INVARIANT ALGEBRAIC SURFACES OF THE BELOUSOV–ZHABOTINSKII SYSTEMS

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Abstract

In this paper we study the Field–Noyes model of the Belousov–Zhabotinskii chemical reaction:

$$\frac{dx}{d\tau} = s(y - xy + x - qx^2), \quad \frac{dy}{d\tau} = s^{-1}(-y - xy + rz), \quad \frac{dz}{d\tau} = w(x - z),$$

from the integrability point of view, and we obtain the necessary and sufficient conditions in order that this system has invariant algebraic surfaces, polynomial first integrals, rational first integrals, and invariants (also called integrals of motion).

1. Introduction and statement of the main results

The Belousov–Zhabotinskii chemical reaction, discovered in 1959 by Belousov [1], is one of the most interesting and best understood chemical oscillators. This reaction has been investigated intensively as a dynamic system (see for instance, [10], [13], [23], [25] and [26]). In 1974 Field and Noyes [8] abstracted a simpler model from the Belousov–Zhabotinskii chemical reaction which appears to retain the important features of the complete system. The differential equations describing the dynamic of the model are

$$\begin{aligned} \frac{dX}{dt} &= k_1AY - k_2XY + k_3BX - 2k_4X^2, \\ \frac{dY}{dt} &= -k_1AY - k_2XY + fk_5Z, \end{aligned}$$

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$$\frac{dZ}{dt} = k_3BX - k_3Z,$$

where $k_i's$, A, B and f are known real constants. These equations can be changed into the following system (see [8]):

$$\begin{aligned}
\frac{dx}{d\tau} &= s(y - xy + x - qx^2) &&= P(x,y,z) \\
\frac{dy}{d\tau} &= s^{-1}(-y - xy + rz) &&= Q(x,y,z), \\
\frac{dz}{d\tau} &= w(x - z) &&= R(x,y,z),
\end{aligned}$$

where s, q, r and w are real constants. For this system Troy [24] in 1977 demonstrated the "threshold phenomenon". In 1981 Klaasen and Troy [14] proved the existence of the travelling wave of this system.

In this paper we investigate this system from the integrability point of view, and we obtain the necessary and sufficient conditions in order that the Belousov–Zhabotinskii system has invariant algebraic surfaces, polynomial first integrals, rational first integrals, and invariants (also called integrals of motion).

Darboux [7] showed how first integrals of planar polynomial ordinary differential equations possessing sufficient invariant algebraic curves can be constructed (see also [5]). This theory depends on the number of invariant algebraic curves. Corollaries 8.1 and 8.2 of [6] show that the method of Darboux can find all elementary first integrals and Liouvillian first integrals, respectively (see also for instance [20] and [22]). For planar quadratic systems, Schlomiuk [21] investigated its particular algebraic integrals, and gived a generic characterization of each condition in terms of possessing two particular algebraic integrals of degrees not exceeding 3.

The version of the Darboux theory of integrability for 3–dimensional polynomial vector fields can be found in [3]. Goriely [11] proved that a system of dimension n has k ($k < n$) independent algebraically first integrals if and only if it has k independent rational first integrals. The rational function h_1/h_2 is a first integral if and only if $h_1 = 0$ and $h_2 = 0$ are invariant algebraic surfaces with the same cofactor (see for instance [19]). These results show that in order to find rational first integrals, it is sufficient to find invariant algebraic surfaces with same cofactor. For some special three dimensional systems, Labrunie [15] characterized all polynomial first integrals of the (a,b,c) Lotka–Volterra system, Moulin Ollagnier [18] obtained necessary and sufficient conditions for this system to have polynomial first integrals. In [19] Moulin Ollagnier studied the rational first integrals of the Lotka–Volterra system. Giacomini, Repetto and Zandron [9] investigated the integrals of motion of three–dimensional non–Hamiltonian dynamical systems. Llibre and Zhang [17] characterized all the invariant algebraic surfaces, the polynomial fist integrals, the rational first integrals, the invariant, and the algebraically integrable of the Rikitake system.

Let $f(x, y, z)$ be a real polynomial in the variables x, y and z. The algebraic surface $f(x, y, z) = 0$ of $\mathbf{R}^3$ is called an *invariant algebraic surface* of the Field–Noyes model of the Belousov-Zhabotinskii chemical reaction if

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = kf, \tag{1}$$

for some real polynomial $k(x, y, z)$, which is called the *cofactor* of $f = 0$. If $f(x, y, z) = 0$ is an invariant algebraic surface, then f is also called a *Darboux polynomial*. From (1) it follows that if an orbit of the Field–Noyes model has a point on the invariant algebraic surface $f(x, y, z) = 0$, then the whole orbit is contained in this surface.

We can easily prove that the degree of the cofactor k is less than or equal to 1. Therefore, we can assume that the cofactor is of the form

$$k(x, y, z) = ax + by + cz + d. \tag{2}$$

We say that a real function $H : \mathbf{R}^3 \times \mathbf{R} \longrightarrow \mathbf{R}$, is a *first integral* of the Belousov–Zhabotinskii system, if it is constant on all solution curves $(x(t), y(t), z(t))$ of the Belousov–Zhabotinskii system, that is, $H(x(t), y(t), z(t), t) \equiv$ constant for all values of t for which the solution $(x(t), y(t), z(t))$ is defined on $\mathbf{R}^3$. In particular, if the first integral H is independent on the time and it is a polynomial, then it is called a *polynomial first integral*. If the first integral H is a rational function independent on the time, then it is called a *rational first integral*. If the first integral H is of the form $g(x, y, z)e^{\sigma t}$ with $g = 0$ an invariant algebraic curve and σ a nonzero constant, then we say that H is an *invariant* (also called *integral of motion*).

An *algebraic function* $H(x, y, z) = C$ is a solution of the algebraic equation

$$f_0 + f_1 C + f_2 C^2 + \cdots + f_{n-1} C^{n-1} + C^n = 0,$$

where $f_i(x, y, z)$ are rational functions, and n is the smallest positive integer for which such a relation holds. Obviously, any rational function is algebraic. A Belousov–Zhabotinskii system is said to be *algebraically integrable* if it has two independent algebraic first integrals.

In this paper, by using the method of characteristic curves for solving linear partial differential equations (this method was first used to investigate the problem about invariant algebraic surfaces by Llibre and Zhang [16]), we obtain the following results, which characterize all Darboux polynomials of the Field and Noyes models.

Theorem 1. *For the Field and Noyes models of Belousov–Zhabotinskii chemical reaction the following statements hold.*

(a) If $s^2q = 1$, then the function f is an irreducible Darboux polynomial if and only if one of the following four cases holds.

(i) If $s^2 = 1$, $w \neq 0$ and $r \neq 0$, then $f = x - 1$ is a Darboux polynomial with the cofactor $k = -s(x + y)$;

(ii) If $s^2 = 1$, $w \neq 0$ and $r = 0$, then $f_1 = x - 1$ and $f_2 = y$ are two Darboux polynomials with the cofactors $k_1 = -sx - sy$ and $k_2 = -sx - s$, respectively;

(iii) If $s^2 = 1$, $w = 0$ and $r \neq 0$, then $f_1 = x - 1$ is a Darboux polynomial with the cofactor $k_1 = -s(x + y)$, and $f_2 = z$ is a polynomial first integral;

(iv) If $s^2 = 1$, $w = 0$ and $r = 0$, then $f_1 = x - 1$ and $f_2 = y$ are two Darboux polynomials with the cofactors $k = -sx - sy$ and $k_2 = -sx - s$ respectively, and $f_3 = z$ is a polynomial first integral.

(b) If $q = 0$, then the function f is an irreducible Darboux polynomial if and only if one of the following four cases holds.

(i) If $r = 0$ and $w = 0$, then $f_1 = y$ is a Darboux polynomial with the cofactor $k_1 = -s^{-1}x - s^{-1}$, and $f_2 = z$ is a polynomial first integral;

(ii) If $r \neq 0$ and $w = 0$, then $f = z$ is a polynomial first integral;

(iii) If $r = 0$ and $sw = 2$, then $f_1 = y$ and $f_2 = z - \frac{2}{s^2+2}(x - s^2y)$ are two Darboux polynomials with the cofactors $k_1 = -s^{-1}x - s^{-1}$ and $k_2 = -2s^{-1}$, respectively;

(iv) If $r \neq 0$ and $(sw - 2)(s^2 + 2) = s^3rw$, then $f = z - \frac{sw}{s^2+2}(x - s^2y)$ is a Darboux polynomial with the cofactor $k = -2s^{-1}$.

(c) If $q \neq 0$ and $s^2q \neq 1$, then the function f is an irreducible Darboux polynomial if and only if one of the following three cases holds.

(i) If $r \neq 0$ and $w = 0$, then $f = z$ is a polynomial first integral;

(ii) If $r = 0$ and $w \neq 0$, then $f = y$ is a Darboux polynomial with the cofactor $k = -s^{-1}(x + 1)$;

(iii) If $r = 0$ and $w = 0$, then $f_1 = y$ is a Darboux polynomial with the cofactor $k_1 = -s^{-1}(x + 1)$, and $f_2 = z$ is a polynomial first integral.

From Theorem 1 we can get the following results.

Corollary 2. *(a) If the Belousov–Zhabotinskii system has a rational first integral H, then $w = 0$ and H must be a rational function of z.*

(b) The Belousov–Zhabotinskii system is not algebraically integrable.

(c) The unique invariant $H(x, y, z, t)$ of the Belousov–Zhabotinskii system is

$$H(x, y, z, t) = \left(z - \frac{sw}{s^2 + 2}(x - s^2 y) \right) e^{2s^{-1}t},$$

and it exists when $q = 0$ and $(sw - 2)(s^2 + 2) = s^3 rw$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1. Section 3 contributes to the proof of Corollary 2.

2. Proof of Theorem 1

We first recall the following result, which will be used later on (for a proof see for instance [5]).

Proposition 3. *Assume that $f(x, y, z)$ is a polynomial function in the real polynomial ring $\mathbf{R}[x, y, z]$. Let $f = f_1^{n_1} \cdots f_m^{n_m}$ be the factorization of f in irreducible factors over $\mathbf{R}[x, y, z]$. Then for the Field–Noyes model of the Belousov–Zhabotinskii chemical reaction, f is a Darboux polynomial with the cofactor k_f if and only if each f_i is a Darboux polynomial with the cofactor k_{f_i} for $i = 1, 2, \ldots, m$. Moreover, $k_f = n_1 k_{f_1} + \cdots + n_m k_{f_m}$.*

We now prove our theorem. Assume that

$$f(x, y, z) = \sum_{i=0}^{n} f_i(x, y, z),$$

is a Darboux polynomial of the Field–Noyes model of the Belousov--Zhabotinskii chemical reaction, where f_i is a homogeneous polynomial of degree i for $i = 0, 1, \ldots, n$. The cofactor is that given in (2).

Substituting f and (2) into equality (1) and identifying the homogeneous components of degree $n + 1$, we obtain

$$s(-xy - qx^2)\frac{\partial f_n}{\partial x} - s^{-1}xy\frac{\partial f_n}{\partial y} = (ax + by + cz)f_n, \tag{3}$$

$$\begin{aligned} s(-xy - qx^2)\frac{\partial f_i}{\partial x} - s^{-1}xy\frac{\partial f_i}{\partial y} = (ax + by + cz)f_i - s(y + x)\frac{\partial f_{i+1}}{\partial x} + \\ s^{-1}(y - rz)\frac{\partial f_{i+1}}{\partial y} - w(x - z)\frac{\partial f_{i+1}}{\partial z} + df_{i+1}, \end{aligned} \tag{4}$$

for $i = n - 1, n - 2, \ldots, 1, 0$.

We claim that $c = 0$. If $c \neq 0$, since f_n is a homogeneous polynomial, then from (3) we obtain that x divides f_n. Let $f_n = xg_{n-1}$, where g_{n-1} is a homogeneous polynomial of degree $n-1$ in x, y and z. We have the following equation

$$s(-xy - qx^2)\frac{\partial g_{n-1}}{\partial x} - s^{-1}xy\frac{\partial g_{n-1}}{\partial y} = [(a+sq)x + (b+s)y + cz]g_{n-1}.$$

It is easy to see that x divides g_{n-1}. So, by recursive calculations, we get that $f_n = \alpha x^n$ with α real constant and

$$(a + nsq)x + (b + ns)y + cz = 0.$$

This equality contradicts the assumption $c \neq 0$. This proves the claim.

In what follows, in order to prove our theorem we will use the method of characteristic curves for solving linear partial differential equations (see for instance, Chapter 2 of [2]). The characteristic equation associated to (3) is

$$\frac{dx}{dy} = \frac{s(xy + qx^2)}{s^{-1}xy} = \frac{s^2(y + qx)}{y}, \qquad \frac{dz}{dy} = 0.$$

Its general solution is

$$-\frac{x}{y} + s^2 \log|y| = c_1, \qquad z = c_2, \qquad \text{if } s^2q = 1,$$

or

$$y^{-s^2 q}\left(x + \frac{s^2}{s^2q - 1}y\right) = c_1, \qquad z = c_2, \qquad \text{if } s^2q \neq 1,$$

where c_1 and c_2 are arbitrary constants.
Case 1: $s^2q = 1$. We consider the change of variables

$$u = -\frac{x}{y} + s^2\log|y|, \qquad v = y, \qquad z = z. \tag{5}$$

Correspondingly, the inverse transformation is

$$x = s^2 v\log|v| - uv, \qquad y = v, \qquad z = z. \tag{6}$$

From equation (3) and the method of characteristic curves, we get the ordinary differential equation

$$-s^{-1}(s^2 v\log|v| - uv)v\frac{d\overline{f}_n}{dv} = \left[a(s^2 v\log|v| - uv) + bv\right]\overline{f}_n, \tag{7}$$

where $\overline{f}_n(u, v, w) = f_n(x, y, z)$, and u and z are fixed. In the following, if we do not say anything, we will always denote by $\overline{R}(u, v, w)$ the function

$R(x,y,z)$, written in the variables u, v and z by using (6). Solving this equation we get

$$\overline{f}_n(u,v,z) = |v|^{-as}(s^2 \log|v| - u)^{-bs^{-1}} \overline{A}(u,z),$$

where $\overline{A}(u,z)$ is an arbitrary function in u and z. By (5) we have

$$f_n(x,y,z) = x^{-bs^{-1}} y^{bs^{-1}-as} \overline{A}\left(-\frac{x}{y} + s^2 \log|y|, z\right).$$

Since $s \neq 0$, in order that f_n be a homogeneous polynomial in x, y and z, we must have $-bs^{-1}$ and $bs^{-1} - as$ nonnegative integers, and $A = a_m z^m$, where m is a suitable nonnegative integer and a_m is an arbitrary nonzero constant, that is

$$f_n(x,y,z) = a_m x^{-bs^{-1}} y^{bs^{-1}-as} z^m,$$

where

$$-as + m = n. \tag{8}$$

Introducing f_n into equation (4) with $i = n-1$ and doing some computations , we get

$$\begin{aligned} s(-xy - qx^2)\frac{\partial f_{n-1}}{\partial x} - s^{-1}xy\frac{\partial f_{n-1}}{\partial y} = (ax+by)f_{n-1} + \\ ba_m x^{-bs^{-1}-1} y^{bs^{-1}-as+1} z^m + \\ \left(b + bs^{-2} - a + wm + d\right) a_m x^{-bs^{-1}} y^{bs^{-1}-as} z^m - \\ rs^{-1}(bs^{-1} - as) a_m x^{-bs^{-1}} y^{bs^{-1}-as-1} z^{m+1} - \\ wma_m x^{-bs^{-1}+1} y^{bs^{-1}-as} z^{m-1}. \end{aligned}$$

Using the transformation (6) we have

$$\begin{aligned} -s^{-1}(s^2 v \log|v| - uv)v\frac{d\overline{f}_{n-1}}{dv} &= \left[a(s^2 v \log|v| - uv) + bv\right] \overline{f}_{n-1} + \\ & ba_m (s^2 v \log|v| - uv)^{-bs^{-1}-1} v^{bs^{-1}-as+1} z^m + \\ & (b + bs^{-2} - a + wm + d) a_m (s^2 v \log|v| - uv)^{-bs^{-1}} v^{bs^{-1}-as} z^m - \\ & rs^{-1}(bs^{-1} - as) a_m (s^2 v \log|v| - uv)^{-bs^{-1}} v^{bs^{-1}-as-1} z^{m+1} - \\ & wma_m (s^2 v \log|v| - uv)^{-bs^{-1}+1} v^{bs^{-1}-as} z^{m-1}. \end{aligned}$$

The corresponding linear homogeneous equation has a general solution

$$\overline{f}^*_{n-1} = v^{-as}(s^2 \log|v| - u)^{-bs^{-1}} \overline{A}^*_{n-1}(u,z),$$

where $\overline{A}^*_{n-1}$ is an arbitrary function in u and z. In order to make use of the method of variation of constant for solving linear ordinary differential equation, we can assume that the solution of the previous linear equation is of the form

$$\overline{f}_{n-1} = v^{-as}(s^2 \log|v| - u)^{-bs^{-1}} \overline{A}_{n-1}(u, v, z).$$

Then $\overline{A}_{n-1}$ satisfies

$$\begin{aligned}
- \quad & s^{-1}(s^2 \log|v| - u)v^2 \frac{d\overline{A}_{n-1}}{dv} = \\
& ba_m(s^2 v \log|v| - uv)^{-1} v z^m + (b + bs^{-2} - a + wm + d)a_m z^m - \\
& rs^{-1}(bs^{-1} - as)a_m v^{-1} z^{m+1} - wma_m(s^2 v \log|v| - uv)z^{m-1}.
\end{aligned}$$

Solving this equation, we obtain

$$\begin{aligned}
\overline{A}_{n-1} \quad = \quad & -bsa_m z^m \int \frac{dv}{(s^2 \log|v| - u)^2 v^2} \\
& -s(b + bs^{-2} - a + wm + d)a_m z^m \int \frac{dv}{(s^2 \log|v| - u)v^2} \\
& +r(bs^{-1} - as)a_m z^{m+1} \int \frac{dv}{(s^2 \log|v| - u)v^3} \\
& +wmsa_m z^{m-1} \log|v| + \overline{B}_{n-1}(u, z),
\end{aligned}$$

where $\overline{B}_{n-1}$ is an arbitrary function in u and z.

Making some computations and combining the formulas 2.325 and 8.214 of [12], we get that

$$\begin{aligned}
& f_{n-1} = x^{-bs^{-1}} y^{bs^{-1} - as} \left[bs^{-1} a_m \frac{z^m}{x} - \right. \\
& s^{-1}(b - a + wm + d)a_m z^m \exp\left(s^{-2}\frac{x}{y} - \log|y| \right) \mathrm{Ei}\left(-s^{-2}\frac{x}{y} \right) + \\
& rs^{-2}(bs^{-1} - as)a_m z^{m+1} \exp\left(2s^{-2}\frac{x}{y} - 2\log|y| \right) \mathrm{Ei}\left(-2s^{-2}\frac{x}{y} \right) + \\
& \left. wmsa_m z^{m-1} \log|y| + B_{m-1}\left(-\frac{x}{y} + s^2 \log|y|, z \right) \right],
\end{aligned}$$

where

$$\mathrm{Ei}(\theta) = \log|\theta| + \sum_{k=1}^{\infty} \frac{\theta^k}{k \cdot k!},$$

is an exponential–integral function. In order that $f_{n-1}(x,y,z)$ be a homogeneous polynomial of degree $n-1$ in x, y and z, we must have $b-a+wm+d=0$, $r(bs^{-1}-as)=0$, $wm=0$ and $B_{m-1}=a_{m-1}z^{m-1}$, because of $s\neq 0$ and $a_m\neq 0$. We distinguish the following two subcases.
Subcase 1: $w\neq 0$. Then $m=0$, $b-a+d=0$ and $r(bs^{-1}-as)=0$. We have

$$f_n=a_0x^{-bs^{-1}}y^{bs^{-1}-as},\qquad f_{n-1}=bs^{-1}a_0x^{-bs^{-1}-1}y^{bs^{-1}-as},$$

we note that $a_{-1}=0$ because f_{n-1} must be a polynomial.

For convenience to computation, we define $l=-bs^{-1}$, then from (8) we have that $bs^{-1}-as=n-l$. From equation (4) with $i=n-2$, we get

$$\begin{aligned}s(-xy-qx^2)\frac{\partial f_{n-2}}{\partial x} \;-\; s^{-1}xy\frac{\partial f_{n-2}}{\partial y} \;=\; (ax+by)f_{n-2}+\\ sl(l-1)a_0x^{l-2}y^{n-l+1}+[s(l-1)-s^{-1}(n-l)-d]la_0x^{l-1}y^{n-l}+\\ s^{-1}rl(n-l)a_0x^{l-1}y^{n-l-1}z.\end{aligned}$$

Working in a similar way to solve f_{n-1}, we can obtain that

$$\begin{aligned}f_{n-2} \;=\; &\overline{f}_{n-2}(u,v,z)=x^ly^{n-l}\overline{A}_{n-2}\left(-\frac{x}{y}+s^2\log|y|,y,z\right)=\\ &x^ly^{n-l}\left[\frac{l(l-1)}{2}a_0x^{-2}-s^{-1}l\left[s(l-1)-s^{-1}(n-1)-d\right]a_0\cdot\right.\\ &\quad\exp\left(2s^{-2}\left(\frac{x}{y}-s^2\log|y|\right)\right)\cdot\\ &\quad\left(-\frac{y}{x}\exp\left(-2s^{-2}\frac{x}{y}\right)-2s^{-2}\mathrm{Ei}\left(-2s^{-2}\frac{x}{y}\right)\right)+\\ &\left.\overline{B}_{n-2}\left(-\frac{x}{y}+s^2\log|y|,z\right)\right].\end{aligned}$$

Moreover, $s(l-1)-s^{-1}(n-1)-d=-b+a-d-s+s^{-1}=-s+s^{-1}$. So, in order that f_{n-2} be a homogeneous polynomial of degree $n-2$ in x, y and z, we must have $-s+s^{-1}=0$, i.e. $s^2=1$, and $\overline{B}_{n-2}\equiv 0$. Then we have

$$f_{n-2}=\frac{l(l-1)}{2}a_0x^{l-2}y^{n-l}.$$

From equation (4), by recursive calculations we get that

$$f_{n-i}=\begin{cases}(-1)^i\begin{pmatrix}l\\ i\end{pmatrix}a_0x^{l-i}y^{n-l}, & \text{for } i=3,4,\ldots,l,\\ 0, & \text{for } i=l+1,l+2,\ldots,n.\end{cases}$$

Hence,

$$f=\sum_{i=0}^{l} f_{n-i}=\sum_{i=0}^{l}(-1)^i \left(\begin{array}{c} l \\ i \end{array}\right) a_0 x^{l-i} y^{n-l}=a_0(x-1)^l y^{n-l}.$$

If $r \neq 0$, then $n-l=0$, i.e. $b-a=0$. The function $f=(x-1)^l$ is a Darboux polynomial with the cofactor $k=-lsx-lsy$. From Proposition 3 we get the proof of statement (i) of Theorem 1(a).

If $r=0$, then $f=(x-1)^l y^{n-l}$ is a Darboux polynomial with the cofactor $k=-nsx-lsy-ns+ls$. Again using Proposition 3 it follows statement (ii) of Theorem 1(a).

Subcase 2: $w=0$. Then $b-a+d=0$, $r(bs^{-1}-as)=0$. Remember that $l=-bs^{-1}$. Now from (8) we have that $bs^{-1}-as=n-l-m$. Moreover, we have

$$\begin{aligned} f_n &= a_m x^l y^{n-l-m} z^m, \\ f_{n-1} &= -l a_m x^{l-1} y^{n-l-m} z^m + a_{m-1} x^l y^{n-l-m} z^{m-1}. \end{aligned}$$

Substituting f_{n-1} into equation (4) with $i=n-2$ and doing some similar computations to Subcase 1, we can obtain the condition $s^2=1$ and

$$f_{n-2}=\left(\begin{array}{c} l \\ 2 \end{array}\right) a_m x^{l-2} y^{n-l-m} z^m - l a_{m-1} x^{l-1} y^{n-l-m} z^{m-1} + a_{m-2} x^l y^{n-l-m} z^{m-2}.$$

By recursive calculations we get that for $i=3,4,\ldots,\max\{l,m\}=M$

$$f_{n-i}(x,y,z)=\sum_{j=0}^{i}(-1)^i \left(\begin{array}{c} l \\ i-j \end{array}\right) a_{m-j} x^{l-i+j} y^{n-l-m} z^{m-j},$$

where $\left(\begin{array}{c} l \\ i-j \end{array}\right)=0$ if $i-j>l$, and $a_{m-j}=0$ if $j>m$. Therefore, we have

$$\begin{aligned} f(x,y,z) &= \sum_{i=0}^{M}\sum_{j=0}^{i}(-1)^i \left(\begin{array}{c} l \\ i-j \end{array}\right) a_{m-j} x^{l-i+j} y^{n-l-m} z^{m-j} \\ &= \sum_{j=0}^{m}\sum_{i=0}^{l}(-1)^i \left(\begin{array}{c} l \\ i \end{array}\right) a_{m-j} x^{l-i} y^{n-l-m} z^{m-j} \\ &= \sum_{j=0}^{m} a_{m-j}(x-1)^l y^{n-l-m} z^{m-j}, \end{aligned}$$

is a Darboux polynomial with the cofactor $k = -(n-m)sx - lsy - (n-l-m)s$.

Since $r(n-l-m) = 0$, when $r \neq 0$, then $n-l-m = 0$. From Proposition 3 we get the proof of statement (iii) of Theorem 1(a). When $r = 0$, again using Proposition 3 we obtain statement (iv) of Theorem 1(a).

Case 2: $s^2 q \neq 1$. We consider the change

$$u = y^{-s^2 q}\left(x + \frac{s^2}{s^2 q - 1} y\right), \qquad v = y, \tag{9}$$

its inverse transformation is

$$x = u v^{s^2 q} - \frac{s^2}{s^2 q - 1} v, \qquad y = v. \tag{10}$$

From (3) we get the ordinary differential equation (for fixed u and z)

$$\frac{d\overline{f}_n}{dv} = \left(-\frac{as}{v} - \frac{bs}{u v^{s^2 q} - \frac{s^2}{s^2 q - 1} v}\right)\overline{f}_n, \tag{11}$$

its general solution is

$$\overline{f}_n = \left(u v^{s^2 q - 1} - \frac{s^2}{s^2 q - 1}\right)^{-bs^{-1}} v^{-as + bs^{-1}(s^2 q - 1)} \overline{A}_n(u, z),$$

where $\overline{A}_n$ is an arbitrary function in u and z. From change (9) we get

$$f_n(x, y, z) = x^{-bs^{-1}} y^{-as + bsq} \overline{A}_n\left(y^{-s^2 q}\left(x + \frac{s^2}{s^2 q - 1} y\right), z\right).$$

In order that f_n be a homogeneous polynomial, we obtain from the above expression of f_n that $\overline{A}_n$ must be a polynomial in $y^{-s^2 q}\left(x + \frac{s^2}{s^2 q - 1} y\right)$ and z, and that $-bs^{-1}$ is a nonnegative integer, let $-bs^{-1} = h$. Moreover, we assume that

$$f_n = x^h y^{-as - hs^2 q} \sum_{i,j} a^n_{i,j} \left[y^{-s^2 q}\left(x + \frac{s^2}{s^2 q - 1} y\right)\right]^i z^j,$$

where $-as - s^2 q(h + i)$ is a nonnegative integer, and $\sum_{i,j}$ is the sum in all nonnegative integers i and j satisfying $h - as - s^2 qh + (-s^2 q + 1)i + j = n$. If there are more than two terms in the above sum, then $(1 - s^2 q)i_0 + j_0 = (1 - s^2 q)i + j$, i.e. $(1 - s^2 q)(i - i_0) + (j - j_0) = 0$. So, $s^2 q(i - i_0)$ is an integer. Without loss of generality, we can assume that $s^2 q = \alpha/\beta$, where

α is an integer, and β is a positive integer. Moreover, we can assume that $i = i_0 + t\beta$, where $i_0 \in [0, \beta)$ is a given nonnegative integer and t are suitable nonnegative integers. Hence, $j = j_0 - (\beta - \alpha)t$. Let $l = -as - hs^2q - i_0 s^2 q$, then it is an integer. The function f_n can be written as

$$f_n = x^h y^l \sum_j a_j^n y^{-\alpha j} \left(x + \frac{s^2}{s^2q - 1} y \right)^{i_0 + j\beta} z^{j_0 - (\beta - \alpha)j}, \tag{12}$$

where the sum $\sum\limits_j$ is for all nonnegative integer j such that $l - \alpha j$ and $j_0 - (\beta - \alpha)j$ are both nonnegative.

Substituting f_n into equation (4) with $i = n - 1$ and then using the changes (9) and (10), working in a similar way to Case 1, we obtain that

$$\overline{f}_{n-1} = \left(uv^{s^2q-1} - \frac{s^2}{s^2q - 1} \right)^{-bs^{-1}} v^{-as + bs^{-1}(s^2q-1)} \overline{A}_{n-1}(u, v, z), \tag{13}$$

where

$$\begin{aligned}
\overline{A}_{n-1} = &\sum_j s^2 h a_j^n u^{i_0 + j\beta} z^{j_0 - (\beta-\alpha)j} \int \frac{dv}{\left(uv^{s^2q} - \frac{s^2}{s^2q-1} v \right)^2} - \\
&\sum_j s \left\{ -sh + s^{-1}(l - \alpha j) + w[j_0 - (\beta - \alpha)j] + d \right\} a_j^n \cdot \\
&\qquad u^{i_0 + j\beta} z^{j_0 - (\beta-\alpha)j} \int \frac{dv}{\left(uv^{s^2q} - \frac{s^2}{s^2q-1} v \right) v} + \\
&\sum_j \frac{s^2q - 2}{s^2q - 1} s^2 (i_0 + j\beta) a_j^n u^{i_0 + j\beta - 1} \cdot \\
&\qquad z^{j_0 - (\beta-\alpha)j} \int \frac{dv}{v^{s^2q} \left(uv^{s^2q} - \dfrac{s^2}{s^2q-1} v \right)} + \\
&\sum_j s^2 (i_0 + j\beta) a_j^n u^{i_0 + j\beta - 1} z^{j_0 - (\beta-\alpha)j} \int \frac{dv}{v^{s^2q+1}} + \\
&\sum_j r(l - \alpha j) a_j^n u^{i_0 + j\beta} z^{j_0 - (\beta-\alpha)j + 1} \int \frac{dv}{\left(uv^{s^2q} - \frac{s^2}{s^2q-1} v \right) v^2} + \\
&\sum_j r(i_0 + j\beta) \frac{s^2}{s^2q - 1} a_j^n u^{i_0 + j\beta - 1} z^{j_0 - (\beta-\alpha)j + 1}.
\end{aligned}$$

$$\int \frac{dv}{v^{s^2q+1}\left(uv^{s^2q}-\dfrac{s^2}{s^2q-1}v\right)}+$$
$$\sum_j sw[j_0-(\beta-\alpha)j]a_j^n u^{i_0+j\beta}z^{j_0-(\beta-\alpha)j-1}\log|v|+\overline{B}_{n-1}(u,z), \tag{14}$$

with $\overline{B}_{n-1}$ an arbitrary function in u and z. Since the expressions of the integrals in $\overline{A}_{n-1}$ depend strongly on the values of $s^2q=\alpha/\beta$, we need to distinguish the following six subcases.
Subcase 1: $\alpha=0$, i.e. $q=0$. Without loss of generality, in (12) we can select $\beta=1$, $i_0=0$ and $j_0=m$, where $h+l+m=n$. Then we have

$$f_n=x^hy^l\sum_{j=0}^{m}a_j^n(x-s^2y)^jz^{m-j},$$

where $h=-bs^{-1}$ and $l=-as$ are both nonnegative integers. Moreover, from (13) and (14), i.e.,

$$f_{n-1}(x,y,z)=\overline{f}_{n-1}(u,v,z)=(u+s^2v)^hv^l\overline{A}_{n-1}(u,v,z),$$

in order to get the homogeneous polynomial f_{n-1} of degree $n-1$, we must have

$$\begin{aligned} rla_0^n &= 0,\\ (-s^2h+l+sd+2m)a_m^n &= 0,\\ \left[-s^2h+l+sw(m-j)+sd+2j\right]a_j^n+rs^2(l+j+1)a_{j+1}^n &= 0,\\ sw(m-j)a_j^n+(s^2+2)(j+1)a_{j+1}^n &= 0, \end{aligned} \tag{15}$$

where $j=0,1,\ldots,m-1$.

From the fourth equation of (15), we get that $a_0^n\neq 0$. Otherwise, all $a_j^n=0$ for $j=0,1,\ldots,m$. Hence, the first equation shows that $rl=0$.

If $w=0$, then $a_j^n=0$ for $j=1,2,\ldots,m$. From the third equation of (15) we get that $-s^2h+l+sd=0$, i.e. $b-a+d=0$. Under these conditions, we have

$$\begin{aligned} f_n &= a_0^nx^hy^lz^m,\\ f_{n-1} &= -ha_0^nx^{h-1}y^lz^m+x^hy^l\sum_{j=0}^{m-1}a_j^{n-1}(x-s^2y)^jz^{m-1-j}. \end{aligned}$$

Working in a similar way to the proof of Case 1, we can obtain that $h = 0$, and the Darboux polynomial is of the form

$$f = y^l \sum_{j=0}^{m} a_j z^j,$$

where $a_m \neq 0$, and a_j for $j = 0, 1, \ldots, m-1$, are arbitrary constants.

If $r = 0$, then $d = a - b$. Since $h = -bs^{-1}$ and $l = -as$, the Darboux polynomial has the cofactor $k = -ls^{-1}x - ls^{-1}$. Using Proposition 3 we prove statement (i) of Theorem 1(b).

If $r \neq 0$, then $l = 0$, i.e. $a = 0$. The Darboux polynomial is a first integral. By Proposition 3 it follows statement (ii) of Theorem 1(b).

We now consider the case $w \neq 0$, then $a_j^n \neq 0$ for $j = 0, 1, \ldots, m$. Moreover, the conditions (15) can be simplified to

$$\begin{aligned} -s^2h + l + sd + 2m &= 0, \qquad (sw-2)(s^2+2) = s^3rw, \\ a_j^n &= \left(-\frac{sw}{s^2+2}\right)^j \binom{m}{j} a_0^m \quad \text{for } j = 1, 2, \ldots, m. \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} f_n &= x^h y^l \sum_{j=0}^{m} a_0^n \left(-\frac{sw}{s^2+2}\right)^j \binom{m}{j} (x - s^2y)^j z^{m-j} \\ &= x^h y^l a_0^n \left[z - \frac{sw}{s^2+2}(x - s^2y)\right]^m, \\ f_{n-1} &= -ha_0^n x^{h-1} y^l \left[z - \frac{sw}{s^2+2}(x - s^2y)\right]^m \\ &\quad + x^h y^l \sum_{j=0}^{m-1} a_j^{n-1} (x - s^2y)^j z^{m-1-j}. \end{aligned}$$

From equation (4) with $i = n-2$ and doing similar arguments for obtaining f_{n-1}, in order to get the homogeneous polynomial f_{n-2} of degree $n-2$, we should have $h = 0$, i.e. $b = 0$, and for $j = 0, 1, \ldots, m-1$

$$\begin{aligned} &\left[-s^2h + l + sw(m-1-j) + sd + 2j\right] a_j^{n-1} + s^2r(j+1)a_{j+1}^{n-1} = 0, \\ &\left[s^2j + s^2h - l - sw(m-1-j) - sd\right] a_j^{n-1} \\ &\qquad - s^2r(j+1)a_{j+1}^{n-1} + sw(m-j)a_{j-1}^{n-1} = 0. \end{aligned} \tag{17}$$

The first condition with $j = m-1$ reduces to

$$\left[-s^2h + l + sd + 2(m-1)\right] a_{m-1}^{n-1} = 0,$$

this means that $a_{m-1}^{n-1} = 0$. Moreover, from conditions (17) we get that

$$j(s^2+2)a_j^{n-1} + sw(m-j)a_{j-1}^{n-1} = 0, \quad j = 1, 2, \ldots, m-1.$$

Hence, $a_j^{n-1} = 0$ for $j = 0, 1, \ldots, m-1$.

By recursive calculations, and under the conditions $rl = 0$ and $l + sd + 2m = 0$, we can get that $f_i \equiv 0$ for $i = n-1, n-2, \ldots, 1$. So, the Darboux polynomial is

$$f = y^l \left[z - \frac{sw}{s^2+2}(x - s^2 y) \right]^m .$$

From condition (16), if $r = 0$, then $sw = 2$ and $d = a - 2ms^{-1}$. The corresponding cofactor is $k = -ls^{-1}x - ls^{-1} - 2ms^{-1}$. So, from Proposition 3 it follows statement (iii) of Theorem 1(b).

If $r \neq 0$, then $l = 0$, i.e. $a = 0$. From (16) and again Proposition 3 we get the proof of statement (iv) of Theorem 1(b).

Subcase 2: $\alpha < 0$ and $\beta > -\alpha$. From (14), in order to get a polynomial solution $f_{n-1}(x, y, z) = \overline{f}_{n-1}(u, v, z)$, we can obtain that for $j = 0, 1, \ldots, m-1$

$$\begin{aligned} \left[-hs^2q + s^2h - (l - \alpha j) - sd + (s^2q - 2)(i_0 + j\beta)\right] a_j^m &= 0, \\ r(l - \alpha j + i_0 + j\beta)a_j^m &= 0, \\ w[j_0 - (\beta - \alpha)j]a_j^m &= 0. \end{aligned} \tag{18}$$

Since $s^2 q \neq 1$, we have $\alpha + (s^2q - 2)\beta \neq 0$. We obtain from the above conditions that there exists a unique j such that $a_j^m \neq 0$. Without loss of generality, we select $j = 0$, and i_0 a suitable nonnegative integer. Then (18) can be reduced to

$$wj_0 = 0, \qquad r(l + i_0) = 0, \qquad -hs^2q + s^2h - l - sd + (s^2q - 2)i_0 = 0. \tag{19}$$

Moreover, we have

$$\begin{aligned} f_{n-1} = & -ha_0^n x^{h-1} y^l \left(x + \frac{s^2}{s^2q - 1} y \right)^{i_0} z^{j_0} \\ & + \frac{s^2q - 2 - s^2}{s^2 q} i_0 a_0^n x^h y^l \left(x + \frac{s^2}{s^2q - 1} y \right)^{i_0 - 1} z^{j_0} \\ & + x^h y^{-as - hs^2q} \sum_{i_1, j_1} a_{i_1, j_1}^{n-1} \left(y^{-s^2q} \left(x + \frac{s^2}{s^2q - 1} y \right) \right)^{i_1} z^{j_1}, \end{aligned} \tag{20}$$

where the sum $\sum_{i_1, j_1}$ is given by all nonnegative integers i_1 and j_1 satisfying $h - as - hs^2q - s^2qi_1 + i_1 + j_1 = n - 1$.

Since $h-as-hs^2q-s^2qi_0+i_0+j_0=n$, we have $(1-s^2q)(i_1-i_0)+j_1-j_0=-1$. So $i_1=i_0+t\beta$, and $j_1=j_0-1-t(\beta-\alpha)$, where t is a suitable integer. Using these notations,

$$\begin{aligned} f_{n-1} &= -ha_0^n x^{h-1}y^l\left(x+\frac{s^2}{s^2q-1}y\right)^{i_0} z^{j_0} \\ &\quad +\frac{s^2q-2-s^2}{s^2q} i_0 a_0^n x^h y^l \left(x+\frac{s^2}{s^2q-1}y\right)^{i_0-1} z^{j_0} \\ &\quad +x^h y^l \sum_j a_j^{n-1} y^{-\alpha j}\left(x+\frac{s^2}{s^2q-1}y\right)^{i_0+j\beta} z^{j_0-1-j(\beta-\alpha)}, \end{aligned}$$

where the sum $\sum_j$ is given by all j such that $l-j\alpha$, $i_0+j\beta$ and $j_0-1-j(\beta-\alpha)$ are all nonnegative integers.

If $r\neq 0$, then $l=i_0=0$ because l and i_0 are nonnegative integers. So, in the expression of f_{n-1} the j is nonnegative. Therefore, if $j_0=0$, then the sum $\sum_j \equiv 0$. From equation (4) with $i=n-2$, we obtain in a similar way to computation of f_{n-1} that in order to get the homogeneous polynomial f_{n-2} we must have $h=0$, $wj_0=0$ and $a_j^{n-1}=0$ for $j\neq 0$. Therefore, we have

$$\begin{aligned} &f_n = a_0^n z^{j_0}, \quad f_{n-1}=a_0^{n-1}z^{j_0-1} \\ &f_{n-2}=\overline{B}_{n-2}(u,z)=\sum_{i_2,j_2}\left(y^{-s^2q}\left(x+\frac{s^2}{s^2q-1}y\right)\right)^{i_2} z^{j_2}, \end{aligned}$$

where the sum $\sum_{i_2,j_2}$ is given by all i_2 and j_2 satisfying $(1-s^2q)i_2+j_2=n-2$, and $n=j_0$.

By recursive calculations we obtain that $f_i=a_0^i z^i$ for $i=1,2,\ldots,n-2$. Since $wn=wj_0=0$, if $w\neq 0$, there are no Darboux polynomials. If $w=0$, the Darboux polynomial of degree n is of the form

$$f=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z=0.$$

Since $h=l=i_0=0$, i.e. $a=b=0$, it follows from (19) that $d=0$. Hence, this Darboux polynomial is also a polynomial first integral. This proves statement (i) of Theorem 1(c) under the condition $-1<s^2q<0$.

If $r=0$, substituting f_{n-1} into equation (4) with $i=n-2$ and working a similar way to the proof of f_{n-1} and the case $r\neq 0$, in order to get the polynomial f_{n-2}, we must have the conditions $h=0$, $i_0=0$, $wa_0^{n-1}=0$ and $a_j^{n-1}=0$ for $j\neq 0$. Therefore, we have

$$f_n = a_0^n y^l z^{j_0}, \quad f_{n-1} = a_0^{n-1} y^l z^{j_0-1},$$
$$f_{n-2} = y^l \sum_{i,j} \left(y^{-s^2 q} \left(x + \frac{s^2}{s^2 q - 1} y \right) \right)^i z^j,$$

where $n = l + j_0$, i and j nonnegative integer and $l + (1 - s^2 q)i + j = n - 2$.

By recursive calculations, we obtain that $f_{n-i} = a_0^{n-i} y^l z^{j_0 - i}$ for $i = 2, 3, \ldots, j_0$. Since $wj_0 = 0$, if $w \neq 0$, then $f_i = 0$ for $i = n-1, n-2, \ldots, 1$. So the Darboux polynomial of degree n is of the form $f = ay^n$ with the cofactor $k = -ns^{-1}(x+1)$ because $n = l = -as$ and $n + sd = 0$ by (19). From Proposition 3 it proves statement (ii) of Theorem 1(c).

If $w = 0$, then the Darboux polynomial of degree n is of the form $f = y^l \sum_{i=0}^{n-l} a_i z^i$ with the cofactor $k = -ls^{-1}(x+1)$. By using Proposition 3 we prove statement (iii) of Theorem 1(c).

Subcase 3: $\beta = -\alpha$, i.e. $s^2 q = -1$. Without loss of generality, in equality (14) we can assume that $\beta = 1$, $\alpha = -1$, $i_0 = 0$ and $j_0 = m$. Hence, $l = -as + h$. In order that f_{n-1} be a polynomial, we must have the following conditions.

$$\begin{aligned} \left[h + s^2 h - (l+j) - sw(m-2j) - sd - 3j\right] a_j^n &= 0, \quad j = 0, 1, \ldots, [m/2], \\ r(l+2j) a_j^n &= 0, \quad j = 0, 1, \ldots, [m/2], \\ rla_0^n = 0, \qquad sw(m - 2[m/2]) a_{[m/2]}^n &= 0, \\ r(l+j) a_j^n + sw(m+2-2j) a_{j-1}^n &= 0, \quad j = 1, 2, \ldots, [m/2]. \end{aligned} \tag{21}$$

If $r = 0$ and $w = 0$, then the above conditions reduce to $(h + s^2 h - l - sd - 4j) a_j^n = 0$ for $j = 0, 1, \ldots, [m/2]$. Hence, there exists a unique j (let $j = j_0$) such that $h + s^2 h - l - sd - 4j_0 = 0$ and $a_j^n = 0$ for $j \neq j_0$. Under these conditions we have

$$\begin{aligned} f_{n-1} \quad = \quad & -h a_{j_0}^n x^{h-1} y^{l+j_0} \left(x - \frac{s^2}{2} y \right)^{j_0} z^{m-2j_0} \\ & + (3+s^2) j_0 a_{j_0}^n x^h y^{l+j_0} \left(x - \frac{s^2}{2} y \right)^{j_0 - 1} z^{m-2j_0} \\ & + \sum_{j=0}^{[(m-1)/2]} a_j^{n-1} h^h y^l \left[y \left(x - \frac{s^2}{2} y \right) \right]^j z^{m-1-2j}. \end{aligned}$$

Substituting f_{n-1} into equation (4) with $i = n-2$, in order to get a polynomial solution f_{n-2}, we should have the conditions $j_0 = 0$ and

$a_j^{n-1} = 0$ for $j \neq 0$. Therefore, we have

$$
\begin{aligned}
f_n &= a_0^n x^h y^l z^m, \qquad f_{n-1} = -h a_0^n x^{h-1} y^l z^m + a_0^{n-1} h^h y^l z^{m-1}, \\
f_{n-2} &= \binom{h}{2} a_0^n x^{h-2} y^l z^m - \frac{s^2+1}{s^2} h a_0^n x^{h-1} y^{l-1} z^m - \\
&\quad h a_0^{n-1} x^{h-1} y^l z^{m-1} + \\
&\quad \sum_{j=0}^{[(m-2)/2]} a_j^{n-2} x^h y^l \left(y \left(x - \frac{s^2}{2} y \right) \right)^j z^{m-2-2j}.
\end{aligned}
$$

From equation (4) with $i = n-3$, in order to obtain a polynomial solution f_{n-3}, we must have $h = 0$ and $a_j^{n-2} = 0$ for $j = 1, 2, \ldots, [(m-2)/2]$. Therefore, we have

$$
\begin{aligned}
f_{n-2} &= a_0^{n-2} y^l z^{m-2}, \\
f_{n-3} &= y^l \sum_{j=0}^{[(m-3)/2]} a_j^{n-3} \left(y \left(x - \frac{s^2}{2} y \right) \right)^j z^{m-3-2j}.
\end{aligned}
$$

By recursive calculations we obtain that $f_{n-j} = a_0^{n-j} y^l z^{m-j}$ for $j = 3, 4, \ldots, m$. Hence, the Darboux polynomial is $f = y^l \sum\limits_{i=0}^{m} a_j z^j$ with the cofactor $k = -ls^{-1}(x+1)$. From Proposition 3 it follows statement (iii) of Theorem 1(c) under the condition $s^2 q = -1$.

If $r = 0$ and $w \neq 0$, then from (21) we obtain that $a_{j-1}^n = 0$ for $j = 1, 2, \ldots, [m/2]$. When m is odd, then $a_{[m/2]}^n = 0$, there are no invariant algebraic surfaces. When m is even, then $a_{[m/2]}^n \neq 0$. We have the condition $h + s^2 h - l - sd - 2m = 0$, and the solution

$$
\begin{aligned}
f_{n-1} &= -h a_{m/2}^n x^{h-1} y^{m/2+l} \left(x - \frac{s^2}{2} y \right)^{m/2} \\
&\quad + \frac{m(3+s^2)}{2} a_{m/2}^n x^h y^{m/2+l} \left(x - \frac{s^2}{2} y \right)^{m/2-1} \\
&\quad + x^h y^l \sum_{j=0}^{m/2-1} a_j^{n-1} \left[y \left(x - \frac{s^2}{2} y \right) \right]^j z^{m-1-2j}.
\end{aligned}
$$

Working in a similar way to the proof of the case $r = 0$ and $w = 0$, in order to get a homogeneous polynomial solution, f_{n-2}, of degree $n-2$ from equation (4) with $i=n-2$, we must have $m=0$ and $a_j^{n-1}=0$ for all j. Furthermore, we obtain that

$$\begin{aligned} f_n &= a_0^n h^h y^l, \qquad f_{n-1}=-ha_0^n h^{h-1}y^l, \\ f_{n-2} &= \left(\begin{array}{c} h \\ 2 \end{array}\right) a_0^n h^{h-2} y^l - hs^{-2}(s^2+1)a_0^n h^{h-1}y^{l-1}. \end{aligned}$$

By solving f_{n-3} from equation (4) with $i=n-3$, we can prove that $h=0$, and so $f_{n-1}=f_{n-2}=f_{n-3}\equiv 0$. Therefore, by recursive calculations we get that the Darboux polynomial of degree n is $f=y^n$ with the cofactor $k=-ns^{-1}(x+1)$.

If $r\neq 0$, then from (21) we get that $l=0$ and $a_j^n=0$ for $j\neq 0$, and so $wm=0$ and $h+s^2h-sd=0$. Moreover, we have

$$\begin{aligned} &f_n = a_0^n x^h z^m, \\ &f_{n-1} = -ha_0^n x^{h-1} z^m + \sum_{j=0}^{[(m-1)/2]} a_j^{n-1} x^h y^j \left(x-\frac{s^2}{2}y\right)^j z^{m-1-2j}. \end{aligned}$$

In order to get the homogeneous polynomial solution f_{n-2} of degree $n-2$ from equation (4) with $i=n-2$, in a similar way to the proof of the case $r=0$ and $w=0$, we obtain that $h=0$, $a_j^{n-1}=0$ for $j\neq 0$ and $wa_0^{n-1}=0$. Furthermore, we have $n=h+m=m$ and

$$\begin{aligned} &f_n = a_0^n z^n, \quad f_{n-1}=a_0^{n-1}z^{n-1}, \\ &f_{n-2} = \sum_{j=0}^{[(n-2)/2]} a_j^{n-2} y^j \left(x-\frac{s^2}{2}y\right)^j z^{n-2-2j}. \end{aligned}$$

By recursive calculations we can prove that $f_j=a_j z^j$ for $j=n-2, n-3,\ldots,1$. Therefore, if $w\neq 0$, then $m=n=0$. There are no Darboux polynomials. If $w=0$, the Darboux polynomial is $f=\sum_{i=1}^{n} a_i z^i$ which is a polynomial first integral. This completes the proof of Theorem 1(c) under the condition $s^2q=-1$.

Subcase 4: $\beta<-\alpha$. Similar to the proof of the case $\alpha<0$ and $\beta>-\alpha$, from (14) in order that f_{n-1} be a polynomial, we obtain the same conditions as those of (18). Furthermore, we get conditions (19) and the function f_{n-1} shown in (20). By doing some similar computations to Subcase 2, we can obtain the same conclusions as those of that subcase.

For the cases $0<\alpha<\beta$ and $\alpha>\beta>0$, by some straightforward calculations we can obtain the same conditions as those of (18). Furthermore, we get the conditions (19) and the function f_{n-1} shown in (20). Moreover, doing some similar computations to Subcase 2 we can obtain the same conclusions as those of that subcase.

This completes the proof of the theorem. ∎

3. Proof of Corollary 2

Before proving the corollary we first recall the following two results.

The first one characterizes the rational and algebraic first integrals of a polynomial vector field.

Proposition 4. *Let* $\mathbf{X}$ *be a polynomial vector field in* $\mathbf{R}^n$. *Then the following statements hold.*

- *(a) If the polynomial functions* f *and* g *are relative prime, then* f/g *is a rational first integral of* $\mathbf{X}$ *if and only if* f *and* g *are both Darboux polynomials with the same cofactor.*
- *(b) The vector field* $\mathbf{X}$ *is algebraically integrable if and only if it has n-1 independent rational first integrals.*

The first statement can be proved easily from the definitions. The second statement is a corollary of Lemma 2.4 of Goriely [11].

The following proposition shows the relationship between the Darboux polynomial and the invariant first integral.

Proposition 5. *A Belousov–Zhabotinskii system has a Darboux polynomial* $f(x, y, z)$ *with a nonzero constant cofactor* k *if and only if the function* $H(x, y, z, t) = f(x, y, z)\exp(-kt)$ *is invariant.*

The proof of Proposition 5 is easy, and follows in the same way that the proof of Proposition 2 of [16], so we omit it.

Proof of statement (a) of Corollary 2: Theorem 1 shows that except for the polynomial first integral $H = z$ the Belousov–Zhabotinskii system has no two Darboux polynomials with the same cofactor. Hence, it follows from statement (a) of Proposition 4 that if the Belousov–Zhabotinskii system has a rational first integral, it must satisfy the condition $w = 0$ and the first integral can only be a rational function of the first integral z. ∎

Proof of statement (b) of Corollary 2: From the proof of statement (a) we know that the unique rational first integrals of the Belousov–Zhabotinskii system are rational function of z. Any two functions in z are not independent in $\mathbf{R}^3$. Therefore, from statement (b) of Proposition 4 we finish the proof of the statement (b) of the corollary. ∎

Proof of statement (c) of Corollary 2: It follows from Theorem 1 and Proposition 5. This completes the proof of the corollary. ∎

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