Monodromy, stability, and bifurcation of a limit cycle from degenerate singular points of certain planar vector fields *

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Abstract

We solve the monodromy and stability problems (except when the principal term of the Dulac development of the return map is the identity) for degenerate singular points of a generic family of vector fields. It is known, that for the most cases, the stability of degenerate monodromic points can be determined integrating the first order variational equations, associated to the edges of the the first polar blow–up polycycle. We are motivated by the fact that this family contains cases, such that the contribution to the stability of the point, *hidden* in the singular points of the first polar blow-up polycycle, compensates the contribution given by the first order variational equations associated to the edges of the polycycle. This property is used to generate a bifurcation of a limit cycle.

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1 Introduction, motivation, and main results

A monodromic singular point of a planar analytic differential equation $\dot{x} = X(x)$, is a singular point p (that we will consider always to be the origin), such that in a neighborhood of p, the solutions of the differential equation turns around the singular point. In this case p is either a focus or a centre. The *monodromy problem* consists in deciding when a singular point is monodromic or not. This is a classical problem in the qualitative theory of dynamical systems. If the differential matrix of the vector field X in p, given by $DX(p)$, is either not degenerated, elementary degenerated (that is, there is at least a non vanishing eigenvalue) or nilpotent, the monodromy problem is solved (see [2] and [3] for more details). If all the eigenvalues of $DX(p)$ vanish the monodromy problem can be solved using the blow–up method, see [9] for instance.

A harder problem is to give algorithms to know the stability of the monodromic singular points. If $DX(p)$ has pure imaginary eigenvalues (different from zero) the problem was already technically solved by Poincaré and Lyapunov. Moussu, see [19], solved the nilpotent case. If $DX(p)$ is identically zero a general method is not known. We give a first approach to the study of the stability of a generic family containing such cases. Let us introduce some notation: Consider Φ to be the set of germs of planar analytic vector fields at the origin. For $X \in \Phi$, we denote by X^1 and X^2 its components. Also denote by X_k or X_k^1 and X_k^2 the corresponding homogeneous components of degree k. For $k \geq 2$, set $\Phi_k = \{X \in \Phi\colon\ X_0 = X_1 = \cdots = X_{k-1} = 0 \text{ and } X_k \neq 0\}$. For $X \in \Phi_k$, $i \geq k$, we set

$$F_i(X)(\theta) = \cos(\theta)X_i^2(\cos(\theta), \sin(\theta)) - \sin(\theta)X_i^1(\cos(\theta), \sin(\theta)),$$
$$R_i(X)(\theta) = \cos(\theta)X_i^1(\cos(\theta), \sin(\theta)) + \sin(\theta)X_i^2(\cos(\theta), \sin(\theta)).$$

A zero of $F_k(X)$ is called a *characteristic direction* of the origin of X. Geometrically, the characteristic directions are the singular points that can appear over the exceptional divisor after performing the first polar blow-up. If there appear no characteristic directions, then the classical Poincaré–Lyapunov theory can be reproduced following the scheme of [1]. This scheme has been used in [4],[6], [13] or [20] for instance. Although other approximations seems to be more efficient for some concrete vector fields. See [7], [8] or [12]. In the monodromic situation, it is known that the return map Π around p, is semi–regular (see [15] for a definition) can be expressed as $\Pi(x) = V_1\,x + o(x)$, for some nonzero constant V_1, called *first generalized Lyapunov constant*. Medvedeva in [18] gives a procedure to compute V_1 for any monodromic singular point p. To apply Medvedeva's result we need to compute all the blow-ups to desingularize the point to decide if it is monodromic, as well as to compute V_1. To our knowledge, this result belongs to the last of series of papers dealing with this problem ([5], [16], [17] and [18]). The stability of the singular point is determined by V_1, when it is not 1. To determine the stability of p when $V_1 = 1$ is still an open problem. This paper is a continuation of the program of investigation started in [10] and [11]. By reasons of uniformity, if it has been possible, we have reproduced the same notation used in the last reference.

Our first motivation is the following: The simplest cases of degenerate monodromic singular points are the ones in which at the end of the desingularization process, there appears a polycycle with hyperbolic corners (i.e. having hyperbolic saddles in the corners). We will call this singular points $\mathcal{H}$–monodromic singular points. In [11], was established that in a generic family of $M_k = \{X \in \Phi_k$, such that the origin is monodromic$\}$, having always $\mathcal{H}$–monodromic points at the origin, the first term of the asymptotic expansion of the return map associated to the origin is

$$V_1 \;=\; \exp\left\{ \text{G.P.V.} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta \right\}, \tag{1}$$

where G.P.V. is an operator defined as follows: Given a function f, continuous in $[0, 2\pi] \setminus \{\theta_1, \theta_2, \ldots, \theta_n\}$. Set $I_\varepsilon = [0, 2\pi] \setminus \cup_{i=1}^{n} (\theta_i - \varepsilon, \theta_i + \varepsilon)$. We define the *Cauchy Global Principal Value* of $\int_0^{2\pi} f(\theta)\, d\theta$, as the following limit (if it exists):

$$\text{G.P.V.} \left\{ \int_0^{2\pi} f(\theta)\, d\theta \right\} := \lim_{\varepsilon \longrightarrow 0} \int_{I_\varepsilon} f(\theta)\, d\theta.$$

Therefore, generically the stability of the origin for vector fields in M_k is given by the sign of

$$\text{G.P.V.} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta, \tag{2}$$

In fact the expression (2) appears as a summand in the expression of $\ln V_1$ for every $\mathcal{H}$–monodromic point. But in general V_1 is not always given by (1). In [10], are given examples of that fact. In these cases there appear elementary degenerate singular points at the end of the desingularization process. Hence, in this cases the origin is not a $\mathcal{H}$–monodromic point. It has been pointed out the necessity of provide examples of $\mathcal{H}$–monodromic singular points such that V_1 is not given by (1). Although this was already done by Medvedeva in [18], we also provide a systematic characterization of one type of the less degenerated families in which this fact occurs.

The main goal of this paper consists in solving the monodromy and the stability problem (mod $V_1 = 1$) of a generic family in M_k, containing S_k —the mentioned family studied in [11])—, such that V_1 is not given by (1). This is done in Theorems 1 and 2 respectively, which are the main results of the paper.

As can be seen in [11], if $X \in \Phi_k$ is monodromic then k is odd, $F_k(X)(\theta)$ is not identically zero and does not changes the sign. In this paper we will consider that $F_k(X)(\theta) \geq 0$ for all $\theta \in [0, 2\pi)$, without loss of generality. This last condition implies that any characteristic direction has even multiplicity.

Set $\mathcal{N}$ the set of characteristic directions θ_*, satisfying the following properties

(a) $R_k(X)(\theta_*) = 0$,

(b) $F_{k+1}(X)(\theta_*) = 0$,

(c) $\left[(F_k(X)'' - 2R_k(X)')F_k(X)''\right](\theta_*) \geq 0$,

(d) $\left[(F_{k+1}(X)' - R_{k+1}(X))^2 - 2(F_k(X)'' - 2R_k(X)')F_{k+2}(X)\right](\theta_*) \leq 0$,

(e) $F_{k+2}(X)(\theta_*) \geq 0$.

Let us denote $\mathcal{A}$ the set of characteristic directions θ_*, satisfying a), b), e), and

(f) $\left[(F_k(X)'' - 2R_k(X)')F_k(X)''\right](\theta_*) > 0$,

(g) $\left[(F_{k+1}(X)' - R_{k+1}(X))^2 - 2(F_k(X)'' - 2R_k(X)')F_{k+2}(X)\right](\theta_*) < 0$.

Finally consider $\mathcal{B}$ the set of characteristic directions θ_*, satisfying a), b), f) and

(h) $F_{k+2}(X)(\theta_*) = 0$,

(i) $\left[(F_{k+1}(X)' - R_{k+1}(X))\right](\theta_*) = 0$,

(j) $\left[(F_k(X)'' - 2R_k(X)')\right](\theta_*) > 0$,

(k) $R_{k+1}(X)(\theta_*) = 0$

(l) $F_{k+3}(X)(\theta_*) = 0$

(m) $\left[(F_k(X)'' - 4R_k(X)')\right](\theta_*) > 0$

(n) $F_{k+4}(X)(\theta_*) > 0$

(o) $\left[(F_{k+2}(X)' - 2R_{k+2}(X))^2 - 2(F_k(X)'' - 4R_k(X)')F_{k+4}(X)\right](\theta_*) < 0$

Theorem 1. *Suppose $X \in \Phi_k$, then the following statements hold:*

(i) If $X \in M_k$, then any characteristic direction $\theta_* \in \mathcal{N}$.

(ii) If any characteristic direction of X, $\theta_* \in \mathcal{A} \cup \mathcal{B}$, then $X \in M_k$.

The proof of Theorem 1, is given in Section 2. Observe that $\mathcal{A} \cup \mathcal{B} \subset \mathcal{N}$. Set $GS_k = \{X \in \Phi_k$ such that any characteristic direction $\theta_* \in \mathcal{A} \cup \mathcal{B}\}$. If $X \in GS_k$ is such that $\mathcal{A} = \emptyset$, then $X \in S_k$, the family studied in [11]. Any vector field in GS_k has a $\mathcal{H}$–monodromic points at the origin.

For any $\theta_j \in \mathcal{B}$, we denote:

$$I(\theta_j) := \text{P.V.} \int_{-\infty}^{+\infty} \frac{(R_k(X)'(\theta_j)m + R_{k+2}(X)(\theta_j))\ dm}{[F_k(X)''/2 - 2R_k(X)'](\theta_j)m^2 + [F_{k+2}(X)' - 2R_{k+2}(X)'](\theta_j)m + F_{k+4}(X)(\theta_j)}.$$

Remember, that given a continuous map $f : \mathbb{R} \to \mathbb{R}$ we denote as P.V. $\int_{-\infty}^{\infty} f(x)dx$, its principal value, defined as the following limit (if it exists): P.V. $\int_{-\infty}^{\infty} f(x)dx := \lim_{\varepsilon\to\infty} \int_{-\varepsilon}^{\varepsilon} f(x)dx$.

Theorem 2. *If $X \in GS_k$, then the following statements hold*

(i) G.P.V. $\displaystyle\int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta$ *exists.*

(ii) For each $\theta_j \in \mathcal{B}$, $I(\theta_j)$ is well defined.

(iii) The return map associated to the origin of (4) has the form: $\Pi(x) = V_1\, x + \mathrm{o}(x)$, *where*

$$V_1 = \exp\left\{ \text{G.P.V.} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta + 2 \sum_{\theta_j \in \mathcal{B}} \frac{F_k(X)'' - 4R_k(X)'}{F_k(X)''}(\theta_j) \times I(\theta_j) \right\}. \quad (3)$$

In Section 3 we proof Theorem 2.

As seen in Theorem 2, for vector fields in GS_k, $\ln V_1$ can be controlled by (2) plus additional terms which depend not only on coefficients in X_k, but also on higher order terms. Roughly speaking, the contribution in $\ln V_1$ hidden in the singular points of the first polar blow-up polycycle, can compensate the contribution given by the first order variational equations associated to the edges of the polycycle, which is (2). Hence we are able to construct explicit examples in which the hidden higher degree contributions can dominate the lower degree contribution. This is done in Section 4.

Our second motivation is that, this last fact permits to construct examples in which a Hopf bifurcation of limit cycle occurs. This is done in Section 5.

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2 Monodromy: Proof of Theorem 1

PROOF OF THEOREM 1. Statement (i) it is proved in [11].

(ii) Assume conditions (a) and (b). From (i), they are necessary conditions to have a monodromic point at the origin. For each $(X^1, X^2) \in \Phi_k$. we associate the differential equation

$$\begin{cases} \dot{x} = & X^1(x,y), \\ \dot{y} = & X^2(x,y). \end{cases} \tag{4}$$

In [11] it is also proved that if $\theta_* \in \mathcal{A}$, then there appear two well–posed hyperbolic saddles in the geometry of the blow–up of $\theta = \theta_*$. Hence there are not characteristic orbits tending or leaving from the origin of (4) with characteristic direction θ_*.

If we prove that, for every characteristic direction $\theta_* \in \mathcal{B}$, there are not orbits tending or leaving from the origin of (4), with characteristic direction θ_*, then we have finished. To do this, we explicit the blow-up process of the characteristic direction $\theta_* \in \mathcal{B}$. We conjugate X by means of a rotation in such a way the characteristic direction θ_* is transformed into the direction $\theta = 0$. We denote by Y the vector field obtained from this conjugation. It can be easily seen that the following relations hold:

(A)

$$\begin{cases} Y^1(x,y) = \cos(\theta_*)X^1(M_{\theta_*}(x,y)) + \sin(\theta_*)X^2(M_{\theta_*}(x,y)), \quad \text{and} \\ Y^2(x,y) = -\sin(\theta_*)X^1(M_{\theta_*}(x,y)) + \cos(\theta_*)X^2(M_{\theta_*}(x,y)), \end{cases}$$

where

$$M_{\theta_*} = \begin{pmatrix} \cos(\theta_*) & -\sin(\theta_*) \\ \sin(\theta_*) & \cos(\theta_*) \end{pmatrix}.$$

(B) For $i \geq k,\ F_i(Y)(\theta) = F_i(X)(\theta + \theta_*)$ and $R_i(Y)(\theta) = R_i(X)(\theta + \theta_*)$.

Consider the following blow-up's

(a) $(u, z) = (x^2/y, y/x)$. This blow-up is obtained by the composition of the two blow-up's $(x, z) = (x, y/x)$ and $(u, z) = (x/z, z)$.

(b) $(x, w) = (x, y/x^2)$. As above, this blow-up is obtained by the composition of the two blow-up's $(x, z) = (x, y/x)$ and $(x, w) = (x, z/x)$.

By the blow-up *(a)*, plus a time re–parameterization, our system is transformed into

$$\begin{cases} \dot{u} &= u[2zY_k^1(1,z) - Y_k^2(1,z)] \\ & \qquad + \frac{\sum_{i>k}^{\infty} 2zY_i^1(uz,uz^2) - Y_i^2(uz,uz^2)}{u^{k-1}z^k} =: U(u,z), \\ \dot{z} &= z[Y_k^2(1,z) - zY_k^1(1,z)] \\ & \qquad + \frac{\sum_{i>k}^{\infty} Y_i^2(uz,uz^2) - zY_i^1(uz,uz^2)}{u^k z^{k-1}} =: Z(u,z). \end{cases}$$

If conditions (a) and (b) hold, then z^2 is a common factor of U and Z. After another time re–parameterization we obtain:

$$\begin{cases} \dot{u} &= (R_k(Y)'(0) - \frac{F_k(Y)''(0)}{2})u + \cdots, \\ \dot{z} &= (\frac{F_k(Y)''(0)}{2})z + \cdots, \end{cases}$$

where the three dots denote second order terms. Using (B), the last system can be re–written in terms of X:

$$\begin{cases} \dot{u} &= (R_k(X)'(\theta_*) - \frac{F_k(X)''(\theta_*)}{2})u + \cdots, \\ \dot{z} &= (\frac{F_k(X)''(\theta_*)}{2})z + \cdots, \end{cases} \tag{5}$$

Hence if $\left(\frac{F_k(X)''(\theta_*)}{2} - R_k(X)'(\theta_*)\right) F_k(X)''(\theta_*) > 0$, that is, (f) holds, the singular point is a hyperbolic saddle.

Using the blow-up *(b)*, we have, again after a time re-parameterization, that our system is transformed into

$$\begin{cases} \dot{x} &= Y_k^1(1,wx) + \dfrac{\sum_{i>k} Y_i^1(x,wx^2)}{x^k}, \\ \dot{w} &= \dfrac{1}{x^2}(Y_k^2(1,wx) - 2xwY_k^1(1,wx)) \\ & \qquad + \frac{\sum_{i>k}^{\infty} Y_i^2(x,wx^2) - 2xwY_i^1(x,wx^2)}{x^{k+2}}. \end{cases}$$

A study of the power series expansion of the expressions appearing in the above system, shows that the last system can be written as

$$\begin{cases} \dot{x} &= Y_k^1(1,wx) + \dfrac{\sum_{i>k}^{\infty} Y_i^1(x,wx^2)}{x^k}, \\ \dot{w} &= F_{k+2}(Y)(0) + (F_{k+1}(Y)' - R_{k+1}(Y))(0)w \\ & \qquad +(\frac{F_k(Y)''}{2} - R_k(Y)')(0)w^2 + x\,\Omega_1(x,w). \end{cases}$$

Expressing $\dot{w}$ in terms of X and using again (B), we have

$$\begin{cases} \dot{x} &= Y_k^1(1,wx) + \dfrac{\sum_{i>k}^{\infty} Y_i^1(x,wx^2)}{x^k} =: P(x,w), \\ \dot{w} &= F_{k+2}(X)(\theta_*) + (F_{k+1}(X)' - R_{k+1}(X))(\theta_*)w \\ & \quad +(\frac{F_k(X)''}{2} - R_k(X)')(\theta_*)w^2 + x\,\Omega_1(x,w) =: Q(x,w). \end{cases} \tag{6}$$

Note that $\{x=0\}$ is invariant under the flow of (6). If (h) and (i) hold then there appears a single degenerate singular point located at $(x,w) = (0,0)$. Condition (j) guarantees the right sense of the flow on $\{x=0\}$. To desingularize $(x,w)=(0,0)$, we perform the following additional blow–up's:

- *(c)* $(v,w) = (x/w, w)$. This blow–up can be obtained directly from (4), performing the blow up: $(v,w) = (x^3/y, y/x^2)$.
- *(d)* $(x,m) = (x, w/x)$, Observe that $(x,m) = (x, y/x^3)$.

Using the blow-up *(c)*, we have, that system (6) is transformed into

$$\begin{cases} \dot{v} &= R_{k+1}(X)(\theta_*)v + F_{k+3}(X)(\theta_*)v^2 \\ & \qquad + \left[2R_k(X)' - \frac{F_k(X)''}{2}\right](\theta_*)vw + \cdots, \\ \dot{w} &= \left[\frac{F_k(X)''}{2} - R_k(X)'\right](\theta_*)w^2 + \cdots. \end{cases} \tag{7}$$

Our purpose is to show that the hypothesis imply the existence of a hyperbolic saddle in the origin of system (7). Condition (k) ensures that the origin is not an elementary degenerate singular point (which would imply that there would be orbits tending or leaving from the origin of system (7)). Conditions (j), (l) and (m), assure that after a new re-parameterization, the origin of (7) is a hyperbolic saddle of system

$$\left\{\begin{array}{rcl} \dot{v} & = & \left[2R_k(X)' - \dfrac{F_k(X)''}{2}\right](\theta_*)v + \cdots =: V(v,w), \\ \dot{w} & = & \left[\dfrac{F_k(X)''}{2} - R_k(X)'\right](\theta_*)w + \cdots =: W(v,w). \end{array}\right. \tag{8}$$

By the blow-up *(d)*, and taking into account that conditions (k) and (l) hold, system (6) becomes

$$\left\{\begin{array}{rcl} \dot{x} & = & \dfrac{Y_k^1(1,x^2m)}{x} + \dfrac{\sum_{i>k} Y_i^1(x,mx^3)}{x^{k+1}} =: \widetilde{X}(x,m), \\ \dot{m} & = & F_{k+4}(X)(\theta_*) + \left[F_{k+2}(X)' - 2R_{k+2}(X)\right](\theta_*)m + \\ & & + \left[\dfrac{F_k(X)}{2} - 2R_k(X)'\right](\theta_*)m^2 + x\,\Omega_2(x,m) =: M(x,m). \end{array}\right. \tag{9}$$

$\{x = 0\}$ is invariant under the flow of (9). If Condition (n) and (o) holds then there are no critical points and the correct sense of the flow on $\{x=0\}$.

Therefore, we have proved that if any characteristic direction θ_* of the origin of X belongs to $\mathcal{A} \cup \mathcal{B}$, then after the above blow-up process the origin of system (4), associated to X, is desingularized obtaining a $\mathcal{H}$–monodromic polycycle. Hence if $X \in GS_k$, then $X \in M_k$. ∎

3 Stability: Proof of Theorem 2

In order to proof Theorem 2 we need to know the asymptotic structure of the transition semi–regular map near a characteristic direction of type $\mathcal{A}$ and $\mathcal{B}$. The first case is done in Lemma 6 of [11], but we restate it for the sake of completeness. See [15] for the definition of a semi–regular map

Lemma 3. *Let $\theta_* \in \mathcal{A} \cup \mathcal{B}$. For $\bar{\varepsilon} > 0$, small enough, set $\varepsilon = \tan(\bar{\varepsilon})$, and let $\mathcal{S}_+ = \{\tan(\theta_* - \bar{\varepsilon})x \le y \le \tan(\theta_* + \bar{\varepsilon})x\}$, and $\mathcal{S}_- = \{\tan(\theta_* + \pi + \bar{\varepsilon})x \le y \le \tan(\theta_* + \pi - \bar{\varepsilon})x\}$. Let $\Delta_{\theta_*}^{\bar{\varepsilon}}$ and $\Delta_{\theta_*+\pi}^{\bar{\varepsilon}}$ be the transition maps of the flow in $\mathcal{S}_+$ and $\mathcal{S}_-$. Then $\Delta_{\theta_*}^{\bar{\varepsilon}}$ and $\Delta_{\theta_*+\pi}^{\bar{\varepsilon}}$ are semi–regular maps, of the form: $\Delta_{\theta_*}^{\bar{\varepsilon}}(r) = D_{\theta_*}^{\bar{\varepsilon}} r + o(r)$, and $\Delta_{\theta_*+\pi}^{\bar{\varepsilon}}(r) = D_{\theta_*+\pi}^{\bar{\varepsilon}} r + o(r)$, where $r^2 = x^2 + y^2$. Moreover* *(i) If $\theta_* \in \mathcal{A}$, then*

$$D^{\bar{\varepsilon}}_{\theta_*+(\pi\pm\pi)/2} = \exp\left\{\mp\alpha\,\mathrm{P.V.}\right.$$
$$\left.\int_{-\infty}^{\infty}\left(\left(\frac{\partial}{\partial x}\frac{P(x,w)}{Q(x,w)}\right)\Big|_{\{x=0\}}\right)dw\right\}\cdot R^{\pm}_{\theta_*}(\bar{\varepsilon}),$$

where $\lim_{\bar{\varepsilon}\longrightarrow 0} R^{\pm}_{\theta_*}(\bar{\varepsilon}) = 1$, and with

$$\alpha = \frac{F_k(X)'' - 2R_k(X)'}{F_k(X)''}(\theta_*).$$

P and Q defined in (6).

(ii) If $\theta_* \in \mathcal{B}$, then

$$D^{\bar{\varepsilon}}_{\theta_*+(\pi\pm\pi)/2} = \exp\left\{\alpha_1\alpha_2\,\mathrm{P.V.}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\frac{\widetilde{X}(x,m)}{M(x,m)}\right)\Big|_{\{x=0\}}dm\pm\right.$$
$$\left.\pm\,\alpha_1\int_{-1/\varepsilon}^{-\varepsilon}+\int_{\varepsilon}^{1/\varepsilon}\left(\frac{\partial}{\partial v}\frac{V(v,w)}{W(v,w)}\right)\Big|_{\{v=0\}}dw\right\}\cdot R^{\pm}_{\theta_*}(\bar{\varepsilon}),$$

where $\lim_{\bar{\varepsilon}\longrightarrow 0} R^{\pm}_{\theta_*}(\bar{\varepsilon}) = 1$, and where

$$\alpha_1 = \frac{F_k(X)'' - 2R_k(X)'}{F_k(X)''}(\theta_*),$$
$$\alpha_2 = \frac{F_k(X)'' - 4R_k(X)'}{F_k(X)'' - 2R_k(X)'}(\theta_*),$$

$V, W, \widetilde{X}$ and M are defined in (8) and (9).

To prove Lemma 3 we need a reformulation of Lemma 8 of [10].

Lemma 4. *Consider system*

$$\begin{cases}\dot{x} = -x(a+f(x,y)) = P(x,y),\\ \dot{y} = y(b+g(x,y)) = Q(x,y),\end{cases}\tag{10}$$

where f and g begin at least with first order terms, and, a and b are positive numbers. Let $\sigma_{\varepsilon,\delta}(y)$ be the transition map of the flow of (10) in the first quadrant, from $\{x=\varepsilon\}$ to $\{y=\delta\}$, being ε and δ positive and small enough. Then $\sigma_{\varepsilon,\delta}(y) = A(\varepsilon,\delta)\,y^{a/b} + o(y^{a/b})$, where $A(\varepsilon,\delta) = R(\varepsilon)S(\delta)\varepsilon\delta^{-a/b}$, such that $\lim_{\varepsilon\to 0} R(\varepsilon) = 1$, and $\lim_{\delta\to 0} S(\delta) = 1$.

PROOF OF LEMMA 3. (i) Proved in [11].

(ii) Assume that $\theta_* \in \mathcal{B}$. Without loss of generality we assume $\theta_* = 0$. The blow-up process *(a)*, *(b)*, *(c)* and *(d)*, introduced in the proof of Theorem 1, desingularizes the direction $\{\theta = \theta_*\}$ into four hyperbolic saddles, p_1, p_2, $\bar{p}_2$ and $\bar{p}_1$ (see Figure 1), with hyperbolic ratios λ_1, λ_2, where $\lambda_1 = 1/\alpha_1$ and $\lambda_2 = 1/\alpha_2$. We will describe $\Delta_{\theta_*}^{\bar{\varepsilon}}$ the transition map of the flow between $\Sigma_1 = \{y = -\varepsilon x\}$ and $\Sigma_8 = \{y = \varepsilon x\}$. $\Delta_{\theta_*+\pi}^{\bar{\varepsilon}}$ can be described analogously. We take the following decomposition

$$\Delta_{\theta_*}^{\bar{\varepsilon}} = \bar{\sigma}_1 \circ \tau_3 \circ \bar{\sigma}_2 \circ \tau_2 \circ \sigma_2 \circ \tau_1 \circ \sigma_1 .$$

In the above expression σ_1 is the transition map of the flow in a neighborhood of p_1, σ_2 is the transition map in a neighborhood of p_2 and τ_1 is the transition map between these neighborhoods. τ_2 is the transition between the mentioned neighborhood of p_2 and a neighborhood of $\bar{p}_2$. $\bar{\sigma}_2$ denotes the transition in the neighborhood of $\bar{p}_2$, $\bar{\sigma}_1$ denotes the transition in the neighborhood of $\bar{p}_1$, and τ_3 is the transition map between these neighborhoods (see again Figure 1). Observe that σ_i and $\bar{\sigma}_i$ are semi–regular maps for $i = 1, 2$ while τ_i are regular maps for $i = 1, 2, 3$.

Let u_0, z_0, v_0 and w_0 be positive constants close enough to zero. To compute σ_1 we work in the coordinates of system (5). This map is the transition between $\Sigma_1 = \{[-u_0, 0) \times \{z = -\varepsilon\}\}$, and $\Sigma_2 = \{\{u = -\delta\} \times [-z_0, 0)\}$. τ_1 is the transition map from Σ_2 to $\Sigma_3 = \{[-v_0, 0) \times \{w = -\varepsilon\}\}$ To compute it, we work in the coordinates of system (8). σ_2, also computed in coordinates of system (8), is the transition from Σ_3 to $\Sigma_4 = \{\{v = -\delta\} \times [-w_0, 0)\}$. τ_2, the transition map from Σ_4 to $\Sigma_5 = \{\{v = \delta\} \times (0, w_0]\}\}$, will be computed in coordinates of system (9). $\bar{\sigma}_2$ is again computed in coordinates of system (8). Is the transition from Σ_5 to $\Sigma_6 = \{(0, v_0] \times \{w = \varepsilon\}\}$. τ_3, computed in coordinates of system (8), is the transition map from Σ_6 to $\Sigma_7 = \{(0, v_0] \times \{w = 1/\varepsilon\}\}$. Finally, in the coordinates of system (5), $\bar{\sigma}_1$ is the transition from Σ_7 to $\Sigma_8 = \{(0, u_0] \times \{z = \varepsilon\}\}$. See Figure 2.

We will use the following notation (The order of the leading term of each σ_i and $\bar{\sigma}_i$ for $i = 1, 2$ is well known, see [15]).

$$\begin{aligned}
&\sigma_1(x) = -a(\varepsilon)\, |x|^{\lambda_1} + o(|x|^{\lambda_1}), && \bar{\sigma}_1(x) = \bar{a}(\varepsilon)\, x^{1/\lambda_1} + o(x^{1/\lambda_1}), \\
&\sigma_2(x) = -b(\varepsilon, \delta)\, |x|^{\lambda_1} + o(|x|^{\lambda_1}), && \bar{\sigma}_2(x) = \bar{b}(\varepsilon, \delta)\, x^{1/\lambda_1} + o(x^{1/\lambda_1}), \\
&\tau_1(x) = t_1(\varepsilon)\, x + o(x), && \tau_2(x) = t_2(\delta)\, x + o(x), \\
&\tau_3(x) = t_3(\varepsilon)\, x + o(x),
\end{aligned}$$

where $a(\varepsilon), \bar{a}(\varepsilon), b(\varepsilon, \delta), \bar{b}(\varepsilon, \delta), t_1(\varepsilon), t_2(\delta)$ and $t_3(\varepsilon)$ are positive coefficients, computed below. In the next computations we use the following notation:

$(a,b)^{(n)}$, where $n\in\{1,2,3,4,5,6\}$, means a point expressed in the original coordinates (x,y) of system (4), if $n=1$; in coordinates $(x,z)=(x,y/x)$ if $n=2$; In coordinates (u,z) of system (5) if $n=3$; Expressed in coordinates (x,w) of system (6) if $n=4$; In coordinates (v,w) of system (8) if $n=5$; And expressed in coordinates (x,m) of system (9) if $n=6$.

Let $u<0$, small enough:

$$
\begin{aligned}
\Delta^{\bar\varepsilon}_{\theta_*}\left((u,-\varepsilon)^{(3)}\right) &= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\circ\sigma_2\circ\tau_1\circ\sigma_1\left((u,-\varepsilon)^{(3)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\circ\sigma_2\circ\tau_1\left((-\varepsilon,-a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}))^{(3)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\circ\sigma_2\circ\tau_1\Big((\varepsilon\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}), \\
&\qquad -a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}))^{(2)}\Big) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\circ\sigma_2\circ\tau_1\left((\varepsilon\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}),-1/\varepsilon)^{(4)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\circ\sigma_2\left((\varepsilon\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}),-\varepsilon)^{(4)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\circ\sigma_2\left((-t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}),-\varepsilon)^{(5)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\left((-\delta,-b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_2\lambda_1}))^{(5)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\Big((\delta\,b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}), \\
&\qquad -b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}))^{(4)}\Big) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\circ\tau_2\left((\delta\,b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}),-1/\delta)^{(6)}\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\left((t_2\,\delta\,b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}),1/\delta)^{(6)}\right) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\Big((t_2\,\delta\,b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}), \\
&\qquad t_2\,b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}))^{(4)}\Big) = \\
&= \bar\sigma_1\circ\tau_3\circ\bar\sigma_2\left((\delta,t_2\,b\,t_1^{\lambda_2}\,a^{\lambda_2}\,|u|^{\lambda_1\lambda_2}+o(|u|^{\lambda_1\lambda_2}))^{(5)}\right) = \\
&= \bar\sigma_1\circ\tau_3\left((\bar b\,t_2^{1/\lambda_2}\,b^{1/\lambda_2}\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}),\varepsilon)^{(5)}\right) = \\
&= \bar\sigma_1\circ\tau_3\left((\varepsilon\,\bar b\,t_2^{1/\lambda_2}\,b^{1/\lambda_2}\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}),\varepsilon)^{(4)}\right) = \\
&= \bar\sigma_1\left((t_3\,\varepsilon\,\bar b\,t_2^{1/\lambda_2}\,b^{1/\lambda_2}\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}),1/\varepsilon)^{(4)}\right) = \\
&= \bar\sigma_1\Big((t_3\,\varepsilon\,\bar b\,t_2^{1/\lambda_2}\,b^{1/\lambda_2}\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}), \\
&\qquad t_3\,\bar b\,t_2^{1/\lambda_2}\,b^{1/\lambda_2}\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}))^{(2)}\Big) = \\
&= \bar\sigma_1\left((\varepsilon,t_3\,\bar b\,t_2^{1/\lambda_2}\,b^{1/\lambda_2}\,t_1\,a\,|u|^{\lambda_1}+o(|u|^{\lambda_1}))^{(3)}\right) = \\
&= \left((\bar a\,t_3^{1/\lambda_1}\,\bar b^{1/\lambda_1}\,t_2^{1/(\lambda_1\lambda_2)}\,b^{1/(\lambda_1\lambda_2)}\,t_1^{1/\lambda_1}\,a^{1/\lambda_1}\,|u|+o(|u|),\varepsilon)^{(3)}\right).
\end{aligned}
$$

Using that $(u,\alpha)^{(3)} = (\alpha u,\alpha)^{(2)} = \left(\alpha u,\alpha^2 u\right)^{(1)}$, we obtain that for $(x,y)^{(1)} = (\varepsilon u,\varepsilon^2 u)^{(1)}$.

$$\Delta_{\theta_*}^{\bar{\varepsilon}}\left((\varepsilon u,\varepsilon^2 u)^{(1)}\right) = (\varepsilon\,\bar{a}\,t_3^{1/\lambda_1}\,\bar{b}^{1/\lambda_1}\,t_2^{1/(\lambda_1\lambda_2)}\,b^{1/(\lambda_1\lambda_2)}\,t_1^{1/\lambda_1}\,a^{1/\lambda_1}\,|u| + o(|u|),$$
$$\varepsilon^2\,\bar{a}\,t_3^{1/\lambda_1}\,\bar{b}^{1/\lambda_1}\,t_2^{1/(\lambda_1\lambda_2)}\,b^{1/(\lambda_1\lambda_2)}\,t_1^{1/\lambda_1}\,a^{1/\lambda_1}\,|u| + o(|u|))^{(1)}.$$

Using polar coordinates, if $r^2 = x^2+y^2$, we have $\Delta_{\theta_*}^{\bar{\varepsilon}}(r) = D_{\theta_*}^{\bar{\varepsilon}}\, r + o(r)$, where

$$D_{\theta_*}^{\bar{\varepsilon}} = a^{1/\lambda_1}\,\bar{a}\,b^{1/\lambda_1\lambda_2}\,\bar{b}^{1/\lambda_1}\,t_1^{1/\lambda_1}\,t_2^{1/\lambda_1\lambda_2}\,t_3^{1/\lambda_1}. \tag{11}$$

Applying Lemma 4, we obtain that

$$a(\varepsilon) = \frac{\varepsilon}{\varepsilon^{\lambda_1}} R_1(\varepsilon), \qquad \bar{a}(\varepsilon) = \frac{\varepsilon}{\varepsilon^{1/\lambda_1}}\bar{R}_1(\varepsilon),$$
$$b(\varepsilon,\delta) = \frac{\varepsilon}{\delta^{\lambda_2}} R_2(\varepsilon) S_2(\delta), \qquad \bar{b}(\varepsilon,\delta) = \frac{\delta}{\varepsilon^{1/\lambda_2}}\bar{R}_2(\varepsilon)\bar{S}_2(\delta),$$

where $\lim\limits_{\varepsilon\longrightarrow 0} R_i(\varepsilon) = \lim\limits_{\varepsilon\longrightarrow 0}\bar{R}_i(\varepsilon) = 1$, for $i = 1,2$, and $\lim\limits_{\delta\longrightarrow 0} S_2(\delta) = \lim\limits_{\delta\longrightarrow 0}\bar{S}_2(\delta) = 1$.

Therefore, from equation (11), we have

$$D_{\theta_*}^{\bar{\varepsilon}} = a(\varepsilon)^{1/\lambda_1}\,\bar{a}(\varepsilon)\,b(\varepsilon,\delta)^{1/\lambda_1\lambda_2}\,\bar{b}(\varepsilon,\delta)^{1/\lambda_1}\,t_1(\varepsilon)^{1/\lambda_1}\,t_2(\delta)^{1/\lambda_1\lambda_2}\,t_3(\varepsilon)^{1/\lambda_1} =$$
$$= t_1(\varepsilon)^{1/\lambda_1}\,t_2(\delta)^{1/\lambda_1\lambda_2}\,t_3(\varepsilon)^{1/\lambda_1}\,R(\varepsilon)\,S(\delta). \tag{12}$$

Now, by integration of the first order variational equations of system (8), associated to $\{v=0\}$ (see [20], pp. 252–254), from Σ_2 to Σ_3, and from Σ_6 to Σ_7, we obtain:

$$t_1(\varepsilon) = \int_{-1/\varepsilon}^{-\varepsilon}\left(\frac{\partial}{\partial v}\frac{V(v,w)}{W(v,w)}\right)\bigg|_{\{v=0\}} dw\,, \quad \text{and}$$

$$t_3(\varepsilon) = \int_{\varepsilon}^{1/\varepsilon}\left(\frac{\partial}{\partial v}\frac{V(v,w)}{W(v,w)}\right)\bigg|_{\{v=0\}} dw\,.$$

Integrating the first order variational equations of system (9), associated to $\{x=0\}$ from Σ_4 to Σ_5, we get

$$t_2(\delta) = \int_{-1/\delta}^{1/\delta}\left(\frac{\partial}{\partial x}\frac{\widetilde{X}(x,m)}{M(x,m)}\right)\bigg|_{\{x=0\}} dm.$$

Observe that $D^{\bar{\varepsilon}}_{\theta_*}$ does not depend on δ, hence

$$D^{\bar{\varepsilon}}_{\theta_*} = \lim_{\delta\to 0} D^{\bar{\varepsilon}}_{\theta_*} = \exp\left\{ \frac{1}{\lambda_1\lambda_2}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\frac{\widetilde{X}(x,m)}{M(x,m)}\right)\Bigg|_{\{x=0\}} dm + \right.$$

$$\left. + \frac{1}{\lambda_1}\int_{-1/\varepsilon}^{-\varepsilon} + \int_{\varepsilon}^{1/\varepsilon}\left(\frac{\partial}{\partial v}\frac{V(v,w)}{W(v,w)}\right)\Bigg|_{\{v=0\}} dw \right\} \cdot R^{+}_{\theta_*}(\bar{\varepsilon}).$$

An easy computation shows that

$$\mathrm{P.V.}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\frac{\widetilde{X}(x,m)}{M(x,m)}\right)\Bigg|_{\{x=0\}} dm \;=\; I(\theta_*)$$

■

Remark 5. *In fact, in the proof of the above result, it is proved that $I(\theta_*)$ is well defined, that is, statement (ii) of Theorem 2.*

PROOF OF THEOREM 2. Consider $X \in GS_k$, and $\{\theta_1, \cdots, \theta_n\}$ the set of characteristic directions of the origin of X. Without loss of generality, we assume that $\{\theta = 0\}$ is not a characteristic direction. Take $\varepsilon > 0$ small enough such that $\{(\theta_j - \varepsilon, \theta_j + \varepsilon)\}_{j\in\{1,\cdots,n\}}$ is a collection of disjoint intervals. Set $S_\varepsilon = \cup_{j=1}^{n}(\theta_j - \varepsilon, \theta_j + \varepsilon)$ and $I_\varepsilon = ([0, 2\pi] \setminus S_\varepsilon)$. Taking polar coordinates (r, θ) given by the change $r^2 = x^2 + y^2$, $\theta = \arctan(y/x)$; and re–scaling the time by $ds/dt = r^{m-1}$, we obtain the new system

$$\left\{\begin{array}{lcl} \dot{r} &=& \Re(r,\theta) = R_k(X)(\theta)r + o(r^2), \\ \dot{\theta} &=& \Theta(r,\theta) = F_k(X)(\theta) + o(r). \end{array}\right. \tag{13}$$

Now, integrating the first order variational equations of the system (13) associated to the orbit $\{r = 0\}$, we have that the transition map T^{ε}_j from $\{\theta = \theta_j + \varepsilon\}$, to $\{\theta = \theta_{j+1} - \varepsilon\}$ (which is regular) is given by

$$T^{\varepsilon}_j(r_0) \;=\; \exp\left\{\int_{\theta_j+\varepsilon}^{\theta_{j+1}-\varepsilon}\frac{R_k(X)(\theta)}{F_k(X)(\theta)}d\theta\right\} r_0 \;+\; o(r_0).$$

Let also T^{ε}_0 be the regular transition map from $\{\theta = 0\}$ to $\{\theta = \theta_1 - \varepsilon\}$, and T^{ε}_n be the regular transition from $\{\theta = \theta_n + \varepsilon\}$ to $\{\theta = 2\pi\}$. Hence, $\Pi = T^{\varepsilon}_n \circ \Delta^{\varepsilon}_n \circ T^{\varepsilon}_{n-1} \cdots T^{\varepsilon}_2 \circ \Delta^{\varepsilon}_2 \circ T^{\varepsilon}_1 \circ \Delta^{\varepsilon}_1 \circ T^{\varepsilon}_0$, is a composition of regular and semi–regular maps with non-vanishing linear leading terms. Therefore, can write $\Pi(x) = V_1 x + o(x)$, where V_1 is the product of the principal terms of the maps T_j^ε and δ_j^ε, for $j = 1, \cdots, n$. Therefore for all $\varepsilon > 0$ small enough:

$$V_1 = \left(\prod_{j=1}^{n} D_j^\varepsilon \right) \cdot \exp\left\{ \int_{I_\varepsilon} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta \right\}. \tag{14}$$

If $\theta_* \in \mathcal{A}$, then for every supplementary characteristic direction the Principal Values appearing in the expressions of $D_{\theta_*}^\varepsilon$ (given in Lemma 3), are the same but with opposite sign, and then they cancel in the expression of V_1. That is

$$D_{\theta_*}^\varepsilon \cdot D_{\theta_*+\pi}^\varepsilon = R_{\theta_*}^+(\varepsilon) \cdot R_{\theta_*}^-(\varepsilon) = R(\varepsilon),$$

where $\lim_{\varepsilon \to 0} R(\varepsilon) = 1$.

On the contrary, if $\theta_* \in \mathcal{B}$, then the contribution of every supplementary characteristic direction is

$$D_{\theta_*}^\varepsilon \cdot D_{\theta_*+\pi}^\varepsilon = \exp\{2\alpha\, I(\theta_*)\} \cdot R(\varepsilon),$$

where $\lim_{\varepsilon \to 0} R(\varepsilon) = 1$, and $\alpha = \dfrac{F_k(X)'' - 4R_k(X)'}{F_k(X)''}(\theta_*)$.

Therefore

$$\lim_{\varepsilon \to 0} \prod_{j=1}^{n} D_j^\varepsilon = \begin{cases} 1 & \text{if } \mathcal{B} = \emptyset, \\ \exp\left\{ 2 \sum_{\theta_j \in \mathcal{B}} \alpha_j\, I(\theta_j) \right\} & \text{otherwise.} \end{cases}$$

Observe that V_1 does not depend on ε. Therefore, taking $\varepsilon \to 0$ in equation (14), we have that the G.P.V. exists and

$$V_1 = \exp\left\{ \text{G.P.V.} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta + 2 \sum_{\theta_j \in \mathcal{B}} \alpha_j\, I(\theta_j) \right\}.$$

∎

4 Examples

Directly from Theorems 1 and 2, we have the next result, that will be useful in the following.

Corollary 6. *Consider $X \in \Phi_k$, with associated system of differential equations*

$$\begin{cases} \dot{x} = & X^1(x,y) = \sum_{i+j\geq k} p_{ij}x^i y^j, \\ \dot{y} = & X^2(x,y) = \sum_{i+j\geq k} q_{ij}x^i y^j, . \end{cases}$$

Then $X \in GS_k$, with $\mathcal{A} = \emptyset$ and $\mathcal{B} = \{\theta = 0\}$, if and only if the following properties hold: (i) $q_{k,0} = 0$, (ii) $q_{k-1,1} - p_{k,0} = 0$, (iii) $q_{k-2,2} - p_{k-1,1} > 0$, *(That is $F_k(X)(0) = F_k(X)'(0) = 0$ and $F_k(X)''(0) > 0$),* (iv) $F_k(X)(\theta) > 0$ *for all* $\theta \neq 0$, *and*

(a)	$p_{k,0} = 0$,	(b)	$q_{k+1,0} = 0$,
(f)	$(q_{k-2,2} - 2p_{k-1,1})(q_{k-2,2} - p_{k-1,1}) > 0$,	(h)	$q_{k+2,0} = 0$,
(i)	$q_{k,1} - 2p_{k+1,0} = 0$,	(j)	$q_{k-2,2} - 2p_{k-1,1} > 0$,
(k)	$p_{k+1,0} = 0$,	(l)	$q_{k+3,0} = 0$,
(m)	$q_{k-2,2} - 3p_{k-1,1} > 0$,	(n)	$q_{k+4,0} = 0$,
(o)	$(q_{k+1,1} - 3p_{k+2,0})^2$ $-4q_{k+4,0}(q_{k-2,2} - 3p_{k-1,1}) < 0$.		

Moreover, in this case

$$V_1 = \exp\left\{ \text{G.P.V.} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta + 2\,\frac{q_{k-2,2} - 3p_{k-1,1}}{q_{k-2,2} - p_{k-1,1}} \times \right. \tag{15}$$

$$\left. \times \text{P.V.} \int_{-\infty}^{+\infty} \frac{p_{k-1,1}\, m + p_{k+2,0}}{(q_{k-2,2} - 3p_{k-1,1})\, m^2 + (q_{k+1,1} - 3p_{k+2,0})\, m + q_{k+4,0}}\, dm \right\}.$$

Remark 7. *(i) Condition* (iv) *of the above result can be tested using Corollary 9 of [11].*

(ii) G.P.V. $\int_0^{2\pi} R_k(X)(\theta)/F_k(X)(\theta) d\theta$ *can be computed using a nonsingular integral, given in Lemma 10 of [11].*

(iii) A computation shows that, assuming conditions (m) *and* (o)*:*

$$\text{P.V.} \int_{-\infty}^{+\infty} \frac{p_{k-1,1}\, m + p_{k+2,0}}{(q_{k-2,2} - 3p_{k-1,1})\, m^2 + (q_{k+1,1} - 3p_{k+2,0})\, m + q_{k+4,0}}\, dm =$$

$$= \pi\, \frac{2p_{k+2,0}(q_{k-2,2} - 3p_{k-1,1}) - p_{k-1,1}(q_{k+1,1} - 3p_{k+2,0})}{(q_{k-2,2} - 3p_{k-1,1})\sqrt{4q_{k+4,0}(q_{k-2,2} - 3p_{k-1,1}) - (q_{k+1,1} - 3p_{k+2,0})^2}}. \tag{16}$$

Example A. Consider

$$\begin{cases} \dot{x} = & \sum_{i+j\geq 3} p_{ij}x^iy^j, \\ \dot{y} = & \sum_{i+j\geq 3} q_{ij}x^iy^j, \end{cases} \tag{17}$$

and assume that
(I) $p_{2,1} = 0$,
(i) $q_{3,0} = 0$, (ii) $q_{2,1} = p_{3,0}$, (iii) $q_{1,2} - p_{2,1} > 0$, (iv) $(q_{0,3} - p_{1,2})^2 + 4p_{0,3}(q_{1,2} - p_{2,1}) < 0$, and

(a)	$p_{3,0} = 0$	(b)	$q_{4,0} = 0$,
(f)	$(q_{1,2} - 2p_{2,1})(q_{1,2} - p_{2,1}) > 0$,	(h)	$q_{5,0} = 0$,
(i)	$q_{3,1} - 2p_{4,0} = 0$,	(j)	$q_{1,2} - 2p_{2,1} > 0$,
(k)	$p_{4,0} = 0$,	(l)	$q_{6,0} = 0$,
(m)	$q_{1,2} - 3p_{2,1} > 0$,	(n)	$q_{7,0} > 0$,
(o)	$(q_{4,1} - 3p_{5,0})^2 - 4q_{7,0}(q_{1,2} - 3p_{2,1}) < 0$.		

Then: (a) Using Corollary 6, we obtain immediately that $X \in GS_3$, with $\mathcal{A} = \emptyset$ and $\mathcal{B} = \{\theta = 0\}$. (b) Using the formulas that appear in [14], pp. 150–151, we have

$$L_1(p_{1,2}, p_{0,3}, q_{1,2}, q_{0,3}) := \text{G.P.V.} \int_0^{2\pi} \frac{R_3(X)(\theta)}{F_3(X)(\theta)} d\theta =$$

$$\int_0^{2\pi} \frac{p_{1,2}\cos^2(\theta) + (p_{0,3} + q_{1,2})\sin(\theta)\cos(\theta) + q_{0,3}\sin^2(\theta)}{q_{1,2}\cos^2(\theta) + (q_{0,3} - p_{1,2})\sin(\theta)\cos(\theta) - p_{0,3}\sin^2(\theta)} d\theta =$$

$$= 2\frac{\pi\,(p_{1,2} + q_{0,3})}{\sqrt{-4q_{1,2}p_{0,3} - (q_{0,3} - p_{1,2})^2}}, \tag{18}$$

if $(q_{0,3} - p_{1,2})^2 + (p_{0,3} + q_{1,2})^2 \neq 0$, or

$$L_1(p_{1,2}, p_{0,3}, q_{1,2}, q_{0,3}) := \pi\,\frac{p_{1,2} + q_{0,3}}{q_{1,2}}, \tag{19}$$

if $(q_{0,3} - p_{1,2})^2 + (p_{0,3} + q_{1,2})^2 = 0$.

On the other hand, from equation (16), we have

$$L_2(q_{1,2}, p_{5,0}, q_{4,1}, q_{7,0}) := 2\,\text{P.V.} \int_{-\infty}^{+\infty} \frac{p_{5,0}}{q_{1,2}\,m^2 + (q_{4,1} - 3p_{5,0})\,m \;+\; q_{7,0}}\, dm$$

$$= \frac{4\pi\,p_{5,0}}{\sqrt{4q_{7,0}q_{1,2} - (q_{4,1} - 3p_{5,0})^2}}. \tag{20}$$

Therefore the first generalized Lyapunov constant of system (17) is given by:

$$V_1 = \exp\{L_1(p_{1,2}, p_{0,3}, q_{1,2}, q_{0,3}) + L_2(q_{1,2}, p_{5,0}, q_{4,1}, q_{7,0})\}. \qquad (21)$$

It is clear that, with an appropriate election of the coefficients, it is possible to obtain $L_1 \cdot L_2 < 0$, and $|L_2| > |L_1|$. In this situation the contribution to the stability given by the first order variational equations associated to the edges of the first polar blow-up polycycle, is compensated by the contribution given by the first variational equations associated to the regular sectors appearing in the desingularization of the characteristic directions. This is more clear in Example B.

Example B. Consider system (17), with $p_{3,0} = p_{2,1} = 0$, $p_{1,2} = 1$, $p_{0,3} = -1$, $p_{4,0} = 0$, $q_{3,0} = q_{2,1} = 0$, $q_{1,2} = 4$, $q_{0,3} = 1$, $q_{4,0} = q_{3,1} = 0$, $q_{5,0} = 0$, $q_{4,1} = -1$ $q_{6,0} = 0$, and $q_{7,0} = 2$. Assume that $32-(1+3p_{5,0})^2 > 0$ (i.e $-(4\sqrt{2}+1)/3 < p_{5,0} < (4\sqrt{2}-1)/3$). Then:

(a) Again from Corollary 6, we have that $X \in GS_3$, with $\mathcal{A} = \emptyset$, and $\mathcal{B} = \{\theta = 0\}$.

(b) From formulas (19) and (20), we have

$$V_1 = \exp\{L_1(1,-1,4,1) + L_2(4, p_{5,0}, -1, 2) \\ = \exp\{\pi + \frac{4\pi\, p_{5,0}}{\sqrt{32-(1+3p_{5,0})^2}}\}.$$

An easy computation shows that the origin is a stable (respectively unstable) focus if $-(4\sqrt{2}+1)/3 < p_{5,0} < -31/25$ (respectively $-31/25 < p_{5,0} < (4\sqrt{2}-1)/3$).

5 Bifurcation of limit cycles

We denote by X_Λ a family of vector fields in GS_k, where Λ is a vector of parameters. From Theorem 2, we have that the origin's first generalized Lyapunov constant of X_Λ can be expressed as $V_1(\Lambda) = \exp\{L_1(\Lambda)+L_2(\Lambda)\}$, where

$$L_1(\Lambda) = \text{G.P.V.} \int_0^{2\pi} \frac{R_k(X)(\theta)}{F_k(X)(\theta)} d\theta, \quad \text{and}$$

$$L_2(\Lambda) = 2 \sum_{\theta_j \in \mathcal{B}} \frac{F_k(X)'' - 4R_k(X)'}{F_k(X)''}(\theta_j)\, I(\theta_j).$$

Lemma 8. *Consider $X_{\Lambda_0} \in GS_k$. If $V_1(\Lambda_0) = 1$ and $\vec{0}$, the origin of X_{Λ_0}, is asymptotically stable (respectively unstable), then at least one stable (respectively unstable) limit cycle can bifurcate from $\vec{0}$, if Λ is a small enough perturbation of Λ_0, such that $V_1(\Lambda) > 1$ (respectively $V_1(\Lambda) < 1$).*

Proof. Suppose that $V_1(\Lambda_0) = 1$ and $\vec{0}$ is asymptotically stable (the proof in the other case is analogous). Using the results in [21], there exists a generalized Lyapunov function V of $\vec{0}$ (see [21] for a definition). A consequence of Sard's Theorem is that for almost every $\varepsilon > 0$ small enough, the curves $\{V = \varepsilon\}$ are compact differential submanifolds of $\mathbb{R}^2$, and then they are closed curves. The vector field keeps being transversal to these curves under small perturbations. By this way we have generated a positive invariant region in $\mathbb{R}^2$ surrounding $\vec{0}$. If the perturbation is such that $V_1(\Lambda_0) > 1$ then $\vec{0}$ changes its stability and turns to be repellor. Since a positive invariant region persists if the perturbation is small enough, then there exists at least one stable limit cycle in this invariant region. ∎

Observe that $L_1(\Lambda)$ only depends on the coefficients coming from $X_k(\Lambda)$, while $L_2(\Lambda)$ depends on the coefficients coming from $X_k(\Lambda)$, $X_{k+2}(\Lambda)$, $X_{k+4}(\Lambda)$. The next result provides ways to construct a bifurcation of limit cycle from the origin of a system X_{Λ_0}. One of these ways (suggested in the next result) is obtained by fixing the coefficients coming from $X_k(\Lambda)$, and perturbing the coefficients coming from $X_{k+2}(\Lambda)$. Hence, directly from Lemma 8, we have:

Corollary 9. *Consider $X_{\Lambda_0} \in GS_k$. If $L_1(\Lambda_0) + L_2(\Lambda_0) = 0$, and $\vec{0}$, the origin of X_{Λ_0}, is asymptotically stable (respectively unstable), then at least one stable (respectively unstable) limit cycle can bifurcate from $\vec{0}$ if Λ is a small enough perturbation of Λ_0, such that $L_2(\Lambda) > -L_1(\Lambda)$ (respectively $L_2(\Lambda) < -L_1(\Lambda)$).*

Next example is a partial consequence of Lemma 8:

Proposition 10. (Example C) *Consider system*

$$\begin{aligned} \dot{x} &= x^2y - y^3 + px^5 - x^{11}, \\ \dot{y} &= 4xy^2 + 9x^7. \end{aligned} \tag{22}$$

The following statements hold:

(i) $X \in GS_3$, *with* $\mathcal{A} = \emptyset$ *and* $\mathcal{B} = \{\theta = 0\}$.

(ii) The origin is a stable (respectively unstable) focus if $-2 < p \leq 0$ (respectively $0 < p < 2$).

(iii) A stable limit cycle bifurcates from the origin of system (22) if $p > 0$, and $|p|$ is small enough.

Proof. (i) is a direct consequence of Corollary 6.
(ii) Formulas (15) and (16), give

$$V_1 = \exp\left\{\frac{10\pi}{9}\frac{p}{\sqrt{4-p^2}}\right\}.$$

Therefore a direct computation shows that if $-2 < p < 0$ the origin is a stable focus (respectively unstable if $0 < p < 2$). We claim that if $p = 0$, the origin of system (22) is asymptotically stable. Indeed, consider $Z(x,y) = (-x^2y - y^3, 4xy^2 + 9x^5)$. It is easy to see that the origin is a reversible centre for the vector field Z. We set $X(x,y) = (x^2y - y^3 + px^5 - x^{11}, 4xy^2 + 9x^7)$. It is easy to check that if $p_{5,0} \leq 0$ $Z \cdot X = 9px^{12} - 4x^{12}y^2 - 9x^{18} < 0$. Hence, we can use the integral curves of the centre associated to Z as level curves of a Lyapunov function for the origin of system (22).

(iii) Is a direct consequence of statement (ii) and Corollary 9. ∎

6 Figures

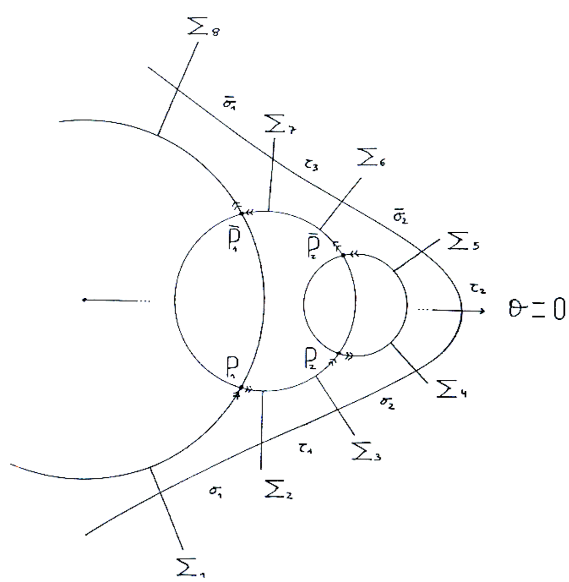

Figure 1: blow-up geometry of a characteristic direction of type $\mathcal{B}$.

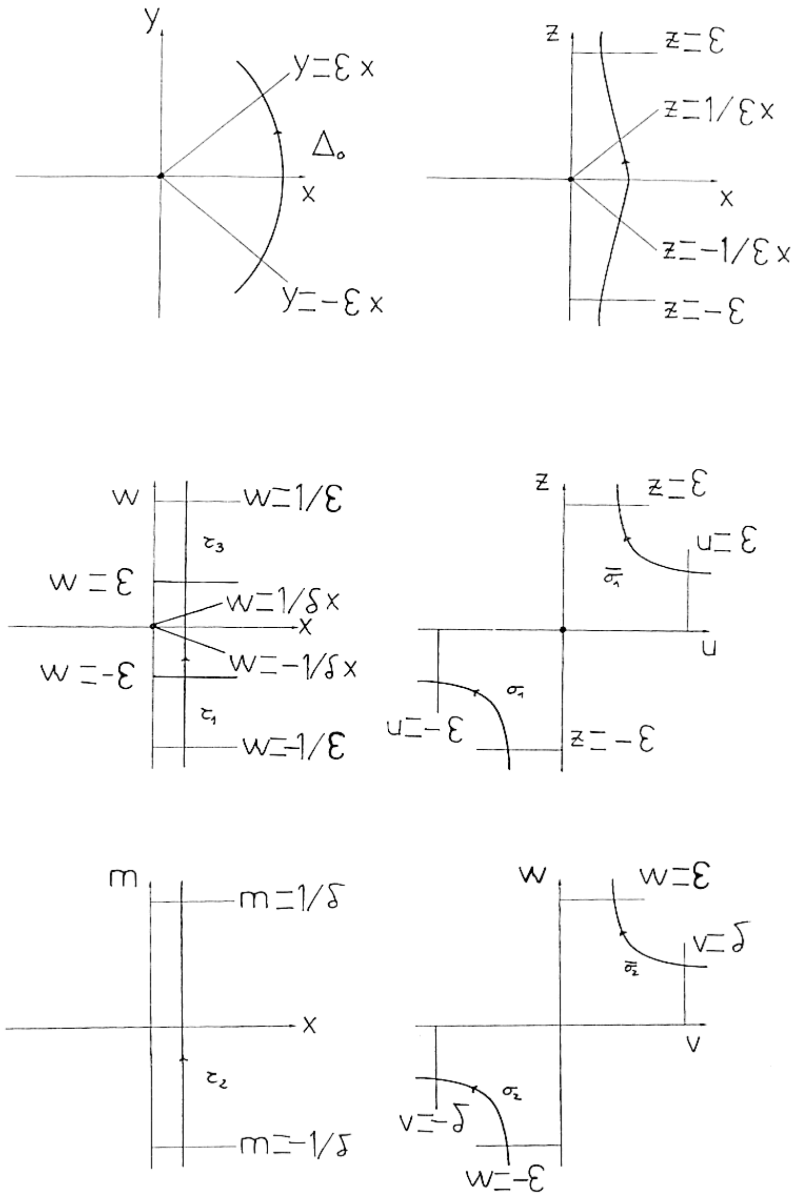

Figure 2: blow-up of a characteristic direction of type $\mathcal{B}$, located at $\{\theta = 0\}$ in local coordinates.

References

[1] M.A.M. Alwash, N.G. Lloyd. *Non-autonomous equation related to polynomial two dimensional systems*, Proc. Roy. Soc. Edinburgh 105A (1987), 129–152.

[2] A. Andreev. *Investigation of the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Translation Amer. Math. Soc. 8 (1958), 187–207.

[3] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maier. 'Theory of bifurcation of dynamic systems on a plane', John Wiley & Sons, New York, 1973.

[4] J.I. Aranda. Ph.D. Thesis. Universidad Complutense de Madrid. Madrid 1998.

[5] F.S. Berezovskaya, N.B. Medvedeva. *A complicated singular point of "centre–focus" type and the Newton diagram*, Sel. Math. *Form.* Soviet. 13 , 1994, 1–15, English translation of Math. Modelirovanie 45 (1990), 45–57 (In Russian).

[6] A. Cima, A. Gasull, F. Mañosas. *Cyclicity of a family of vector fields*, J. Math. Anal. Appl. 196 (1995), 921–937.

[7] L.A. Cherkas, *Conditions for a Liénard equation to have a centre*, Differential Equations 12 (1976), 201–206.

[8] C.J. Christopher, N.G. Lloyd, J.M. Pearson, *On a Cherka's method for centre conditions*, Nonlinear World 2 (1995), 459–469.

[9] F. Dumortier. *Singularities of vector fields on the plane*, J. Differential Equations 23 (1977) 53–106.

[10] A. Gasull, J. Llibre, V. Mañosa, F. Mañosas. *The focus-centre problem for a type of degenerate systems*, Nonlinearity 13 (2000), 699–730.

[11] A. Gasull, V. Mañosa, F. Mañosas. *Monodromy and stability of a generic class of degenerate planar critical points*, Prepublicacions Departament de Matemàtiques UAB (2000).

[12] A. Gasull, J. Torregrosa. *Center Problem for several differential Equations via Cherka's Method*, J. Math. Anal. Appl. 228 (1998), 322–343.

[13] J. Giné. Private communication (2000).

[14] I.S. Grandshteyn, I.M. Ryzhik. 'Table of integrals series and products', Academic Press, New York, 1980.

[15] Y.S. Il'Yashenko. 'Finiteness theorems for limit cycles', Translations of Math. Monographs 94, Amer. Math. Soc., Providence, 1991.

[16] N.B. Medvedeva. *Principal term of the monodromy transformation of a monodromic singular point is linear*, Siberian Math. J. 33 (1992), 280–288.

[17] N.B. Medvedeva. *The principal term of an asymptotics of monodromy transformation: calculation from the Newton diagram*, Proc. Steklov Inst. Math. 213 (1996), 212–223.

[18] N.B. Medvedeva. *The principal term of the asymptotic expansion of the monodromy transformation: calculation in blowing–up geometry*, Siberian Math. J. 38 (1997), 114–126.

[19] R. Moussu. *Symétrie et forme normale des centres et foyers dégénérés*, Ergodic Theory. Dynam. Systems 2 (1982), 241–251.

[20] V.V. Nemytskii, V.V. Stepanov, 'Qualitative theory of differential equations', Dover, New York, 1989.

[21] F.W. Wilson, J.A. Yorke. *Lyapunov Function and Isolating Blocks*, J. Differential Equations 13 (1973), 106–123.