ON THE DIFFERENTIABILITY OF FIRST INTEGRALS OF TWO DIMENSIONAL FLOWS

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Abstract

By using techniques of differential geometry we answer the following open problem proposed by Chavarriga, Giacomini, Gin, and Llibre in [5]. For a given two–dimensional flow what is the maximal order of differentiability of a first integral on a canonical region in function of the order of differentiability of the flow? Moreover, we prove that for every planar polynomial differential system there exist finitely many invariant curves and singular points γ_i, $i = 1, 2, \ldots, l$, such that $\mathbf{R}^2 \setminus \left(\bigcup_{i=1}^{l} \gamma_i \right)$ has finitely many connected open components, and that on each of these connected sets the system has an analytic first integral. For a homogeneous polynomial differential system in $\mathbf{R}^3$, there exist finitely many invariant straight lines and invariant conical surfaces such that their complement in $\mathbf{R}^3$ is the union of finitely many open connected components, and that on each of these connected open components the system has an analytic first integral.

1. Introduction and statement of the main results

Let

$$\mathbf{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

be a C^r vector field with $r \geq 1$, defined on a two dimensional manifold M. Here, $r \geq 1$ means that $r = 1, 2, \ldots, \infty$, or ω. Of course, when $r = \omega$, the flow is analytic. In all this paper a *two dimensional manifold* means a two dimensional differentiable manifold, connected and without boundary, but not necessarily compact nor orientable.

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A map $\phi : D \to M$ or simply (M, ϕ) is the C^r *local flow* with $r \geq 1$ associated to a vector field $\mathbf{X}$, with $D = \{(t,p) : p \in M, t \in I_p\}$ and I_p is the maximum open interval of the real line where the flow $\phi(t,p) = \phi_p(t)$ of $\mathbf{X}$ passing through $p \in M$ when $t = 0$ is defined, if it verifies

$$\frac{d\phi_p(t)}{dt} = \mathbf{X}(\phi_p(t)), \quad \text{and}$$

(i) $\phi(0,p) = p$ for all $p \in M$;

(ii) $\phi(t, \phi(s,p)) = \phi(t+s,p)$ for all $p \in M$, and all s and t such that s, $t+s \in I_p$;

(iii) $\phi_p(-t) = \phi_p^{-1}(t)$ for all $p \in M$ such that t, $-t \in I_p$.

Two flows (M, ϕ) and (M', ϕ') are C^k *equivalent* with $1 \leq k \leq r$ if there is a C^k diffeomorphism of M onto M' which takes orbits of ϕ onto orbits of ϕ', preserving or reversing simultaneously the sense of all orbits.

Let ϕ be a C^r local flow on the two dimensional manifold M for $r \geq 1$. The flow (M, ϕ) is C^k *parallel* if it is C^k–equivalent, with $1 \leq k \leq r$, to either the *strip*, the *annular*, the *spiral*, or the *toral* flow. More precisely, these flows are respectively:

- $(\mathbf{R}^2, \phi)$ with the flow ϕ defined by $x' = 1$, $y' = 0$;
- $(\mathbf{R}^2 \setminus \{0\}, \phi)$ with the flow ϕ defined (in polar coordinates) by $r' = 0$, $\theta' = 1$;
- $(\mathbf{R}^2 \setminus \{0\}, \phi)$ with the flow ϕ defined by $r' = r$, $\theta' = 0$;
- $(\mathbf{S}^1 \times \mathbf{S}^1, \phi)$ with rational flow ϕ (i.e. the flow $x' = 1$, $y' = 0$ over the usual covering space $\mathbf{R}^2$ with rational slope; note in particular that all rational flows on the torus are equivalent).

Let $p \in M$. We denote by $\gamma(p)$ the *orbit* of the flow (M, ϕ) passing through p (i.e. $\gamma(p) = \{\phi(t,p) : t \in I_p\}$), and by $\gamma^+(p)$ (respectively $\gamma^-(p)$) the *positive semiorbit* (respectively *negative semiorbit*), i.e. $\gamma^+(p) = \{\phi(t,p) : t \in I_p \text{ and } t \geq 0\}$ (respectively $\gamma^-(p) = \{\phi(t,p) : t \in I_p \text{ and } t \leq 0\}$).

We define by

$$\alpha(p) = \mathrm{cl}(\gamma^-(p)) - \gamma^-(p) \qquad (\text{respectively } \omega(p) = \mathrm{cl}(\gamma^+(p)) - \gamma^+(p)),$$

the *α–limit* of $\gamma^-(p)$ (respectively *ω–limit* of $\gamma^+(p)$). Here, cl(A) denotes the closure of the subset A of M.

An open neighbourhood U of an orbit $\gamma(p)$ of the C^r flow (M, ϕ) is said to be a *C^k parallel neighbourhood* with $1 \le k \le r$ if (U, ϕ) is C^k equivalent to a parallel flow for some $k \ge 1$.

An orbit $\gamma(p)$ is a *separatrix* of the flow (M, ϕ) if it is not contained in a parallel neighbourhood (U, ϕ) satisfying the following two assumptions:

(a) for any $q \in U$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$;

(b) $\mathrm{cl}(U) \backslash U$ consists of $\alpha(p)$, $\omega(p)$ and exactly two orbits $\gamma(a)$, $\gamma(b)$ of ϕ with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

We denote by Σ the union of all separatrices of the flow (M, ϕ). Then Σ is a closed invariant subset of M. Every connected component of the complement of Σ in M, with the restricted flow, is called a *canonical region* of ϕ.

We say that a C^k function $H : M \backslash \Sigma \longrightarrow \mathbf{R}$ with $k \ge 0$ is a *weak first integral* of the vector field $\mathbf{X}$, if H is constant on every orbit of the flow (M, ϕ) defined by the vector field $\mathbf{X}$ located in every canonical region, and H is nonconstant on any open subset of $M \backslash \Sigma$. As usual, a C^0 function means a continuous function. If $k \ge 1$, this definition is equivalent to

$$\mathbf{X}H = \frac{\partial H}{\partial x} P + \frac{\partial H}{\partial y} Q \equiv 0,$$

on $M \backslash \Sigma$. Moreover, if $\Sigma = \emptyset$, then H is called a *first integral.*

In [5] the authors obtained the following result.

Theorem 1. *Every local flow ϕ on a two dimensional manifold M has continuous first integrals on every canonical region of ϕ.*

Meanwhile, they proposed the following:

Open problem. What is the maximal order of differentiability of the first integrals on a canonical region of a given two–dimensional flow ϕ in function of the order of differenciability of the flow?

In this paper we answer the open problem. The results are the following.

Theorem 2. *Let ϕ be a C^r flow on a two dimensional manifold M with $r \ge 1$, and let Σ be the union of all separatrices of ϕ. Then*

(1) Every canonical region of (M, ϕ) is C^r parallel.

(2) The flow ϕ restricted to every canonical region has a C^r (respectively C^∞, C^ω) first integral for $r \in \mathbf{N}$ (respectively, $r = \infty, \omega$).

As usual $\mathbf{N}$ denotes the set of positive integers. Statement (1) in the case C^0 parallel was proved by Neumann [12], see Section 2.

We consider planar differential systems

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where P and Q are C^r functions for $r \geq 0$ defined in an open subset U of $\mathbf{R}^2$. If P and Q are polynomials in the variables x and y, we say that (1) is a *polynomial system*.

In the next two theorems we use that any analytic vector field on $\mathbf{S}^2$ has finitely many limit cycles as it was proved by Il'Yashenko [9] and Ecalle [6]. The following results improve Theorem 2 for planar polynomial differential systems.

Theorem 3. *For every planar polynomial system there exist finitely many invariant curves and singular points γ_i, $i = 1, 2, \ldots, l$, such that $\mathbf{R}^2 \setminus \left(\bigcup_{i=1}^{l} \gamma_i \right)$ has finitely many connected open sets, and on each of these connected sets the system has an analytic first integral.*

Next we consider the homogeneous polynomial vector field

$$\mathbf{Q}(\mathbf{x}) = (Q_1(\mathbf{x}), Q_2(\mathbf{x}), Q_3(\mathbf{x})), \qquad \mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3.$$

We obtain the following results related with its first integrals.

Theorem 4. *For a homogeneous polynomial vector field $\mathbf{Q}(\mathbf{x})$ in $\mathbf{R}^3$ having each component the same degree , there exist finitely many invariant straights lines and invariant conical surfaces such that their complement in $\mathbf{R}^3$ is the union of finitely many invariant open connected components, and on each of these connected open components the vector field has an analytic first integral.*

This paper is organized as follows. In Section 2, we prove Theorem 2. In Section 3, we recall the Poincar, compactification of planar polynomial vector fields. Finally, in Sections 4 and 5 we prove Theorems 3 and 4, respectively.

2. Proof of Theorem 2

The following result due to Neumann [12] plays a main role in the proof of our Theorem 2, since its proof is short we give it here.

Lemma 5. *The flow of (M, ϕ) on every canonical region is a C^0–parallel flow given by either a strip, an annular, a spiral, or a toral flow.*

Proof: Let U be a canonical region of the C^r flow (M, ϕ). We denote the flow ϕ on U by $(U, \phi' = \phi|_U)$. Since there are no separatrices in U, the set consisting of orbits homeomorphic with $\mathbf{S}^1$ is open, and similarly the set consisting of orbits homeomophic with $\mathbf{R}$ is open. Hence U consists entirely of closed orbits or entirely of line orbits.

We claim that two orbits of ϕ' can be separated with disjoint parallel neighborhoods. To prove this, we suppose that $\gamma(p)$ and $\gamma(q)$ are distinct orbits (closed or not) which cannot be separated. Then, for any parallel neighborhood N_p of p, we have $q \in \mathrm{cl}(N_p)$; i.e., $q \in \bigcap_{N_p} \mathrm{cl}(N_p) = \alpha(p) \cup \gamma(p) \cup \omega(p)$. This means that $q \in \alpha(p)$ (or $q \in \omega(p)$), but it is impossible because $q \in N_q \subset U$ and $\alpha(p) \cup \omega(p) \subset \mathrm{cl}(N_q) \setminus N_q \not\subset U$.

It follows that the quotient space U/ϕ', obtained by collapsing orbits of (U, ϕ') to points, is a (Hausdorff) one dimensional manifold. Hence the natural projection $\pi : U \to U/\phi'$ is a locally trivial fibering, there are only four possibilities, the four classes of parallel flows described above. This proves the lemma. ∎

Proof of Theorem 2. Let M be a two dimensional differentiable manifold and ϕ be a C^r flow defined on it by the vector field $\mathbf{X}$. If the manifold M is compact, the flow ϕ is complete; i.e. for every $q \in M$, $\phi_q(t) = \phi(t, q)$ is defined for all $t \in \mathbf{R}$.

In general the manifold M is not compact and the flow need not to be complete. We will show now that the restriction of the flow to each canonical region can be parametrized in such a way that it becomes complete.

Let U be a noncompact canonical region. It follows from the above Lemma that U is a smooth orientable manifold. The differentiable structure of U underlies a unique real analytic structure. This follows from [11] and [8]. Moreover one can consider on U a complex structure which will be necessarily compatiable with ite real analytic structure. Such a complex structure on U can be constructed by putting any Riemannian metric on U and by considering local coordinates which are compatiable with a fixed orientation and isothermal with respect to the metric. The coordinate changes are then holomorphic (see for instance [3]).

The Riemann uniformization theorem states that the universal covering space $\widetilde{U}$ of U is holomorphically equivalent to the complex line $\mathbb{C}$ or to the unit disc $\mathbb{D}$. In both cases the (holomorphic) projection map $\widetilde{U} \to U$ induces a real analytic geodesically complete Riemannian metric on U: the projection of the Euclidean metric in case $\widetilde{U} \cong \mathbb{C}$ or the projection of the hyperbolic metric in case $\widetilde{U} \cong \mathbb{D}$.

We claim that the vector field $\mathbf{Y} = \mathbf{X}/\|\mathbf{X}\|$ on U, where $\|\cdot\|$ stands for the norm associated to the metric constructed above, is complete and of class C^r, where $r = 1, \ldots, \infty, \omega$ is the degree of derivability of $\mathbf{X}$.

Let ψ denote the flow on U associated to $\mathbf{Y}$. It follows from the above Lemma that the positive (respectively negative) semiorbit of ψ through a given point $p \in U$ is closed, and therefore $\psi(t,p)$ is defined for each time, or is not contained in any compact subset of U. Since the metric is complete in the second situation the semiorbit will have an infinite length. But by construction the time parameter of ψ is just the arc-length of the orbit, hence $\psi(t,p)$ is also defined for each time. This proves the completeness of ψ.

Let (U,ψ) be a canonical region of M where ψ denotes either the complete C^r flow constructed above in the case U is noncompact or the original flow ϕ in case U is compact. For $p \in U$ let Σ_p be a local tranversal section of the flow ψ with $p \in \Sigma_p$. Let V_p be the set $\{\psi(t,q) \in U : t \in \mathbf{R} \text{ and } q \in \Sigma_p\}$. Then if the canonical region (U,ψ) is a strip or a spiral, respectively an annular or a toral, the flow ψ defines a C^r diffeomorphism:

$$\begin{array}{rccl} \eta_p^{-1} = \psi : & \Sigma_p \times \mathbf{R} & \longrightarrow & V_p, \\ & (q,t) & \longrightarrow & \psi(t,q), \end{array}$$

respectively

$$\begin{array}{rccl} \eta_p^{-1} = \psi : & \Sigma_p \times \mathbf{S}^1 & \longrightarrow & V_p, \\ & (q,t) & \longrightarrow & \psi(tT(q),q), \end{array}$$

where $T(q)$ denotes the minimal period for which the flow ψ with initial point q moves along $\mathbf{S}^1$. Moreover, in both cases the inverse map η_p is also a C^r diffeomorphism.

Take p, $q \in U$ such that $V_p \cap V_q \neq \emptyset$. We have that the map $\eta_q \circ \eta_p^{-1}$ is a C^r diffeomophism. This proves that the quotient space U/ψ has a natural structure of C^r manifold of dimension 1 which is C^r equivalent to either $\mathbf{R}$ or $\mathbf{S}^1$. Moreover the projection $U \to U/\psi$ is a locally trivial fibre bundle of class C^r with fibre $\mathbf{R}$ or $\mathbf{S}^1$. Lemma 5 implies that this fibre bundle is trivial and, according to the different possibilities, the flow (U,ψ) is C^r equivalent to either the strip, annular, spiral or the toral flow. This proves that the canonical region U of M is C^r parallel.

We now prove statement (2). From statement (1) it follows that for the C^r flow ψ associated with the vector field $\mathbf{Y}$, in every canonical region U of M there exists a C^r diffeomorphism h from (U,ψ) onto (V,ξ) which takes orbits of ψ onto orbits of ξ preserving or reversing simultaneously the sense of all orbits, where (V, ξ) is one of the four parallel flows. For the strip and toral flows the first integral is $H(x, y) = y$, for the annular flow the first integral is $H(r, \theta) = r$, and for the spiral flow the first integral is $H(r, \theta) = \theta$. Moreover, we obtain that $h^{-1} \circ H$ is a C^r first integral of the C^r flow ψ on the canonical region U. This proves the theorem. ∎

3. Poincaré compactification of planar polynomial vector fields

Let $\mathbf{X}$ be a planar real polynomial vector field of degree n. The *Poincaré compactified vector field* $p(\mathbf{X})$ corresponding to $\mathbf{X}$ is a vector field induced on $\mathbf{S}^2$ as follows (see for instance [7] and [1]). Let $\mathbf{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbf{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (called the *Poincaré sphere*) and $T_y\mathbf{S}^2$ be the tangent space to $\mathbf{S}^2$ at point y. Consider the central projections $f_+ : T_{(0,0,1)}\mathbf{S}^2 \longrightarrow \mathbf{S}^2_+ = \{y \in \mathbf{S}^2 : y_3 > 0\}$ and $f_- : T_{(0,0,1)}\mathbf{S}^2 \longrightarrow \mathbf{S}^2_- = \{y \in \mathbf{S}^2 : y_3 < 0\}$. These maps define two copies of $\mathbf{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathbf{X}'$ the vector field $Df_+ \circ \mathbf{X}$ and $Df_- \circ \mathbf{X}$ defined on $\mathbf{S}^2$ except on its equator $\mathbf{S}^1 = \{y \in \mathbf{S}^2 : y_3 = 0\}$. Obviously $\mathbf{S}^1$ is identified to the infinity of $\mathbf{R}^2$. The *Poincaré compactification* $p(\mathbf{X})$ is the only analytic extension of $y_3^{n-1}\mathbf{X}'$ to $\mathbf{S}^2$. Under the flow of the compactified vector field $p(\mathbf{X})$, the equator $\mathbf{S}^1$ is invariant. On $\mathbf{S}^2 \backslash \mathbf{S}^1$ there are two symmetric copies of $\mathbf{X}$, and knowing the behaviour of $p(\mathbf{X})$ around $\mathbf{S}^1$, we know the behaviour of $\mathbf{X}$ near infinity. The projection of the closed northern hemisphere of $\mathbf{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc*, it is denoted by $\mathbf{D}^2$. The singular points of $p(\mathbf{X})$ on $\mathbf{S}^1$ are called the *infinite* singular points of $\mathbf{X}$.

As $\mathbf{S}^2$ is a differentiable manifold, for computing the expression of $p(\mathbf{X})$, we can consider the six local charts $U_i = \{y \in \mathbf{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbf{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$, and the diffeomorphisms $F_i : U_i \longrightarrow \mathbf{R}^2$ and $G_i : V_i \longrightarrow \mathbf{R}^2$ defined as the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$, then z represents different things according to the local charts under consideration. Some straightforward calculations give for $p(\mathbf{X})$ the following expressions:

$$z_2^n \Delta(z) \left[Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right),\ -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) \right] \quad \text{in} \quad U_1,$$

$$z_2^n \Delta(z) \left[P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right),\ -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right] \quad \text{in} \quad U_2,$$

$$\Delta(z)[P(z_1, z_2),\ Q(z_1, z_2)] \qquad \text{in} \quad U_3,$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{n-1}{2}}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbf{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(\mathbf{X})$. Thus we obtain a polynomial vector field of degree at most $n + 1$ in each local chart.

4. Some preliminaries and the proof of Theorem 3.

We first recall some basic results, which will be used in the later on.

Poincaré–Bendixson Theorem (see Anosov and Arnold [2] or Roussarie [13] Chapter 1). *Let $\mathbf{X}$ be a vector field on a compact surface S of genus 0. Suppose that every singular point of $\mathbf{X}$ is isolated. Then the ω–limit set of the phase curves of $\mathbf{X}$ must be one of the following types:*

- *a singular point;*
- *a periodic orbit;*
- *a polycycle, i.e. a union of finitely many singular points $p_1, \ldots, p_m$, and finitely many orbits $\gamma_1, \ldots, \gamma_m$ such that $\alpha(\gamma_i) = p_i$ for $i = 1, \ldots, n$ and $\beta(\gamma_i) = p_{i+1}$ for $i = 1, \ldots, m-1$ and $\beta(\gamma_m) = p_1$; eventually some of the points p_i can be identified.*

Theorem 6 (see Il' Yashenko [9], or Lefschetz [10] Chapter X). *Any isolated singular point of a two dimensional analytic system can only be one of following three cases:*

- *a center;*
- *a focus;*
- *a singular point whose neighbourhood is a union of finitely many hyperbolic, elliptic or parabolic sectors.*

In order to prove Theorem 3, we need the following result.

Proposition 7. *Assume that P and Q are polynomials in the variables x and y, and that they are relative prime. If $yP_n(x,y) - xQ_n(x,y) \not\equiv 0$, where $n = max\{deg(P), deg(Q)\}$, P_n and Q_n are the homogeneous polynomials of degree n of P and Q respectively. Then system (1) has finitely many separatrices and canonical regions in the Poincar, disc.*

Proof: From the assumptions of the proposition we know that polynomial system (1) has finitely many singular points (finite and infinite). So they are isolated. Since a polynomial system is C^ω, by Theorem 6 we get that its singular points are centers, foci, or union of finitely many hyperbolic, elliptic or parabolic sectors.

Again as a polynomial system is C^ω, from Poincar,–Bendixson Theorem we obtain that in the Poincar, disc the α–limit and ω–limit of every orbit for system (1) is a singular point (finite or infinite), a periodic orbit, or a polycycle.

We claim that each separatrix of polynomial system (1) is one of the following types:

· a singular point (finite or infinite);

· a limit cycle;

· a boundary of a hyperbolic sector of a singular point.

We now prove the claim. By the definition of separatrix it is easy to prove that every singular point is a separatrix, and that each limit cycle is also a separatrix.

A boundary orbit γ of a hyperbolic sector is a separatrix because there are no parallel neighbourhoods which contain γ and satisfy the condintions (a) and (b) . Otherwise, all orbits in the parallel neighborhood have the same α–limit and the same ω–limit, and this is in contradiction with the fact that the orbit γ is in the boundary of a hyperbolic sector.

From Poincaré–Bendixson Theorem and Theorem 6, it is easy to obtain that the separatrices of system (1) can only be singular points, limit cycles, or the boundary orbits of a hyperbolic sector. This proves the claim.

We recall that under our assumptions system (1) has finitely many singular points. Il'Yashenko [9] proved that a given analytic system on $\mathbf{S}^2$ has finitely many limit cycles (see also Ecalle [6]). Since a polynomial system is analytic, we get from Theorem 6 that the number of boundaries of hyperbolic sectors of all singular points (finite and infinite) for system (1) is finite. Therefore, system (1) has finitely many separatrices.

Since the boundary of every canonical region is formed by separatrices, system (1) has finitely many canonical regions. This completes the proof of the theorem. ∎

Proof of Theorem 3: If the polynomial system (1) has infinitely many singular points in the finite plane, then the polynomials P and Q must have a common real factor. Let $B(x, y)$ be the maximum common real factor of P

and Q, and $P = B\overline{P}(x,y)$ and $Q = B\overline{Q}(x,y)$. Then, from the definition of first integral it follows that system (1) and the system

$$\dot{x} = \overline{P}(x,y), \qquad \dot{y} = \overline{Q}(x,y), \tag{2}$$

have the same first integrals. Obviously, system (2) has finitely many finite singular points. So, in the following we only consider system (2) with finitely many finite singular points.

Since the vector field $\mathbf{X}$ associated to system (2) is polynomial, according to Section 3 $\mathbf{X}$ can be compactified to the vector field $p(\mathbf{X})$ defined on the Poncar, sphere $\mathbf{S}^2$. On $\mathbf{S}^2$ the compactified vector field is also analytic. Meanwhile, the finite singular points and separatrices of the vector field $\mathbf{X}$ in the finite plane correspond to the singular points and separatrices of $p(\mathbf{X})$ on $\mathbf{S}^2$ excepting the equator.

If the vector field $\mathbf{X}$ has infinitely many singular points at infinity, then the compactified vector field $p(\mathbf{X})$ has infinitely many singular points on the equator. From the expression of the compactified vector field as it is shown in Section 3 we know that the two polynomials in z_1 and z_2 must have the common factor z_2. Rescaling the time we can elimilate the maximum common factor in z_2, then the new vector field has finitely many singular points on the correspondig local chart. Obviously, the new vector field has the same first integrals than $p(\mathbf{X})$. In what follows, without loss of generality, we can assume that compactified vector field has finitely many singular points in every local chart.

From Proposition 7 the compactified vector field has finitely many separatrices on $\mathbf{S}^2$. We denote by Γ_i for $i = 1, 2, \ldots, l$ the separatrices of $p(\mathbf{X})$. Then $\mathbf{S}^2 \backslash \left(\bigcup_{i=1}^{l} \Gamma_i \right)$ is a union of finitely many connected open sets. Every open set is a canonical region. From Theorem 2 and the compactification of the vector fields on $\mathbf{S}^2$ we obtain that the vector field $p(\mathbf{X})$ has an analytic first integral H on every canonical region.

Since the boundary of the canonical regions of $p(\mathbf{X})$ on $\mathbf{S}^2$ correspond to finitely many invariant curves $\{\Gamma_i, i = 1, \ldots, k\}$ formed by separatrices or curves of singular points, and isolated singular points of $\mathbf{X}$ on $\mathbf{R}^2$, let Σ be the union of $\bigcup_{i=1}^{l} \Gamma_i$ with the isolated singular points, then $\mathbf{R}^2 \backslash \Sigma$ is a finite union of connected open components. Since the diffeomorphism f_+ from $\mathbf{R}^2$ to $\mathbf{S}^2_+$ is analytic (see Section 3), $H \circ f_+$ is an analytic first integral of $\mathbf{X}$ in the corresponding connected open component. This proves the theorem. ∎

5. Proof of Theorem 4.

Consider the homogeneous polynomial vector field $\mathbf{Q}(\mathbf{x}) = (Q_1(\mathbf{x}), Q_2(\mathbf{x}), Q_3(\mathbf{x}))$, where $Q_i(\mathbf{x})$ for $i = 1, 2, 3$, are homogeneous polynomials in $\mathbf{x} = (x_1, x_2, x_3)$ of the same degree. Since the vectors defined by $\mathbf{Q}(\mathbf{x})$ are parallel at all points on the straight line passing through the origin $(0, 0, 0)$, we project all the vectors $\mathbf{Q}(\mathbf{x})$ along every straight line passing through the origin onto the sphere $\mathbf{S}^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, and we get a tangent vector field $\mathbf{Q}_T(\mathbf{x})$ on $\mathbf{S}^2$. Obviously, we have

$$\mathbf{Q}_T(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) - \langle \mathbf{x}, \mathbf{Q}(\mathbf{x}) \rangle \cdot \mathbf{x}, \qquad \mathbf{x} \in \mathbf{S}^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors, $\langle \mathbf{x}, \mathbf{Q}(\mathbf{x}) \rangle \cdot \mathbf{x}$ denotes the projection of $\mathbf{Q}(\mathbf{x})$ on the direction $\mathbf{x}$ (see for instance, Ye [14] §22). Moreover, the vector field $\mathbf{Q}_T(\mathbf{x})$ is symmetric or anti–symmetric with respect to the origin.

Define

$$\Pi_3 = \{\mathbf{x} \in \mathbf{R}^3 : x_3 = 1\}.$$

Then Π_3 is the tangent plane to $\mathbf{S}^2$ at $(0, 0, 1)$. We now define a vector field in Π_3 as follows:

$$\begin{aligned} \mathbf{W}_Q(\mathbf{x}) &= (W_1(\mathbf{x}), W_2(\mathbf{x}), 0) \\ &= (-x_1 Q_3(\mathbf{x}) + Q_1(\mathbf{x}), -x_2 Q_3(\mathbf{x}) + Q_2(\mathbf{x}), 0), \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, 1)$. It follows from Ye [14] that the vector field $\mathbf{W}_Q(\mathbf{x})$ is tangent to the intersection curve of $\Omega(\gamma) = \{tp : t \in \mathbf{R}, p \in \gamma\}$ with Π_3, where γ is a trajectory of $\mathbf{Q}_T(\mathbf{x})$ on $\mathbf{S}^2$. Then $\Omega(\gamma)$ is a surface constructed by straight lines connecting the origin with all points of γ. Furthermore, this surface is invariant under the flow of $\mathbf{Q}(x)$, and it is called an *invariant conical surface*. This conical surface is symmetric with respect to the origin.

It is easy to prove that for an arbitrary point $P \in \mathbf{S}^2$, if P is a singular point of $\mathbf{Q}_T(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$, then the straight line l connecting O and P is formed by singular points of $\mathbf{Q}(\mathbf{x})$. If P is a singular point of $\mathbf{Q}_T(\mathbf{x})$, but not a singular point of $\mathbf{Q}(\mathbf{x})$, then l is an invariant straight line of $\mathbf{Q}(\mathbf{x})$ with a unique singular point O on l.

If the vector field $W_Q(\mathbf{x})$ has infinitely many singular points, it must have common factors. By rescaling the time we eliminate the maximum common factor, then the new vector field through the central projection produces a vector field $\overline{\mathbf{P}}_T(\mathbf{x})$ on $\mathbf{S}^2$ with finitely many singular points outside the equator. If the vector field $\overline{\mathbf{P}}_T(\mathbf{x})$ has infinitely many singular points on the equator, working in a similar way to the proof of Theorem 3, we can get a vector field $\mathbf{P}_T(\mathbf{x})$ on $\mathbf{S}^2$ such that on every local chart has finitely many singular points.

Using the arguments of the proofs of Theorems 2 and 3 we obtain that the vector field $\mathbf{P}_T(\mathbf{x})$ on $\mathbf{S}^2$ has finitely many canonical region on $\mathbf{S}^2$, and that in every canonical region the vector field $\mathbf{P}_T(\mathbf{x})$ has an analytical first integral.

We claim that the boundary of every canonical region of $\mathbf{P}_T(\mathbf{x})$ is invariant under the flow of $\mathbf{Q}_T(\mathbf{x})$, and that the first integral of $\mathbf{P}_T(\mathbf{x})$ in every canonical region is also a first integral of $\mathbf{Q}_T(\mathbf{x})$. The first statement follows easily. We now prove the second statement. For any first integral $H(\mathbf{x})$ and any point $\mathbf{p}$ in a canonical region U of $\mathbf{P}_T(\mathbf{x})$, if $\mathbf{Q}_T(\mathbf{p}) \neq 0$ then the vector is parallel to the vector $\mathbf{P}_T(\mathbf{p}) \neq 0$, so along every orbit in U different from a singular point of $\mathbf{Q}_T(\mathbf{x})$ $H(\mathbf{x})$ is also a constant. If $\mathbf{p}$ is a singular point of $\mathbf{Q}_T(\mathbf{x})$ in U, on this special orbit $H(\mathbf{x})$ is clearly a constant. Thus, we obtain an analytical first integral of $\mathbf{Q}_T(\mathbf{x})$ on each of finitely many open connected canonical regions of $\mathbf{S}^2$.

Since the boundaries of the canonical regions are invariant under the flow of $\mathbf{Q}_T(\mathbf{x})$, the surfaces constructed by the rays connecting O and all the points on the boundaries form finitely many invariant straight lines and invariant conical surfaces. In $\mathbf{R}^3$ the complement of these invaraint straight lines and conical surfaces are finitely many open connected conical tubes. If $H(\mathbf{x})$ is an analytic first integral of $\mathbf{Q}_T(\mathbf{x})$ in a canonical region, then $F(\mathbf{x}) = H(\mathbf{x}/\|\mathbf{x}\|)$ is an analytic first integral of $\mathbf{Q}(\mathbf{x})$ in the corresponding conical tube, where $\|\cdot\|$ denotes the norm induced by the Euclidean distance. This completes the proof of the theorem. ∎

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