INVARIANT ALGEBRAIC SURFACES OF THE RIKITAKE SYSTEM

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Abstract

In this paper we use the method of characteristic curves for solving linear partial differential equations to study the invariant algebraic surfaces of the Rikitake system

$$\dot{x} = -\mu x + y(z+\beta), \quad \dot{y} = -\mu y + x(z-\beta), \quad \dot{z} = \alpha - xy.$$

Our main results are the following. First, we show that the cofactor of any invariant algebraic surface is of the form $rz + c$, where r is an integer. Second, we characterize all invariant algebraic surfaces. Moreover, as a corollary we characterize all values of the parameters for which the Rikitake system has a rational or algebraic first integral.

1. Introduction and statement of the main results

We consider the Rikitake systems

$$\begin{array}{rclcl} \dot{x} &=& -\mu x + y(z+\beta) &=& P(x,y,z), \\ \dot{y} &=& -\mu y + x(z-\beta) &=& Q(x,y,z), \\ \dot{z} &=& \alpha - xy &=& R(x,y,z), \end{array}$$

which is a simple model for describing the earth's magnetohydrodynamic dynamo (see for instance [2]), where x, y and z are real variables; α, β and μ are real parameters. These systems have been investigated as dynamical systems. For instance, Barge [1] gave conditions for which the system has two invariant surfaces. Hardy and Steeb [8] derived the conditions to find

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periodic orbits by using an ellipsoid bounding condition. Plunian, Marty and Alemany [12] studied its chaotic behaviour. Sachdev and Ramanan [14] discussed its singularity structure. From the integrability point of view, using the Painlevé method Steeb, Kunick and Strampp [13] studied their integrability. Hu and Yan [9] tested the complete integrability by finding regular mirror system near movable singularities. Labrunie and Conte [10] developed a geometrical method to find some invariant algebraic surfaces of these systems. Figueiredo, Rocha and Brenig [5] used an algebraic method to get similar results as those of [10].

Let $f(x,y,z)$ be a real polynomial in the variables x, y and z. The algebraic surface $f(x,y,z)=0$ of $\mathbf{R}^3$ is called an *invariant algebraic surface* of the Rikitake system if

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = kf, \tag{1}$$

for some real polynomial $k(x,y,z)$, which is called the *cofactor* of $f=0$. If $f(x,y,z)=0$ is an invariant algebraic surface, then f is also called a *Darboux polynomial*. From (1) it follows that if an orbit of the Rikitake system has a point on the invariant algebraic surface $f(x,y,z)=0$, then the whole orbit is contained in this surface.

We claim that the degree of the cofactor k is less than or equal to 1. The claim follows from the fact that in (1) $\deg(k)+\deg(f)=\max\{\deg(f)-1+\deg(P), \deg(f)-1+\deg(Q), \deg(f)-1+\deg(R)\} \leq \deg(f)+1$. Therefore, without loss of generality, we can assume that the cofactor is of the form

$$k(x,y,z) = px + qy + rz + c. \tag{2}$$

We say that a real function

$$\begin{aligned} H:\quad & \mathbf{R}^3 \times \mathbf{R} \longrightarrow \mathbf{R}, \\ & (x,y,z,t) \longmapsto H(x,y,z,t), \end{aligned}$$

is a *first integral* of the Rikitake system, if it is constant on all solution curves $(x(t),y(t),z(t))$ of the Rikitake system, that is, $H(x(t),y(t),z(t),t) \equiv$ constant for all values of t for which the solution $(x(t),y(t),z(t))$ is defined on $\mathbf{R}^3$. In particular, if the first integral H is independent on the time and it is a polynomial, then it is called a *polynomial first integral*. If the first integral H is a rational function independent on the time, then it is called a *rational first integral*.

We say that two first integrals independent on the time $H_1(x,y,z)$ and $H_2(x,y,z)$ are *independent*, if their gradients are linear independent vectors for all point $(x,y,z) \in \mathbf{R}^3$ except perhaps for a set of zero Lebesgue measure.

If a Rikitake system has two independent first integrals, then we say that it is *completely integrable*. We note that in this case the orbits of the Rikitake system are contained in the curves $\{H_1(x,y,z) = h_1\} \cap \{H_2(x,y,z) = h_2\}$, where h_1 and h_2 vary in $\mathbf{R}$.

An *algebraic function* $H(x,y,z) = C$ is a solution of the algebraic equation

$$f_0 + f_1 C + f_2 C^2 + \cdots + f_{n-1} C^{n-1} + C^n = 0,$$

where $f_i(x,y,z)$ are rational functions, and n is the smallest positive integer for which such a relation holds. Obviously, any rational function is algebraic. The Rikitake system is said to be *algebraically integrable* if it has two independent algebraic first integrals.

So far as we know, only one irreducible Darboux polynomial, i.e., $f = x^2 - y^2$ with the constant cofactor $k = -2\mu$ and the condition $\beta = 0$, has been found for the Rikitake systems (see for instance, [5] and [10]).

In this paper, by using the method of characteristic curves for solving linear partial differential equations, we obtain the following results. The first one gives the character of the cofactor of each invariant algebraic surface for the Rikitake system.

Proposition 1. *If $f(x,y,z)$ is a Darboux polynomial of the Rikitake system, then we can obtain that the cofactor is of the form $k = rz + c$ with r an integer number, and that the homogeneous component of the highest degree of f is of the form $(x+y)^r A(x^2+z^2, y^2+z^2)$ if r is nonnegative, or $(x-y)^{-r} A(x^2+z^2, y^2+z^2)$ if r is nonpositive, where A is a homogeneous polynomial in the variables x^2+z^2 and y^2+z^2.*

From Proposition 1 we obtain immediately the following corollary.

Corollary 2. *If $f(x,y,z)$ is a Darboux polynomial with a constant cofactor of the Rikitake system, then f has even degree.*

Our next result shows the relationship between invariant algebraic surfaces and first integrals of the Rikitake system.

Proposition 3. *A Rikitake system has a Darboux polynomial $f(x,y,z)$ with a constant cofactor k if and only if the function $H(x,y,z,t) = f(x,y,z)$ $\exp(-kt)$ is a first integral.*

In this paper the first integrals of the form given in Proposition 3 with $k \neq 0$ are called *invariants*.

The following proposition is known, for a proof see [4].

Proposition 4. *Assume that $f(x,y,z)$ is a polynomial function in the real polynomial ring $\mathbf{R}[x,y,z]$. Let $f = f_1^{n_1} \cdots f_m^{n_m}$ be the factorization of f in irreducible factors over $\mathbf{R}[x,y,z]$. Then for the Rikitake system, f is a Darboux polynomial with cofactor k_f if and only if each f_i is a Darboux polynomial with cofactor k_{f_i} for $i = 1, 2, \ldots, m$. Moreover, $k_f = n_1 k_{f_1} + \cdots + n_m k_{f_m}$.*

The next theorem is our main result, in it we characterize all Darboux polynomials of the Rikitake system.

Theorem 5. *The Rikitake system has invariant algebraic surfaces if and only if one of the following three cases holds.*

(a) *If $\mu = \alpha = 0$, then $H_1 = x^2 + z^2 + 2\beta z$ and $H_2 = y^2 + z^2 - 2\beta z$ are two polynomial first integrals. Consequently, in this case the Rikitake system is completely integrable.*

(b) *If $\mu = \beta = 0$ and $\alpha \neq 0$, then $H = x^2 - y^2$ is a polynomial first integral.*

(c) *If $\beta = 0$, then the Darboux polynomials are $f = x + y$ with the cofactor $k = z - \mu$ and $f = x - y$ with the cofactor $k = -z - \mu$.*

From Theorem 5 we get easily the following corollary.

Corollary 6. *(a) There are Rikitake systems having irreducible polynomial first integrals of any even degree.*

(b) The Rikitake systems have no polynomial first integrals of odd degree.

(c) The unique irreducible invariant for the Rikitake systems is $(x^2 - y^2)\exp(-2\mu t)$ when $\beta = 0$.

The following proposition characterizes the rational and algebraic first integrals of a polynomial vector field.

Proposition 7. *Let $\mathbf{X}$ be a polynomial vector field in $\mathbf{R}^n$. Then the following statements hold.*

(a) *If the polynomial functions f and g are relative prime, then f/g is a rational first integral of $\mathbf{X}$ if and only if f and g are both Darboux polynomials with the same cofactor.*

(b) *The vector field $\mathbf{X}$ is algebraically integrable if and only if it has n-1 independent rational first integrals.*

The first statement can be proved easily from the definitions. The second statement is a corollary of Lemma 2.4 of Goriely [6].

From Theorem 5 and Proposition 7 we can get the following result.

Corollary 8. *(a) The Rikitake system has a rational first integral if and only if either $\mu = \alpha = 0$, or $\mu = \beta = 0$ and $\alpha \neq 0$.*

(b) The Rikitake system is algebraically integrable if and only if $\mu = \alpha = 0$. Moreover, under this condition the Rikitake system has a solution given by the following implicit functions

$$x^2 + z^2 + 2\beta z = h_1, \qquad y^2 + z^2 - 2\beta z = h_2,$$

$$\pm \int \frac{dz}{\sqrt{h_1+\beta^2-(z+\beta)^2}\sqrt{h_2+\beta^2-(z-\beta)^2}} = t + h_3,$$

which is the elliptic integral of first kind (see [7]), where h_1, h_2 and h_3 are integrating constants.

This paper is organized as follows. In Section 2, we prove Propositions 1, 3 and 4. The proof of Theorem 5 is given in Section 3. Finally, in Section 4 we summarize the results of this paper.

2. Proof of Propositions 1, 3 and 4

Proof of Proposition 1. Assume that

$$f(x,y,z) = \sum_{i=0}^{n} f_i(x,y,z),$$

is a Darboux polynomial of the Rikitake system, where f_i is a homogeneous polynomial of degree i for $i = 0, 1, \ldots, n$. The cofactor is that given in (2).

Substituting f and (2) into equality (1) and identifying the homogeneous components of degree $n+1$, we obtain

$$yz\frac{\partial f_n}{\partial x} + xz\frac{\partial f_n}{\partial y} - xy\frac{\partial f_n}{\partial z} = (px + qy + rz)f_n. \qquad (3)$$

In what follows, in order to prove our proposition we will use the method of characteristic curves for solving linear partial differential equations (see for instance, Chapter 2 of [3]). The characteristic equation associated to (3) is

$$\frac{dx}{dz} = -\frac{z}{x}, \qquad \frac{dy}{dz} = -\frac{z}{y},$$

its general solution is

$$x^2 + z^2 = c_1, \qquad y^2 + z^2 = c_2,$$

where c_1 and c_2 are arbitrary constants.

We consider the change of variables

$$u = x^2 + z^2, \qquad v = y^2 + z^2, \qquad w = z. \qquad (4)$$

Correspondingly, the inverse transformation is

$$x = \pm\sqrt{u - w^2}, \qquad y = \pm\sqrt{v - w^2}, \qquad z = w. \qquad (5)$$

From equation (3) we get the ordinary differential equation

$$
\begin{aligned}
-(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})\frac{d\overline{f}_n}{dw} = \\
\left[p(\pm\sqrt{u-w^2})+q(\pm\sqrt{v-w^2})+rw\right]\overline{f}_n,
\end{aligned}
$$

where $\overline{f}_n(u,v,w)=f_n(x,y,z)$, and u and v are fixed. In the following, if we do not say anything, we will always denote by $\overline{R}(u,v,w)$ the function $R(x,y,z)$, written in the variables u, v and w by using (5).

Solving this equation we get that for $xy>0$

$$
\begin{aligned}
\overline{f}_n &= \overline{A}(u,v)\left|2\sqrt{(u-w^2)(v-w^2)}+2w^2-(u+v)\right|^{-r/2}\cdot \\
&\quad \exp\left(-p\left(\pm\arcsin\frac{w}{\sqrt{v}}\right)\right)\exp\left(-q\left(\pm\arcsin\frac{w}{\sqrt{u}}\right)\right),
\end{aligned}
$$

for $xy<0$

$$
\begin{aligned}
\overline{f}_n &= \overline{A}(u,v)\left|2\sqrt{(u-w^2)(v-w^2)}+2w^2-(u+v)\right|^{r/2}\cdot \\
&\quad \exp\left(-p\left(\pm\arcsin\frac{w}{\sqrt{v}}\right)\right)\exp\left(-q\left(\pm\arcsin\frac{w}{\sqrt{u}}\right)\right),
\end{aligned}
$$

where $\overline{A}(u,v)$ is an arbitrary function in u and v. Correspondingly, for $xy>0$ we have

$$
\begin{aligned}
f_n &= A(x^2+z^2,y^2+z^2)(x-y)^{-r}\cdot \\
&\quad \exp\left(-p\left(\pm\arcsin\frac{z}{\sqrt{y^2+z^2}}\right)\right)\exp\left(-q\left(\pm\arcsin\frac{z}{\sqrt{x^2+z^2}}\right)\right),
\end{aligned}
$$

and for $xy<0$ we have

$$
\begin{aligned}
f_n &= A(x^2+z^2,y^2+z^2)(x+y)^{r}\cdot \\
&\quad \exp\left(-p\left(\pm\arcsin\frac{z}{\sqrt{y^2+z^2}}\right)\right)\exp\left(-q\left(\pm\arcsin\frac{z}{\sqrt{x^2+z^2}}\right)\right).
\end{aligned}
$$

In order that f_n is a homogeneous polynomial, we must have $p=q=0$, the function A a homogeneous polynomial in x^2+z^2 and y^2+z^2, and r a convenient integer. More precisely, if r is a nonnegative (respectively nonpositive) integer, then $f_n=(x+y)^rA(x^2+z^2,y^2+z^2)$ (respectively $f_n=(x-y)^{-r}A(x^2+z^2,y^2+z^2)$. This completes the proof of the proposition. ∎

Proof of Proposition 3. The proof of this proposition is easy, and follows in the same way that the proof of Proposition 2 of [11]. Since the proof is short we give it.

Assume that $f(x,y,z)$ is a Darboux polynomial of the Rikitake system with the constant cofactor k. Then from the definition of the Darboux polynomial

$$\frac{df}{dt} = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R \equiv kf.$$

Therefore, we have

$$\frac{dH}{dt} = \exp(-kt)\frac{df}{dt} - kf\exp(-kt) \equiv 0,$$

that is, $H(x,y,z,t)$ is a first integral. Consequently, the proof follows from the above equation. This proves the proposition. ∎

Proof of Proposition 4. Sufficiency. Since f_i, for $i=1,\dots,m$, is a Darboux polynomial with cofactor k_{f_i}, we have

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial(f_1^{n_1}\dots f_m^{n_m})}{\partial x}P + \frac{\partial(f_1^{n_1}\dots f_m^{n_m})}{\partial y}Q + \frac{\partial(f_1^{n_1}\dots f_m^{n_m})}{\partial z}R \\
&= \sum_{i=1}^{m} n_i f_i^{n_i-1}\frac{\partial f_i}{\partial x}\prod_{\substack{1\le j\le n\\ j\neq i}} f_j^{n_j}P + \sum_{i=1}^{m} n_i f_i^{n_i-1}\frac{\partial f_i}{\partial y}\prod_{\substack{1\le j\le n\\ j\neq i}} f_j^{n_j}Q \\
&\quad + \sum_{i=1}^{m} n_i f_i^{n_i-1}\frac{\partial f_i}{\partial z}\prod_{\substack{1\le j\le n\\ j\neq i}} f_j^{n_j}R \\
&= \sum_{i=1}^{m} n_i f_i^{n_i-1}\left(\frac{\partial f_i}{\partial x}P + \frac{\partial f_i}{\partial y}Q + \frac{\partial f_i}{\partial z}R\right)\prod_{\substack{1\le j\le n\\ j\neq i}} f_j^{n_j} \\
&= \sum_{i=1}^{m} n_i k_{f_i} f_i^{n_i}\prod_{\substack{1\le j\le n\\ j\neq i}} f_j^{n_j} = \sum_{i=1}^{m} n_i k_{f_i}\prod_{1\le j\le n} f_j^{n_j} = \sum_{i=1}^{m} n_i k_{f_i} f.
\end{aligned}$$

This proves that $f = f_1^{m_1}\dots f_m^{n_m}$ is a Darboux polynomial with the cofactor $k_f = n_1 k_{f_1} + \dots + n_m k_{f_m}$.

Necessity. Assume that f is a Darboux polynomial with the cofactor k_f, and $f = f_1^{n_1}\cdots f_m^{n_m}$ is the factorization of f in irreducible factors over

$\mathbf{R}[x,y,z]$. Then from this last equality we get

$$\begin{aligned}\frac{df}{dt} &= \sum_{i=1}^{m} n_i f_i^{n_i-1}\left(\frac{\partial f_i}{\partial x}P + \frac{\partial f_i}{\partial y}Q + \frac{\partial f_i}{\partial z}R\right)\prod_{\substack{1\le j\le n\\ j\neq i}} f_j^{n_j}\\ &= k_f f = k_f \prod_{1\le j\le n} f_j^{n_j}.\end{aligned}$$

Since f_i and f_j are coprime for $1 \le i,j \le m$ and $i \neq j$, we have for every given l $(1 \le l \le m)$, that f_l divides $\frac{\partial f_l}{\partial x}P + \frac{\partial f_l}{\partial y}Q + \frac{\partial f_l}{\partial z}R$ in $\mathbf{R}[x,y,z]$. Let

$$k_{f_l} = \frac{1}{f_l}\left(\frac{\partial f_l}{\partial x}P + \frac{\partial f_l}{\partial y}Q + \frac{\partial f_l}{\partial z}R\right).$$

This means that f_l is a Darboux polynomial with the cofactor k_{f_l}. Moreover, we have $k_f = \sum_{i=1}^{m} n_i k_{f_i}$. This completes the proof of the proposition. ∎

3. The proof of Theorem 5

According to Proposition 1 we first consider the case in which the cofactor is a constant. Assume that

$$f(x,y,z) = \sum_{i=0}^{n} f_i(x,y,z),$$

is a Darboux polynomial of the Rikitake system with the constant cofactor $k(x,y,z) = c$, where f_i is a homogeneous polynomial of degree i for $i = 0,1,\ldots,n$. From Corollary 2 we can assume that $n = 2m$, where m is a positive integer.

Substituting f and $k = c$ into equation (1) and identifying the terms of the same degree, we obtain

$$yz\frac{\partial f_{2m}}{\partial x} + xz\frac{\partial f_{2m}}{\partial y} - xy\frac{\partial f_{2m}}{\partial z} = 0, \tag{6}$$

$$\begin{aligned}&yz\frac{\partial f_{2m-1}}{\partial x} + xz\frac{\partial f_{2m-1}}{\partial y} - xy\frac{\partial f_{2m-1}}{\partial z} =\\ &\qquad (\mu x - \beta y)\frac{\partial f_{2m}}{\partial x} + (\mu y + \beta x)\frac{\partial f_{2m}}{\partial y} + c f_{2m},\end{aligned} \tag{7}$$

$$\begin{aligned}&yz\frac{\partial f_i}{\partial x} + xz\frac{\partial f_i}{\partial y} - xy\frac{\partial f_i}{\partial z} =\\ &\qquad (\mu x - \beta y)\frac{\partial f_{i+1}}{\partial x} + (\mu y + \beta x)\frac{\partial f_{i+1}}{\partial y} + c f_{i+1} - \alpha\frac{\partial f_{i+2}}{\partial z},\end{aligned} \tag{8}$$

$$cf_0 - \alpha \frac{\partial f_1}{\partial z} = 0. \tag{9}$$

for $i = 2m-2, 2m-3, \ldots, 1, 0$.

From Proposition 1 and its proof we get that the solution of (6) is

$$f_{2m} = \sum_{i=0}^{m} a_i^m (x^2+z^2)^{m-i}(y^2+z^2)^i$$

where a_i^m is a real constant for $i = 0, 1, \ldots, m$.

Introducing f_{2m} into equation (7) and doing some calculations, we have

$$\begin{aligned} yz\frac{\partial f_{2m-1}}{\partial x} + xz\frac{\partial f_{2m-1}}{\partial y} - xy\frac{\partial f_{2m-1}}{\partial z} = \\ \sum_{i=0}^{m}(2m\mu + c)a_i^m(x^2+z^2)^{m-i}(y^2+z^2)^i \\ - \sum_{i=0}^{m-1} 2\mu[(m-i)a_i^m + (i+1)a_{i+1}^m](x^2+z^2)^{m-i-1}(y^2+z^2)^i z^2 \\ - \sum_{i=0}^{m-1} 2\beta[(m-i)a_i^m - (i+1)a_{i+1}^m](x^2+z^2)^{m-i-1}(y^2+z^2)^i xy \end{aligned}$$

Using the transformations (4) and (5), from this last equation we get the following ordinary differential equation

$$\begin{aligned} \frac{d\overline{f}_{2m-1}}{dw} = -\sum_{i=0}^{m}(2m\mu + c)a_i^m u^{m-i} v^i \frac{1}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})} \\ + \sum_{i=0}^{m-1} 2\mu[(m-i)a_i^m + (i+1)a_{i+1}^m]u^{m-i-1}v^i w^2 \frac{1}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})} \\ + \sum_{i=0}^{m-1} 2\beta[(m-i)a_i^m - (i+1)a_{i+1}^m]u^{m-i-1}v^i. \end{aligned}$$

Solving this equation we obtain

$$\begin{aligned} \overline{f}_{2m-1} = -\sum_{i=0}^{m}(2m\mu + c)a_i^m u^{m-i} v^i \int \frac{dw}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})} \\ + \sum_{i=0}^{m-1} 2\mu[(m-i)a_i^m + (i+1)a_{i+1}^m]u^{m-i-1}v^i \int \frac{w^2 dw}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})} \end{aligned}$$

$$+\sum_{i=0}^{m-1} 2\beta[(m-i)a_i^m-(i+1)a_{i+1}^m]u^{m-i-1}v^i w+\overline{f}^*_{2m-1}(u,v),$$

where $\overline{f}^*_{2m-1}$ is an arbitrary function in u and v.

An easy computation gives

$$\int \frac{w^2dw}{\sqrt{u-w^2}\sqrt{v-w^2}}=-\int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}}dw+u\int \frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}}.$$

Since

$$\int \frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}} \qquad \text{and} \qquad \int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}}dw,$$

are elliptic integrals of the first and second kind respectively(see for instance [7]), in order that f_{2m-1} is a homogeneous polynomial of degree $2m-1$, we must have $\overline{f}^*_{2m-1}(x^2+z^2,y^2+z^2)\equiv 0$ and

$$\begin{aligned}(2m\mu+c)a_i^m &= 0, \qquad i=0,1,\ldots,m,\\ \mu[(m-i)a_i^m+(i+1)a_{i+1}^m] &= 0, \qquad i=0,1,\ldots,m-1.\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}f_{2m-1} &= \sum_{i=0}^{m-1}[(m-i)a_i^m-(i+1)a_{i+1}^m](x^2+z^2)^{m-i-1}(y^2+z^2)^i(2\beta z)\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{1}(-1)^j\begin{pmatrix} m-i-j\\ 1-j\end{pmatrix}\begin{pmatrix} i+1\\ j\end{pmatrix}\cdot \qquad (11)\\ &\qquad a_{i+j}^m(x^2+z^2)^{m-1-i}(y^2+z^2)^i(2\beta z).\end{aligned}$$

From the first equation of (10) we have $c=-2m\mu$. Otherwise, $a_i^m=0$ for $i=0,1,\ldots,m$, and then $f_{2m}\equiv 0$. By the second equation of (10) we get

$$\begin{aligned}&\mu=0 \qquad \text{or}\\ &\mu\neq 0 \text{ and } (m-i)a_i^m+(i+1)a_{i+1}^m=0 \text{ for } i=0,1,\ldots,m-1.\end{aligned} \tag{12}$$

Case 1. $\mu=0$. Then $c=0$. Introducing f_{2m} and f_{2m-1} into equation (8) with $i=2m-2$ and doing some calculations, we obtain

$$yz\frac{\partial f_{2m-2}}{\partial x}+xz\frac{\partial f_{2m-2}}{\partial y}-xy\frac{\partial f_{2m-2}}{\partial z}=$$

$$
\begin{aligned}
- & \sum_{i=0}^{m-2} 4\beta^2 \left[(m-i)(m-i-1)a_i^m - 2(m-i-1)(i+1)a_{i+1}^m \right. \\
& \qquad \left. +(i+2)(i+1)a_{i+2}^m\right] (x^2+z^2)^{m-i-2}(y^2+z^2)^i xyz \\
- & \sum_{i=0}^{m-1} 2\alpha \left[(m-i)a_i^m + (i+1)a_{i+1}^m\right] (x^2+z^2)^{m-i-1}(y^2+z^2)^i z.
\end{aligned}
$$

From this last equation we get the following ordinary differential equation taking into account the changes (4) and (5),

$$
\begin{aligned}
\frac{\overline{f}_{2m-2}}{dw} = & \sum_{i=0}^{m-2} 4\beta^2 \left[(m-i)(m-i-1)a_i^m - \right. \\
& \left. 2(m-i-1)(i+1)a_{i+1}^m + (i+2)(i+1)a_{i+2}^m\right] u^{m-i-2}v^i w + \\
& \sum_{i=0}^{m-1} 2\alpha \left[(m-i)a_i^m + (i+1)a_{i+1}^m\right] \cdot \\
& \quad u^{m-i-1}v^i \frac{w}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})}.
\end{aligned}
$$

Since

$$
\int \frac{2w dw}{\sqrt{u-w^2}\sqrt{v-w^2}} = \log\left|2\sqrt{u-w^2}\sqrt{v-w^2} + 2w^2 - (u+v)\right|, \quad (13)
$$

in order that $f_{2m-2}(x,y,z) = \overline{f}_{2m-2}(u,v,w)$ is a homogeneous polynomial in x, y and z, we must have

$$
\alpha\left[(m-i)a_i^m + (i+1)a_{i+1}^m\right] = 0, \quad \text{for } i = 0,1,\ldots,m-1. \quad (14)
$$

Therefore,

$$
\begin{aligned}
f_{2m-2} \quad = \quad & \sum_{i=0}^{m-2}\sum_{j=0}^{2}(-1)^j \begin{pmatrix} m-i-j \\ 2-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot \\
& \quad a_{i+j}^m (x^2+z^2)^{m-i-2}(y^2+z^2)^i (2\beta z)^2 \\
& + \sum_{i=0}^{m-1} a_i^{m-1}(x^2+z^2)^{m-1-i}(y^2+z^2)^i
\end{aligned}
$$

where a_i^{m-1} is a real constant for $i = 0,1,\ldots,m-1$. The second line of the expression of f_{2m-2} is an arbitrary polynomial in the variables u and v which appears after the integration of $d\overline{f}_{2m-2}/dw$.

Subcase 1. $\alpha = 0$. Introducing f_{2m-2} into equation (8) with $i = 2m-3$, we get

$$yz\frac{\partial f_{2m-3}}{\partial x} + xz\frac{\partial f_{2m-3}}{\partial y} - xy\frac{\partial f_{2m-3}}{\partial z} = -\beta y\frac{\partial f_{2m-2}}{\partial x} + \beta x\frac{\partial f_{2m-2}}{\partial y}$$

$$= -\sum_{i=0}^{m-3} 2\beta \left[(m-2-i)\sum_{j=0}^{2}(-1)^j \begin{pmatrix} m-i-j \\ 2-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} a_{i+j}^m - \right.$$

$$\left. (i+1)\sum_{j=0}^{2}(-1)^j \begin{pmatrix} m-i-1-j \\ 2-j \end{pmatrix} \begin{pmatrix} i+1+j \\ j \end{pmatrix} a_{i+1+j}^m \right] \cdot$$

$$(x^2+z^2)^{m-3-i}(y^2+z^2)^i xy(2\beta z)^2 -$$

$$\sum_{i=0}^{m-2} 2\beta \left[(m-1-i)a_i^{m-1} - (i+1)a_{i+1}^{m-1} \right] (x^2+z^2)^{m-i-2}(y^2+z^2)^i xy$$

$$= -\sum_{i=0}^{m-3} 6\beta \sum_{j=0}^{3}(-1)^j \begin{pmatrix} m-i-j \\ 3-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^m (x^2+z^2)^{m-3-i}(y^2+z^2)^i xy(2\beta z)^2 -$$

$$\sum_{i=0}^{m-2} 2\beta \sum_{j=0}^{1}(-1)^j \begin{pmatrix} m-1-i-j \\ 1-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-1}(x^2+z^2)^{m-2-i}(y^2+z^2)^i xy$$

In the above computations we used the following

Lemma 6. *For any nonnegative integers m, s and i satisfying $m > s+i$, the following equality hold.*

$$(m-s-i)\sum_{j=0}^{s}(-1)^j \begin{pmatrix} m-i-j \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} a_{i+j} -$$

$$(i+1)\sum_{j=0}^{s}(-1)^j \begin{pmatrix} m-i-1-j \\ s-j \end{pmatrix} \begin{pmatrix} i+1+j \\ j \end{pmatrix} a_{i+1+j}$$

$$= (s+1)\sum_{j=0}^{s+1}(-1)^j \begin{pmatrix} m-i-j \\ s+1-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} a_{i+j}$$

Proof: By straightforward computations we have

$$
\begin{aligned}
&(m-s-i)\sum_{j=0}^{s}(-1)^j \binom{m-i-j}{s-j}\binom{i+j}{j} a_{i+j} - \\
&\qquad (i+1)\sum_{j=0}^{s}(-1)^j \binom{m-i-1-j}{s-j}\binom{i+1+j}{j} a_{i+1+j} \\
&= (m-s-i)\binom{m-i}{s} a_i + \\
&\qquad (m-s-i)\sum_{j=1}^{s}(-1)^j \binom{m-i-j}{s-j}\binom{i+j}{j} a_{i+j} + \\
&\qquad (i+1)\sum_{j=1}^{s}(-1)^j \binom{m-i-j}{s+1-j}\binom{i+j}{j-1} a_{i+j} + \\
&\qquad (i+1)(-1)^{s+1}\binom{i+1+s}{s} a_{i+1+s} \\
&= (s+1)\binom{m-i}{s+1} a_i + (-1)^{s+1}(s+1)\binom{i+s+1}{s+1} a_{i+s+1} + \\
&\qquad \sum_{j=1}^{s}(-1)^j \Bigg[(m-s-i)\binom{m-i-j}{s-j}\binom{i+j}{j} + \\
&\qquad (i+1)\binom{m-i-j}{s+1-j}\binom{i+j}{j-1}\Bigg] a_{i+j} \\
&= (s+1)\binom{m-i}{s+1} a_i + (-1)^{s+1}(s+1)\binom{i+s+1}{s+1} a_{i+s+1} + \\
&\qquad \sum_{j=1}^{s}(-1)^j \Bigg[(s+1-j)\binom{m-i-j}{s+1-j}\binom{i+j}{j} + \\
&\qquad \binom{m-i-j}{s+1-j}\binom{i+j}{j} j\Bigg] a_{i+j} \\
&= (s+1)\sum_{j=0}^{s+1}(-1)^j \binom{m-i-j}{s+1-j}\binom{i+j}{j} a_{i+j}.
\end{aligned}
$$

This proves the lemma. ∎

Using the transformations (4) and (5) and working in a similar way to solving f_{2m-1}, we obtain

$$f_{2m-3} = \sum_{i=0}^{m-3} \sum_{j=0}^{3} (-1)^j \left(\begin{array}{c} m-i-j \\ 3-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$

$$a_{i+j}^{m} (x^2+z^2)^{m-3-i} (y^2+z^2)^i (2\beta z)^3 +$$

$$\sum_{i=0}^{m-2} \sum_{j=0}^{1} (-1)^j \left(\begin{array}{c} m-1-i-j \\ 1-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$

$$a_{i+j}^{m-1} (x^2+z^2)^{m-2-i} (y^2+z^2)^i (2\beta z).$$

Substituting f_{2m-3} into equation (8) with $i = 2m-4$ and doing some calculations which are similar to the proof of f_{2m-2}, we have

$$yz \frac{\partial f_{2m-4}}{\partial x} + xz \frac{\partial f_{2m-4}}{\partial y} - xy \frac{\partial f_{2m-4}}{\partial z}$$

$$= -\sum_{i=0}^{m-4} 8\beta \sum_{j=0}^{4} (-1)^j \left(\begin{array}{c} m-i-j \\ 4-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$

$$a_{i+j}^{m} (x^2+z^2)^{m-4-i} (y^2+z^2)^i xy (2\beta z)^3 -$$

$$\sum_{i=0}^{m-3} 4\beta \sum_{j=0}^{2} (-1)^j \left(\begin{array}{c} m-1-i-j \\ 2-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$

$$a_{i+j}^{m-1} (x^2+z^2)^{m-3-i} (y^2+z^2)^i xy (2\beta z).$$

Working in a similar way to solving f_{2m-2} we obtain that

$$f_{2m-4} = \sum_{i=0}^{m-4} \sum_{j=0}^{4} (-1)^j \left(\begin{array}{c} m-i-j \\ 4-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$

$$a_{i+j}^{m} (x^2+z^2)^{m-4-i} (y^2+z^2)^i (2\beta z)^4 +$$

$$\sum_{i=0}^{m-3} \sum_{j=0}^{2} (-1)^j \left(\begin{array}{c} m-1-i-j \\ 2-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$

$$a_{i+j}^{m-1} (x^2+z^2)^{m-3-i} (y^2+z^2)^i (2\beta z)^2 +$$

$$\sum_{i=0}^{m-2} a_i^{m-2} (x^2+z^2)^{m-2-i} (y^2+z^2)^i.$$

Introducing f_{2m-4} into equation (8) with $i = 2m-5$ and in a similar way to the proof of f_{2m-3} we have

$$f_{2m-5} = \sum_{i=0}^{m-5}\sum_{j=0}^{5}(-1)^j \left(\begin{array}{c} m-i-j \\ 5-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m}(x^2+z^2)^{m-5-i}(y^2+z^2)^i(2\beta z)^5 +$$
$$\sum_{i=0}^{m-4}\sum_{j=0}^{3}(-1)^j \left(\begin{array}{c} m-1-i-j \\ 3-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m-1}(x^2+z^2)^{m-4-i}(y^2+z^2)^i(2\beta z)^3 +$$
$$\sum_{i=0}^{m-2}\sum_{j=0}^{1}(-1)^j \left(\begin{array}{c} m-2-i-j \\ 1-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m-2}(x^2+z^2)^{m-3-i}(y^2+z^2)^i(2\beta z).$$

By recursive computations we can get for $s = 3, 4, \dots, m-1$

$$f_{2m-2s} = \sum_{i=0}^{m-2s}\sum_{j=0}^{2s}(-1)^j \left(\begin{array}{c} m-i-j \\ 2s-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m}(x^2+z^2)^{m-2s-i}(y^2+z^2)^i(2\beta z)^{2s} +$$
$$\sum_{i=0}^{m-2s+1}\sum_{j=0}^{2s-2}(-1)^j \left(\begin{array}{c} m-1-i-j \\ 2s-2-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m-1}(x^2+z^2)^{m-2s+1-i}(y^2+z^2)^i(2\beta z)^{2s-2} +$$
$$\sum_{i=0}^{m-2s+2}\sum_{j=0}^{2s-4}(-1)^j \left(\begin{array}{c} m-2-i-j \\ 2s-4-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m-2}(x^2+z^2)^{m-2s+2-i}(y^2+z^2)^i(2\beta z)^{2s-4} +$$
$$\cdots\cdots\cdots\cdots +$$
$$\sum_{i=0}^{m-s-1}\sum_{j=0}^{2}(-1)^j \left(\begin{array}{c} m-s-1-i-j \\ 2-j \end{array} \right) \left(\begin{array}{c} i+j \\ j \end{array} \right) \cdot$$
$$a_{i+j}^{m-s-1}(x^2+z^2)^{m-s-3-i}(y^2+z^2)^i(2\beta z)^2 +$$
$$\sum_{i=0}^{m-s} a_i^{m-s}(x^2+z^2)^{m-s-i}(y^2+z^2)^i$$

$$= \sum_{h=0}^{s} \sum_{i=0}^{m-2s+h} \sum_{j=0}^{2(s-h)} (-1)^j \begin{pmatrix} m-h-i-j \\ 2(s-h)-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-2s+h-i} (y^2+z^2)^i (2\beta z)^{2(s-h)},$$

and

$$f_{2m-2s-1} = \sum_{h=0}^{s} \sum_{i=0}^{m-2s+h-1} \sum_{j=0}^{2(s-h)+1} (-1)^j \begin{pmatrix} m-h-i-j \\ 2(s-h)+1-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-2s+h-1-i} (y^2+z^2)^i (2\beta z)^{2(s-h)+1}.$$

We note that in the above two sums, if $l < 0$, then the sum $\sum_{i=o}^{l} A_i = 0$ for any A_i. Unifying the expressions of f_{2m-2s} and $f_{2m-2s-1}$ we get that for $s = 0, 1, \ldots, 2m-1$

$$f_{2m-s} = \sum_{h=0}^{[s/2]} \sum_{i=0}^{m-s+h} \sum_{j=0}^{s-2h} (-1)^j \begin{pmatrix} m-h-i-j \\ s-2h-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-s+h-i} (y^2+z^2)^i (2\beta z)^{s-2h}.$$

Here, $[\cdot]$ denotes the *integer part function*. Therefore, we have

$$\begin{aligned} f &= f_{2m} + f_{2m-1} + \cdots + f_2 + f_1 \\ &= \sum_{s=0}^{2m-1} \sum_{h=0}^{[s/2]} \sum_{i=0}^{m-s+h} \sum_{j=0}^{s-2h} (-1)^j \begin{pmatrix} m-h-i-j \\ s-2h-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot \end{aligned}$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-s+h-i} (y^2+z^2)^i (2\beta z)^{s-2h}.$$

For every given $h \in \{0, 1, \ldots, m-1\}$, we know from the calculations of f_s for $s = 1, 2, \ldots, 2m$ that a_i^{m-h} for $i = 0, 1, \ldots, m-h$ appear in f_j with $1 \le j \le 2m-2h$. Hence, in the above expression the sum of the terms containing a_i^{m-h} for $i = 0, 1, \ldots, m-h$ is

$$\sum_{s=2h}^{2m-1} \sum_{i=0}^{m-s+h} \sum_{j=0}^{s-2h} (-1)^j \begin{pmatrix} m-h-i-j \\ s-2h-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-s+h-i} (y^2+z^2)^i (2\beta z)^{s-2h}$$

$$= \sum_{s=0}^{2m-1-2h} \sum_{i=0}^{m-h-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} m-h-i-j \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-s-h-i} (y^2+z^2)^i (2\beta z)^s$$

$$= \sum_{s=0}^{m-h} \sum_{i=0}^{m-h-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} m-h-i-j \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-h} (x^2+z^2)^{m-s-h-i} (y^2+z^2)^i (2\beta z)^s.$$

Therefore, adding the previous expressions for $h = 0, 1, \ldots, m-1$ we obtain

$$f = \sum_{s=0}^{m} \sum_{i=0}^{m-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} m-(i+j) \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m} (x^2+z^2)^{m-s-i} (y^2+z^2)^i (2\beta z)^s +$$

$$\sum_{s=0}^{m-1} \sum_{i=0}^{m-1-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} m-1-(i+j) \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-1} (x^2+z^2)^{m-1-s-i} (y^2+z^2)^i (2\beta z)^s +$$

$$\sum_{s=0}^{m-2} \sum_{i=0}^{m-2-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} m-2-(i+j) \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{m-2} (x^2+z^2)^{m-2-s-i} (y^2+z^2)^i (2\beta z)^s +$$

$$\cdots\cdots\cdots\cdots +$$

$$\sum_{s=0}^{1} \sum_{i=0}^{1-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} 1-(i+j) \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \cdot$$

$$a_{i+j}^{1} (x^2+z^2)^{1-s-i} (y^2+z^2)^i (2\beta z)^s.$$

Since in the sum

$$\sum_{i=0}^{m-s} \sum_{j=0}^{s} (-1)^j \begin{pmatrix} m-(i+j) \\ s-j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} a_{i+j}^{m} (x^2+z^2)^{m-s-i} (y^2+z^2)^i (2\beta z)^s,$$

the term containing a_h^m for $h \in \{0, 1, \ldots, m\}$ is

$$\sum_{j=0}^{h} (-1)^j \begin{pmatrix} m-h \\ s-j \end{pmatrix} \begin{pmatrix} h \\ j \end{pmatrix} a_h^m (x^2+z^2)^{m-s-(h-j)} (y^2+z^2)^{h-j} (2\beta z)^s,$$

where if $s-j < 0$ or $s-j > m-h$, then $\begin{pmatrix} m-h \\ s-j \end{pmatrix} = 0$. So in the polynomial f the sum of all terms containing a_h^m is

$$\sum_{i=0}^{m-h}\sum_{j=0}^{h}(-1)^j \left(\begin{array}{c} m-h \\ i \end{array}\right)\left(\begin{array}{c} h \\ j \end{array}\right) a_h^m (x^2+z^2)^{m-h-i}(y^2+z^2)^{h-j}(2\beta z)^{i+j}.$$

Therefore,

$$\begin{aligned}
&\sum_{s=0}^{m}\sum_{i=0}^{m-s}\sum_{j=0}^{s}(-1)^j \left(\begin{array}{c} m-(i+j) \\ s-j \end{array}\right)\left(\begin{array}{c} i+j \\ j \end{array}\right)\cdot \\
&\qquad\qquad a_{i+j}^m (x^2+z^2)^{m-s-i}(y^2+z^2)^{i}(2\beta z)^{s} \\
&= \sum_{h=0}^{m} a_h^m \sum_{i=0}^{m-h}\sum_{j=0}^{h}(-1)^j \left(\begin{array}{c} m-h \\ i \end{array}\right)\left(\begin{array}{c} h \\ j \end{array}\right)\cdot \\
&\qquad\qquad (x^2+z^2)^{m-h-i}(y^2+z^2)^{h-j}(2\beta z)^{i+j} \\
&= \sum_{h=0}^{m} a_h^m \sum_{i=0}^{m-h}\left(\begin{array}{c} m-h \\ i \end{array}\right)(x^2+z^2)^{m-h-i}(2\beta z)^{i}\cdot \\
&\qquad\qquad \sum_{j=0}^{h}(-1)^j \left(\begin{array}{c} h \\ j \end{array}\right)(y^2+z^2)^{h-j}(2\beta z)^{j} \\
&= \sum_{h=0}^{m} a_h^m (x^2+z^2+2\beta z)^{m-h}(y^2+z^2-2\beta z)^{h}.
\end{aligned}$$

Working in a similar way to the above calculations we obtain

$$\begin{aligned}
f =& \sum_{i=0}^{m} a_i^m (x^2+z^2+2\beta z)^{m-i}(y^2+z^2-2\beta z)^{i} + \\
& \sum_{i=0}^{m-1} a_i^{m-1} (x^2+z^2+2\beta z)^{m-1-i}(y^2+z^2-2\beta z)^{i} + \\
& \sum_{i=0}^{m-2} a_i^{m-2} (x^2+z^2+2\beta z)^{m-2-i}(y^2+z^2-2\beta z)^{i} + \\
& \cdots\cdots\cdots\cdots + \\
& \sum_{i=0}^{2} a_i^{2} (x^2+z^2+2\beta z)^{2-i}(y^2+z^2-2\beta z)^{i} + \\
& a_0^1 (x^2+z^2+2\beta z) + a_1^1 (y^2+z^2-2\beta z) \\
=& \sum_{h=1}^{m}\sum_{i=0}^{h} a_i^h (x^2+z^2+2\beta z)^{h-i}(y^2+z^2-2\beta z)^{i}.
\end{aligned}$$

By the arbitrariness of m and a_i^h, we get the two polynomial first integrals $H_1 = x^2+z^2+2\beta z$ and $H_2 = y^2+z^2-2\beta z$. This proves statement (a) of the theorem.

Subcase 2. $\alpha \neq 0$ and $(m-i)a_i^m + (i+1)a_{i+1}^m = 0$ for $i = 0, 1, \ldots, m-1$. So we have

$$a_i^m = (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} a_0^m, \qquad i = 1, 2, \ldots, m. \tag{15}$$

Hence

$$f_{2m} = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} a_0^m (x^2+z^2)^{m-i}(y^2+z^2)^i = a_0^m (x^2-y^2)^m.$$

Moreover, from (11) and (15) we have

$$\begin{aligned} f_{2m-1} &= \sum_{i=0}^{m-1} 2(m-i)a_i^m (x^2+z^2)^{m-1-i}(y^2+z^2)^i (2\beta z) \\ &= 2\sum_{i=0}^{m-1} (m-i)(-1)^i \begin{pmatrix} m \\ i \end{pmatrix} a_0^m (x^2+z^2)^{m-1-i}(y^2+z^2)^i (2\beta z) \\ &= 2\sum_{i=0}^{m-1} (-1)^i m \begin{pmatrix} m-1 \\ i \end{pmatrix} a_0^m (x^2+z^2)^{m-1-i}(y^2+z^2)^i (2\beta z) \\ &= 4\beta m a_0^m (x^2-y^2)^{m-1} z. \end{aligned}$$

Substituting f_{2m-1} and f_{2m} into equation (8) with $i = 2m-2$, we get

$$\begin{aligned} & yz\frac{\partial f_{2m-2}}{\partial x} + xz\frac{\partial f_{2m-2}}{\partial y} - xy\frac{\partial f_{2m-2}}{\partial z} \\ & \qquad = -\beta y\frac{\partial f_{2m-1}}{\partial x} + \beta x\frac{\partial f_{2m-1}}{\partial y} - \alpha\frac{\partial f_{2m}}{\partial z} \\ & \qquad = -16 a_0^m m(m-1)\beta^2 (x^2-y^2)^{m-2} xyz. \end{aligned}$$

Using the transformation (4) and (5) and working in a similar way to the proof in Subcase 1, we obtain

$$\begin{aligned} f_{2m-2} = 16 a_0^m \frac{m(m-1)}{2!}\beta^2 (x^2-y^2)^{m-2} z^2 \\ + \sum_{i=0}^{m-1} a_i^{m-1} (x^2+z^2)^{m-1-i}(y^2+z^2)^i. \end{aligned}$$

Substituting f_{2m-2} and f_{2m-1} into equation (8) with $i = 2m-3$, we get

$$\begin{aligned} yz\frac{\partial f_{2m-3}}{\partial x} &+ xz\frac{\partial f_{2m-3}}{\partial y} - xy\frac{\partial f_{2m-3}}{\partial z} \\ = &- \beta y\frac{\partial f_{2m-2}}{\partial x} + \beta x\frac{\partial f_{2m-2}}{\partial y} - \alpha\frac{\partial f_{2m-1}}{\partial z} \end{aligned}$$

$$
\begin{aligned}
= \quad & - \quad 64a_0^m\beta^3(m-2)\begin{pmatrix} m \\ 2 \end{pmatrix}(x^2-y^2)^{m-3}z^2xy \\
& - \quad \sum_{i=0}^{m-2} 2\beta\left[(m-1-i)a_i^{m-1}-(i+1)a_{i+1}^{m-1}\right]\cdot \\
& \qquad (x^2+z^2)^{m-2-i}(y^2+z^2)^i xy \\
& - \quad 4ma_0^m\alpha\beta(x^2-y^2)^{m-1}.
\end{aligned}
$$

In a similar way to the computations in Subcase 1, we obtain

$$
\begin{aligned}
&\overline{f}_{2m-3} = 64a_0^m\beta^3\begin{pmatrix} m \\ 3 \end{pmatrix}(u-v)^{m-3}w^3 + \\
&\sum_{i=0}^{m-2} 2\beta\left[(m-1-i)a_i^{m-1}-(i+1)a_{i+1}^{m-1}\right]u^{m-2-i}v^iw + \\
&4ma_0^m\alpha\beta(u-v)^{m-1}\int\frac{dw}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})} + \overline{f}^*_{2m-3}(u,v)\,.
\end{aligned}
$$

In order that $f_{2m-3}(x,y,z) = \overline{f}_{2m-3}(u,v,w)$ is a homogeneous polynomial in x, y and z of degree $2m-3$, we must have $\overline{f}^*_{2m-3}(u,v) = 0$ and $\beta = 0$. Hence, we obtain

$$
f_{2m-3}(x,y,z) \equiv 0.
$$

Equation (8) with $i = 2m-4$ now is

$$
\begin{aligned}
yz\frac{\partial f_{2m-4}}{\partial x} \quad & + \quad xz\frac{\partial f_{2m-4}}{\partial y} - xy\frac{\partial f_{2m-4}}{\partial z} \\
& = \quad -\alpha\frac{\partial f_{2m-2}}{\partial z} \\
& = \quad -\sum_{i=0}^{m-1} 2\alpha\left[(m-1-i)a_i^{m-1}\right. \\
& \qquad \left.+(i+1)a_{i+1}^{m-1}\right](x^2+z^2)^{m-2-i}(y^2+z^2)^iz.
\end{aligned}
$$

In order to get a homogeneous polynomial solution of degree $2m-4$ of this equation, from the integrating formula (13) we must have

$$
(m-1-i)a_i^{m-1} + (i+1)a_{i+1}^{m-1} = 0, \qquad i = 0,1,\ldots,m-1.
$$

Hence

$$
\begin{aligned}
f_{2m-2} \quad &= \quad a_0^{m-1}(x^2-y^2)^{m-1}, \\
f_{2m-4} \quad &= \quad \sum_{i=0}^{m-2} a_i^{m-2}(x^2+z^2)^{m-2-i}(y^2+z^2)^i.
\end{aligned}
$$

By recursive calculations we can obtain that

$$f_{2m-2s} = a_0^{m-s}(x^2-y^2)^{m-s}, \quad f_{2m-2s-1} \equiv 0, \text{ for } s = 0, 1, \ldots, m-1,$$

where $a_0^m \neq 0$, a_0^i for $i = 1, 2, \ldots, m-1$ is an arbitrary constant. Therefore, we have

$$f = \sum_{i=1}^{m} a_0^i (x^2-y^2)^i.$$

So, $H = x^2 - y^2$ is a polynomial first integral. This proves statement (b) of the theorem.

Case 2. $\mu \neq 0$ and $(m-i)a_i^m + (i+1)a_{i+1}^m = 0$ for $i = 0, 1, \ldots, m-1$. Then, we have

$$f_{2m} = a_0^m (x^2-y^2)^m, \qquad f_{2m-1} = 4\beta m a_0^m (x^2-y^2)^{m-1} z.$$

Equation (8) with $i = 2m-2$ can be written as

$$\begin{aligned} & yz\frac{\partial f_{2m-2}}{\partial x} + xz\frac{\partial f_{2m-2}}{\partial y} - xy\frac{\partial f_{2m-2}}{\partial z} \\ &= (\mu x - \beta y)\frac{\partial f_{2m-1}}{\partial x} + (\mu y + \beta x)\frac{\partial f_{2m-1}}{\partial y} - 2m\mu f_{2m-1} - \alpha\frac{\partial f_{2m}}{\partial z} \\ &= -8\beta\mu m a_0^m (x^2-y^2)^{m-1} z - 16\beta^2 m(m-1) a_0^m (x^2-y^2)^{m-2} xyz. \end{aligned}$$

Working in a similar way to the proof of Case 1, we get

$$\begin{aligned} f_{2m-2}(x,y,z) &= \overline{f}_{2m-2}(u,v,w) \\ &= 4\mu m\beta a_0^m (u-v)^{m-1} \int \frac{dw^2}{(\pm\sqrt{u-w^2})(\pm\sqrt{v-w^2})} + \\ & \quad 16\beta^2 \begin{pmatrix} m \\ 2 \end{pmatrix} a_0^m (u-v)^{m-2} w^2 + \overline{f}^*_{2m-2}(u,v), \end{aligned}$$

where $\overline{f}^*_{2m-2}$ is an arbitrary function in u and v. Since f_{2m-2} is a polynomial in x, y and z, we get from (13) that $\beta = 0$ and

$$f_{2m-2}(x,y,z) = \sum_{i=0}^{m-1} a_i^{m-1} (x^2+z^2)^{m-1-i} (y^2+z^2)^i.$$

Substituting f_{2m-2} and f_{2m-1} into equation (8) with $i = 2m-3$, we get

$$yz\frac{\partial f_{2m-3}}{\partial x} + xz\frac{\partial f_{2m-3}}{\partial y} - xy\frac{\partial f_{2m-3}}{\partial z}$$

$$= \mu x \frac{\partial f_{2m-2}}{\partial x} + \mu y \frac{\partial f_{2m-2}}{\partial y} - 2m\mu f_{2m-2} - \alpha \frac{\partial f_{2m-1}}{\partial z}$$

$$= -2\mu \sum_{i=0}^{m-1} a_i^{m-1}(x^2+z^2)^{m-1-i}(y^2+z^2)^i - 2\mu \sum_{i=0}^{m-2} \big[(m-1-i)a_i^{m-1}$$

$$+(i+1)a_{i+1}^{m-1}\big](x^2+z^2)^{m-2-i}(y^2+z^2)^i z^2.$$

Working in a similar way to the proof of Case 1, in order to get a homogeneous polynomial solution of degree $2m-3$ of this equation, we must have $a_i^{m-1} = 0$, $i = 0, 1, \ldots, m-1$, and then $f_{2m-2} \equiv 0$ and $f_{2m-3} \equiv 0$.

By recursive calculations we can obtain from equation (8) and (9) that $f_i \equiv 0$ for $i = 2m-4, 2m-5, \ldots, 2, 1$. Therefore,

$$f = a_0^m (x^2 - y^2)^m,$$

whose cofactor is $k = -2m\mu$.

From Proposition 4, it follows that the irreducible Darboux polynomials of the Rikitake system are $f = x + y$ with the cofactor $k = z - \mu$ and $f = x - y$ with the cofactor $k = -z - \mu$. This proves statement (c) of the theorem under the conditions $\beta = 0$ and $\mu \neq 0$.

Now we consider the case in which the cofactor is nonconstant. According to the proof of Proposition 1, without loss of generality, we can assume that f is a Darboux polynomial of degree $2m+r$ (respectively $2m-r$) with cofactor $k = rz + c$ if r is a positive (respectively negative) integer, and that

$$f = \sum_{i=0}^{2m+r} f_i, \qquad \text{(respectively } f = \sum_{i=0}^{2m-r} f_i),$$

where f_i is a homogeneous polynomial of degree i, and

$$f_{2m+r} = (x+y)^r \sum_{i=0}^{m} a_i^m (x^2+z^2)^{m-i}(y^2+z^2)^i, \tag{16}$$

respectively

$$f_{2m-r} = (x-y)^{-r} \sum_{i=0}^{m} a_i^m (x^2+z^2)^{m-i}(y^2+z^2)^i. \tag{17}$$

First we consider the case $r > 0$. Substituting f and k into equation (1) and identifying the terms of same degrees, we get

$$yz \frac{\partial f_{2m+r}}{\partial x} + xz \frac{\partial f_{2m+r}}{\partial y} - xy \frac{\partial f_{2m+r}}{\partial z} = rz f_{2m+r}, \tag{18}$$

$$yz\frac{\partial f_{2m+r-1}}{\partial x}+xz\frac{\partial f_{2m+r-1}}{\partial y}\;-\;xy\frac{\partial f_{2m+r-1}}{\partial z}$$

$$=rzf_{2m+r-1}+(\mu x-\beta y)\frac{\partial f_{2m+r}}{\partial x}+(\mu y+\beta x)\frac{\partial f_{2m+r}}{\partial y}+cf_{2m+r}, \quad (19)$$

$$yz\frac{\partial f_i}{\partial x}+xz\frac{\partial f_i}{\partial y}-xy\frac{\partial f_i}{\partial z}$$

$$=rzf_i+(\mu x-\beta y)\frac{\partial f_{i+1}}{\partial x}+(\mu y+\beta x)\frac{\partial f_{i+1}}{\partial y}+cf_{i+1}-\alpha\frac{\partial f_{i+2}}{\partial z}, \quad (20)$$

$$cf_0-\alpha\frac{\partial f_1}{\partial z}=0, \quad (21)$$

for $i=2m+r-2,2m+r-3,\ldots,2,1,0$.

From Proposition 1 equation (18) has a solution of the form (16). Introducing (16) into equation (19) and doing some computations we obtain

$$yz\frac{\partial f_{2m+r-1}}{\partial x}+xz\frac{\partial f_{2m+r-1}}{\partial y}\;-\;xy\frac{\partial f_{2m+r-1}}{\partial z}$$

$$=rzf_{2m+r-1}+(x+y)^r\sum_{i=0}^{m}[(r+2m)\mu+c]a_i^m(x^2+z^2)^{m-i}(y^2+z^2)^i+$$

$$(x+y)^{r-1}(x-y)\sum_{i=0}^{m}r\beta a_i^m(x^2+z^2)^{m-i}(y^2+z^2)^i-$$

$$(x+y)^r\sum_{i=0}^{m-1}2\mu\left[(m-i)a_i^m+(i+1)a_{i+1}^m\right](x^2+z^2)^{m-i-1}(y^2+z^2)^iz^2-$$

$$(x+y)^r\sum_{i=0}^{m-1}2\beta\left[(m-i)a_i^m-(i+1)a_{i+1}^m\right](x^2+z^2)^{m-i-1}(y^2+z^2)^ixy.$$

We consider the transformations (4) and (5). As in the proof of Proposition 1, we now select $xy<0$. Without loss of generality, we can assume that $x=\sqrt{u-w^2}$ and $y=-\sqrt{v-w^2}$. From the above equation we get

$$\sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}_{2m+r-1}}{dw}=rw\overline{f}_{2m+r-1}+$$

$$(\sqrt{u-w^2}-\sqrt{v-w^2})^r\sum_{i=0}^{m}[(r+2m)\mu+c]a_i^mu^{m-i}v^i+$$

$$(\sqrt{u-w^2}-\sqrt{v-w^2})^{r-1}(\sqrt{u-w^2}+\sqrt{v-w^2})\sum_{i=0}^{m}r\beta a_i^mu^{m-i}v^i-$$

$$
(\sqrt{u-w^2}-\sqrt{v-w^2})^r \sum_{i=0}^{m-1} 2\mu\left[(m-i)a_i^m+(i+1)a_{i+1}^m\right]u^{m-i-1}v^i w^2 +
$$

$$
(\sqrt{u-w^2}-\sqrt{v-w^2})^r \sum_{i=0}^{m-1} 2\beta\left[(m-i)a_i^m-(i+1)a_{i+1}^m\right]\cdot
$$

$$
u^{m-i-1}v^i\sqrt{u-w^2}\sqrt{v-w^2}. \tag{22}
$$

This is a linear ordinary differential equation in f_{2m+r-1}. The corresponding homogeneous equation

$$
\sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}^*_{2m+r-1}}{dw} = rw\overline{f}^*_{2m+r-1},
$$

has a general solution

$$
\overline{f}^*_{2m+r-1} = (\sqrt{u-w^2}-\sqrt{v-w^2})^r\overline{A}^*_{2m-1}(u,v),
$$

where $\overline{A}^*_{2m-1}(u,v)$ is an arbitrary function in u and v. In order to use the method of variation of constants, we assume that

$$
\overline{f}_{2m+r-1} = (\sqrt{u-w^2}-\sqrt{v-w^2})^r\overline{A}_{2m-1}(u,v,w),
$$

is a solution of (22), then $\overline{A}_{2m-1}(u,v,w)$ satisfies

$$
\frac{d\overline{A}_{2m-1}}{dw} = \sum_{i=0}^{m}[(r+2m)\mu+c]a_i^m u^{m-i}v^i\frac{1}{\sqrt{u-w^2}\sqrt{v-w^2}} +
$$

$$
\sum_{i=0}^{m} r\beta a_i^m u^{m-i}v^i\frac{\sqrt{u-w^2}+\sqrt{v-w^2}}{\sqrt{u-w^2}\sqrt{v-w^2}(\sqrt{u-w^2}-\sqrt{v-w^2})} -
$$

$$
\sum_{i=0}^{m-1} 2\mu\left[(m-i)a_i^m+(i+1)a_{i+1}^m\right]u^{m-i-1}v^i\frac{w^2}{\sqrt{u-w^2}\sqrt{v-w^2}} +
$$

$$
\sum_{i=0}^{m-1} 2\beta\left[(m-i)a_i^m-(i+1)a_{i+1}^m\right]u^{m-i-1}v^i.
$$

Since

$$
\int\frac{\sqrt{u-w^2}+\sqrt{v-w^2}}{\sqrt{u-w^2}\sqrt{v-w^2}(\sqrt{u-w^2}-\sqrt{v-w^2})}dw =
$$

$$
\frac{u+v}{u-v}\int\frac{dw}{\sqrt{u-w^2}\sqrt{v-w^2}} - \frac{2}{u-v}\int\frac{w^2dw}{\sqrt{u-w^2}\sqrt{v-w^2}} + \frac{2}{u-v}\int dw.
$$

In order that $A_{2m-1}(x,y,z) = \overline{A}_{2m-1}(u,v,w)$ is a homogeneous polynomial of degree 2m-1, we must have

$$\begin{array}{rcll} [(r+2m)\mu + c]a_i^m &=& 0, & i = 0,1,\ldots,m, \\ r\beta a_i^m &=& 0, & i = 0,1,\ldots,m, \\ 2\mu\left[(m-i)a_i^m + (i+1)a_{i+1}^m\right] &=& 0, & i = 0,1,\ldots,m-1. \end{array} \qquad (23)$$

Therefore, we obtain that

$$c = -(r+2m)\mu, \qquad \beta = 0,$$

otherwise $a_i^m = 0$ for $i = 0,1,\ldots,m$, and so $f_{2m+r} \equiv 0$. Hence, $A_{2m-1}(x,y,z) = \overline{A}_{2m-1}(u,v,w) = \overline{A}_{2m-1}(u,v) \equiv 0$, and then

$$f_{2m+r-1}(x,y,z) = \overline{f}_{2m+r-1}(u,v,w) \equiv 0.$$

Substituting f_{2m+r-1} and f_{2m+r} into (20) with $i = 2m+r-2$, we get

$$\begin{aligned} & yz\frac{\partial f_{2m+r-2}}{\partial x} + xz\frac{\partial f_{2m+r-2}}{\partial y} - xy\frac{\partial f_{2m+r-2}}{\partial z} \\ &= rzf_{2m+r-2} - \alpha\frac{\partial f_{2m+r}}{\partial z} \\ &= rzf_{2m+r-2} - (x+y)^r\sum_{i=0}^{m-1} 2\alpha\left[(m-i)a_i^m + (i+1)a_{i+1}^m\right]\cdot \\ &\qquad (x^2+z^2)^{m-1-i}(y^2+z^2)^i z. \end{aligned}$$

Working in a similar way to the proof of f_{2m+r-1}, from this equation we obtain the ordinary differential equation

$$\begin{aligned} \sqrt{u-w^2}\sqrt{v-w^2}\frac{d\overline{f}_{2m+r-2}}{dw} &= rw\overline{f}_{2m+r-2} - (\sqrt{u-w^2} - \sqrt{v-w^2})^r\cdot \\ & \sum_{i=0}^{m-1} 2\alpha\left[(m-i)a_i^m + (i+1)a_{i+1}^m\right]u^{m-1-i}v^i w. \end{aligned}$$

The corresponding homogeneous equation has the general solution

$$\overline{f}^*_{2m+r-2} = (\sqrt{u-w^2} - \sqrt{v-w^2})^r\overline{A}^*_{2m-2}(u,v).$$

Let

$$\overline{f}_{2m+r-2} = (\sqrt{u-w^2} - \sqrt{v-w^2})^r\overline{A}_{2m-2}(u,v,w)$$

be a solution of the previous linear ordinary differential equation. Then the function $\overline{A}_{2m-2}$ satisfies the following equation

$$\frac{d\overline{A}_{2m-2}}{dw} = -\sum_{i=0}^{m-1} 2\alpha \left[(m-i)a_i^m + (i+1)a_{i+1}^m\right] u^{m-1-i} v^i \frac{w}{\sqrt{u-w^2}\sqrt{v-w'^2}}.$$

In order that $A_{2m-2}(x,y,z) = \overline{A}_{2m-2}(u,v,w)$ is a homogeneous polynomial in x, y and z, we should have

$$\alpha \left[(m-i)a_i^m + (i+1)a_{i+1}^m\right] = 0, \qquad i = 0,1,\ldots,m-1, \tag{24}$$

and $\overline{A}_{2m-2}(u,v,w) = \overline{A}_{2m-2}(u,v) = A_{2m-2}(x^2+z^2, y^2+z^2)$. Therefore,

$$f_{2m+r-2} = (x+y)^r \sum_{i=0}^{m-1} a_i^{m-1} (x^2+z^2)^{m-1-i} (y^2+z^2)^i.$$

Introducing f_{2m+r-2} and f_{2m+r-1} into equation (20) with $i = 2m+r-3$ and doing some computations, we get

$$\begin{aligned}
& yz\frac{\partial f_{2m+r-3}}{\partial x} + xz\frac{\partial f_{2m+r-3}}{\partial y} - xy\frac{\partial f_{2m+r-3}}{\partial z} \\
&= rzf_{2m+r-3} + \mu x\frac{\partial f_{2m+r-2}}{\partial x} + \mu y\frac{\partial f_{2m+r-2}}{\partial y} - (r+2m)\mu f_{2m+r-2} \\
&= rzf_{2m+r-3} - (x+y)^r \sum_{i=0}^{m-1} 2\mu a_i^{m-1}(x^2+z^2)^{m-1-i}(y^2+z^2)^i - \\
&\quad (x+y)^r \sum_{i=0}^{m-2} 2\mu \left[(m-1-i)a_i^{m-1} + (i+1)a_{i+1}^{m-1}\right] \cdot \\
&\qquad (x^2+z^2)^{m-2-i}(y^2+z^2)^i z^2.
\end{aligned}$$

Working in a similar way to the proof of f_{2m+r-1}, we obtain that

$$\begin{array}{rcll} \mu a_i^{m-1} & = & 0, & i = 0,1,\ldots,m-1, \\ \mu\left[(m-1-i)a_i^{m-1} + (i+1)a_{i+1}^{m-1}\right] & = & 0, & i = 0,1,\ldots,m-2, \end{array} \tag{25}$$

and so

$$f_{2m+r-3}(x,y,z) \equiv 0.$$

Equation (20) with $i = 2m+r-4$ now can be written as

$$yz\frac{\partial f_{2m+r-4}}{\partial x} \quad + \quad xz\frac{\partial f_{2m+r-4}}{\partial y} - xy\frac{\partial f_{2m+r-4}}{\partial z}$$

$$
\begin{aligned}
&= rzf_{2m+r-4} - \alpha \frac{\partial f_{2m+r-2}}{\partial z} \\
&= rzf_{2m+r-4} - (x+y)^r \sum_{i=0}^{m-2} 2\alpha \left[(m-1-i)a_i^{m-1} + \right. \\
&\qquad \left. (i+1)a_{i+1}^{m-1} \right] (x^2+z^2)^{m-2-i}(y^2+z^2)^i z.
\end{aligned}
$$

In a similar way to the proof of f_{2m+r-2} we get

$$
\alpha \left[(m-1-i)a_i^{m-1} + (i+1)a_{i+1}^{m-1} \right] = 0, \qquad i = 0, 1, \ldots, m-2, \tag{26}
$$

and

$$
f_{2m+r-4} = (x+y)^r \sum_{i=0}^{m-2} a_i^{m-2}(x^2+z^2)^{m-2-i}(y^2+z^2)^i.
$$

By recursive calculations we obtain

$$
\begin{aligned}
f_{2m+r-2s+1} &= 0, \qquad s = 3, 4, \ldots, m, \\
f_{2m+r-2s} &= (x+y)^r \sum_{i=0}^{m-s} a_i^{m-s}(x^2+z^2)^{m-s-i}(y^2+z^2)^i, \\
&\qquad\qquad s = 3, 4, \ldots, m, \\
f_j &= 0, \qquad j = 0, 1, 2, \ldots, r-1,
\end{aligned}
$$

with conditions

$$
\mu a_0^0 = 0, \tag{27}
$$

and for $s = 2, 3, \ldots, m-1$

$$
\begin{aligned}
\mu a_i^{m-s} &= 0, \quad i = 0, 1, \ldots, m-s, \\
\mu \left[(m-s-i)a_i^{m-s} + (i+1)a_{i+1}^{m-s} \right] &= 0, \quad i = 0, 1, \ldots, m-s-1, \\
\alpha \left[(m-s-i)a_i^{m-s} + (i+1)a_{i+1}^{m-s} \right] &= 0, \quad i = 0, 1, \ldots, m-s-1.
\end{aligned} \tag{28}
$$

Summing up the above results, we get from conditions (23)–(28) that if f is a Darboux polynomial of degree $2m+r$ with a nonconstant cofactor, then one of the following three cases holds:

(1) $\beta = \mu = \alpha = 0$, and

$$
f = (x+y)^r \sum_{s=0}^{m} \sum_{i=0}^{m-s} a_i^{m-s}(x^2+z^2)^{m-s-i}(y^2+z^2)^i,
$$

is a Darboux polynomial with the cofactor $k = rz$;

(2) $\beta = \mu = 0$, $\alpha \neq 0$ and

$$f = (x+y)^r \sum_{s=0}^{m} a_0^{m-s}(x^2 - y^2)^{m-s},$$

is a Darboux polynomial with the cofactor $k = rz$;

(3) $\beta = 0$, $\mu \neq 0$ and

$$f = (x+y)^r (x^2 - y^2)^m,$$

is a Darboux polynomial with the cofactor $k = rz - (r + 2m)\mu$.

Working in a similar way as in the proof of the case r a positive integer, when r is a negative integer we get that if f is a Darboux polynomial of degree $2m - r$ with a nonconstant cofactor, then one of the following three cases holds:

(1) $\beta = \mu = \alpha = 0$, and

$$f = (x-y)^{-r} \sum_{s=0}^{m} \sum_{i=0}^{m-s} a_i^{m-s}(x^2 + z^2)^{m-s-i}(y^2 + z^2)^i,$$

is a Darboux polynomial with the cofactor $k = rz$;

(2) $\beta = \mu = 0$, $\alpha \neq 0$ and

$$f = (x-y)^{-r} \sum_{s=0}^{m} a_0^{m-s}(x^2 - y^2)^{m-s},$$

is a Darboux polynomial with the cofactor $k = rz$;

(3) $\beta = 0$, $\mu \neq 0$ and

$$f = (x-y)^{-r} (x^2 - y^2)^m,$$

is a Darboux polynomial with the cofactor $k = rz - (-r + 2m)\mu$.

From Proposition 4 and statements (a) and (b) of this theorem, we obtain that if f is an irreducible Darboux polynomial of the Rikitake system, then $\beta = 0$, and $f = x + y$ with the cofactor $k = z - \mu$ and $f = x - y$ with the cofactor $k = -z - \mu$.

This proves the "only if " part of the theorem. The "if " part follows from an easy computation. This completes the proof of the theorem. ∎

4. Conclusion

In this paper we characterize the Darboux polynomials, the polynomial first integrals, the rational first integrals, the invariant, and the algebraic integrability of the Rikitake systems. Thus the main results are the following

(a) The Rikitake system has Darboux polynomials if and only if $\beta = 0$. The irreducible Darboux polynomials are $f_1 = x + y$ with the cofactor $k_1 = z - \mu$, and $f_2 = x - y$ with the cofactor $k_2 = -z - \mu$.

(b) The Rikitake system has a polynomial first integral if and only if either $\mu = \alpha = 0$, or $\mu = \beta = 0$ and $\alpha = 0$.

  (i) If $\mu = \alpha = 0$, the generators of polynomial first integrals are $H_1 = x^2 + z^2 + 2\beta z$ and $H_2 = y^2 + z^2 - 2\beta z$.

  (ii) If $\mu = \beta = 0$ and $\alpha = 0$, the generator of polynomial first integral is $H = x^2 - y^2$.

(c) The Rikitake system has a rational first integral if and only if either $\mu = \alpha = 0$, or $\mu = \beta = 0$ and $\alpha = 0$.

(d) The unique irreducible invariant (also called integral of motion) is $(x^2 - y^2)\exp(-2\mu t)$ when $\beta = 0$.

(e) The Rikitake system is algebraically integrable if and only if $\mu = \alpha = 0$.

We remark that Labrunie and Conte [10] proved that $(x^2 - y^2)\exp(-2\mu t)$ is an invariant of the Rikitake system when $\beta = 0$. Here we prove that it is unique.

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