Brownian Motion Reflected on Brownian Motion

Krzysztof Burdzy* David Nualart†

1. Introduction. We will investigate some aspects of the local time, parabolic measure and excursion theory of Brownian motion reflected on Brownian motion.

Reflected Brownian motion in a domain with a time-varying boundary has appeared in several articles (Bass and Burdzy [BB1], Cranston and Le Jan [CLJ], El Karoui and Karatzas [EKK1,2], Knight [K] and Soucaliuc, Toth and Werner [STW]) although it had never been the main subject of study until a recent paper of Burdzy, Chen and Sylvester [BCS]. The last article contains a number of theorems on reflected Brownian motion and the corresponding heat equation in domains with smooth space-time boundaries. If the boundary of a space-time domain is of class C^3 then practically all results on reflected Brownian motion in fixed domains can be proved in the new setting. [BCS] also shows that various singularities appear in domains which have rough boundaries. The critical shape of the moving boundary seems to be the square root. Analysts observed some time ago that rough boundaries cause difficulties in the parabolic potential theory. For example, the results of Hofmann and Lewis [HL] or Lewis and Murray [LM] (see also references in these papers) show that if the boundary of a time-varying domain is given by a Hölder function with exponent 1/2 then the parabolic measure is absolutely continuous with respect to the Lebesgue measure but this is not necessarily true in less smooth domains.

We decided to examine a particular domain with a rough time-dependent boundary, namely, a domain bounded by a Brownian path. We chose a Brownian path because Brownian trajectories are known to have rough behavior and so are qualitatively different from smooth graphs. On the other hand, there exists a large body of literature describing their local properties;

*Research partially supported by NSF grant DMS-0071486 (USA) and SAB1999-0171 (Spain).

†Research partially supported by the grant PB96-0087 from the DGES.

hence there are good reasons to believe that one can prove many concrete results on Brownian motion reflected on Brownian motion. It is well known that a typical Brownian path is Hölder with exponent $1/2+\delta$ for any $\delta > 0$ but fails to be Hölder with exponent $1/2$. Hence we will be dealing with the situation very close to the "critical modulus of continuity."

Brownian motion reflected on Brownian motion appeared in recent papers by Soucaliuc, Toth and Werner [STW] and Burdzy, Chen and Sylvester [BCS]; we will describe some of those results later in our article. We parenthetically mention an article ([B2]) with a title which is similar to the present one, namely, "A three-dimensional Brownian path reflected on a Brownian path is a free Brownian path;" the similarity does not extend beyond the title, though.

2. Results on global distributions. Our original goal was to study local properties of reflected Brownian motion but it turned out that some global results were needed as technical elements of the proofs—we hope that they will have some interest of their own.

First we will review some facts about reflected processes. Let us consider a space-time domain with continuous deterministic moving boundary $g(t)$, i.e., $D = \{(t,x) : t \geq 0, x \leq g(t)\}$. [BCS] contains several definitions of Brownian motion X_t in D reflected on $g(t)$. The definition which is most relevant to our present project uses the "Skorohod Lemma" [BCS, Lemma 3.13] or [KS, Lemma 3.6.14]. Suppose that B_t is a Brownian motion with $B_0 \leq g(0)$. Then there exists a unique continuous non-decreasing process L_t with $L_0 = 0$, such that $X_t = B_t - L_t \leq g(t)$ for all $t \geq 0$, and L_t may increase only when X_t is on the boundary of the domain D, i.e.,

$$\int_0^\infty \mathbf{1}_{(-\infty,g(s))}(X_s) dL_s = 0.$$

By convention, the reflected process will stay below or at the moving boundary, unless stated otherwise. The definition applies in the obvious way to any random function $g(t)$ which is continuous with probability 1.

In our paper all stochastic processes will be denoted by capital letters such as B or X except that we will use g to denote a random moving reflecting boundary to emphasize that, most of the time, g should be thought of as a "fixed Brownian path."

Theorem 2.1. *Suppose that $g(t)$ and B_t are independent Brownian motions starting from $g(0) = B_0 = 0$. Consider the Brownian motion X_t reflected on $g(t)$, obtained from B_t by the means of the Skorohod lemma.*

Let L_t denote the local time of X_t on $g(t)$ and let $\xi = \inf\{t: L_t = 1\}$. Define processes $\widetilde{X}_t = X_{\xi-t} - X_\xi$ and $\widetilde{g}(t) = g(\xi - t) - g(\xi)$. The pair of processes $\{(-\widetilde{X}_t, -\widetilde{g}(t)), t \in [0,\xi]\}$ has the same distribution as $\{(g(t), X_t), t \in [0,\xi]\}$.

Proof. Let $Z^1_t = (g(t) - B_t)/\sqrt{2}$ and $Z^2_t = (g(t) + B_t)/\sqrt{2}$ for $t \geq 0$. Let L^Y_t be the unique non-decreasing continuous process, constructed according to the Skorohod lemma, such that $Y^1_t = Z^1_t + L^Y_t \geq 0$ for all $t \geq 0$, and such that L^Y_t does not increase when $Y^1_t > 0$. Let $K^1_t = (Y^1_t + Z^2_t)/\sqrt{2}$ and $K^2_t = (-Y^1_t + Z^2_t)/\sqrt{2}$ and note that the condition $Y^1_t = 0$ is equivalent to $K^1_t = K^2_t$.

Consider the following vectors,

$$\mathbf{n} = (\sqrt{2}/2, -\sqrt{2}/2), \qquad \mathbf{m} = (0, -\sqrt{2}), \qquad \mathbf{q} = (-\sqrt{2}/2, -\sqrt{2}/2).$$

It is elementary to check that

$$(K^1_t, K^2_t) = (g(t), B_t) + \mathbf{n}\, L^Y_t. \tag{2.1}$$

We now define a vector process

$$(R^1_t, R^2_t) = (g(t), B_t) + \mathbf{m}\, L^Y_t. \tag{2.2}$$

Since $\mathbf{m} = \mathbf{n} + \mathbf{q}$, we obtain from (2.1) and (2.2)

$$(R^1_t, R^2_t) = (K^1_t, K^2_t) + \mathbf{q}\, L^Y_t. \tag{2.3}$$

The vector $\mathbf{q}$ is parallel to the diagonal so the statement $R^1_t = R^2_t$ is equivalent to $K^1_t = K^2_t$ and, therefore, it is equivalent to $Y^1_t = 0$.

We will show that in fact $(R^1_t, R^2_t) = (g(t), X_t)$, i.e.,

$$(g(t), X_t) = (K^1_t, K^2_t) + \mathbf{q}\, L^Y_t. \tag{2.4}$$

Since the first component of the vector $\mathbf{m}$ is equal to 0, so is the first component of the last term in (2.2) and hence $R^1_t = g(t)$.

We have $Y^1_t \geq 0$, so the process (K^1_t, K^2_t) always stays on or below the diagonal. The same is true of (R^1_t, R^2_t) because the last term in (2.3) is a vector parallel to the diagonal. In other words, $R^2_t \leq R^1_t = g(t)$ for all t. From (2.2) we obtain

$$R^2_t = B_t + (-\sqrt{2})L^Y_t.$$

The process $t \to (-\sqrt{2})L^Y_t$ is monotone and decreases only when $Y^1_t = 0$, i.e., only when $R^2_t = R^1_t = g(t)$. By the uniqueness part of the Skorohod lemma, the process R_t^2 is the path of B_t reflected on $g(t)$, i.e., it is equal to X_t. This completes the proof of (2.4).

As a by-product of our argument we have $\sqrt{2}L_t^Y = L_t$, where L_t has been defined before the statement of Theorem 2.1.

Let $\widetilde{Y}_t^1 = Y_{\xi-t}^1 - Y_\xi^1$ for $t \in [0,\xi]$ and apply the same notation convention for other processes reversed at time ξ, except for $\widetilde{L}_t^Y$ which is defined as $-(L_{\xi-t}^Y - L_\xi^Y)$. It is easy to see that the processes $\{Y_t^1, t \in [0,\xi]\}$ and $\{\widetilde{Y}_t^1, t \in [0,\xi]\}$ have identical distributions. This follows, for example, from excursion theory. Each of these processes can be assembled from excursions generated by the Itô Poisson point process of excursions on the local time interval $[0,1]$. The Poisson point process of excursions and the individual excursions are symmetric with respect to time reversal.

Note that Z_t^1 and Z_t^2 are independent Brownian motions. It follows that Y_t^1 and Z_t^2 are independent. Since ξ may be defined in terms of Y_t^1, i.e., $\xi = \inf\{t : \sqrt{2}L_t^Y = 1\}$, it follows that ξ is a random variable independent of Z_t^2. The process $\widetilde{Y}_t^1$ is defined in terms of Y_t^1 so this and the previous remark imply that $\widetilde{Z}_t^2$ is a Brownian motion independent of $\widetilde{Y}_t^1$. We conclude that the vector processes (Y_t^1, L_t^Y, Z_t^2) and $(\widetilde{Y}_t^1, \widetilde{L}_t^Y, \widetilde{Z}_t^2)$ have identical distributions.

Next observe that K_t^1 and K_t^2 are deterministic functions of Y_t^1 and Z_t^2. The processes $\widetilde{K}_t^1$ and $\widetilde{K}_t^2$ are defined in terms of $\widetilde{Y}_t^1$ and $\widetilde{Z}_t^2$ in the analogous way.

Formula (2.4) can be used to determine the trajectories of $g(t)$ and X_t from the paths of K_t^1, K_t^2 and L_t^Y. It follows easily from the construction of K_t^1 and K_t^2 that (K_t^1, K_t^2, L_t^Y) has the same distribution as $(-K_t^2, -K_t^1, L_t^Y)$. This implies that

$$(-X_t, -g(t)) = (-K_t^2, -K_t^1) - \mathbf{q}\, L_t^Y \tag{2.5}$$

has the same distribution as

$$(K_t^1, K_t^2) - \mathbf{q}\, L_t^Y. \tag{2.6}$$

We obtain directly from the definitions of time reversed processes and (2.4),

$$(\widetilde{g}(t), \widetilde{X}_t) = (\widetilde{R}_t^1, \widetilde{R}_t^2) = (\widetilde{K}_t^1, \widetilde{K}_t^2) - \mathbf{q}\, \widetilde{L}_t^Y. \tag{2.7}$$

By comparing (2.5), (2.6) and (2.7), we see that $(\widetilde{g}(t), \widetilde{X}_t)$ has the same distribution as $(-X_t, -g(t))$. □

For future reference we summarize the construction of $(g(t), X_t)$ contained in the proof of Theorem 2.1. See [VW] for information about reflected

Brownian motion with oblique angle of reflection used in this construction. Consider a half-space $H = \{(y_1, y_2) : y_1 \geq y_2\}$ and let $(g(t), X_t)$ be a two-dimensional Brownian motion in H with oblique reflection on the boundary, whose vector of reflection $\mathbf{m}$ is equal to $(0, -\sqrt{2})$.

The process $(g(t), X_t)$ satisfies the following two-dimensional Skorohod representation. There exist a two-dimensional Brownian motion $(g(t), B_t)$ and a continuous additive functional $\widehat{L}_t^1$ of $(g(t), B_t)$ increasing only on ∂H such that a.s. for all $t \geq 0$,

$$(g(t), X_t) = (g(t), B_t) + \int_0^t \mathbf{m} \, d\widehat{L}_s^1. \tag{2.8}$$

The process $\widehat{L}_t^1$ can be identified with the local time L_t^Y in the proof of Theorem 2.1. It follows from (2.8) that for almost every trajectory of $g(t)$, the process X_t is a Brownian motion reflected on $g(t)$ in the sense of the "deterministic" Skorohod lemma. Indeed, $X_t = B_t - L_t^1$, where $L_t^1 = \sqrt{2}\widehat{L}_t^1$; we always have $B_t - L_t^1 \leq g(t)$; the process L_t^1 is continuous, non-decreasing and it does not increase when $X_t \neq g(t)$.

A recent paper by Soucaliuc, Toth and Werner [STW] contains strikingly counter-intuitive and beautiful theorems about Brownian motion reflected on Brownian motion. Here is a simple version of one of their results. Suppose that B_t and W_t are independent Brownian motions with $B_0 = W_0 = 0$. Let $g(t) = W_t - W_1$ for $t \in [0, 1]$ and let X_t be the path B_t reflected on $g(t)$ according to the Skorohod lemma. The trajectory of X_t lies above or below $g(\cdot)$ depending on whether $g(0) < 0$ or $g(0) > 0$. Then $\{X_t, 0 \leq t \leq 1\}$ has the distribution of the standard Brownian motion. The proof given in [STW] has combinatorial nature. We have learnt (private communication) that F. Soucaliuc and W. Werner have a new proof based on ideas similar to those in our proof of Theorem 2.1.

We will prove another "global" result about Brownian motion reflected on Brownian motion. It will be needed in the next section.

Let B_t be a Brownian motion with $B_0 = 0$. Suppose that C_t is a 3-dimensional Bessel process independent of B_t and starting from 0. A process with the same distribution as $\{(B_t + C_t)/\sqrt{2}, t \geq 0\}$ will be called a BMB-process.

Theorem 2.2. *Let X_t be defined as in Theorem 2.1. The process $-X_t$ is a BMB-process.*

Proof. Recall the definitions and the notation from the proof of Theorem 2.1. From (2.4) and the definitions of K_t^2 and Y_t^1 we have

$$\begin{aligned} -X_t &= -K_t^2 + \frac{1}{\sqrt{2}} L_t^Y \\ &= -\frac{1}{\sqrt{2}}(-Y_t^1 + Z_t^2) + \frac{1}{\sqrt{2}} L_t^Y \\ &= -\frac{1}{\sqrt{2}}(-(Z_t^1 + L_t^Y) + Z_t^2) + \frac{1}{\sqrt{2}} L_t^Y \\ &= \frac{1}{\sqrt{2}}((Z_t^1 + 2L_t^Y) - Z_t^2). \end{aligned}$$

The definition of L_t^Y in the proof of Theorem 2.1 and the Skorohod lemma imply that $L_t^Y = -\inf\{Z_s^1 : 0 \le s \le t\}$. Since Z_t^1 and Z_t^2 are independent Brownian motions, the processes $Z_t^1 + 2L_t^Y$ and $-Z_t^2$ are also independent and the second one is a Brownian motion. The first process, $Z_t^1 + 2L_t^Y = Z_t^1 - 2\inf\{Z_s^1 : 0 \le s \le t\}$ is a 3-dimensional Bessel process, by Pitman's theorem ([P]). The theorem now follows from the definition of a BMB-process. □

3. Two different local times on the boundary. A natural question about any reflected Brownian motion is whether one can explicitly represent its local time on the boundary using intuitive notions. We will examine four possible definitions of the local time.

In principle, we may obtain four different continuous additive functionals in this way, although classical results on the local time of standard Brownian motion at 0 show that all "natural" definitions of local time agree. Two of the local times defined below (L_t^1 and L_t^4) are elements of analytic formulae and so they can be used, in theory at least, to calculate various probabilities. For this reason, these two local times seem to have the greatest significance—it turns out that they are different from each other.

From now on, $g(t)$ and X_t will denote the processes constructed in the paragraph preceding (2.8). As a by-product of that construction, we have obtained our first local time, $L_t^1 = \sqrt{2}\widehat{L}_t^1$.

The second definition of a local time of X_t on $g(t)$ is the following.

$$L_t^2 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[g(s)-\varepsilon, g(s)]}(X_s) ds. \tag{3.1}$$

The existence of the limit on the right hand side will be proved below.

The third definition of a local time will be based on excursion count. We will call a piece $\{X(s), s \in (u_l, u_r)\}$ of a trajectory of reflected Brownian motion an excursion of X_s from $g(s)$ if $X(u_l) - g(u_l) = X(u_r) - g(u_r) = 0$ but $X_s \neq g(s)$ for all $s \in (u_l, u_r)$. Let N_t^ε be the number of excursions of X_s from $g(s)$ with $u_l \in [0, t]$ whose maximum distance from g exceeds ε, i.e., $\sup_{s \in (u_l, u_r)} |g(s) - X_s| > \varepsilon$. We will show below that

$$L_t^3 = \lim_{\varepsilon \to 0} \varepsilon N_t^\varepsilon \tag{3.2}$$

exists and so defines a local time.

The last definition of the local time will be based on the notion of an exit system which is a rigorous approach to the excursion theory. For the original full discussion of exit systems we refer to Maisonneuve [M]. Our own presentation, although somewhat informal, has to be rather technical due to the nature of the subject. Suppose for the moment that $g(t)$ is a fixed continuous trajectory.

An excursion law $H^{(t,x)}$ is a σ-finite measure on $C_*[0, \infty)$, the set of functions on $[0, \infty)$ with values in $D = \{(t, x) : t \geq 0, x \leq g(t)\}$ which are continuous until a lifetime ζ (finite or infinite) and are equal to the "coffin state" Δ on $[\zeta, \infty)$. Every excursion law is strong Markov on $(0, \infty)$ with the transition probabilities of the space-time Brownian motion killed upon hitting the boundary $g(t)$ of D.

The $H^{(t,x)}$-measure of the set of trajectories not starting from (t, x) is equal to 0. An exit system is a pair $(\widehat{L}^4, \widehat{H})$ consisting of a continuous additive functional $\widehat{L}_t^4$ of Brownian motion X_t reflected on $g(t)$ and a family of excursion laws $\widehat{H}^{(t,x)}$, one for every (t, x) of the form $(t, g(t))$. A pair $(\widehat{L}^4, \widehat{H})$ is called an exit system if it satisfies the exit system formula (3.3) below. Let $\mu(t) = \inf\{s > 0 : \widehat{L}_s^4 > t\}$, $\eta_t = \inf\{s > t : X(s) = g(s)\} - t$ and

$$e_t(s) = \begin{cases} X(t+s) & \text{if } 0 \leq s < \eta_t \text{ and } X_t = g(t), \\ \Delta & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E^{(t,x)} \sum_{0<t<\infty} Z_t f(e_t) &= E^{(t,x)} \int_0^\infty Z_t \widehat{H}^{(t,X_t)}(f) d\widehat{L}_t^4 \\ &= E^{(t,x)} \int_0^\infty Z_{\mu(t)} \widehat{H}^{(\mu(t), X_{\mu(t)})}(f) dt, \end{aligned} \tag{3.3}$$

for all $(t, x) \in D$, all positive predictable processes Z and positive measurable functions f defined on $C_*[0, \infty)$ which vanish on paths equal identically to Δ. The existence of an exit system $(\widehat{L}^4, \widehat{H})$, in particular, the existence of an additive continuous functional $\widehat{L}^4_t$, follow from the results of Maisonneuve [M].

We note that there is a variety of exit systems, for example, if $(\widehat{L}^4, \widehat{H})$ is an exit system, so is $(2\widehat{L}^4, (1/2)\widehat{H})$. We will specify a single "natural" exit system. First, we let A denote the event that an excursion $e_t(\,\cdot\,)$ deviates from g by more than 1 unit, i.e.,

$$A = \{|g(t+s) - e_t(s)| > 1 \text{ for some } s \geq 0\}. \tag{3.4}$$

We take an arbitrary exit system $(\widehat{L}^4, \widehat{H})$. It is easy to see that $\widehat{H}^{(t,x)}(A) > 0$ if and only if the excursion law $\widehat{H}^{(t,x)}$ is not identically zero. If $\widehat{H}^{(t,x)}$ vanishes identically, we let $H^{(t,x)}$ be the same zero measure. For other excursion laws we let $H^{(t,x)}(\,\cdot\,) = \widehat{H}^{(t,x)}(\,\cdot\,)/\widehat{H}^{(t,x)}(A)$ and then we define a new continuous additive functional,

$$L^4_t = \int_0^t H^{(s,X_s)}(A) d\widehat{L}^4_s. \tag{3.5}$$

The local time L^4_t does not increase at (t,x) such that $H^{(t,x)}$ is identically zero. In view of (3.3), the excursions of type A form a Poisson point process with intensity 1 on the L^4_t-local time scale.

We are interested in a random boundary function $g(t)$—a Brownian path. The results on exit systems apply to almost all paths $g(t)$ because Brownian paths are almost surely continuous.

Recall that the existence of local times L^1_t and L^4_t follows from the known results on the existence of obliquely reflected Brownian motion and exit systems.

Theorem 3.1. *(i) The limits in the definitions (3.1) and (3.2) of L^2_t and L^3_t exist a.s. It follows that for almost every path $g(t)$, the convergence in (3.1) and (3.2) holds with (conditional) probability 1.*

(ii) With probability 1, $L^1_t = L^2_t = L^3_t$ for all $t \geq 0$.

(iii) With probability 1, the random measures dL^1_t and dL^4_t are mutually singular.

Proof. Recall the notation from the proof of Theorem 2.1. Note that $Y^1_t = (g(t) - X_t)/\sqrt{2}$. The process Y^1_t is a Brownian motion reflected at 0. The local time L^Y_t of Y^1_t at 0 is equal to $\widehat{L}^1_t$ by (2.8). Let $\widehat{N}^\varepsilon_t$ be the number of downcrossings of $[0,\varepsilon]$ by Y^1_s on the interval $[0,t]$. Then, by Theorems 3.6.17 and 6.2.23 of [KS],

$$\lim_{\varepsilon \downarrow 0} \varepsilon \widehat{N}^\varepsilon_t = \widehat{L}^1_t,$$

for all $t \geq 0$, a.s. Recall the definition of N_t^ε and note that $\widehat{N}_t^\varepsilon$ differs from $N_t^{\sqrt{2}\varepsilon}$ by at most 1. This implies that a.s., for all $t \geq 0$,

$$L_t^1/\sqrt{2} = \widehat{L}_t^1 = \lim_{\varepsilon \downarrow 0} \varepsilon \widehat{N}_t^\varepsilon = \lim_{\varepsilon \downarrow 0} \varepsilon N_t^{\sqrt{2}\varepsilon} = \lim_{\varepsilon \downarrow 0} (1/\sqrt{2}) \varepsilon N_t^\varepsilon = L_t^3/\sqrt{2}.$$

This completes the proof of the existence of L_t^3 and its equivalence to L_t^1.

The limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0 \leq Y_s^1 \leq \varepsilon\}}(\omega) ds$$

exists for almost all ω's and is equal to $2\widehat{L}_t^1$, by Definition 3.6.3 and Theorems 3.6.11 and 3.6.17 of [KS]. Hence for almost all ω,

$$\begin{aligned} L_t^2(\omega) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0 \leq g(s) - X_s \leq \varepsilon\}}(\omega) ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{0 \leq Y_s^1 \leq \varepsilon/\sqrt{2}\}}(\omega) ds = (1/\sqrt{2}) 2\widehat{L}_t^1(\omega) = L_t^1(\omega). \end{aligned}$$

This completes the proof of parts (i) and (ii) of the theorem. Part (iii) is a corollary of Theorem 3.2 below. □

Remark 3.1. For a fixed boundary, i.e., $g(t) \equiv 0$, the normalizations of the local times do not match in the same way as in Theorem 3.1 (ii). We will show that $L_t^1 = (1/2)L_t^2 = L_t^3$. If $g(t) \equiv 0$ then the process $-X_t$ is a classical reflected Brownian motion. In this case, the equality of L_t^1 and L_t^3 follows from Theorems 3.6.17 and 6.2.23 of [KS]. Definition 3.6.3 and Theorem 3.6.17 of [KS] imply that $L_t^2 = 2L_t^1$. By piecing together parts of a Brownian path and a flat function one can obtain a function $g(t)$ for which L_t^1 is not a constant multiple of L_t^2. We do not know whether $L_t^1 \equiv L_t^3$ for every (deterministic) function $g(t)$.

Let $C[0,\infty)$ be the family of continuous functions on $[0,\infty)$ equipped with the topology of uniform convergence on compact intervals. We denote a generic element of $C[0,\infty)$ by $\omega(t)$ and we let $\mathcal{F}_t$ be the smallest σ-field which makes the coordinate mappings $\omega \to \omega(s)$ Borel-measurable for every $s \leq t$. We will call a set $A \subset C[0,\infty)$ a "local property" if it belongs to $\mathcal{F}_{0+} = \bigcap_{t>0} \mathcal{F}_t$.

Let B_t be a Brownian motion with $B_0 = 0$. We will call $A \subset C[0,\infty)$ a Brownian local property if A is a local property and $P(\{B_t, 0 \leq t < \infty\} \in A^c) = 0$.

The following definition is related to BMB-processes defined in the previous section. Suppose that C_t is a 3-dimensional Bessel process independent of B_t and starting from 0. We will say that $A \subset C[0,\infty)$ is a BMB local property if A is a local property and

$$P(\{(B_s + C_s)/\sqrt{2}, 0 \leq s \leq \infty\} \in A^c) = 0.$$

By Blumenthal's 0-1 Law, if a local property A is not a Brownian local property then A^c is; the same holds for BMB local properties.

For a deterministic function $g(t) : [0,\infty) \to \mathbf{R}$ let $g^+_s(t) = g(s+t) - g(s)$ for $t \geq 0$, and let $g^-_s(t) = g(s-t) - g(s)$ for $t \leq s$ and $g^-_s(t) = g(0) - g(s)$ for $t \geq s$.

Recall the pair of processes $(g(t), X_t)$ defined in (2.8).

Theorem 3.2. *(i) Fix any Brownian local property A_1 and any BMB local property A_2. For almost every trajectory of $g(t)$ the following is true. With probability 1, the random measure dL^1_t does not charge the set of s such that $g^+_s(\,\cdot\,) \in A^c_1$ or $g^-_s(\,\cdot\,) \in A^c_2$.*

(ii) If A is a BMB local property then, for almost all trajectories of $g(t)$, the measure dL^4_t does not charge the set of s such that $g^+_s(\,\cdot\,) \in A^c$ or $g^-_s(\,\cdot\,) \in A^c$, a.s.

(iii) There exist a Brownian local property A_1 and a BMB local property A_2 such that $A_1 \cap A_2 = \emptyset$.

Proof. Consider a Brownian local property A_1. Fix any $a > 0$ and let $\xi_a = \inf\{t : L^1_t = a\}$. The random time ξ_a is a stopping time for $Y_t = (g(t), X_t)$ so the process $\{g(t+\xi_a) - g(\xi_a), t \geq 0\}$ is a Brownian motion. This implies that with probability 1, the function $\{g(t+\xi_a) - g(\xi_a), t \geq 0\}$ has the Brownian local property A_1. Hence, for almost every trajectory $g(t)$, we have $\{g(t+\xi_a) - g(\xi_a), t \geq 0\} \in A_1$ a.s. For the rest of this paragraph, we fix a trajectory of $g(t)$ for which the last statement is true. By Fubini's theorem, with probability 1, $\{g(t+\xi_a) - g(\xi_a), t \geq 0\} \in A_1$ for almost all $a > 0$. Let U be a random variable with the exponential distribution, independent of the processes considered so far. Consider $R = \xi_U$ and note that a.s., R does not belong to the set of s such that $\{g(t+s) - g(s), t \geq 0\} \in A^c_1$. If the random measure dL^1_t charged with positive probability a set of times s such that $\{g(t+s) - g(s), t \geq 0\} \in A^c_1$ then R would belong to A^c_1 with positive probability, which is not the case. We conclude that dL^1_t does not charge the set of s such that $g^+_s(\,\cdot\,) \in A^c_1$.

Next consider a BMB local property A_2. We will assume without loss of generality that $X_0 = g(0) = 0$. Let $\widetilde{X}_t = X_{\xi_1 - t} - X_{\xi_1}$, $\widetilde{g}(t) = g(\xi_1 - t) - g(\xi_1)$ and $\widetilde{L}^1_t = -(L^1_{\xi_1 - t} - L^1_{\xi_1})$. According to Theorem 2.1, $\{(-\widetilde{X}_t, -\widetilde{g}(t)), t \in$

$[0,\xi_1]\}$ has the same distribution as $\{(g(t), X_t), t \in [0,\xi_1]\}$. Fix some $a \in (0,1)$ and set $\widetilde{\xi}_{1-a} = \inf\{t : \widetilde{L}^1_t = 1-a\}$. Note that $\xi_1 - \xi_a = \widetilde{\xi}_{1-a}$. It is easy to see that $\widetilde{L}^1_t$ is adapted to the filtration generated by $(-\widetilde{X}_t, -\widetilde{g}(t))$, for example, using the equivalent definition (3.1) of the local time. Hence, $\widetilde{\xi}_{1-a}$ is a stopping time for $(-\widetilde{X}_t, -\widetilde{g}(t))$. Since $\{-\widetilde{X}_t, t \in [0,\xi_1]\}$ is a Brownian motion and $\{-\widetilde{g}(t), t \in [0,\xi_1]\}$ is a Brownian motion reflected on $-\widetilde{X}_t$, the same is true of $\{-\widetilde{X}_{\widetilde{\xi}_{1-a}+t} + \widetilde{X}_{\widetilde{\xi}_{1-a}}, t \in [0,\xi_1 - \widetilde{\xi}_{1-a}]\}$ and $\{-\widetilde{g}(\widetilde{\xi}_{1-a}+t) + \widetilde{g}(\widetilde{\xi}_{1-a}), t \in [0,\xi_1 - \widetilde{\xi}_{1-a}]\}$, by the strong Markov property applied at $\widetilde{\xi}_{1-a}$. It follows from Theorem 2.2 that $\{\widetilde{g}(\widetilde{\xi}_{1-a}+t) - \widetilde{g}(\widetilde{\xi}_{1-a}), t \in [0,\xi_1 - \widetilde{\xi}_{1-a}]\}$ is a BMB process and so $\{\widetilde{g}(\widetilde{\xi}_{1-a}+t) - \widetilde{g}(\widetilde{\xi}_{1-a}), t \in [0,\xi_1 - \widetilde{\xi}_{1-a}]\} \in A_2$, a.s. This is equivalent to saying that, for a fixed $a \in (0,1)$, a.s., $\{g(\xi_a - t) - g(\xi_a), t \in [0,\xi_a]\} \in A_2$. We can use this to deduce that dL^1_t does not charge the set of s such that $g^-_s(\,\cdot\,) \in A^c_2$, just like in the first part of the proof.

We will next analyze the support of dL^4_t. Recall the definition (3.5) of L^4_t. In a sense, we have eliminated from the support of L^4_t all times s from which an excursion cannot start. More precisely, if we can show that excursions of X_t from $g(t)$ start only at points having a certain property, it will follow that the measure dL^4_t must be supported on the set of points with this property.

Suppose that A is a BMB local property.

The end of an excursion of $-\widetilde{g}(t)$ from $-\widetilde{X}_t$ corresponds in an obvious way to the start of an excursion of X_t from $g(t)$. Fix any rational number $u > 0$ and consider the end T_u of an excursion of $-\widetilde{g}(t)$ from $-\widetilde{X}_t$ straddling u (there might no such excursion, for example, if $-\widetilde{g}(u) = -\widetilde{X}_u$ or $u > \xi_1$; in such a case we let $T_u = \xi_1$). The random variable T_u is a stopping time relative to $(-\widetilde{X}_t, -\widetilde{g}(t))$ so $\{\widetilde{g}(T_u+t) - \widetilde{g}(T_u), t \in [0,\xi_1 - T_u]\} \in A$, a.s., just like in the earlier part of the proof. The same argument as before proves that dL^4_t does not charge the set of s such that $g^-_s(\,\cdot\,) \in A^c$.

Recall the notation from the proof of Theorem 2.1. An excursion of X_t from $g(t)$ starting at time s corresponds to an excursion of Y^1_t from 0 starting at the same time, in view of (2.1) and (2.3). The local properties of $\{Y^1_{t+s} - Y^1_s, t \geq 0\}$ are those of the 3-dimensional Bessel process. Hence, $\{(Y^1_{t+s} - Y^1_s, Z^2_{t+s} - Z^2_s), t \geq 0\}$ has the local properties of a pair of independent processes—a 3-dimensional Bessel process and a Brownian motion (see Theorem 5.2 in [B1]). Since the time s is the starting point of an excursion, there exists $\delta > 0$ such that for $t \in [0,\delta]$,

$$g(s+t) - g(s) = K^1_{s+t} - K^1_s = (Y^1_{t+s} - Y^1_s + Z^2_{t+s} - Z^2_s)/\sqrt{2}.$$

This shows that if s is a starting point of an excursion then $g^+_s(\,\cdot\,) \in A$ a.s. The proof of (ii) is complete.

Finally we will prove part (iii). Let

$$A_1 = \left\{ \omega : \liminf_{t \downarrow 0} \frac{\omega(t)}{\sqrt{2t \log \log(1/t)}} = -1 \right\},$$

and

$$A_2 = \left\{ \omega : \liminf_{t \downarrow 0} \frac{\omega(t)}{\sqrt{2t \log \log(1/t)}} \geq -\frac{1}{\sqrt{2}} \right\}.$$

Obviously, $A_1 \cap A_2 = \emptyset$. The classical local Law of Iterated Logarithm for Brownian motion implies that A_1 is a local Brownian property. It will now suffice to show that A_2 is a BMB local property.

Recall the definition of a BMB-process. We take a Brownian motion B_t and an independent 3-dimensional Bessel process C_t, both starting from 0. Then $\{(B_t + C_t)/\sqrt{2}, t \geq 0\}$ is a BMB-process. Since $C_t \geq 0$ for all $t \geq 0$, a.s., we have using the LIL,

$$\liminf_{t \downarrow 0} \frac{(B_t + C_t)/\sqrt{2}}{\sqrt{2t \log \log(1/t)}} \geq \liminf_{t \downarrow 0} \frac{1}{\sqrt{2}} \frac{B(t)}{\sqrt{2t \log \log(1/t)}} = -\frac{1}{\sqrt{2}}.$$

This means that A_2 is a BMB local property. The proof of (iii) is complete. □

When the reflecting boundary is sufficiently smooth, the time-reversed reflected Brownian path has a distribution absolutely continuous with respect to reflected Brownian path run in the reversed direction; see Theorem 3.12 of [BCS] for a rigorous statement of the result. Our Theorem 3.2 (i) implies that if we reverse in time a path of Brownian motion reflected on a (fixed) path $g(t)$ of Brownian motion then the resulting trajectory will be fundamentally different from the Brownian path running in the negative direction of time and reflected on the same (fixed) Brownian boundary $g(t)$. This is because the definition of the local time L_t^2 is "time-reversible," i.e., if (3.1) is used to define a random measure dL_t^2 and the same definition is applied to the time-reversed path, then the measure dL_t^2 and its time-reversed counterpart are supported on the same subset of the time axis. However, the points in the support of $dL_t^2 = dL_t^1$ have different properties to the left and right.

One would like to characterize local times L_t^1 and L_t^4 by specifying their Revuz measures (see [R]). This does not seem to be an easy task for several reasons. First, the results contained in Theorem 3.2 show that the Revuz measures of L_t^1 and L_t^4 are not equal to the "projection" λ of the Lebesgue measure on $g(t)$, defined by $\lambda(\{(t, g(t)) : s \le t \le u\}) = u - s$ for all $s \le u$. The second reason why it may be hard to find an explicit characterization for the Revuz measures of L^1_t and L^4_t is that the transition density for X_t is unbounded near (some points of) the reflecting boundary $g(t)$—see our next theorem. Good estimates for the transition densities of the Brownian motion seem to be the key of the Revuz measure-type characterization for the local time spent on a curve by (non-reflecting) Brownian motion (see Remark 2.4 of [BB2]).

Theorem 3.3. *For definiteness, assume that $g(0) = X_0 = 0$. Almost all trajectories $g(t)$ have the following property. Let $u(t,x) = u_g(t,x)$ be defined by*

$$u(t,x)dx = P(X_t \in dx \mid g(\,\cdot\,)).$$

Every non-empty open interval of the real line contains a point s such that

$$\limsup_{x \uparrow g(s)} u(s,x) = \infty.$$

Proof. It follows from the results of Davis [D], using the invariance of Brownian path properties under time-reversal, that for almost all trajectories of $g(t)$, and every non-empty open interval (t_1, t_2), there exist $s \in (t_1, t_2)$ and $\delta > 0$ such that

$$g(s-t) - g(s) > (1/2)\sqrt{t} \qquad \text{for all } 0 < t < \delta.$$

Because of the lack of domain monotonicity for the heat equation solution singularities and the form of results stated in [BCS], we will use the following obviously weaker assertion; there exist $s \in (t_1, t_2)$ and $\delta > 0$ such that

$$g(s-t) - g(s) > \sqrt{t}|\log t|^{-1} \qquad \text{for all } 0 < t < \delta. \tag{3.6}$$

Fix some trajectory $g(t)$ and $s > 0$ satisfying (3.6) and let $g_1(t)$ be a continuous function such that $g_1(0) = 0$, $g_1(t) \le g(t)$ for all t, and

$$g_1(s-t) - g_1(s) = \sqrt{t}|\log t|^{-1} \qquad \text{for all } 0 < t < \delta.$$

Let X^1_t be the Brownian motion reflected on $g_1(t)$, i.e., inside the space-time domain $D = \{(t,x) : t \ge 0, x \le g_1(t)\}$. Assume that $X^1_0 = 0$ and let $u_1(t,x)$ be its density, i.e., $u_1(t,x)dx = P(X^1_t \in dx)$. Recall the process X_t conditioned on our "fixed" trajectory $g(t)$. Since $g_1(t) \le g(t)$, we may assume, in view of Corollary 3.15 of [BCS], that $X^1_t \le X_t$ for all t a.s. This implies that

$$P(X^1_s > y) \le P(X_s > y), \tag{3.7}$$

for $y < g(s)$. By Theorem 4.5 of [BCS], there are no heat atoms on the path $g(t)$ so $P(X_s = g(s)) = 0$. This and (3.7) imply that

$$\int_y^{g(s)} u_1(s,x)dx \le \int_y^{g(s)} u(s,x)dx, \tag{3.8}$$

for every $y < g(s)$. By Theorem 4.10 (ii) and Lemma 4.8 (i) of [BCS] and the remarks following that lemma,

$$\limsup_{x \uparrow g(s)} u_1(s,x) = \infty.$$

Lemma 4.8 (ii) and (iii) of [BCS] and the remarks following it imply that in fact

$$\lim_{x \uparrow g(s)} u_1(s,x) = \infty.$$

This and (3.8) easily imply that

$$\limsup_{x \uparrow g(s)} u(s,x) = \infty.$$

Note, however, that our argument does not imply that

$$\lim_{x \uparrow g(s)} u(s,x) = \infty.$$

We leave it as an open problem to determine whether there exist points s with this property. □

In the language of [BCS], a point $(s, g(s))$ with the property stated in Theorem 3.3 would be called a "singularity" of the heat equation solution. Another type of singularity is when a point $(s, g(s))$ is a "heat atom," i.e., when $P(X_s = g(s)) > 0$. It has been proved in [BCS] that with probability 1, there are no heat atoms on the paths of Brownian motion $g(t)$. Singularities seem to be much harder to understand than heat atoms in view of the results presented in [BCS]. In particular, there is no domain monotonicity for singularities. This means that there exist deterministic functions $g(t), g_1(t), g_2(t)$ and a time $s > 0$ such that $g_1(s) = g(s) = g_2(s)$, $g_1(t) \le g(t) \le g_2(t)$ for $t \le s$, the heat equation has a singularity at $(s, g(s))$ in the space-time domain bounded by $g(t)$ but there is no singularity at the same point in the domains corresponding to $g_1(t)$ and $g_2(t)$.

We end this section with a theorem stated without proof. Part of Theorem 3.1 applies not only in the context of Brownian motion reflected on

Brownian motion but it can also be applied to the excursion theory of the standard (non-reflected) Brownian motion from the graph of an independent Brownian motion. See [BB2] for results on the local time of Brownian motion on deterministic rough curves.

The local times L_t^2, L_t^3 and L_t^4 have their analogues in the non-reflecting context but L_t^1 does not. Let $\widehat{g}(t)$ and $\widehat{X}_t$ be independent Brownian motions. It is not necessary to assume that $\widehat{g}(0) \geq \widehat{X}_0$. Upon close inspection of the definitions of L_t^2, L_t^3 and L_t^4, it turns out that they apply verbatim to the case when the processes $g(t)$ and X_t are independent Brownian motions. Hence, we let $\widehat{L}_t^2, \widehat{L}_t^3$ and $\widehat{L}_t^4$ be the local times of $\widehat{X}_t$ on $\widehat{g}(t)$, defined in the same way as L_t^2, L_t^3 and L_t^4 were defined relative to $g(t)$ and X_t.

Theorem 3.4. *(i) The limits in the definitions of $\widehat{L}_t^2$ and $\widehat{L}_t^3$ exist a.s. It follows that for almost every path $\widehat{g}(t)$, the convergence in these expressions holds with (conditional) probability 1.*

(ii) With probability 1, $2\widehat{L}_t^2 = \widehat{L}_t^3$ for all $t \geq 0$.

(iii) With probability 1, the random measures $d\widehat{L}_t^2$ and $d\widehat{L}_t^4$ are mutually singular.

Proof. The theorem can be proved using the same techniques as applied earlier in this section. Specifically, one has to analyze local time on and the excursions from the diagonal line made by the two-dimensional Brownian motion $(\widehat{g}(t), \widehat{X}_t)$. The details are left to the reader. $\square$

4. Support of the parabolic measure. Consider a (deterministic) continuous function $g(t)$ and the corresponding space-time domain $D = \{(t,x) : 0 \leq t \leq 1, x \leq g(t)\}$. Suppose that $g(0) = 0$ and let X_t be the reflected Brownian motion in D with $X_0 = -1$. Let $\tau = \inf\{t \in (0,1) : X_t = g(t)\}$ and $\sigma = \sup\{t \in (0,1) : X_t = g(t)\}$, with the convention that $\inf \emptyset = 1$ and $\sup \emptyset = 0$. The parabolic measure μ_τ on the boundary of D is defined by $\mu_\tau(A) = P(X_\tau \in A)$. The last exit measure is given by $\mu_\sigma(A) = P(X_\sigma \in A)$. We will examine the "projections" of both measures on $\mathbf{R}$, namely, $\mu_\tau^{\mathbf{R}}(K) = \mu_\tau(\{(t, g(t)) : t \in K\})$ and $\mu_\sigma^{\mathbf{R}}(K) = \mu_\sigma(\{(t, g(t)) : t \in K\})$, for $K \subset \mathbf{R}$.

Now assume that $g(t)$ is a Brownian motion. The measures $\mu_\tau^{\mathbf{R}}$ and $\mu_\sigma^{\mathbf{R}}$ have different support, a.s. This follows from Theorem 3.2 (iii) and the following result.

Theorem 4.1. *Fix any Brownian local property A_1 and any BMB local property A_2. For almost every trajectory of $g(t)$ the following is true. With* *probability 1, the measure $\mu_\tau^{\mathbf{R}}$ does not charge the set of s such that $g_s^+(\,\cdot\,) \in A_1^c$ or $g_s^-(\,\cdot\,) \in A_2^c$. The measure $\mu_\sigma^{\mathbf{R}}$ does not charge the set of s such that $g_s^+(\,\cdot\,) \in A_2^c$ or $g_s^-(\,\cdot\,) \in A_2^c$.*

Proof. The starting points of excursions were analyzed in the proof of Theorem 3.2 (ii). The last assertion of the present theorem follows from that argument because σ is the starting point of an excursion of X_t from $g(t)$.

The random variable τ is a stopping time for the process $(g(t), X_t)$. The argument in the first part of the proof of Theorem 3.2 (i) applies to any stopping time in place of ξ_a so it shows that $\{g(\tau+t) - g(\tau), t \geq 0\} \in A_1$ a.s. This means that the measure $\mu_\tau^{\mathbf{R}}$ does not charge the set of s such that $g_s^+(\,\cdot\,) \in A_1^c$.

It remains to examine the behavior of $g(t)$ just before time τ. This has been done in the proof of Theorem 3.2 (ii). □

We will use Theorem 4.1 to obtain information about the size of the support of $\mu_\tau^{\mathbf{R}}$ and $\mu_\sigma^{\mathbf{R}}$. The results of Lewis and Murray [LM] and Hofmann and Lewis [HL] show that the parabolic measure $\mu_\tau^{\mathbf{R}}$ is absolutely continuous with respect to the Lebesgue measure if the boundary function $g(t)$ is Hölder with exponent $1/2$. Brownian paths barely fail to be Hölder with exponent $1/2$ but it is known that their modulus of continuity is the function $h(\delta) = c\sqrt{2\delta \log(1/\delta)}$ for any $c > 1$ (see [KS], Section 2.9.F). Of course, there exist (deterministic) functions $g(t)$ which are not Hölder with exponent $1/2$ but for which $\mu_\tau^{\mathbf{R}}$ is absolutely continuous with respect to the Lebesgue measure.

We will show that this is not the case for a typical Brownian path. In addition, we will give an "upper bound" for the exact Hausdorff measure of the support of $\mu_\tau^{\mathbf{R}}$.

Let us recall the definition of the Hausdorff measure. Suppose that $\phi : (0,1) \to \mathbf{R}$ is an increasing continuous function with $\phi(\delta) \to 0$ as $\delta \downarrow 0$, and such that for some $c > 0$ and all $\delta \in (0,1)$ we have $\phi(2\delta) < c\phi(\delta)$. For a family $\mathcal{K}$ of bounded sets $K \subset \mathbf{R}$, we let $|\mathcal{K}| = \sup\{\mathrm{diam}(K) : K \in \mathcal{K}\}$. Then we define the ϕ-Hausdorff measure of a set M by

$$\phi\text{--}m(M) = \lim_{\varepsilon \downarrow 0} \inf_{\mathcal{K}} \left\{ \sum_{K \in \mathcal{K}} \phi(\mathrm{diam}(K)) : |\mathcal{K}| < \varepsilon, M \subset \bigcup_{K \in \mathcal{K}} K \right\}.$$

Theorem 4.2. *Fix any $\gamma < \infty$ and let $\phi(\delta) = \delta|\log(1/\delta)|^\gamma$. Then for almost all Brownian trajectories $g(t)$ the following is true. There exist sets M_τ and M_σ such that $\phi\text{--}m(M_\tau) = \phi\text{--}m(M_\sigma) = 0$, and $\mu_\tau^{\mathbf{R}}(M_\tau^c) = \mu_\sigma^{\mathbf{R}}(M_\sigma^c) = 0$.*

Proof. Our argument is based on estimates used in the proof of the Law of Iterated Logarithm; see [KS]. Suppose that B_t is a standard Brownian motion with $B_0 = 0$. We will first estimate the probability that $B_t \leq (1/\sqrt{2})\sqrt{2t\log\log(1/t)}$ for all $t \in (\varepsilon, 1/(2e))$.

The following standard estimates may be found in [KS], Problem 2.9.22,

$$\begin{aligned}\frac{y\sqrt{t}}{\sqrt{2\pi}(t+y^2)}\exp(-y^2/(2t)) &\leq P(B_t \geq y) \\ &= \int_y^\infty \frac{1}{\sqrt{2\pi t}}\exp(-x^2/(2t))dx \leq \frac{\sqrt{t}}{\sqrt{2\pi}y}\exp(-y^2/(2t)).\end{aligned} \tag{4.1}$$

Using the last estimate, we have for $t < 1/6$ and large $a > 0$,

$$\begin{aligned}&P\left(B_t \leq -a\sqrt{2t\log\log(1/t)}\right) \\ &\quad\leq \frac{\sqrt{t}}{\sqrt{2\pi}a\sqrt{2t\log\log(1/t)}}\exp\left(-\left(a\sqrt{2t\log\log(1/t)}\right)^2/(2t)\right) \\ &\quad= \frac{1}{\sqrt{2\pi}a\sqrt{2\log\log(1/t)}}\left(\log(1/t)\right)^{-a^2} \\ &\quad\leq \left(\log(1/t)\right)^{-a^2}.\end{aligned} \tag{4.2}$$

We will use several parameters whose values will be specified later in the proof. We will need a (large) positive integer k, and constants $0 < c_1, c_2 < \infty$. We let $a = c_1\sqrt{\log k}$ and $\theta = c_2/a^2 = c_3/\log k$. Then for $n \geq 1$ and large k,

$$\begin{aligned}P(B_{\theta^{n+1}} \leq -a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}) &\leq \left(\log(1/\theta^{n+1})\right)^{-a^2} \\ &= ((n+1)\log(\log k/c_3))^{-c_1^2\log k} \\ &\leq (n+1)^{-c_1^2\log k},\end{aligned}$$

which implies that

$$\begin{aligned}&P\left(\bigcup_{n=2}^\infty\left\{B_{\theta^{n+1}} \leq -a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}\right\}\right) \\ &\quad\leq \sum_{n=2}^\infty P(B_{\theta^{n+1}} \leq -a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}) \\ &\quad\leq \sum_{n=2}^\infty (n+1)^{-c_1^2\log k}.\end{aligned}$$

Since

$$\sum_{n=2}^{\infty}(n+1)^{-\beta} \le \int_3^{\infty}(x-1)^{-\beta}dx = \frac{2^{-\beta+1}}{\beta-1},$$

we obtain

$$P\left(\bigcup_{n=2}^{\infty}\left\{B_{\theta^{n+1}} \le -a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}\right\}\right)$$
$$\le \frac{2^{-c_1^2\log k+1}}{c_1^2\log k-1} = \frac{2k^{-c_1^2\log 2}}{c_1^2\log k-1}. \quad (4.3)$$

Fix some small $\delta > 0$, such that $(1+\delta)/\sqrt{2} < 1$. We can use the lower bound in (4.1) to see that

$$\begin{aligned}
&P(B_{\theta^n} - B_{\theta^{n+1}} \ge ((1+\delta)/\sqrt{2})\sqrt{2\theta^n\log\log(1/\theta^n)} \\
&\qquad + a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}) \\
&\ge P(B_{\theta^n} - B_{\theta^{n+1}} \ge ((1+\delta)/\sqrt{2}+a\sqrt{\theta})\sqrt{2\theta^n\log\log(1/\theta^{n+1})}) \\
&\ge \frac{((1+\delta)/\sqrt{2}+a\sqrt{\theta})\sqrt{2\theta^n\log\log(1/\theta^{n+1})}\sqrt{\theta^n(1-\theta)}}{\theta^n(1-\theta)+((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2(\sqrt{2\theta^n\log\log(1/\theta^{n+1})})^2} \\
&\qquad \times \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2(\sqrt{2\theta^n\log\log(1/\theta^{n+1})})^2}{2\theta^n(1-\theta)}\right) \\
&\ge \frac{((1+\delta)/\sqrt{2}+a\sqrt{\theta})\sqrt{2\log\log(1/\theta^{n+1})}\sqrt{1-\theta}}{(1-\theta)+2((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2\log\log(1/\theta^{n+1})} \\
&\qquad \times \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2\log\log(1/\theta^{n+1})}{(1-\theta)}\right). \quad (4.4)
\end{aligned}$$

Choose a value for $c_2 > 0$ so small that $a\sqrt{\theta} < 1-(1+\delta)/\sqrt{2}$ and so $(1+\delta)/\sqrt{2}+a\sqrt{\theta} < 1$. Then for large k, the right hand side of (4.4) is bounded below by

$$\begin{aligned}
&\frac{1}{4\sqrt{2\pi}\sqrt{\log\log(1/\theta^{n+1})}}\exp\left(-\frac{((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2\log\log(1/\theta^{n+1})}{(1-\theta)}\right) \\
&= \frac{1}{4\sqrt{2\pi}\sqrt{\log\log(1/\theta^{n+1})}}(\log(1/\theta^{n+1}))^{-((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2/(1-\theta)}. \quad (4.5)
\end{aligned}$$

Find α and α_1 such that $((1+\delta)/\sqrt{2}+a\sqrt{\theta})^2 < \alpha < \alpha_1 < 1$. Then for large k (i.e., small θ), and $n \ge 1$, the expression in (4.5) is greater than

$$\frac{1}{4\sqrt{2\pi}\sqrt{\log\log(1/\theta^{n+1})}}(\log(1/\theta^{n+1}))^{-\alpha} \ge (\log(1/\theta^{n+1}))^{-\alpha_1}.$$

Combining this with (4.4) and (4.5), and using independence of Brownian increments, we obtain for large k,

$$
\begin{aligned}
P\Bigg(\bigcap_{n=2}^{k}\Big\{B_{\theta^n} - B_{\theta^{n+1}} &< ((1+\delta)/\sqrt{2})\sqrt{2\theta^n \log\log(1/\theta^n)} \\
&\quad + a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}\Big\}\Bigg) \\
&\le \prod_{n=2}^{k}\left(1 - (\log(1/\theta^{n+1}))^{-\alpha_1}\right) \\
&= \exp\left(\sum_{n=2}^{k}\log\left(1 - (\log(1/\theta^{n+1}))^{-\alpha_1}\right)\right) \\
&\le \exp\left(\sum_{n=2}^{k} -(\log(1/\theta^{n+1}))^{-\alpha_1}\right) \\
&= \exp\left(\sum_{n=2}^{k} -((n+1)\log(\log k/c_3))^{-\alpha_1}\right) \\
&\le \exp\left(-(\log(\log k/c_3))^{-\alpha_1}\int_3^k x^{-\alpha_1}\,dx\right) \\
&\le \exp\left(-(\log(\log k/c_3))^{-\alpha_1}(1/(1-\alpha_1))(k^{-\alpha_1+1} - 3^{-\alpha_1+1})\right).
\end{aligned}
$$

Since $\alpha_1 < 1$, we can find $\alpha_2 > 0$ such that the last expression is less than $\exp(-k^{\alpha_2})$, for large k, and so

$$
\begin{aligned}
P\Bigg(\bigcap_{n=2}^{k}\Big\{B_{\theta^n} - B_{\theta^{n+1}} &< ((1+\delta)/\sqrt{2})\sqrt{2\theta^n \log\log(1/\theta^n)} \\
&\quad + a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}\Big\}\Bigg) \le \exp(-k^{\alpha_2}), \qquad (4.6)
\end{aligned}
$$

for large k.

If

$$B_{\theta^{n+1}} > -a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}$$

and

$$B_{\theta^n} - B_{\theta^{n+1}} \ge ((1+\delta)/\sqrt{2})\sqrt{2\theta^n\log\log(1/\theta^n)} + a\sqrt{2\theta^{n+1}\log\log(1/\theta^{n+1})}$$

then

$$B_{\theta^n} \ge ((1+\delta)/\sqrt{2})\sqrt{2\theta^n\log\log(1/\theta^n)}.$$

It follows, in view of (4.3) and (4.6), that

$$
\begin{aligned}
&P\left(\sup_{\theta^k\le t\le 1/2}\frac{B_t}{((1+\delta)/\sqrt{2})\sqrt{2t\log\log(1/t)}}\le 1\right)\\
&\quad\le P\left(\sup_{\theta^k\le t\le \theta^2}\frac{B_t}{((1+\delta)/\sqrt{2})\sqrt{2t\log\log(1/t)}}\le 1\right)\\
&\quad\le P\left(\bigcap_{n=2}^{k}\left\{B_{\theta^n}\le((1+\delta)/\sqrt{2})\sqrt{2\theta^n\log\log(1/\theta^n)}\right\}\right)\\
&\quad\le \exp(-k^{\alpha_2})+\frac{2k^{-c_1^2\log 2}}{c_1^2\log k-1}\le\frac{4k^{-c_1^2\log 2}}{c_1^2\log k-1},
\end{aligned}
\tag{4.7}
$$

for large k.

Let $\Delta=\theta^{2k}$. For sufficiently large $c_4>0$ we obtain from (4.2) for large k,

$$
\begin{aligned}
&P\left(\sup_{0<t\le\Delta}B_t\ge c_4\sqrt{2\Delta\log\log(1/\Delta)}\right)\\
&\quad=2P(B_\Delta\ge c_4\sqrt{2\Delta\log\log(1/\Delta)})\\
&\quad\le 2\left(\log(1/\Delta)\right)^{-c_4^2}\\
&\quad=2\left(\log(1/\theta^{2k})\right)^{-c_4^2}\\
&\quad=2\left(2k\log(\log k/c_3)\right)^{-c_4^2}\\
&\quad\le k^{-c_4^2}.
\end{aligned}
\tag{4.8}
$$

For large k we have

$$
\begin{aligned}
&((1+\delta)/\sqrt{2})\sqrt{2\theta^k\log\log(1/\theta^k)}\\
&\qquad>((1+\delta/2)/\sqrt{2})\sqrt{2\theta^k\log\log(1/\theta^k)}+c_4\sqrt{2\theta^{2k}\log\log(1/\theta^{2k})}\\
&\qquad=((1+\delta/2)/\sqrt{2})\sqrt{2\theta^k\log\log(1/\theta^k)}+c_4\sqrt{2\Delta\log\log(1/\Delta)},
\end{aligned}
$$

so for large k and all $t\ge\theta^k$,

$$
\begin{aligned}
&((1+\delta)/\sqrt{2})\sqrt{2t\log\log(1/t)}\\
&\qquad>((1+\delta/2)/\sqrt{2})\sqrt{2t\log\log(1/t)}+c_4\sqrt{2\Delta\log\log(1/\Delta)}.
\end{aligned}
$$

It follows that if for some $s\in[0,\Delta]$ and all $t\in(\Delta,1/2]$ we have

$$B_t-B_s\le((1+\delta/2)/\sqrt{2})\sqrt{2(t-s)\log\log(1/(t-s))}$$

then either

$$B_t \le ((1+\delta)/\sqrt{2})\sqrt{2t \log\log(1/t)}$$

for all $t \in [\theta^k, 1/2]$ or

$$B_t \ge c_4 \sqrt{2\Delta \log\log(1/\Delta)}$$

for some $t \in (0, \Delta]$. In view of (4.7) and (4.8) we obtain for large k,

$$\begin{aligned} &P\left(\exists s \in [0,\Delta]\ \forall t \in (0,1/2] : B_{s+t} - B_s \right. \\ &\quad \left. \le\ ((1+\delta/2)/\sqrt{2})\sqrt{2t \log\log(1/t)}\right) \\ &\quad \le P\left(\exists s \in [0,\Delta]\ \forall t \in (\Delta,1/2] : B_t - B_s \right. \\ &\quad \left. \le\ ((1+\delta/2)/\sqrt{2})\sqrt{2(t-s) \log\log(1/(t-s))}\right) \\ &\quad \le \frac{4k^{-c_1^2 \log 2}}{c_1^2 \log k - 1} + k^{-c_4^2}. \end{aligned}$$

Since c_1 and c_4 are arbitrarily large, we have for any $c_5 < \infty$ and sufficiently large k,

$$\begin{aligned} &P(\exists s \in [0,\Delta]\ \forall t \in (0,1/2] : B_{s+t} - B_s \\ &\qquad \le ((1+\delta/2)/\sqrt{2})\sqrt{2t \log\log(1/t)}) \le k^{-c_5}. \end{aligned} \tag{4.9}$$

Recall that $\Delta = \theta^{2k}$ and $\theta = c_3/\log k$. Consider a large integer m and let k be the integer part of $1 + \log m/\log\log m$. It is elementary to verify that $1/m \le \Delta$ for large m. Since c_5 in (4.9) is arbitrarily large, we can take an arbitrarily large c_6 and then apply (4.9) with $c_5 > c_6$ to obtain for sufficiently large m,

$$\begin{aligned} &P(\exists s \in [0,1/m]\ \forall t \in (0,1/2] : B_{s+t} - B_s \\ &\qquad \le ((1+\delta/2)/\sqrt{2})\sqrt{2t \log\log(1/t)}) \\ &\qquad \le (\log m/\log\log m)^{-c_5} \le (\log m)^{-c_6}. \end{aligned} \tag{4.10}$$

Let N_m be the number of intervals of the form $(j/m, (j+1)/m]$, $j = 0, 1, \ldots, m-1$, such that for some $s \in (j/m, (j+1)/m]$ and all $t \in (0, 1/2]$ we have

$$B_{s+t} - B_s \le ((1+\delta/2)/\sqrt{2})\sqrt{2t \log\log(1/t)}. \tag{4.11}$$

Then (4.10) implies that $EN_m \le m(\log m)^{-c_6}$, for large m. We have

$$P(N_m \ge m(\log m)^{-c_6+1}) \le \frac{EN_m}{m(\log m)^{-c_6+1}} \le \frac{1}{\log m}.$$

We take m_j to be the integer part of $1 + e^{j^2}$ so that

$$P(N_{m_j} \geq m_j(\log m_j)^{-c_6+1}) \leq \frac{1}{\log m_j} \leq \frac{1}{j^2}.$$

By the Borel-Cantelli Lemma, with probability 1, only a finite number of events $\{N_{m_j} \geq m_j(\log m_j)^{-c_6+1}\}$ occur.

Let W be the set of all $s \in (0,1]$ such that (4.11) holds for all $t \in (0,1/2]$. The set W is covered by N_m intervals of length $1/m$. For sufficiently large j, W is covered by at most $m_j(\log m_j)^{-c_6+1}$ intervals of length $1/m_j$. Let $\phi(r) = r(\log(1/r))^{c_6-2}$ and note that

$$m_j(\log m_j)^{-c_6+1}\phi(1/m_j) = \frac{1}{\log m_j}.$$

Since $1/\log m_j \to 0$ as $j \to \infty$, the definition of the Hausdorff measure shows that $\phi{-}m(W) = 0$. This is true for an arbitrarily large c_6.

Let W_n be the set of all $s \in (0,1]$ such that (4.11) holds for all $t \in (0,1/n]$. For any fixed integer $n \geq 2$, one can show that $\phi{-}m(W_n) = 0$, a.s., in a way completely analogous to the case $n = 2$ presented above. It follows that, with probability 1, $\phi{-}m\left(\bigcup_{n\geq 2} W_n\right) = 0$. By symmetry and time-reversal, if we let $\widehat{W}_n$ be the set of all $s \in (0,1]$ such that

$$B_{s-t} - B_s \geq -((1+\delta/2)/\sqrt{2})\sqrt{2t\log\log(1/t)},$$

holds for all $t \in (0, s \wedge 1/n]$, then $\phi{-}m\left(\bigcup_{n\geq 2} \widehat{W}_n\right) = 0$, a.s. It will suffice to find sets $M_\tau \subset \bigcup_{n\geq 2} \widehat{W}_n$ and $M_\sigma \subset \bigcup_{n\geq 2} \widehat{W}_n$ such that $\mu_\tau^{\mathbf{R}}(M_\tau^c) = \mu_\sigma^{\mathbf{R}}(M_\sigma^c) = 0$.

The event

$$A_2 = \left\{\omega : \liminf_{t\downarrow 0} \frac{\omega(t)}{\sqrt{2t\log\log(1/t)}} \geq -\frac{1}{\sqrt{2}}\right\}$$

is one of the BMB local properties according to the proof of Theorem 3.2 (iii). Let $M_\tau = M_\sigma = \{s \in (0,1] : g_s^-(\,\cdot\,) \in A_2\}$. By Theorem 4.1, the measures $\mu_\tau^{\mathbf{R}}$ and $\mu_\sigma^{\mathbf{R}}$ do not charge the set of s such that $g_s^-(\,\cdot\,) \in A_2^c$, i.e., $\mu_\tau^{\mathbf{R}}(M_\tau^c) = \mu_\sigma^{\mathbf{R}}(M_\sigma^c) = 0$. One can easily verify that $M_\tau = M_\sigma \subset \bigcup_{n\geq 2} \widehat{W}_n$. $\square$

5. Conditional non-intersection probabilities. Recall various local times L_t^1, L_t^2, L_t^3 and L_t^4 from Section 3. This section is an attempt to shed some light on the mysterious fact that L_t^3 is not a constant multiple of L_t^4 although both local times are defined in terms of excursions of X_t from $g(t)$.

First we will argue that L_t^3 has been defined by counting the "wrong" excursions. Recall the event $A = \{|g(t+s) - e_t(s)| > 1 \text{ for some } s \geq 0\}$ from (3.4) and other notation used in the definition of L_t^4. The excursion laws $H^{(t,x)}$ have been normalized so that $H^{(t,x)}(A) = 1$, unless $H^{(t,x)}$ is identically equal to zero. For $\varepsilon \in [0,1]$, let M_ε be the set of (t,x) such that

$$P(\inf\{s > t : |X_t - g(t)| = 1\} < \inf\{s > t : X_t = g(t)\} \mid X_t = x) = \varepsilon.$$

Let

$$A_\varepsilon = \{e_t(s) \in M_\varepsilon \text{ for some } s \geq 0\}.$$

By the strong Markov property of $H^{(t,x)}$ applied at the hitting time of M_ε, we have $H^{(t,x)}(A_\varepsilon) = 1/\varepsilon$. It follows that the excursions of X_t from $g(t)$ which hit M_ε arrive according to a Poisson point process with intensity $1/\varepsilon$, on the time scale defined by the local time L_t^4. Let $\widehat{N}_s^\varepsilon$ be the number of excursions of X_t from $g(t)$ which started before time s and which hit the set M_ε. Then standard techniques show that for a sequence ε_n which decreases to 0 sufficiently fast we have $L_t^4 = \lim_{n\to\infty} \varepsilon_n \widehat{N}_t^{\varepsilon_n}$, for all $t \geq 0$, a.s. Hence, one can define L_t^4 using excursions but one has to count excursions which hit M_ε, not excursions which depart from $g(t)$ by ε units. Since

$$\lim_{n\to\infty} \varepsilon_n \widehat{N}_t^{\varepsilon_n} = L_t^4 \neq L_t^1 = \lim_{n\to\infty} \varepsilon_n N_t^{\varepsilon_n},$$

it is clear that most excursions included in one count do not make the other list. It seems that for a typical excursion which departs ε units from $g(t)$, the probability that it will hit M_1 is much different from ε. One may wonder what this probability is.

For technical reasons, we will not address the last question but a different one which we nevertheless believe is "morally equivalent" to it. We will estimate the probability that Brownian motion starting ε units away from $g(t)$ will not hit the trajectory of $g(t)$ for one unit of time. The answer, of course, depends on the trajectory of $g(t)$. We will try to quantify this dependence.

It will be convenient to present the next result using a product probability space $\Omega = C[0,1] \times C[0,1]$. Its generic element will be denoted $\omega = (\omega_1, \omega_2)$. We will use $g(t)$ and X_t to denote the canonical stochastic processes, i.e., $g(t)(\omega) = \omega_1(t)$ and $X_t(\omega) = \omega_2(t)$.

Theorem 5.1. *Let P^ε be a probability measure on Ω which makes $g(t)$ and X_t independent Brownian motions with $g(0) = 0$ and $X_0 = -\varepsilon$.*

(i) For some $0 < c_1 < c_2 < \infty$ *and all* $\varepsilon \in (0,1)$,

$$c_1\varepsilon \le P^\varepsilon(X_t \ne g(t), 0 \le t \le 1) \le c_2\varepsilon.$$

(ii) Let

$$q(\omega_1) = P^\varepsilon(X_t \ne g(t), 0 \le t \le 1 \mid g(\,\cdot\,) = \omega_1).$$

Then for any $\gamma_1, \gamma_2 > 0$ *with* $\gamma_1 + \gamma_2 < 1$, *and all sufficiently small* $\varepsilon > 0$,

$$P^\varepsilon(q > \varepsilon|\log\varepsilon|^{-\gamma_1}) < |\log\varepsilon|^{-\gamma_2}. \tag{5.1}$$

The proof of the above result hinges on the estimate in the next theorem. That theorem is concerned with distributions of stochastic processes which are not necessarily absolutely continuous with respect to each other. We will give a meaning to the Radon-Nikodym derivative for such measures. Suppose that Q and P are probability measures. The measure Q can be represented as Q_1+Q_2, the sum of two non-negative measures, such that Q_1 is absolutely continuous with respect to P and Q_2 is singular with respect to P. Then we define the Radon-Nikodym derivative dQ/dP as dQ_1/dP. Note that the integral of dQ/dP with respect to P may take any value in $[0,1]$.

Theorem 5.2. *Suppose that* B_t *and* X_t *are standard Brownian motions and* Y_t *is a process which does not take positive values. Assume that* B_t, X_t *and* Y_t *are independent and* $B_0 = X_0 = Y_0 = -\varepsilon < 0$. *If* Q_1 *denotes the distribution of* $\{(B_t + Y_t)/\sqrt{2}, t \in [0,1]\}$ *on* $C[0,1]$ *and* Q_2 *is the analogous distribution of* $\{(B_t + X_t)/\sqrt{2}, t \in [0,1]\}$ *then for any* $\gamma_1, \gamma_2 > 0$ *with* $\gamma_1 + \gamma_2 < 1$, *for sufficiently small* $\varepsilon > 0$ *we have*

$$Q_2(dQ_1/dQ_2 > |\log\varepsilon|^{-\gamma_1}) < |\log\varepsilon|^{-\gamma_2}.$$

Remark 5.1. Before we prove Theorems 5.1 and 5.2, we would like to describe a seemingly natural strategy to obtain a good estimate of dQ_1/dQ_2 in Theorem 5.2 which in fact leads to a vicious circle of ideas. We will only sketch the argument and leave the details to the reader.

We will use the notation of Theorem 5.2 except that the processes will start above the axis—in other words, we will flip them to the positive side. Let Y_t be a Brownian motion starting from $\varepsilon > 0$ under a probability measure P. We will apply the Girsanov theorem following the presentation in [RW], Theorem (38.5), Ch. IV. Let $\xi_t^\delta = (1/Y_t)\mathbf{1}_{\{|Y_t|>\delta\}}$ for $\delta > 0$ and

$$M_t^\delta = \exp\left(\int_0^t \xi_s^\delta dY_s - (1/2)\int_0^t (\xi_s^\delta)^2 ds\right).$$

Define a probability measure Q^δ on $C[0,\infty)$ by declaring that its Radon-Nikodym derivative on $C[0,t]$ with respect to P is equal to $dQ^\delta/dP = M_t^\delta$. Then under Q^δ, the process $Y_t - \int_0^t \xi_s^\delta ds$ is a Brownian motion. When we let $\delta \downarrow 0$, the functions ξ_t^δ converge to $\xi_t = 1/Y_t$ and processes M_t^δ converge to

$$M_t = \mathbf{1}_{\{Y_s>0, 0\le s\le t\}} \exp\left(\int_0^t \xi_s dY_s - (1/2)\int_0^t \xi_s^2 ds\right), \tag{5.2}$$

because $\int_0^t \xi_s^2 ds = \infty$ if the path of Y hits 0 before time t (this is one of the facts which needs to be verified by the reader). Let Q be defined by $dQ/dP = M_t$ on $C[0,t]$. The process $Y_t - \int_0^t \xi_s ds$ is a Brownian motion under Q. We will call this process V_t, i.e.,

$$V_t = Y_t - \int_0^t \xi_s ds = Y_t - \int_0^t (1/Y_s) ds.$$

This can be written as

$$Y_t = V_t + \int_0^t (1/Y_s) ds.$$

Since the process V_t is a Brownian motion under Q, the process Y_t satisfying the above equation is a three-dimensional Bessel process ([RY], page 446) under Q. We see that (5.2) is the Radon-Nikodym derivative of the distribution of the three-dimensional Bessel process with respect to the Brownian motion distribution. We can simplify (5.2) as follows. By Itô's lemma,

$$\begin{aligned}\log Y_t - \log Y_0 &= \int_0^t (1/Y_s) dY_s - (1/2)\int_0^t (1/Y_s)^2 ds \\ &= \int_0^t \xi_s dY_s - (1/2)\int_0^t \xi_s^2 ds.\end{aligned}$$

Thus

$$M_t = \frac{Y_t}{Y_0}\mathbf{1}_{\{Y_s>0, 0\le s\le t\}}.$$

Consider processes B_t and Y_t. Suppose that under Q_1 these processes are independent, $B_0 = Y_0 = \varepsilon > 0$, B_t is a Brownian motion and Y_t is a three-dimensional Bessel process. Let Q_2 be a probability measure which makes B_t and Y_t independent Brownian motions starting from ε. Since B_t is independent of Y_t and has the same distribution under both measures,

$$\frac{dQ_1}{dQ_2}((B_s, Y_s), 0 \le s \le t) = \frac{dQ_1}{dQ_2}(Y_s, 0 \le s \le t) = M_t.$$

Let $W_t = (B_t + Y_t)/\sqrt{2}$ and $R_t = (-B_t + Y_t)/\sqrt{2}$. The goal of Theorem 5.2 is to find an estimate for the following Radon-Nikodym derivative,

$$\begin{aligned}
\frac{dQ_1}{dQ_2}(W_s, 0 \le s \le t) &= E^{Q_2}(M_t \mid W_s, 0 \le s \le t) \\
&= E^{Q_2}\left(\frac{Y_t}{Y_0}\mathbf{1}_{\{Y_s>0, 0\le s\le t\}} \mid W_s, 0 \le s \le t\right) \\
&= E^{Q_2}\left(\frac{W_t + R_t}{W_0 + R_0}\mathbf{1}_{\{W_s+R_s>0, 0\le s\le t\}} \mid W_s, 0 \le s \le t\right) \\
&= \frac{1}{2\varepsilon}E^{Q_2}\left((W_t + R_t)\mathbf{1}_{\{W_s+R_s>0, 0\le s\le t\}} \mid W_s, 0 \le s \le t\right). \quad (5.3)
\end{aligned}$$

Under Q_2, the processes W_t and R_t are independent Brownian motions so $W_t + R_t$ takes mostly moderate values. Hence it is natural to expect that the last line in (5.3) is as hard to estimate as

$$P^{Q_2}(W_s + R_s > 0, 0 \le s \le t \mid W_s, 0 \le s \le t).$$

This conditional probability is analogous to the one which is estimated in Theorem 5.1 (ii). The estimate in Theorem 5.1 (ii) is based on an estimate of dQ_1/dQ_2 in Theorem 5.2 and thus the vicious circle of ideas is closed. We will use an argument unrelated to the Girsanov theorem to prove Theorem 5.2.

Proof of Theorem 5.2. Let B_t denote the standard Brownian motion with $B_0 = 0$. We will need several parameters whose values will be specified later. Suppose $\alpha \in (0, 1)$, consider a small $\varepsilon > 0$, and let k be the largest integer with $\alpha^{k/2} \ge \varepsilon$. Let $c > 0$ and $a = c\sqrt{\log k}$. We will write $t_n = \alpha^n$. Using the well known formula for the distribution of the maximum of Brownian motion and (4.1), we obtain for all $n \le k$, assuming k is sufficiently large,

$$\begin{aligned}
P\left(\max_{0\le s\le t_n} B_s \ge a\sqrt{t_n}\right) &= P\left(\max_{0\le s\le t_n} B_s/\sqrt{t_n} \ge a\right) \\
&= 2P\left(B_{t_n}/\sqrt{t_n} \ge a\right) \\
&= 2P\left(B_1 \ge a\right)
\end{aligned}$$

$$\leq \frac{2}{a\sqrt{2\pi}} \exp(-a^2/2)$$
$$= \frac{\sqrt{2}}{c\sqrt{\log k}\sqrt{\pi}} \exp(-(c\sqrt{\log k})^2/2)$$
$$= \frac{\sqrt{2}}{c\sqrt{\pi \log k}} k^{-c^2/2}. \tag{5.4}$$

We will assume that $c^2/2 > 1$ from now on. We obtain from (5.4), for some fixed $\beta_1 > 0$ and large k,

$$P\left(\bigcup_{0\leq n\leq k}\left\{\max_{0\leq s\leq t_n} B_s \geq a\sqrt{t_n}\right\}\right) \leq \frac{\sqrt{2}}{c\sqrt{\pi\log k}} k^{1-c^2/2} \leq |\log\varepsilon|^{-\beta_1}. \tag{5.5}$$

Next consider $b \in (0,1)$. Let $\delta, \delta_1 > 0$ be arbitrarily small constants. We have

$$\sqrt{2}\varepsilon < \delta ab\sqrt{t_{n-1} - t_n}$$

for large k and all $n \leq k$. We obtain for large k, using the lower bound in (4.1),

$$P\left(\max_{t_n\leq s\leq t_{n-1}} B_s - B_{t_n} > \sqrt{2}\varepsilon + ab\sqrt{t_{n-1}-t_n}\right)$$
$$\geq P\left(\max_{t_n\leq s\leq t_{n-1}} B_s - B_{t_n} > (1+\delta)ab\sqrt{t_{n-1}-t_n}\right)$$
$$= P\left(\max_{t_n\leq s\leq t_{n-1}} \frac{B_s - B_{t_n}}{\sqrt{t_{n-1}-t_n}} > (1+\delta)ab\right)$$
$$= 2P\left(\frac{B_{t_{n-1}} - B_{t_n}}{\sqrt{t_{n-1}-t_n}} > (1+\delta)ab\right)$$
$$= 2P\left(B_1 > (1+\delta)ab\right)$$
$$\geq (1-\delta_1)\frac{2}{(1+\delta)ab\sqrt{2\pi}} \exp(-(1+\delta)^2a^2b^2/2)$$
$$= (1-\delta_1)\frac{2}{(1+\delta)bc\sqrt{\log k}\sqrt{2\pi}} \exp(-(1+\delta)^2b^2(c\sqrt{\log k})^2/2)$$
$$= \frac{(1-\delta_1)\sqrt{2}}{(1+\delta)bc\sqrt{\pi\log k}} k^{-(1+\delta)^2b^2c^2/2}.$$

This implies for large k,

$$P\left(\bigcup_{1\leq n\leq k}\left\{\max_{t_n\leq s\leq t_{n-1}} B_s - B_{t_n} > \sqrt{2}\varepsilon + ab\sqrt{t_{n-1}-t_n}\right\}\right)$$

$$\begin{aligned}
&\geq 1-\left(1-\frac{(1-\delta_1)\sqrt{2}}{(1+\delta)bc\sqrt{\pi\log k}}k^{-(1+\delta)^2b^2c^2/2}\right)^k \\
&= 1-\exp\left[k\log\left(1-\frac{(1-\delta_1)\sqrt{2}}{(1+\delta)bc\sqrt{\pi\log k}}k^{-(1+\delta)^2b^2c^2/2}\right)\right] \\
&\geq 1-\exp\left[k\left(-\frac{(1-\delta_1)\sqrt{2}}{(1+\delta)bc\sqrt{\pi\log k}}k^{-(1+\delta)^2b^2c^2/2}\right)\right] \\
&= 1-\exp\left(-\frac{(1-\delta_1)\sqrt{2}}{(1+\delta)bc\sqrt{\pi\log k}}k^{1-(1+\delta)^2b^2c^2/2}\right). \qquad (5.6)
\end{aligned}$$

We will choose the parameters so that $1-(1+\delta)^2b^2c^2/2>0$.

It is time for us to specify the values of the parameters or, to be more precise, the ranges of their values. First we fix some $b\in(1/\sqrt{2},1)$. Next we choose $\alpha\in(0,1)$ so small that

$$1<\sqrt{\frac{1}{\alpha}-1}-\frac{1}{b\sqrt{2\alpha}}.$$

Set $\gamma=\sqrt{\frac{1}{\alpha}-1}-\frac{1}{b\sqrt{2\alpha}}$, and choose $\delta>0$ such that $\delta<\min(\gamma-1,\frac{1}{b}-1)$. We find some c satisfying

$$\max\left(\frac{\sqrt{2}}{b\gamma},\sqrt{2}\right)<c<\frac{\sqrt{2}}{b(1+\delta)}.$$

Finally we choose c_1 such that $\sqrt{2}<c_1<cb\gamma$, and let $d=c_1\sqrt{\log k}$.

The following inequality is completely analogous to (5.5). We have replaced the maximum with the minimum, a with $-d$ and c with c_1.

$$P\left(\bigcup_{0\leq n\leq k}\left\{\min_{0\leq s\leq t_n}B_s\leq -d\sqrt{t_n}\right\}\right)\leq\frac{\sqrt{2}}{c_1\sqrt{\pi\log k}}k^{1-c_1^2/2}. \qquad (5.7)$$

Assume that the following events hold

$$\bigcup_{1\leq n\leq k}\left\{\max_{t_n\leq s\leq t_{n-1}}B_s-B_{t_n}>\sqrt{2}\varepsilon+ab\sqrt{t_{n-1}-t_n}\right\} \qquad (5.8)$$

and

$$\bigcap_{0\leq n\leq k}\left\{\min_{0\leq s\leq t_n}B_s>-d\sqrt{t_n}\right\}. \qquad (5.9)$$

Then for some $n \in \{1, 2, \ldots, k\}$ and $s \in [t_n, t_{n-1}]$,

$$\begin{aligned}
B_s = (B_s - B_{t_n}) + B_{t_n} &> \sqrt{2}\varepsilon + ab\sqrt{t_{n-1} - t_n} - d\sqrt{t_n} \\
&= \sqrt{2}\varepsilon + ab\sqrt{\alpha^{n-1} - \alpha^n} - d\sqrt{\alpha^n} \\
&= \sqrt{2}\varepsilon + \sqrt{\alpha^{n-1}}(ab\sqrt{1-\alpha} - d\sqrt{\alpha}) \\
&= \sqrt{2}\varepsilon + \sqrt{\alpha^{n-1}}(ab\sqrt{1-\alpha} - a(c_1/c)\sqrt{\alpha}) \\
&\geq \sqrt{2}\varepsilon + \sqrt{\alpha^{n-1}}\left(ab\sqrt{1-\alpha} - a\left(b\sqrt{\frac{1}{\alpha} - 1} - \frac{1}{\sqrt{2\alpha}}\right)\sqrt{\alpha}\right) \\
&= \sqrt{2}\varepsilon + \sqrt{\alpha^{n-1}}a/\sqrt{2} \\
&= \sqrt{2}\varepsilon + \sqrt{t_{n-1}}a/\sqrt{2}.
\end{aligned}$$

We see that if the events (5.8) and (5.9) occurred then the following event holds,

$$\bigcup_{1 \leq n \leq k} \left\{ \max_{t_n \leq s \leq t_{n-1}} B_s > \sqrt{2}\varepsilon + (a/\sqrt{2})\sqrt{t_{n-1}} \right\}.$$

Recall that $c_1 > \sqrt{2}$ and $1 - (1+\delta)^2 b^2 c^2/2 > 0$. Then, in view of (5.6) and (5.7), for some fixed $\delta_2, \beta_2 > 0$ and large k,

$$\begin{aligned}
&P\left(\bigcup_{1 \leq n \leq k} \left\{ \max_{t_n \leq s \leq t_{n-1}} B_s > \sqrt{2}\varepsilon + (a/\sqrt{2})\sqrt{t_{n-1}} \right\}\right) \\
&\geq 1 - \exp\left(-\frac{(1-\delta_1)\sqrt{2}}{(1+\delta)bc\sqrt{\pi \log k}} k^{1-(1+\delta)^2 b^2 c^2/2}\right) - \frac{\sqrt{2}}{c_1\sqrt{\pi \log k}} k^{1-c_1^2/2} \\
&\geq 1 - k^{1-(1-\delta_2)c_1^2/2} \\
&\geq 1 - |\log \varepsilon|^{-\beta_2}. \qquad (5.10)
\end{aligned}$$

Let us examine possible values of exponents β_1 and β_2 in (5.5) and (5.10). We can choose b very close to $1/\sqrt{2}$ and so c can be very close to 2.

It follows that (5.5) holds for any $\beta_1 < 1$. Given any $b \in (1/\sqrt{2}, 1)$, α can be chosen very close to 0 and then c_1 can be made arbitrarily large. We conclude that (5.10) holds for any $\beta_2 < \infty$.

Now recall processes B_t, X_t and Y_t from the statement of the theorem. The processes B_t and X_t are standard Brownian motions and Y_t is a process which does not take positive values. We have assumed that B_t, X_t and Y_t are independent and $B_0 = X_0 = Y_0 = -\varepsilon < 0$. We immediately obtain from (5.5) that for $\beta_1 < 1$ and small $\varepsilon > 0$,

$$P\left(\bigcup_{0\le n\le k}\left\{\max_{0\le s\le t_n}(B_s+Y_s)/\sqrt{2}\ge (a/\sqrt{2})\sqrt{t_n}\right\}\right)\le |\log\varepsilon|^{-\beta_1}. \quad (5.11)$$

Since $(B_t + X_t)/\sqrt{2}$ is a standard Brownian motion starting from $-\sqrt{2}\varepsilon$, (5.10) implies

$$P\left(\bigcup_{1\le n\le k}\left\{\max_{t_n\le s\le t_{n-1}}(B_s+X_s)/\sqrt{2}\ge (a/\sqrt{2})\sqrt{t_{n-1}}\right\}\right) \ge 1-|\log\varepsilon|^{-\beta_2}, \quad (5.12)$$

for any fixed $\beta_2 < \infty$ and small $\varepsilon > 0$.

Consider the following event,

$$A=\bigcup_{1\le n\le k}\left\{\omega\in C[0,1]:\max_{t_n\le s\le t_{n-1}}\omega_s\ge (a/\sqrt{2})\sqrt{t_{n-1}}\right\}.$$

Recall measures Q_1 and Q_2 from the statement of the theorem. We can rewrite (5.11) and (5.12) as

$$Q_1(A)\le |\log\varepsilon|^{-\beta_1}$$

and

$$Q_2(A)\ge 1-|\log\varepsilon|^{-\beta_2}.$$

Let $B = \{dQ_1/dQ_2 > |\log\varepsilon|^{-\gamma_1}\}$ and assume that $Q_2(A\cap B)\ge |\log\varepsilon|^{-\gamma_2}$. Then

$$\begin{aligned}|\log\varepsilon|^{-\beta_1}&\ge Q_1(A)\ge Q_1(A\cap B)\\ &\ge \int_{A\cap B}\frac{dQ_1}{dQ_2}(\omega)dQ_2(\omega)\ge |\log\varepsilon|^{-\gamma_1-\gamma_2}. \quad (5.13)\end{aligned}$$

We can take β_1 arbitrarily close to 1, in particular, we can assume that $\gamma_1+\gamma_2<\beta_1$ because $\gamma_1+\gamma_2<1$. Then (5.13) gives us a contradiction for $\varepsilon > 0$ small and we have to conclude that $Q_2(A\cap B) < |\log\varepsilon|^{-\gamma_2}$. Hence,

$$\begin{aligned}Q_2(dQ_1/dQ_2 > |\log\varepsilon|^{-\gamma_1}) &= Q_2(B) = Q_2(A\cap B)+Q_2(A^c\cap B)\\ &\le Q_2(A\cap B)+Q_2(A^c) < |\log\varepsilon|^{-\gamma_2}+|\log\varepsilon|^{-\beta_2}.\end{aligned}$$

The exponent β_2 can be taken arbitrarily large so

$$Q_2(dQ_1/dQ_2 > |\log\varepsilon|^{-\gamma_1}) < |\log\varepsilon|^{-\gamma_2},$$

for any pair γ_1, γ_2 with $\gamma_1 + \gamma_2 < 1$, and sufficiently small $\varepsilon > 0$. □

Proof of Theorem 5.1. (i) The process $Z_t = (X_t - g(t))/\sqrt{2}$ is a Brownian motion starting from $-\varepsilon/\sqrt{2}$. The event in question is equivalent to $A = \{Z_t < 0, 0 \le t \le 1\}$. By the gambler's ruin problem, the probability that Z_t will hit -1 before hitting 0 is equal to $\varepsilon/\sqrt{2}$. Standard arguments can be used to show that this implies that $P(A)$ is of order ε.

(ii) In addition to Z_t defined in part (i) of the proof, we will need $W_t = (X_t + g(t))/\sqrt{2}$. Under P^ε, the processes Z_t and W_t are independent Brownian motions starting from $-\varepsilon/\sqrt{2}$. Let Q^ε denote the probability measure P^ε conditioned by A, the event defined in part (i). Under Q^ε, the processes Z_t and W_t are independent, W_t is a Brownian motion starting from $-\varepsilon/\sqrt{2}$ and $-Z_t$ is a three-dimensional Bessel process; hence Z_t is a non-positive process. Note that $g_t = (W_t - Z_t)/\sqrt{2}$.

The estimate in part (i) of the theorem yields

$$\begin{aligned} P^\varepsilon\left(g > c_2\varepsilon|\log\varepsilon|^{-\gamma_1}\right) &< P^\varepsilon\left(P^\varepsilon(A \mid g) > P^\varepsilon(A)|\log\varepsilon|^{-\gamma_1}\right) \\ &= P^\varepsilon\left(E_{P^\varepsilon}\left(\frac{dQ^\varepsilon}{dP^\varepsilon} \mid g\right) > |\log\varepsilon|^{-\gamma_1}\right). \qquad (5.14) \end{aligned}$$

Consider the probabilities

$$Q_1 = Q^\varepsilon \circ \left(\frac{W+Z}{\sqrt{2}}\right)^{-1}$$

and

$$Q_2 = P^\varepsilon \circ \left(\frac{W+Z}{\sqrt{2}}\right)^{-1}.$$

For any Borel set G in $C[0\infty)$ we have

$$\begin{aligned} \int_{\{g\in G\}} E_{P^\varepsilon}\left(\frac{dQ^\varepsilon}{dP^\varepsilon} \mid g\right) dP^\varepsilon &= Q^\varepsilon(g \in G) = Q^\varepsilon\left(\frac{W-Z}{\sqrt{2}} \in G\right) \\ &= Q^\varepsilon\left(\frac{W+Z}{\sqrt{2}} \in -G-\varepsilon\right) = Q_1(-G-\varepsilon) \\ &= \int_{-G-\varepsilon} \frac{dQ_1}{dQ_2} dQ_2 = \int_{\{g\in G\}} \frac{dQ_1}{dQ_2}(-\omega-\varepsilon)dP^\varepsilon(\omega). \end{aligned}$$

Hence, a.s.,

$$E_{P^\varepsilon}\left(\frac{dQ^\varepsilon}{dP^\varepsilon} \mid g\right)(\omega) = \frac{dQ_1}{dQ_2}(-\omega-\varepsilon)$$

and we deduce

$$P^\varepsilon\left(E_{P^\varepsilon}\left(\frac{dQ^\varepsilon}{dP^\varepsilon} \mid g\right) > |\log\varepsilon|^{-\gamma_1}\right) = Q_2\left(\frac{dQ_1}{dQ_2} > |\log\varepsilon|^{-\gamma_1}\right). \qquad (5.15)$$

Theorem 5.2 can be applied to measures Q_2 and Q_1, and from (5.14) and (5.15) we obtain for any fixed $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$, and small $\varepsilon > 0$,

$$P^\varepsilon\left(q > c_2\varepsilon|\log\varepsilon|^{-\gamma_1}\right) < Q_2\left(\frac{dQ_1}{dQ_2} > |\log\varepsilon|^{-\gamma_1}\right) < |\log\varepsilon|^{-\gamma_2}.$$

By slightly adjusting the values of γ_1 and γ_2 we can eliminate c_2 from the formula. This completes the proof of (5.1). □

Acknowledgments. We would like to thank Rich Bass, Pat Fitzsimmons and John Sylvester for very valuable advice. The first author wishes to express his deep gratitude for the opportunity to visit the Centre de Recerca Matemàtica (Institut d'Estudis Catalans) where the paper was partly written.

References

[BB1] R. Bass and K. Burdzy (1999) Stochastic bifurcation models *Ann. Probab.* **27**, 50–108.

[BB2] R. Bass and K. Burdzy (2000) The supremum of Brownian local times on Hölder curves, preprint.

[B1] K. Burdzy (1987) *Multidimensional Brownian Excursions and Potential Theory,* Longman, London.

[B2] K. Burdzy (1993) A three-dimensional Brownian path reflected on a Brownian path is a free Brownian path. *Letters Math. Phys.* **27**, 239–241.

[BCS] K. Burdzy, Z. Chen and J. Sylvester (2000) Heat equation and reflected Brownian motion in time dependent domains (preprint).

[CLJ] M. Cranston and Y. Le Jan, Simultaneous boundary hitting for a two point reflecting Brownian motion. *Seminaire de Probabilités, XXIII*, 234–238, Lecture Notes in Math., 1372, Springer, Berlin, (1989)

[D] B. Davis (1983) On Brownian slow points. *Z. Wahrsch. verw. Gebiete* **64**, 359–367.

[EKK1] N. El Karoui and I. Karatzas (1991a) A new approach to the Skorohod problem, and its applications. *Stochastics Stochastics Rep.* **34**, 57–82.

[EKK2] N. El Karoui and I. Karatzas (1991b) Correction: "A new approach to the Skorohod problem, and its applications". *Stochastics Stochastics Rep.* **36**, 265.

[HL] S. Hofmann and J.L. Lewis (1996) L^2 solvability and representation by caloric layer potentials in time-varying domains. *Ann. Math.* **144**, 349–420.

[KS] I. Karatzas and S. Shreve (1994) *Brownian Motion and Stochastic Calculus, Second Edition*. Springer-Verlag, New York.

[K] F. Knight (1999) On the path of an inert object impinged on one side by a Brownian particle (preprint)

[LM] J.L. Lewis and M.A.M. Murray (1995) The method of layer potentials for the heat equation in time-varying domains. *Memoir AMS 545*, vol. 114.

[M] B. Maisonneuve (1975) Exit systems. *Ann. Probab* **3**, 399–411.

[P] J. Pitman (1975) One-dimensional Brownian motion and the three-dimensional Bessel process. *Adv. in Appl. Probab.* **7**, 511–526.

[R] D. Revuz (1970) Mesures associees aux fonctionelles addiives de Markov. I. *Trans. AMS* **148**, 501–531.

[RY] D. Revuz and M. Yor (1999) *Continuous Martingales and Brownian Motion. Third Edition*. Springer, Berlin.

[RW] L.C.G. Rogers and D. Williams (1987) *Diffusions, Markov Processes and Martingales. Vol. 2. Itô Calculus*. Wiley, New York.

[STW] F. Soucaliuc, B. Toth and W. Werner (2000) Reflection and coalescence between independent one-dimensional Brownian paths. *Ann. Inst. H. Poincare* **36**, 509–545.

[VW] S.R.S. Varadhan and R.J. Williams (1985) Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38**, 405–443.

Krzysztof Burdzy
Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195-4350, USA
e-mail: burdzy@math.washington.edu

David Nualart
Facultat de Mathemàtiques
Universitat de Barcelona
Gran Via 585,
08007 Barcelona, Spain
e-mail:nualart@mat.ub.es