partial x}(0,0) \neq 0.$$

Therefore, the function $\bar{\rho}(x,y) := \frac{\bar{P}_1(x,y)}{\bar{P}_2(x,y)}$ is analytical and has definite sign in some neighbourhood of the origin. Let $\rho = |\bar{\rho}|$, then the proof of the lemma is completed. ∎

Proof of Proposition 27. It is a corollary of Lemmas 28 and 29. ∎

Proof of Theorem 8. Consider the following one parameter family of analytical systems:

$$\dot{x} = f(x) + \epsilon g(x, \epsilon), \quad x \in \mathbf{R}^2. \tag{76}$$

We assume that for $\epsilon = 0$ system (76) has a homoclinic orbit γ_0 of a hyperbolic saddle of infinite codimension, i.e. there is a continuous family of periodic orbits tending to γ_0. By Proposition 27, there exists a positive analytical function $\rho(x)$ defined in some neighbourhood of γ_0 such that $\text{div}(\rho f) \equiv 0$. Now we consider the following perturbed Hamiltonian system which is orbitally equivalent to system (76):

$$\dot{x} = \rho(x) f(x) + \epsilon \rho(x) g(x, \epsilon). \tag{77}$$

We assume, without loss of generality, that there exists an analytical function H defined in some neighbourhood of γ_0 such that $JDH = \rho f$, $\gamma_0 \subset H^{-1}(0)$, and for $0 < h \ll 1$, $\gamma_h \subset H^{-1}(h)$ is a family of periodic orbits with $\gamma_h \to \gamma_0$ as $h \searrow 0$. Let $M_k(h)$ denote the k-th Melnikov function of system (77).

Case 1. $M_i(h) \equiv 0$, $\forall i \geq 1$. Then for $|\epsilon| \ll 1$, system (77) has a period annulus and hence has no limit cycles near γ_0.

Case 2. There exists an integer $k \geq 1$ such that $M_i(h) \equiv 0$, $0 \leq i \leq k-1$ and $M_k(h) = a_k(h) + b_k(h) \ln h$ is not identically vanishing. Then by Theorem 7, system (77) in some neighbourhood of γ_0 has at most $2\min\{m(a_k), m(b_k)\}$ limit cycles for $|\epsilon| \ll 1$.

This completes the proof of Theorem 8. ∎

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