C^1- approximation and extension of subharmonic functions

M.S. Melnikov,* P.V. Paramonov† and J. Verdera‡

1 Introduction

Assume that a function f is analytic on the interior of some compact X in the plane and has some regularity on X. For instance, f could be continuous on X or, more generally of class $C^m(X)$, $m = 0, 1, 2, \ldots$ In recent decades the problem of deciding whether f can be approximated in the norm of $C^m(X)$ by functions analytic on neighbourhoods of X has been widely studied. The same can be said of the case in which X is assumed to be a compact subset of $\mathbb{R}^N$ and f to be harmonic on its interior. Of course, in this setting one requires the approximating functions to be *harmonic* on neighbourhoods of X. The interested reader is referred to the papers [P1], [P3], [Ve1], [Ve2].

Particularly difficult problems arise in the analytic uniform approximation problem in the plane and in the harmonic C^1 approximation problem in $\mathbb{R}^N$, $N \geq 2$. Not only are the main known results hard to prove but it turns out that they do not provide *convenient* criteria for the approximation. The reason is that the possibility of approximation is controlled by certain capacities which unfortunately are extremely difficult to handle. In particular one does not know if they are semiadditive, that is, if an inequality of the type

$$\text{Capacity }(A \cup B) \leq \text{Const}(\text{Capacity }(A) + \text{Capacity }(B))$$

holds for all pairs of compact sets A and B. The absence of such an inequality is a serious complication.

The capacity of a set is defined in terms of potentials of distributions compactly supported on the set. Recent work of Tolsa [T1] shows that if

*Partially supported by the grant DGES PB96-1183

†Partially supported by the grants: SAB 1998-0049 of Ministerio de Educación y Cultura (Spain), RFFR 00-01-00618 and RFFR 00-15-96008 (Russia)

‡Partially supported by the grant DGES PB96-1183

in the analytic case one replaces distributions by positive measures then the corresponding capacity is semiadditive. Using the $T(1)$-Theorem for Calderón-Zygmund operators on non-homogeneous spaces, we show in this paper that the same holds in the context of Lipschitz harmonic functions in all dimensions. It is then natural to ask whether the semiadditivity of the Lipschitz harmonic capacity defined in terms of positive measures yields good criteria for approximation. The final answer is yes, *provided we restrict our attention to subharmonic functions.*

We proceed now to state our main results precisely. The main capacity we will be dealing with is defined as follows (see [MP]). Denote by Φ the standard fundamental solution of the Laplacean in $\mathbb{R}^N$. Given a set $E \subset \mathbb{R}^N$ set

$$\gamma_+(E) = \sup \mu(E)\,, \tag{1.1}$$

where the supremum is taken over all positive Radon measures μ, with compact support contained in E, such that the function $f = \Phi * \mu$ satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq C|x-y|\,, \quad x, y \in \mathbb{R}^N\,, \tag{1.2}$$

with constant $C = 1$.

Notice that condition (1.2) is equivalent to the requirement that the distributions $\partial f/\partial x_j$, $1 \leq j \leq N$, be functions in $L^\infty(\mathbb{R}^N)$ and $|\nabla f(x)| \leq C$, $x \in \mathbb{R}^N$.

Since $\Delta f = \mu$ in the distributional sense, the supremum in (1.1) is equal to

$$\sup \langle \Delta f, 1 \rangle \tag{1.3}$$

where the brackets stand for the action of the distribution Δf on the function 1 and the supremum is taken over all functions f satisfying (1.2) with $C = 1$, which are harmonic outside a compact subset of E and subharmonic on $\mathbb{R}^N$. If we drop the subharmonicity condition, then (1.3) essentially gives the Lipschitz harmonic capacity κ' of [P1] and [MP].

Before stating our main result we establish some notational conventions. Given a ball B we denote by kB the ball with the same center and k times the radius. Given a set $E \subset \mathbb{R}^N$ we set, for each scalar or vector valued function f, $||f||_E = \sup\{|f(x)| : x \in E\}$. Also, if f is differentiable we define

$$\operatorname{osc}(\nabla f, E) = \sup\{|\nabla f(x) - \nabla f(y)| : x, y \in E\}\,.$$

When $E = \mathbb{R}^N$ we will let $||f||$ stand for $||f||_{\mathbb{R}^N}$.

Theorem 1.1. *Let f be subharmonic and continuously differentiable on $\mathbb{R}^N$ with Riesz measure $\mu = \Delta f$. Let X be a compact subset of $\mathbb{R}^N$. Then the following are equivalent.*

(i) There exists a sequence of functions f_n, subharmonic and continuously differentiable on $\mathbb{R}^N$, harmonic on some neighbourhood (depending on n) of X, such that

$$\nabla f_n \to \nabla f \quad \textit{uniformly on } \mathbb{R}^N .$$

(ii) For each open bounded set D we have

$$\mu(D) \leq C \operatorname{osc}(\nabla f; 2B)\gamma_+(D \setminus X) ,$$

where B is a ball containing D.

(iii) There exist $k \geq 1$ and $\varepsilon(t) \to 0$ as $t \to 0$ such that for each open ball B of radius $r \leq 1$ that intersects X, one has

$$\mu(B) \leq \varepsilon(r)\gamma_+(kB \setminus X) .$$

(iv) There exist $k \geq 1$ and $\varepsilon(t) \to 0$ as $t \to 0$ such that for each open ball B of radius $r \leq 1$ that intersects X, one has

$$0 \leq \frac{1}{\sigma(\partial B)} \int_{\partial B} f(x)\, d\sigma(x) - \frac{1}{|B|} \int_B f(x)\, dx \leq \varepsilon(r) r^{2-N} \gamma_+(kB \setminus X) ,$$

where σ is the surface measure on ∂B and $|B|$ is the N-dimensional volume of B.

Remark 1.2. *(a) It is easily seen that once one knows (i) to hold, then a constant depending on n can be added to f_n so that moreover $f_n \to f$ uniformly on compacta of $\mathbb{R}^N$. This applies also to condition (i) in Corollary 1.3 below.*

(b) Condition (ii) has no analog in the context of uniform analytic approximation in the plane or of C^1 harmonic approximation in $\mathbb{R}^N$. Instead, there is a precedent to condition (iv) in [P2, Corollary 2.3.].

(c) It is not difficult to see that (iv) with $k = 1$ is also equivalent to any of the condition in Theorem 1.1.

There is a continuous version of the capacity γ_+ that plays a role in the next result. Let

$$\alpha_+(E) = \sup \mu(E)$$

where the supremum is taken over all positive measures μ, compactly supported in E, such that $f = \Phi * \mu$ is continuously differentiable on $\mathbb{R}^N$ and $||\nabla f||_{\mathbb{R}^N} \leq 1$.

Corollary 1.3. *Let $X \subset \mathbb{R}^N$ be compact. Then the following are equivalent.*

(i) *For each continuously differentiable subharmonic function on* $\mathbb{R}^N$*, which is harmonic on the interior* $\overset{\circ}{X}$ *of* X*, one can find a sequence of continuously differentiable subharmonic functions* f_n *on* $\mathbb{R}^N$*, each* f_n *being harmonic on some neighbourhood (depending on* n*) of* X*, such that*

$$\nabla f_n \to \nabla f \quad \textit{uniformly on } \mathbb{R}^N .$$

(ii) $\alpha_+(D \setminus X) = \alpha_+(D \setminus \overset{\circ}{X})$*, for each open bounded set* D*.*

(iii) *There exist* $k \geq 1$ *and* $C > 0$ *such that*

$$\alpha_+(B \setminus \overset{\circ}{X}) \leq C \alpha_+(kB \setminus X) ,$$

for each open ball B*.*

It is worth mentioning that in $\mathbb{R}^3$ our approximation results have very natural physical sense. If we consider μ as some "mass distribution" on the "rigid body" $\operatorname{spt}(\mu)$ then, by Newton's classical law of gravitation, the vector-field $-4\pi\gamma\nabla(\Phi * \mu)$ is precisely the *gravitational* field, induced by μ (here $\Phi(x) = -1/(4\pi|x|)$ and γ is the universal gravitational constant). Therefore, we answer the following question: when it is possible to "remove" a mass distribution from $\mathbb{R}^3 \setminus \overset{\circ}{X}$ to $\mathbb{R}^3 \setminus X$ (or even to ∞) in order that the corresponding gravitational field changes as little as we wish on all of $\mathbb{R}^3$ (respectively on X, if one considers *harmonic polynomial* approximating vector fields). See [F] and references therein for some concrete applications of approximation theory to the theory of gravitation.

When one tries to obtain geometric criteria from Theorem 1.1 and Corollary 1.3, one is led, as in classical uniform rational approximation ([Me1] and [Vi]) to the problem of estimating certain integrals. More concretely, we are led to estimating the flux of a function through the boundary of a ball. Given a function of class C^1 on $\overline{B}$, where B is an open ball, we wish to estimate

$$\int_{\partial B} \nabla f \cdot \vec{\nu} d\sigma \tag{1.4}$$

$\vec{\nu}$ being the unit outward normal to ∂B and σ the surface measure on ∂B. Notice that if f is harmonic on B then (1.4) vanishes by Gauss' Theorem. Thus one is led to suspect that the quantity $\kappa(\operatorname{spt}(\Delta f) \cap B)$ will play a role in estimating (1.4). Here κ is the C^1-harmonic capacity introduced in [P1]. In that paper one asks about the existence of some constant C depending only on N such that

$$|\int_{\partial B} \nabla f \cdot \vec{\nu} d\sigma| \leq C \operatorname{osc}(\nabla f, B)\, \kappa(\operatorname{spt}(\Delta f) \cap B) . \tag{1.5}$$

Although we cannot prove (1.5), we do get an inequality of the above type if we further assume that f is subharmonic on B and κ is replaced by α_+.

Theorem 1.4. *Let f be of class C^1 on $\overline{B}$ and subharmonic on B. Then*

$$|\int_{\partial B} \nabla f \cdot \nu \, d\sigma| \leq C \operatorname{osc}(\nabla f, B)\, \alpha_+(\operatorname{spt}(\Delta f) \cap B)\,,$$

where C is a positive constant depending only on N.

In proving Theorem 1.4 we are led to consider the following question: given a function of class C^1 on $\overline{B}$ and subharmonic on B, is it possible to extend f to a subharmonic function of class C^1 on the whole of $\mathbb{R}^N$?

We give a positive answer in Theorem 5.1. The reader is referred to [G] and [Ga] where the above extension problem is mentioned.

The necessary conditions for approximation in Theorem 1.1 are discussed in Section 3. In particular it is there proved that (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv) ((ii) $\Rightarrow$ (iii) is obvious). The sufficient conditions for approximation, in particular (iv)$\Rightarrow$(i), are considered in Section 4. In Section 5 a proof of Theorem 1.4 is presented, including a discussion of the extension result for C^1 subharmonic functions from balls. In Section 6 we show that the semiadditivity properties of α_+ (in some special cases) give geometric sufficient conditions for the approximation. A Mergelyan type theorem is deduced from recent results of P. Jones [J] showing that for planar continua $\gamma_+(K)$ is comparable to the diameter of K. Section 2 is devoted to a discussion of the $T(1)$-Theorem.

The notation used in the paper is the standard one and usually is self-explaining. For example, ∂X is the topological boundary of the set X, $C_0^1(\Omega)$ is the space of continuously differentiable functions with compact support contained in Ω and $B(a,r)$ is the open ball with center a and radius r.

The letter C stands for a quantity that, although it may depend on some parameters like dimension, does not depend on the relevant variables under consideration and thus plays the role of a constant.

2 The $T(1)$-Theorem

The $T(1)$-Theorem is an important result in Calderón-Zygmund theory due to David and Journé [DJ]. It states that for an operator T belonging to some special family described below, T is bounded on L^2 if and only if the image under T of the constant function 1 belongs to the space BMO (see Remark 2.2 below). The special family of operators to which our attention must be restricted consists of singular integrals associated to kernels $K(x,y)$ satisfying the antisymmetry condition $K(y,x) = -K(x,y)$ and the standard growth and smoothness requirements of Calderón-Zygmund theory, namely,

$$|K(x,y)| \leq C|x-y|^{-(N-1)}\,, \tag{2.1}$$

and

$$|K(x,y) - K(x',y)| \leq C\frac{|x-x'|}{|x-y|^N}\,, \quad \text{for } |x'-x| < \frac{1}{2}|x-y|\,. \tag{2.2}$$

Notice that if (2.1) and (2.2) hold then K is a $N-1$ dimensional kernel. In fact we are interested only in kernels of the Riesz type

$$K(x,y) = \frac{x_i - y_i}{|x-y|^N}\,, \qquad 1 \leq i \leq N\,.$$

To define a singular integral operator associated to an antisymmetric kernel K satisfying (2.1) and (2.2) we need to specify an underlying positive measure μ. Fixed $\varepsilon > 0$ and given a compactly supported $f \in L^2(\mu)$, one defines the truncated operator T_ε by

$$T_\varepsilon f(x) = \int_{|y-x|>\varepsilon} f(y)K(x,y)\,d\mu(y)\,, \quad x \in \mathbb{R}^N\,.$$

The existence of the limit of $T_\varepsilon f(x)$ as $\varepsilon \to 0$ is not a simple problem even for smooth functions f because of the generality of the underlying measure μ. Therefore one avoids assuming that T is defined even on some small class of functions and decides to work with the whole family of operators T_ε, $\varepsilon > 0$, looking for estimates independent of ε. For example, one says that the operator T with kernel K is bounded on $L^2(\mu)$ provided one has the inequality

$$\int |T_\varepsilon f|^2 d\mu \leq C \int |f|^2 d\mu\,,$$

with a constant C independent of ε and of the compactly supported function f in $L^2(\mu)$. As it was shown by Semmes [D, p. 56], a necessary condition for the boundedness of T on $L^2(\mu)$ is that

$$\mu(B) \leq C(\text{radius } B)^{N-1}\,, \qquad \text{for each ball } B. \tag{2.3}$$

Here is the version of the $T(1)$-Theorem we will use:

Theorem 2.1. *Let $K(x,y)$ be an antisymmetric kernel, smooth off the diagonal, and satisfying* (2.1) *and* (2.2)*. Let μ be a positive measure satisfying the growth condition* (2.3)*. Then the following are equivalent* *(i) The operator T associated to K is bounded on $L^2(\mu)$.*

(ii) $T(1) \in BMO_2^2(\mu)$.

Remark 2.2. *The space $BMO_\lambda^2\,(\mu), \lambda \geq 1$, consists of those functions in $L^2_{\rm loc}(\mu)$ such that, for some constant C and all cubes Q*

$$\int_Q |f - f_Q|^2 d\mu \leq C\mu(\lambda Q)\,.$$

where $f_Q = \frac{1}{\mu(Q)} \int_Q f\, d\mu$.

Clearly $L^\infty(\mu) \subset BMO_2^2(\mu)$ and therefore T is bounded on $L^2(\mu)$ provided $T(1) \in L^\infty(\mu)$. It is only this fact about BMO that will be used in the sequel. The reader should be aware of the fact that there is some technical difficulty in defining $T(1)$ (see [D, p. 27]). We will not discuss any detail here.

If μ satisfies the doubling condition $\mu(2Q) \leq c\mu(Q)$, for all cubes Q, then the preceding result is due to David, Journé and Semmes [DJS].

It has been ascertained only recently [NTV1] that the $T(1)$-Theorem holds in the generality in which it has been stated above, that is, without the doubling condition on μ (see also [T1], [T2] and [Ve5]).

We will use the $T(1)$-Theorem as follows. Our hypothesis will yield that $T_\varepsilon(1) \in L^\infty(\mu)$, with $L^\infty(\mu)$-norms bounded uniformly in ε. From that is a simple matter to deduce, using the definition of $T(1)$, that $T(1) \in L^\infty(\mu)$. Therefore (ii) in Theorem 2.1 holds and, consequently, T is bounded on $L^2(\mu)$. By the Calderón-Zygmund theory without doubling conditions [NTV2], the operators T_ε send boundedly the space $M(\mathbb{R}^N)$ of finite measures on $\mathbb{R}^N$ into the space $L^{1,\infty}(\mu)$ of weak $L^1(\mu)$ functions, with constants independent of ε. In other words, for each finite Radon measure ν,

$$\mu\{x\colon |T_\varepsilon(\nu)(x)| > t\} \leq Ct^{-1}||\nu||\,, \tag{2.4}$$

where C is independent of ε, t and ν, and

$$T_\varepsilon \nu(x) = \int_{|y-x|>\varepsilon} K(x,y)\, d\nu(y)\,, \quad x \in \mathbb{R}^N\,.$$

It is the inequality (2.4) that will be relevant for our approximation problem. For some other instances where (2.4) leads to approximation theorems for analytic and harmonic functions see [Ve1] and [Ve2]. In [MP] one finds an application of these results to the study of some special geometric properties of κ' and γ_+.

3 Necessary conditions for approximation

We need several lemmas.

Lemma 3.1. *Let μ be a positive Radon measure such that $f = \Phi * \mu$ is continuously differentiable on $\mathbb{R}^N$. Then for each open ball B with radius r*

$$\mu(B) \le C \operatorname{osc}(\nabla f, B) r^{N-1} .$$

Thus $\mu(E) = 0$ provided the $N-1$-dimensional Hausdorff measure of the Borel set E is finite.

Proof. To apply conveniently Gauss' formula we need to regularize μ. Let $\varphi \in C_0^\infty(B(0,1))$, $0 \le \varphi \le 1$, $\int \varphi(x)dx = 1$. For $\delta > 0$ set $\varphi_\delta(x) = \varphi(\delta^{-1}x)\delta^{-N}$, $\mu_\delta = \varphi_\delta * \mu$ and $f_\delta = \varphi_\delta * f$. Since $\Delta f_\delta = \mu_\delta$, Gauss' formula gives

$$\int_{\partial B} (\nabla f_\delta \cdot \vec{\nu}) d\sigma = \int_B \Delta f_\delta(x)\, dx = \mu_\delta(B)$$

where $\vec{\nu}$ is the outward unit normal vector to ∂B. Notice that we can replace ∇f_δ by $\nabla f_\delta - \nabla f_\delta(a)$, where a is the center of the ball B, in the preceding identity.

Thus

$$\mu_\delta(B) \le C \operatorname{osc}(\nabla f_\delta, B) r^{N-1} .$$

Letting $\delta \to 0$ we obtain

$$\mu(B) \le \liminf_{\delta \to 0} \mu_\delta(B) \le C \operatorname{osc}(\nabla f, B) r^{N-1} ,$$

as desired.

Let E be a Borel set of finite $N-1$-dimensional Hausdorff measure. To prove that $\mu(E) = 0$ one can assume without loss of generality that E is compact. Then

$$\mu(B(a,r)) \le \varepsilon(r) r^{N-1} \quad a \in E\,, \quad 0 < r < 1\,,$$

for some function $\varepsilon(r) \to 0$ as $r \to 0$ and hence $\mu(E) = 0$. □

Remark 3.2. *If in the preceding lemma f is assumed only to have a bounded gradient, then we get, with the same proof, the weaker conclusion $\mu(B) \le Cr^{N-1}$.*

Lemma 3.3. *Let μ be a positive Radon measure with compact support and such that $f = \Phi * \mu$ has bounded first order distributional derivatives. Then, for some positive constant $C = C(N)$, one has*

$$\mu(E) \le C \gamma_+(E) ||\nabla f||$$

for each bounded Borel set E.

Proof. The distributions $\partial f/\partial x_i = \partial\Phi/\partial x_i * \mu$ are functions in $L^\infty(\mathbb{R}^N)$ by hypothesis. We assume, without loss of generality, that $||\nabla f|| = 1$. Because of the Remark 3.2 the measure μ satisfies the growth condition (2.3).

Let T_i $1 \le i \le N$, be the singular integral operator associated to the kernel $c_N \frac{x_i - y_i}{|x-y|^N}$, where c_N is chosen so that $\partial\Phi/\partial x_i = c_N \frac{x_i}{|x|^N}$, and to the underlying measure μ.

We claim that $T_i(1) \in L^\infty(\mu)$, $1 \le i \le N$, which is equivalent to the existence of a constant C such that

$$|T_{i,\varepsilon}(1)(x)| \le C\,, \qquad x \in \mathbb{R}^N\,, \quad \varepsilon > 0\,, \tag{3.1}$$

where

$$T_{i,\varepsilon}(1)(x) = \int_{|x-y|>\varepsilon} \partial_i\Phi(x-y)\, d\mu(y)\,.$$

Set $B = B(0,1)$. Since $\partial_i\Phi(x)$ is locally integrable the function $\partial_i\Phi * \frac{\chi_B}{|B|}$ is continuous everywhere. We are going to prove the identity

$$\left(\partial_i\Phi * \frac{\chi_B}{|B|}\right)(x) = \begin{cases} \partial_i\Phi(x)\,, & |x| \ge 1\,, \\ c_N x_i\,, & |x| \le 1\,. \end{cases}$$

Since $\partial_i\Phi$ is harmonic off the origin, the mean value property gives $(\partial_i\Phi * \frac{\chi_B}{|B|})(x) = \partial_i\Phi(x)$, for $|x| \ge 1$. On the other hand, $\partial_i\Phi * \frac{\chi_B}{|B|} = \Phi * \partial_i(\frac{\chi_B}{|B|})$ is harmonic on B, because $\partial_i\chi_B$ is supported on ∂B, and takes continuously the boundary values $c_N x_i$. Thus

$$\left(\partial_i\phi * \frac{\chi_B}{|B|}\right)(x) = c_N x_i \qquad \text{on} \quad B\,.$$

Setting $K_\varepsilon = \partial_i\Phi * \frac{\chi_{B(0,\varepsilon)}}{|B(0,\varepsilon)|}$, we then have

$$K_\varepsilon = \begin{cases} \partial_i\Phi(x)\,, & |x| > \varepsilon\,, \\ c_N\varepsilon^{-N} x_i\,, & |x| \le \varepsilon. \end{cases}$$

Therefore, for some constant $C = C(N)$ and all $x \in \mathbb{R}^N$,

$$\left|T_{i,\varepsilon}(1)(x) - \int K_\varepsilon(x-y)\, d\mu(y)\right| \le C\mu(B(x,\varepsilon))\varepsilon^{-(N-1)} \le C\,.$$

Clearly

$$\int K_\varepsilon(x-y)\, d\mu(y) = \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} (\partial_i\Phi * \mu)(y)\, dy\,,$$

which yields

$$\left| \int K_\varepsilon(x-y)\, d\mu(y) \right| \le ||\partial_i \Phi * \mu|| \,, \quad x \in \mathbb{R}^N \,,$$

and, hence, (3.1).

By the discussion on the $T(1)$-Theorem of Section 1 we conclude that the T_i send boundedly $M(\mathbb{R}^N)$ into $L^{1,\infty}(\mu)$. Assume without loss of generality that E is compact. Applying Uy's Theorem (see [MP], [T1], [Ve3] or [Ve4]) we find a μ-measurable function h, supported on E such that $0 \le h \le 1$, $\mu(E) \le C \int h\, d\mu$ and

$$||T_i(h\mu)|| \le C \,, \qquad 1 \le i \le N \,.$$

Strictly speaking Uy's Theorem cannot be applied to $T_{i,\varepsilon}$ because its kernel is not continuous. It can be applied, however, to the operator with continuous kernel K_ε, which differs from $T_{i,\varepsilon}$ by a term controlled by the fractioned maximal operator $\sup_{r>1} r^{-(N-1)} \, | \int_{B(x,r)} h\, d\mu|$. For more details the reader is referred to [MP], [T1], [Ve3] and [Ve4].

Therefore $||\nabla(\Phi * h\mu)|| \le C$ and so, by the definition of γ_+,

$$\int h\, d\mu \le C \gamma_+(E) \,. \qquad \square$$

Given $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $f \in L^1_{\text{Loc}}(\mathbb{R}^N)$ let

$$V_\varphi(f) = \Phi * \varphi \Delta f$$

be the Vitushkin operator associated to φ (see [B], [Ve2] or [P1]). We need the following estimate for V_φ.

Lemma 3.4. *For each $\varphi \in C_0^1(B(a,r))$ the following statements hold.*

(a) If f is continuously differentiable, then

$$||\nabla(V_\varphi(f))|| \le C \operatorname{osc}(\nabla f, B(a,r)) ||\nabla \varphi|| r \,.$$

(b) If the distributional first order derivatives of f are bounded, then

$$||\nabla(V_\varphi(f))|| \le C ||\nabla f|| ||\nabla \varphi|| r \,.$$

Remark 3.5. *The reader should compare the preceding statement with [Ve2, Lemma 2.1] and also with similar estimates in [P1, Lemma 4.2].*

Proof. We shall prove only (a), (b) is similar. From regularization arguments it is enough to assume that $f \in C^\infty(\mathbb{R}^N)$. Finally, because of the maximum principle we only need to estimate $|\nabla(V_\varphi(f))(x)|$ for x in $B(a,r)$. Fix such an x and set

$$F(y) = f(y) - f(x) - \nabla f(x) \cdot (y-x)\,.$$

For $y \in B(a,r)$ we then have

$$|F(y)| \leq M|y-x| \quad \text{and} \quad |\nabla F(y)| \leq M\,, \tag{3.2}$$

M being $\operatorname{osc}(\nabla f, B(a,r))$.

Denoting by $\langle T, \psi\rangle$ the action of the distribution T on the smooth function ψ one gets

$$\nabla(V_\varphi(f))(x) = \nabla\langle \Phi(x-y), \varphi(y)\Delta F(y)\rangle\,.$$

Convolving the identity

$$\Delta(\varphi F) = F\Delta\varphi + \varphi\Delta F + 2\nabla\varphi \cdot \nabla F$$

with Φ, evaluating at x and taking into account that $F(x) = 0$ we conclude that

$$\langle \Phi(x-y), \varphi(y)\Delta F(y)\rangle = -\langle \Phi(x-y), F\Delta\varphi\rangle - 2\langle \Phi(x-y), \nabla\varphi\cdot\nabla F\rangle\,.$$

Now

$$F\partial_j^2\varphi = \partial_j(F\partial_j\varphi) - \partial_j F\partial_j\varphi$$

and thus

$$\langle \Phi(x-y), \varphi(y)\Delta F(y)\rangle = \sum_{j=1}^{n}\langle \partial_j\Phi(x-y), F(y)\partial_j\varphi(y)\rangle - \langle \Phi(x-y), \nabla\varphi\cdot\nabla F\rangle\,.$$

After taking gradient with respect to the x variable we realize that we have to estimate terms of the form

$$\langle \partial_k\partial_j\Phi(x-y), F(y)\partial_j\varphi(y)\rangle \tag{3.3}$$

and

$$\langle \partial_k\Phi(x-y), \partial_j\varphi(y)\partial_j F(y)\rangle\,. \tag{3.4}$$

The absolute value of (3.4) is clearly not greater than

$$C||\nabla\varphi||M\int_{B(a,r)}|x-y|^{-N+1}\,dy \leq CM||\nabla\varphi||r\,.$$

Estimating (3.3) seems at first glance a more delicate task because

$$|\partial_k \partial_j \Phi(x-y)| \simeq |x-y|^{-N}\,.$$

However, as it is well known [S], the convolution of the distribution $\partial_k \partial_j \Phi$ with the continuous compactly supported function ψ at the point x is given by the principal value integral

$$\lim_{\varepsilon \to 0} \int_{B(x,\varepsilon)} \partial_k \partial_j \Phi(x-y)\psi(y)\, dy$$

provided $\psi(y) - \psi(x)$ vanishes fast enough as $y \to x$. In the case at hand $\psi = F\partial_j \varphi$, which vanishes faster than $|y-x|$ at the point x, because of (3.4). Thus the principal value integral is absolutely convergent and

$$\int |\partial_k \partial_j \Phi(x-y) F(y) \partial_j \varphi(y)|\; dy \le$$

$$\le CM||\nabla \varphi|| \int_{B(a,r)} |x-y|^{-N+1} dy \le CM||\nabla\varphi|| r\,.$$

□

Proof of (i)⇒ (ii) in Theorem 1. Let f_n be as in condition (i) and set $\mu_n = \Delta f_n$. Consider an open bounded set D. Let B be a ball of radius $r = \operatorname{diam}(D)$ containing D. Take $\varphi \in C_0^1(2B)$, $\varphi = 1$ on B, $|\nabla \varphi| \le Cr^{-1}$. By Lemma 3.4

$$||\nabla(V_\varphi(f))|| \le C \operatorname{osc}(\nabla f, 2B) \quad \text{and} \quad ||\nabla(V_\varphi f_n) - \nabla(V_\varphi f)|| \xrightarrow{n\to\infty} 0\,. \tag{3.5}$$

Let Ω be any smoothly bounded domain with closure contained in D. Then, denoting by $\vec{\nu}$ the unit normal vector to $\partial\Omega$,

$$\mu_n(\Omega) = \int_\Omega \varphi d\mu_n = \int_{\partial\Omega} (\nabla V_\varphi f_n \cdot \vec{\nu})\, d\sigma$$

$$\xrightarrow{n\to\infty} \int_{\partial\Omega} (\nabla V_\varphi f \cdot \vec{\nu})\, d\sigma = \int_\Omega \varphi d\mu = \mu(\Omega)\,.$$

Because of Lemma 3.3 with μ replaced by μ_n,

$$\int_\Omega \varphi d\mu_n \le C||\nabla V_\varphi f_n|| \gamma_+(\Omega \cap spt\mu_n) \le C||\nabla V_\varphi f_n|| \gamma_+(D \setminus X)\,,$$

and so we get, letting $n \to \infty$ and using (3.5),

$$\mu(\Omega) = \int_\Omega \varphi d\mu \le C \operatorname{osc}(\nabla f, 2B) \gamma_+(D \setminus X)\,,$$

which gives (ii) in Theorem 1.1 after taking supremum on Ω. □

Proof of (iii)⇒ (iv) in Theorem 1. We will find convenient to consider an auxiliary conditions that will turn out to be equivalent to (i)-(iv) (this will become evident after Section 4).

Let $\varphi \geq 0$ be a bounded Borel function, with positive integral and support contained in $\overline{B}(0,1)$. The condition we wish to consider, is the following.

There exists a $k \geq 1$ and a function $\varepsilon(t) \to 0$ as $t \to 0$, such that

$$\int \varphi\left(\frac{x-a}{r}\right) d\mu(x) \leq C\varepsilon(r)\gamma_+(B(a,kr)\setminus X)\,, \tag{3.6}$$

provided $0 < r < 1$ and the ball $B(a,r)$ contains some point in X.

A simple computation shows that (iv) is (3.6) for the function defined by

$$\varphi(x) = 1 - |x|^2\,, \quad |x| \leq 1 \quad \text{and} \quad \varphi(x) = 0\,, \quad |x| > 1.$$

On the other hand, it is obvious that (iii) applied to $B = B(a,r)$ gives (3.6) with $C = ||\varphi||$ □

In the next Section one shows that (3.6) for a single function φ implies (i). Therefore (iii) or (iv) imply (i).

4 Sufficient conditions for approximation

In this Section we prove that condition (i) in the statement of Theorem 1.1 follows from the assumption that (3.6) holds for a specific function φ. We use Vitushkin's scheme for the approximation [Vi], such as adapted to the harmonic case in [B] and [Ve2], and an idea introduced in [P2] to deal with a single function φ.

Given $\delta > 0$ consider a covering $\{B_j\}$ of $\mathbb{R}^N$ by open balls of radius δ and a partition of unity $\{\psi_j\}$ such that

$$\begin{array}{l}\text{for some positive integer } M \text{ and for each } x \in \mathbb{R}^N \\ \text{the number of balls } B_j \text{ such that } 2B_j \cap B(x,\delta) \neq \emptyset \\ \text{is less than } M,\end{array} \tag{4.1}$$

$$\psi_j \in C_0^\infty(B_j)\,, \quad 0 \leq \psi_j\,, \quad |\nabla \psi_j| \leq C\delta^{-1} \quad \text{and} \quad \sum_j \psi_j = 1. \tag{4.2}$$

Details for the construction of the above covering and partition of unity can be found in [P1] or [Ve2] (we can assume that the set of indices j is $\mathbf{Z}^N$ and that M, C depend only on N).

Set $\varphi_\delta(x) = \varphi\left(\frac{x}{\delta}\right)\delta^{-N}(\int \varphi(y)dy)^{-1}$ and $\varphi_j = \varphi_\delta * \psi_j$ so that (4.2) holds with ψ_j replaced by φ_j and B_j by $2B_j$. We claim that

$$\int \varphi_j d\mu \le C\varepsilon(\delta)\gamma_+((k+1)B_j \setminus X)\,, \tag{4.3}$$

where μ is as in condition (i) of Theorem 1.1 and $\varepsilon(\delta)$ is the function in (3.6). To show (4.3) write

$$\int \varphi_j d\mu = \int \left(\int \varphi_\delta(x-y)d\mu(x)\right)\psi_j(y)\,dy$$

and estimate the inner integral for y in B_j using (3.6):

$$\begin{aligned}
&\int \varphi_\delta(x-y)\,d\mu(x) = \\
&= C\delta^{-N}\int \varphi\left(\frac{x-y}{\delta}\right)d\mu(x) \le C\delta^{-N}\omega(\delta)\gamma_+(B(y,k\delta)\setminus X) \le \\
&\le C\delta^{-N}\omega(\delta)\gamma_+((k+1)B_j)\setminus X)\,.
\end{aligned}$$

Hence

$$\int \varphi_j d\mu \le C\delta^{-N}\omega(\delta)\gamma_+((k+1)B_j\setminus X)\int \psi_j(y)dy \le C\omega(\delta)\gamma_+((k+1)B_j\setminus X)\,,$$

as announced.

We now turn our attention to the Vitushkin's localization procedure. Take $\mu_j = \varphi_j\mu$ and set $f_j = \Phi * \mu_j$. By Lemma 3.4 each f_j is continuously differentiable on $\mathbb{R}^N$. Set $J = \{j : 2B_j \cap X \ne \emptyset\}$. If $j \in J$ then $||\nabla f_j|| \le C\omega(\delta)$ because of Lemma 3.4, where $\omega(\delta)$ stands for the modulus of continuity of f restricted to $\{x \in \mathbb{R}^N : \operatorname{dist}(x,X) \le 4\delta\}$. Moreover $\Delta f_j = \varphi_j\mu \ge 0$ and so f_j is subharmonic on $\mathbb{R}^N$, as f was.

Let a_j be the center of B_j, $j \in J$. The function f_j admits an expression of the form

$$f_j(x) = ||\mu_j||\Phi(x-a_j) + d_j(x)\,, \quad x \notin 2B_j\,,$$

where $d_j(x)$ satisfies, for $|x-a_j| > 4\delta$, the estimate

$$\begin{aligned}
|\nabla d_j(x)| &= \left|\int (\nabla\Phi(x-y) - \nabla\Phi(x-a_j))\,d\mu_j(y)\right| \\
&\le C\delta||\mu_j||\,|x-a_j|^{-N}\,.
\end{aligned}$$

From (4.3) and the definition of γ_+ we conclude that we can find a positive measure ν_j supported on $(k+1)B_j \setminus X$ such that $||\nu_j|| = ||\mu_j||$, and $||\nabla g_j|| \leq C\varepsilon(\delta)$, where g_j stands for $\Phi * \nu_j$. Regularizing ν_j we can also assume that g_j is continuously differentiable on $\mathbb{R}^N$.

As we discussed above for f_j, one has an expression for g_j of the form

$$g_j(x) = ||\nu_j||\Phi(x - a_j) + \widetilde{d}_j(x)\,, \quad x \notin (k+1)B_j\,,$$

with

$$|\nabla \widetilde{d}_j(x)| \leq C\delta ||\nu_j||\, |x - a_j|^{-N}\,, \quad |x - a_j| > 2(k+1)\delta\,.$$

Thus $||\nabla f_j - \nabla g_j|| \leq C(\omega(\delta) + \varepsilon(\delta))$ and

$$|\nabla f_j(x) - \nabla g_j(x)| \leq \frac{C\delta ||\mu_j||}{|x - a_j|^N}\,, \qquad \text{for} \quad |x - a_j| > 2(k+1)\delta\,.$$

Set $g = f - \sum_{j \in J} f_j + \sum_{j \in J} g_j$. Then

$$g \in C^1(\mathbb{R}^N), \quad \Delta g = \mu(1 - \sum_{j \in J} \varphi_j) + \sum_{j \in J} \nu_j$$

is nonnegative and is equal to 0 on some vicinity U of X. Hence g is subharmonic everywhere and harmonic on U. It suffices to show that for some $\eta(\delta) \to 0$ as $\delta \to 0$,

$$|\nabla f(x) - \nabla g(x)| \leq C\eta(\delta)\,, x \in \mathbb{R}^N\,. \tag{4.4}$$

Let p be the least integer larger than $2(k+1)+1$. Fix $x \in \mathbb{R}^N$ and define

$$J_0 = \{j \in J \colon\, |x - a_j| < p\delta\}\,,$$
$$J_n = \{j \in J \colon\, n\delta \leq |x - a_j| < (n+1)\delta\}\,, \quad n = p, p+1, \ldots$$

Then

$$|\nabla f(x) - \nabla g(x)| \leq$$
$$\leq \sum_{j \in J_0} |\nabla f_j(x) - \nabla g_j(x)| + \sum_{n=p}^{\infty} \sum_{j \in J_n} \frac{C\delta ||\mu_j||}{|x - a_j|^N} \equiv T_0 + \sum_{n=p}^{\infty} T_n\,.$$

Because of (4.1), the number of indexes in J_0 is not greater than some constant depending only on p. Thus $T_0 \leq C(\omega(\delta) + \varepsilon(\delta))$.

Since $||\mu_j|| \leq \mu(2B_j)$, we get, by (4.1),

$$T_n \leq \sum_{j \in J_n} \frac{C\delta\mu(2B_j)}{n^N \delta^N} \leq C\delta^{1-N} n^{-N} (\mu(B(x, (n+3)\delta) - \mu(B(x, (n-2)\delta))\,,$$

and therefore, applying Lemma 3.1,

$$\sum_{n=p}^{\infty} T_n \leq C\delta^{1-N} p^{-N} (\omega(\delta) + \varepsilon(\delta))\delta^{N-1} +$$

$$+ C\delta^{1-N} \sum_{n=p}^{\infty} \mu(B(x,(n+3)\delta)) \left(\frac{1}{n^N} - \frac{1}{(n+5)^N} \right) \leq$$

$$\leq Cp^{-N}(\omega(\delta) + \varepsilon(\delta)) + C \sum_{n=p}^{\infty} \frac{\omega(n\delta)}{n^2} \equiv \eta(\delta)\,.$$

This shows (4.4), because $\sum_{n=p}^{\infty} \omega(n\delta) n^{-2} \to 0$ as $\delta \to 0$.

5 C^1-extension of subharmonic functions and the flux estimate

The first result of this Section shows that one can extend functions that are subharmonic on an open ball B and of class C^1 on the closed ball $\overline{B}$ to subharmonic functions on $\mathbb{R}^N$ which are also of class C^1. As usual, the extension can be done with control on the norms.

Theorem 5.1. *Let f be subharmonic on an open ball B and of class C^1 on $\overline{B}$. Then there is a function F, subharmonic and of class C^1 on $\mathbb{R}^N$, such that $F = f$ on $\overline{B}$, and $||\nabla F||_{\mathbb{R}^N} \leq C||\nabla f||_B$, where C is a positive constant depending only on N.*

We need a lemma.

Lemma 5.2. *Let f and B be as in the statement of Theorem 5.1. Then, given $\varepsilon > 0$, there is a function F_ε of class C^1 on $\mathbb{R}^N$ such that*

i) *F_ε is subharmonic on $\mathbb{R}^N$ and $f - F_\varepsilon$ is subharmonic on B.*

ii) $(\operatorname{spt} \Delta F_\varepsilon) \cap B \subset \operatorname{spt} \Delta f$.

iii) *$||f - F_\varepsilon||_B + ||\nabla f - \nabla F_\varepsilon||_B < \varepsilon$, $||\nabla F_\varepsilon||_{\mathbb{R}^N} \leq C||\nabla f||_B$, $||F_\varepsilon||_{\mathbb{R}^N} \leq C||f||_B$ if $N \geq 3$, and $||F_\varepsilon||_{2B} \leq C||f||_B$ if $N = 2$, C being a positive constant depending only on N.*

Remark 5.3. *For $N = 2$ the inequality $||F_\varepsilon||_{\mathbb{R}^N} \leq C||f||_B$ cannot hold because there are no non-constant subharmonic bounded functions on the whole plane.*

Proof of Lemma 5.2. The proof is inspired on ideas of reflection and approximation coming from [Me1].

Without loss of generality we can assume that B is the open ball of radius 1 centered at the origin and that f is a compactly supported function of class C^1 on $\mathbb{R}^N$. We claim that there exists a functions f_ε of class C^1 on $\mathbb{R}^N$, such that $f - f_\varepsilon$ is subharmonic on B,

$$||f - f_\varepsilon||_{2B} < \varepsilon\,, \qquad ||\nabla f - \nabla f_\varepsilon||_{\mathbb{R}^N} < \varepsilon\,, \qquad \operatorname{spt}(\Delta f_\varepsilon) \cap B \subset \operatorname{spt}(\Delta f)$$

and f_ε is harmonic on some neighbourhood of ∂B. The existence of the f_ε follows from a standard construction, which we describe below for the reader's convenience, whose effect is pushing the singularities of f from a small neighbourhood of ∂B to the complement of $\overline{B}$.

Fix $\delta > 0$ small enough and consider a covering $\{B_j\}$ of $\mathbb{R}^N$ by open balls of radius δ and an associated partition of unity $\{\psi_j\}$ satisfying (4.1) and (4.2).

Set $f_j = \Phi * \psi_j \Delta f$, so that $f = \sum_j f_j$.

Let a_j be the center of B_j. Then one has an expression of the form

$$f_j(x) = c_j \Phi(x - a_j) + d_j(x)\,, \quad x \notin 2B_j\,,$$

where

$$c_j = \langle \psi_j \Delta f, 1\rangle = -\int \nabla\psi \cdot (\nabla f - \nabla f(a_j))\, dx\,.$$

Hence

$$|c_j| \le C\delta^{N-1}\omega(\nabla f, \delta)\,.$$

Let J be the set of indexes j such that $2B_j \cap \partial B \neq \emptyset$. For each $j \in J$ one can find a ball B_j^* of radius δ, contained in $5B_j$, such that $2B_j^* \cap B = \emptyset$.

Take $\chi_j \in C_0^\infty(B_j^*)$ satisfying $\int \chi_j = 1$ and $0 \le \chi_j \le C\delta^{-N}$. If we set $g_j = c_j(\Phi * \chi_j)$, we obtain $|\nabla g_j| \le Cw(\nabla f, \delta)$ and

$$|\nabla f_j(x) - \nabla g_j(x)| \le C\frac{w(\nabla f, \delta)\delta^N}{|x - a_j|^N}\,, \quad x \notin 6B_j\,.$$

The function defined by

$$f_\varepsilon = \sum_{j \notin J} f_j + \sum_{j \in J} g_j$$

satisfies all requirements of the claim provided δ is small enough as can be ascertained following the arguments of Section 4. Note that f is harmonic in δ-neighbourhood of ∂B.

Now set $R = 1 + \frac{\delta}{2}$ and $M = ||\nabla f_\varepsilon||_{\mathbb{R}^N}$. Define a function g on $\mathbb{R}^N$ by means of the Kelvin transform: $g(x) = f_\varepsilon(x)$, $|x| \leq R$, and

$$g(x) = \left(\frac{R}{|x|}\right)^{N-2} f_\varepsilon\left(\frac{xR^2}{|x|^2}\right), \quad |x| > R.$$

Thus g is subharmonic on $\mathbb{R}^N \setminus \partial B(0,R)$ and is harmonic on

$$\mathbb{R}^N \setminus (\partial B(0,R) \cup S_\varepsilon \cup (S_\varepsilon)^*),$$

where $(S_\varepsilon)^*$ is the reflection of $S_\varepsilon = \operatorname{spt}(\Delta f_\varepsilon) \cap B(0,R)$ with respect to the sphere $\partial B(0,R)$. We need to modify $g(x)$ because it is not (in general) of class C^1 on $\mathbb{R}^N$ and is not necessarily subharmonic near $\partial B(0,R)$. The idea is to add to g a fixed function with a positive jump of the normal derivative on $\partial B(0,R)$, big enough to compensate the possible negative jump of the normal derivative of g on $\partial B(0,R)$. For $N \geq 3$ set $g_1(x) = g(x)$ if $|x| \leq R$, and

$$g_1(x) = g(x) + 2MR\left(1 - \left(\frac{R}{|x|}\right)^{N-2}\right), \quad \text{if} \quad |x| > R.$$

If $N = 2$ the term $1 - (\frac{R}{|x|})^{N-2}$ must be replaced by $-\log\frac{R}{|x|}$. Notice that g_1 is a Lipschitz subharmonic function on $\mathbb{R}^N$ and satisfies all properties required to F_ε in the statement of Lemma 5.2, except for being of class C^1. We overcome this difficulty by regularizing g_1 in the neighbourhood of $\partial B(0,R)$. Take a *radial* function $\varphi \in C_0^1(B(0,\frac{\delta}{4}))$, $\varphi \geq 0$, $\int \varphi = 1$ and define

$$F_\varepsilon(x) = g_1(x), \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad F_\varepsilon(x) = (\varphi * g_1)(x), \qquad \text{if } |x| > 1.$$

Since φ is radial and g_1 is harmonic on $1 < |x| < 1 + \frac{\delta}{2}$, we get $g_1(x) = (\varphi * g_1)(x)$, $1 < |x| < 1 + \frac{\delta}{4}$. Therefore F_ε is of class C^1 on $\mathbb{R}^N$ and the proof is complete. □

Proof of Theorem 5.1. Apply Lemma 5.2 to f and $\varepsilon = 1/2$ to obtain F_1. Apply again Lemma 5.2 to $f - F_1$ and $\varepsilon = 1/2^2$ to obtain F_2. We then get inductively functions $F_1, \ldots, F_n, \ldots$ so that

$$||f - (F_1 + \cdots + F_n)||_B + ||\nabla(f - (F_1 + \cdots + F_n))||_B < 1/2^n$$

and

$$||F_n||_{2B} + ||\nabla F_n||_{\mathbb{R}^N} \leq C 1/2^n.$$

Thus, the function $F = \sum_{n=1}^{\infty} F_n$ is the desired extension. □

We proceed now to the proof of Theorem 1.4.

Proof of Theorem 1.4. It is clearly enough, in view of the extension Theorem 5.1, to prove the following lemma, which should be compared to Lemma 3.3. □

Lemma 5.4. *Let μ be positive compactly supported Radon measure on $\mathbb{R}^N$ such that $f = \Phi * \mu$ is of class C^1 and $||\nabla f|| \leq 1$. Then for any bounded open set D one has*

$$\mu(D) \leq C\alpha_+(D \cap \operatorname{spt}\mu)\,,$$

where $C = C(N)$.

Proof. For $\delta > 0$ write

$$D_\delta = \{x \in D : \operatorname{dist}(x, \partial D) > 2\delta\}\,,$$

so that $\mu(D) \leq 2\mu(D_\delta)$ provided δ is sufficiently small. Consider a covering of $\mathbb{R}^N$ by balls $\{B_j\}$, of radius δ and a partition of unity $\{\psi_j\}$ satisfying (4.1) and (4.2). Let J be the set of indexes j such that $\overline{B}_j \subset D$. Then $\sum_{j\in J} \psi_j = 1$ on D_δ. Set $\mu_j = \psi_j \mu$ and $f_j = \Phi * \mu_j$. By Lemma 3.4 and (4.2) $||\nabla f_j|| \leq C\omega(\nabla f, \delta)$, where $\omega(\nabla f, \delta)$ is the modulus of continuity of ∇f at level δ. By Lemma 3.1, $||\mu_j|| \leq C\omega(\nabla f, \delta)\delta^{N-1}$.

Let a_j stand for the center of B_j. Take r_j such that $||\mu_j|| = \nu_j(\partial B(a_j, r_j))$, where ν_j is the surface measure on $\partial B(a_j, r_j)$. We can clearly take δ so small that the balls $B(a, 3r_j)$ are disjoint. Set $K = \cup_{j\in J}\partial B(a_j, r_j)$ and define

$$\nu = \sum_{j\in J} \nu_j + \sum_{j\notin J} \mu_j\,.$$

It is easily checked that $||\nabla\Phi * \nu_j|| \leq C$, and thus, arguing as in Section 4, we get

$$||\sum_{j\in J}(\nabla\Phi * \nu_j - \nabla\Phi * \mu_j)|| \leq C\,,$$

and so $||\nabla\Phi * \nu|| \leq C$. Applying Lemma 3.3 to the function $\Phi * \nu$ we obtain $\nu(K) \leq C\gamma_+(K)$, which yields $\mu(D) \leq C\gamma_+(K))$.

To complete the proof we only need to show that $\gamma_+(K) \leq C\alpha_+(\operatorname{spt}\mu \cap D)$. Take a positive measure σ supported on K satisfying $\gamma_+(K) \leq 2||\sigma||$ and $||\nabla\Phi * \sigma|| \leq 1$.

Let $\varphi_j \in C_0^\infty(B(a_j, 3r_j))$, $\varphi_j = 1$ on $B(a_j, 2r_j)$ and $||\nabla\varphi_j|| \leq Cr_j^{-1}$. If σ_j denotes the restriction of σ to $\partial B(a_j, r_j)$, then $\sigma_j = \varphi_j\sigma$. Hence, because of Lemma 3.4 (b), $||\nabla\Phi * \sigma_j|| \leq C$ and, by Remark 3.2,

$$||\sigma_j|| \leq Cr_j^{N-1} \leq C||\mu_j||\,.$$

Let λ_j satisfy $||\sigma_j|| = \lambda_j ||\mu_j||$, so that the λ_j are bounded uniformly in j. Set $\sigma_0 = \sum_{j \in J} \lambda_i \mu_j$. Then

$$2||\sigma_0|| = 2||\sigma|| \geq \gamma_+(K)\,, \quad \operatorname{spt} \sigma_0 \subset D \cap \operatorname{spt} \mu\,,$$

and

$$||\nabla \Phi * \sigma - \nabla \Phi * \sigma_0|| \leq C\,,$$

again by the argument of Section 4.

Now $||\nabla \Phi * \sigma_0|| \leq C$ and $\nabla \Phi * \sigma_0$ is continuous because each $\nabla \Phi * \mu_j$ is. Therefore

$$\gamma_+(K) \leq C \alpha_+(D \cap \operatorname{spt} \mu)\,.$$

□

We close this Section by mentioning that, to the best of our knowledge, it is an open problem to describe those compact sets X in $\mathbb{R}^N$ such that each function in $C^1_{\text{jet}}(X)$ (see [P1] or [S] for a definition) that is subharmonic on $\overset{\circ}{X}$ can be extended to a continuously differentiable subharmonic function on $\mathbb{R}^N$.

6 Properties of γ_+, α_+ and related criteria for approximation

One of the typical results of this Section is the following.

Theorem 6.1. *Let $X \subset \mathbb{R}^N$ be compact and assume that the boundary of X has finite $N-1$-dimensional Hausdorff measure. If f is a subharmonic continuously differentiable function on $\mathbb{R}^N$, harmonic on the interior of X, then there exist subharmonic continuously differentiable on $\mathbb{R}^N$ functions f_n, each f_n being harmonic on some neighbourhood (depending on n) of X, such that*

$$\nabla f_n \to \nabla f \quad \text{as} \quad n \to \infty, \text{ uniformly on } \mathbb{R}^N\,.$$

The result above follows readily from Corollary 1.3 and the semiadditivity inequalities below.

Theorem 6.2. *Let E_1 and E_2 be bounded Borel subsets of $\mathbb{R}^N$. There is a positive constant C depending only on N such that the following two statements hold.*

(a)

$$\gamma_+(E_1 \cup E_2) \leq C(\gamma_+(E_1) + \gamma_+(E_2))\,.$$

(b) If E_2 is open and either the $N-1$-dimensional Hausdorff measure of E_1 is finite or $\gamma_+(E_1)=0$ then

$$\alpha_+(E_1 \cup E_2) \le C\alpha_+(E_2)\,.$$

Proof. Take a positive Radon measure μ, supported on $E_1 \cup E_2$, such that

$$\gamma_+(E_1 \cup E_2) \le 2\mu(E_1 \cup E_2)$$

and $||\nabla f|| \le 1$, where $f = \Phi * \mu$. By Lemma 3.3 applied to E_1 and E_2 we obtain

$$\mu(E_j) \le C\gamma_+(E_j)\,, \qquad j = 1, 2,$$

which completes the proof of (a).

To show (b) take a positive Radon measure μ, with compact support contained in $E_1 \cup E_2$, such that $\alpha_+(E_1 \cup E_2) \le 2\mu(E_1 \cup E_2)$ and $f = \Phi * \mu$ is a continuously differentiable function on $\mathbb{R}^N$ with $||\nabla f|| \le 1$. Now, $\mu(E_1) = 0$ if E_1 has finite $N-1$-dimensional Hausdorff measure because of Lemma 3.1 and $\mu(E_1) = 0$ if $\gamma_+(E_1) = 0$ because of Lemma 3.3.

One more application of Lemma 3.3 gives

$$\mu(E_1 \cup E_2) = \mu(E_2) \le C\gamma_+(E_2)\,.$$

But $\gamma_+(E_2) = \alpha_+(E_2)$ because E_2 is open, as it can be easily seen by a simple regularization argument.

□

Proof of Theorem 6.1. If B is any open ball then

$$\alpha_+(B \setminus \overset{\circ}{X}) = \alpha_+((B \cap \partial X) \cup (B \setminus X)) \le C\alpha_+(B \setminus X)\,,$$

according to Theorem 6.2(b). Now it suffices to apply Corollary 1.3 (iii). □

In dimension $N = 2$ the capacity γ_+ is related to a geometric quantity called Menger curvature [Me2], [T1], [Ve5] and, at least in principle, this should make easier to apply the criterion of Theorem 1.1.

Recent unpublished work of P. Jones [J] shows that if K is a continuum in the plane, then

$$\gamma_+(K) \ge C\,\mathrm{diam}(K)\,. \tag{6.1}$$

Thus, if X is a compact in the plane with connected complement one obtains that

$$\gamma_+(B(a,\delta) \setminus X) \ge C\delta\,, \quad a \in \partial X\,,$$

and this readily gives

$$\alpha_+(B(a,\delta)\setminus \overset{\circ}{X}) \le \delta \le C\alpha_+(B(a,2\delta)\setminus X)\,, \qquad a\in\mathbb{C}.$$

By Corollary 1.3 (iii) and the well known method of translation of poles a subharmonic $C^1_{\rm jet}$ version of the Mergelyan theorem follows.

In fact, working with subharmonic functions is not necessary in the previous argument and so the genuine C^1-harmonic version of the Mergelyan theorem follows from Jones' result (see [P1] where this problem was posed and reduced to (6.1)).

But we do need subharmonicity to prove the following criterion. Recall that the inner boundary of a compact set $X\subset\mathbb{R}^2$ is $\partial_i X = \partial X\setminus \cup_k \partial U_k$, where the U_k are the connected components of $\mathbb{R}^2\setminus X$.

Theorem 6.3. *Let $X\subset\mathbb{R}^2$ be compact and assume that its inner boundary $\partial_i X$ has either finite one dimensional Hausdorff measure or $\gamma_+(\partial_i X)=0$. Then for each subharmonic function f on $\mathbb{R}^2$, of class C^1 on $\mathbb{R}^2$, which is harmonic on the interior of X, there exist subharmonic functions f_n on $\mathbb{R}^2$, of class C^1 on $\mathbb{R}^2$, each f_n being harmonic on some neighbourhood (depending on n) of X such that*

$$\nabla f_n \to \nabla f \quad \textit{as} \quad n\to\infty\,, \quad \textit{uniformly on } \mathbb{R}^N\,.$$

Proof. Let U_k be the connected components of the complement of X, so that $\partial X\setminus\cup_k\partial U_k$ is the inner boundary of X. It can be shown that for each k one can find a $\delta_k>0$ such that, if $a\in\partial U_k$ and $\delta\le\delta_k$, then there is a continuum of diameter at least $\delta/2$ in $B(a,\delta)\cap U_k$. This follows by fixing a point z_k in U_k and then joining z_k to any point in $B(a,\delta/2)\cap U_k$ by means of a polygonal curve. Applying (6.1) to the above continuum one gets, for $a\in\partial U_k$ and $\delta\le\delta_k$,

$$\alpha_+(B(a,\delta)\setminus \overset{\circ}{X}) \le \delta \le C\alpha_+(B(a,\delta)\cap U_k) \le C\alpha_+(B(a,\delta)\setminus X)\,.$$

Using this condition and the standard approximation scheme as in [Vi] one shows that for each $\varepsilon>0$ there exists a function f_ε subharmonic and of class C^1 on $\mathbb{R}^2$ such that $||\nabla f_\varepsilon - \nabla f||_{\mathbb{R}^2}<\varepsilon$ and f_ε is harmonic on $\overset{\circ}{X}$ and moreover on some neighbourhood of $\partial X\setminus\partial_i X$. Thus if we set $\mu_\varepsilon=\Delta f_\varepsilon$, then $\operatorname{spt}\mu_\varepsilon\subset(\mathbb{R}^2\setminus X)\cup(\partial X_i)$. Now fix an open disc B and set $\mu=\Delta f$. If ε is small enough $\mu(B)\le 2\mu_\varepsilon(B)$ and $\operatorname{osc}(\nabla f,B)\le 2\operatorname{osc}(\nabla f_\varepsilon,B)$.

Applying Lemma 5.4 it is not difficult to check that

$$\mu_\varepsilon(B)\le C\operatorname{osc}(\nabla f_\varepsilon,B)\alpha_+((B\setminus X)\cup\partial_i X)\,.$$

On the other hand, because of Theorem 6.2 (b) and the hypothesis on $\partial_i X$, we have

$$\alpha_+((B \setminus X) \cup \partial_i X) \le C\alpha_+(B \setminus X)\,.$$

So finally

$$\mu(B) \le C\,\mathrm{osc}(\nabla f, B)\alpha_+(B \setminus X)\,,$$

which completes the proof in view of Theorem 1.1 (iii). □

References

[B] T. Bagby, *Approximation in the mean by solutions of elliptic equations*, Trans. Amer. Math. Soc. **281** (1984), 761–784.

[D] G. David, *Wavelets and singular integrals on curves and surfaces*, Lecture Notes in Math. **1465**, Springer-Verlag, 1991.

[DJ] G. David and J. L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. **120** (1984), 371–397.

[DJS] G. David, J. L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana **4** (1985), 1–56.

[F] W. Freeden, *Basis systems and their role in the approximation of the Earth's gravitational field*, Gerlands Beitr. Geophysik, Leipzig. **94**:1 (1985), 19–34.

[G] S. J. Gardiner, *Harmonic approximation*, London Math. Soc., Lectures Notes series, Vol. 221, Cambridge University Press, Cambridge, 1995.

[Ga] P. M. Gauthier, *Subharmonic extension and approximation*, Canadian Math. Bull. **37** (1994), 46–53.

[J] P. W. Jones, personal communication.

[MP] P. Mattila, P.V. Paramonov, *On geometric properties of harmonic* Lip_1*-capacity*, Pacific J. Math. **171**:2 (1995), 469–490.

[Me1] M. Melnikov, *A bound for the Cauchy integral along an analytic curve*, Math. Sb. **71**(113) (1966), 503–515 (Russian), Amer. Math. Soc. Translations 80:2 (1969), 243–256.

[Me2] M. Melnikov, *Analytic capacity: discrete approach and the curvature of measure*, Sbornik: Mathematics **186**:6 (1995), 827–846.

[NTV1] F. Nazarov, S. Treil and A. Volberg, *Cauchy integral and Calderón-Zygmund operators on non homogeneous spaces*, Internat. Math. Res. Notices **15** (1997), 703–726.

[NTV2] F. Nazarov, S. Treil and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on non homogeneous spaces*, Internat. Math. Res. Notices **9** (1998), 463–487.

[P1] P. V. Paramonov, *On harmonic approximation in the C^1-norm*, Math. USSR Sbornik **71**:1 (1992), 183–207.

[P2] P. V. Paramonov, *Some new criteria for uniform approximability of functions by rational fractions*, Sbornik: Mathematics **186**:9 (1995), 1325–1340.

[P3] P. V. Paramonov, *C^m- approximation by harmonic polynomials on compact sets in $\mathbb{R}^n$*, Russian Acad. Sci. Sb. Math. **78**:1 (1994), 231–251.

[S] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.

[T1] X. Tolsa, *L^2-boundedness of the Cauchy integral operator for continuous measures*, Duke Math. J. **98**:2 (1999), 269–304.

[T2] X. Tolsa, *Littlewood-Paley Theory and the $T(1)$-Theorem with non doubling measures*, Preprint (2000).

[Ve1] J. Verdera, *On C^m rational approximation*, Proc. Amer. Math. Soc. **97** (1986), 621–625.

[Ve2] J. Verdera, *C^m approximation by solutions of elliptic equations, and Calderón-Zygmund operators*, Duke Math. J. **55** (1987), 157–187.

[Ve3] J. Verdera, *A weak type inequality for Cauchy transforms of finite measures*, Publ. Mat. UAB **36** (1992), 1029–1034.

[Ve4] J. Verdera, *Removability, capacity and approximation, in "Complex Potential Theory"*, NATO ASI Series, Kluwer Academic Publ., Dordrecht, 1994, pp. 419–473.

[Ve5] J. Verdera, *On the $T(1)$-Theorem for the Cauchy Integral*, Ark. Mat. **38** (2000), 183–199.

[Vi] A. G. Vitushkin, *The analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk **22**:6 (1967), 141–199 (Russian); English translation in Russian Math. Surveys **22** (1967), 139–200.

M.S. Melnikov.
Departament de Matematiques,
Universitat Autonoma de Barcelona,
08193 Bellaterra (Barcelona), Spain.
e-mail: melnikov@mat.uab.es

P. V. Paramonov.
Mechanics and Mathematics Faculty,
Moscow State University,
119899 Moscow, Russia.
e-mail: petr@paramonov.msk.ru

J. Verdera.
Departament de Matematiques,
Universitat Autonoma de Barcelona,
08193 Bellaterra (Barcelona), Spain.
e-mail: verdera@mat.uab.es