ON THE INVARIANT ALGEBRAIC SURFACES OF THE LORENZ SYSTEMS

Jaume Llibre* and Xiang Zhang**

*Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 – Bellaterra, Barcelona, Spain. Email: jllibre@mat.uab.es

** Department of Mathematics, Nanjing Normal University, Nanjing 210097, P. R. China. Email: xzhang@pine.njnu.edu.cn[1]

Abstract

In this paper we use the method of characteristic curves for solving linear partial differential equations to study the invariant algebraic surfaces of the Lorenz systems

$$\dot{x} = s(y-x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy.$$

Our main results about the Lorenz systems are the following. First, we show that the cofactor of any invariant algebraic surface is constant. Second, we characterize all homogeneous invariant algebraic surfaces of even degree. Finally, we prove that the Lorenz systems have no invariant algebraic surfaces of odd degree (this is our main result).

1. Introduction and statement of the main results

We consider the famous Lorenz systems [5]

$$\begin{array}{rcccl} \dot{x} & = & s(y-x) & = & P(x,y,z), \\ \dot{y} & = & rx - y - xz & = & Q(x,y,z), \\ \dot{z} & = & -bz + xy & = & R(x,y,z), \end{array}$$

where x, y and z are real variables; and s, r and b are real parameters. These systems have been thoroughly investigated as dynamical systems (see for instance, [8]), and were intensively studied using different integrability theories (for example, see [2], [4], [6], [7], [9], [10]).

[1]The current address: Centre de Recerca Matemàtica, Universitat Autònoma de Barcelona, Apartat 50, E–08193 Bellaterra, Barcelona, Spain. Email: zhang@crm.es

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Let $f(x,y,z)$ be a real polynomial in the variables x, y and z. The algebraic surface $f(x,y,z)=0$ of $\mathbf{R}^3$ is called an *invariant algebraic surface* of the Lorenz system if

$$\frac{\partial f}{\partial x}P+\frac{\partial f}{\partial y}Q+\frac{\partial f}{\partial z}R=kf, \tag{1}$$

for some real polynomial $k(x,y,z)$, which is called the *cofactor* of $f=0$. It is easy to prove that the degree of k is less than or equal to 1. Therefore, we can assume that the cofactor is of the form

$$k(x,y,z)=px+qy+lz+c. \tag{2}$$

We say that a real function

$$\begin{aligned} H: \quad & \mathbf{R}^3\times\mathbf{R}\longrightarrow\mathbf{R}, \\ & (x,y,z,t)\longmapsto H(x,y,z,t), \end{aligned}$$

is a *first integral* of the Lorenz system, if it is constant on all solution curves $(x(t),y(t),z(t))$ of the Lorenz system, i.e., $H(x(t),y(t),z(t),t)\equiv$ constant for all values of t for which the solution $(x(t),y(t),z(t))$ is defined on $\mathbf{R}^3$. In particular, if the first integral H is independent on the time and it is a polynomial (respectively, rational function), then it is called a *polynomial first integral* (respectively, *rational first integral*).

Up to date we only know six irreducible invariant algebraic surfaces for the Lorenz systems:

invariant algebraic surface	cofactor	parameters
$x^2-2sz=0$	$-2s$	$b=2s$
$-rx^2+\frac{1}{3}y^2+\frac{2}{3}xy+x^2z-\frac{3}{4}x^4=0$	$-4/3$	$b=0,\ s=1/3$
$y^2+z^2=0$	-2	$b=1,\ r=0$
$4(1-r)z+rx^2+y^2-2xy+x^2z-\frac{1}{4}x^4=0$	-4	$b=4,\ s=1$
$-rx^2+y^2+z^2=0$	-2	$b=1,\ s=1$
$\frac{1}{s}(2s-1)^2x^2+sy^2-(4s-2)xy+x^2z-\frac{1}{4s}x^4=0$	$-4s$	$b=6s-2,\ r=2s-1$

The first three invariant algebraic surfaces of the previous list were found by Segur [7] in 1982 using Painlevé method (see also [9] and [11]). The next three ones were found by K£s [4] in 1983 using the method of Carlemann imbedding. In addition to, it is easy to prove that the function $H=(y^2+z^2)/(x^2-z)^2$ is a rational first integral under the conditions $b=1$, $s=1/2$ and $r=0$. As far as we know, there are no other new results about the invariant algebraic surfaces of the Lorenz systems.

Using the perturbation of the above known invariant algebraic surfaces, in 1997 Giacomini and Neukirch [3] constructed families of two–dimensional surfaces transverse to the flow of the Lorenz systems, in which everyone separates the phase space $\mathbf{R}^3$ and hence can be used to describe the location of the global attractor of the flow.

In this paper, by using the method of characteristic curves for solving linear partial differential equations, we obtain the following results. The first one shows that the cofactor of an invariant algebraic surface for the Lorenz system must be constant.

Proposition 1. *The cofactor of every invariant algebraic surface of a Lorenz system is a constant.*

Our next result shows the relationship between invariant algebraic surfaces and first integrals of the Lorenz system.

Proposition 2. *A Lorenz system has an invariant algebraic surface* $f(x, y, z) = 0$ *with the constant cofactor* k*, if and only if the function* $H(x, y, z, t) = f(x, y, z)e^{-kt}$ *is a first integral depending on the time.*

The following result characterizes all homogeneous invariant algebraic surfaces of even degree for the Lorenz systems.

Proposition 3. *Assume that* $s \neq 0$*. Then a Lorenz system has a homogeneous invariant algebraic surface* $f_{2k}(x, y, z) = 0$ *of degree* $2k$ *if and only if either* $b = 1$*,* $r = 0$ *and* $f_{2k}(x, y, z) = (y^2 + z^2)^k$ *with the cofactor* $c = -2k$*; or* $b = 1$*,* $s = 1$ *and* $f_{2k}(x, y, z) = (y^2 + z^2 - rx^2)^k$ *with the cofactor* $c = -2k$*.*

Our main result is the following.

Theorem 4. *For* $s \neq 0$ *the following statements hold for the Lorenz systems.*

(a) There are no invariant algebraic surfaces of odd degree.

(b) There are no polynomial first integrals of odd degree.

We remark that when $s = 0$, Lorenz systems have the first integral $H(x, y, z) = x$, and then on each plane $x =$ constant the system becomes linear.

This paper is organized as follows. In Section 2, we recall the method of characteristic curves for solving linear partial differential equations, it is the main tool of this paper. In Section 3, we prove Propositions 1, 2 and 3. The proof of Theorem 4 is given in Section 4.

2. The method of characteristic curves.

This section states the method of characteristic curves for solving linear partial differential equations (see for instance, Chapter 2 of [1]), which is a main tool of this paper.

Consider the following first–order linear partial differential equation

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \quad u = u(x,y). \tag{3}$$

A curve in the xy–plane is called a *characteristic curve* for the partial differential equation (3), if at each point (x_0, y_0) on the curve, the vector $(a(x_0,y_0), b(x_0,y_0))$ is tangent to the curve.

If the characteristic curve is the graph of function $y(x)$ (assuming that $a(x,y) \neq 0$), then

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{4}$$

This ordinary differential equation is known as the *characteristic equation* of (3). The solution curves of the characteristic equation are the characteristic curves of equation (3).

Suppose that (4) has a solution in the implicit form $h(x,y) = d$, where d is an arbitrary constant. We consider the change of variables

$$\mu = h(x,y), \qquad \nu = y, \tag{5}$$

and we write its inverse transformation as $x = p(\mu,\nu)$, $y = q(\mu,\nu)$ (of course, sometimes the explicit inverse transformation cannot be obtained). Then linear partial differential equation (3) becomes an ordinary differential equation in ν (for fixed μ)

$$\overline{b}(\mu,\nu)\overline{u}_\nu + \overline{c}(\mu,\nu)\overline{u} = \overline{f}(\mu,\nu), \tag{6}$$

where $\overline{b}$, $\overline{c}$, $\overline{u}$ and $\overline{f}$ are b, c, u and f, written in terms of μ and ν.

If $\overline{u} = \overline{u}(\mu,\nu)$ is a solution of (6), then by transformation (5)

$$u(x,y) = \overline{u}(h(x,y),y),$$

is a solution of the linear partial differential equation (3). Moreover, the general solution of (6) is that of (3), written in terms of x and y by using (5).

3. Proof of Propositions 1, 2 and 3.

Proof of Proposition 1: Assume that

$$f(x,y,z)=\sum_{i=0}^{n} f_i(x,y,z), \tag{7}$$

is an algebraic solution of equation (1) with a cofactor of the form (2), where f_i is a homogeneous polynomial of degree i for $i=0,1,2,\cdots,n$.

Substituting (2) and (7) into (1) we obtain

$$s(y-x)\sum_{i=0}^{n}\frac{\partial f_i}{\partial x}+(rx-y-xz)\sum_{i=0}^{n}\frac{\partial f_i}{\partial y} + (-bz+xy)\sum_{i=0}^{n}\frac{\partial f_i}{\partial z}=(px+qy+lz+c)\sum_{i=0}^{n} f_i.$$

Comparing the coefficients of terms of degree $n+1$ of this equation, we get

$$-xz\frac{\partial f_n}{\partial y}+xy\frac{\partial f_n}{\partial z}=(px+qy+lz)f_n. \tag{8}$$

Suppose that $l^2+q^2\neq 0$. Since f_n is a polynomial, we obtain that $x\,|f_n$, that is, $f_n=xg_{n-1}$, where g_{n-1} is a homogeneous polynomial of degree $n-1$. So, the polynomial g_{n-1} satisfies

$$-xz\frac{\partial g_{n-1}}{\partial y}+xy\frac{\partial g_{n-1}}{\partial z}=(px+qy+lz)g_{n-1}.$$

Similarly, we have $f_n=xg_{n-1}=x^2g_{n-2}$, where g_{n-2} is a homogeneous polynomial of degree $n-2$. Recursively, we obtain that

$$f_n=ax^n,$$

where a is a nonzero constant. But it is impossible for equation (8). This contradiction implies that $l=q=0$.

Now equation (8) can be simplified to

$$-z\frac{\partial f_n}{\partial y}+y\frac{\partial f_n}{\partial z}=pf_n. \tag{9}$$

In what follows, we will use the method of characteristic curves for solving linear partial differential equations to prove that $p=0$.

Consider the characteristic equation associated to (9)

$$\frac{dz}{dy} = -\frac{y}{z},$$

it has solutions $y^2 + z^2 = d$, where d is an arbitrary nonnegative constant. We consider the change of variables

$$\mu = y^2 + z^2, \qquad \nu = z, \tag{10}$$

its inverse transformation is

$$y = \pm\sqrt{\mu - \nu^2}, \qquad z = \nu. \tag{11}$$

Then equation (9) becomes the ordinary differential equation

$$\pm\sqrt{\mu - \nu^2}\frac{d\overline{f}_n}{d\nu} = p\overline{f}_n, \tag{12}$$

where $\overline{f}_n(x, \mu, \nu) = f_n(x, y, z)$ and μ is fixed. Equation (12) has solution

$$\overline{f}_n(x, \mu, \nu) = A(x, \mu) \exp\left(\pm p \arcsin \frac{\nu}{\sqrt{\mu}}\right),$$

where $A(x, \mu)$ is an arbitrary function in x and μ. Hence,

$$f_n(x, y, z) = A(x, y^2 + z^2) \exp\left(\pm p \arcsin \frac{z}{\sqrt{y^2 + z^2}}\right).$$

Since f_n is a homogeneous polynomial of degree n, we must have $p = 0$, and $A(x, y^2 + z^2)$ must be a homogeneous polynomial of degree n. This completes the proof of the proposition. ∎

Proof of Proposition 2: Assume that $f(x, y, z) = 0$ is an invariant algebraic surface of the Lorenz system. Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R \equiv kf.$$

Therefore, we have

$$\frac{dH}{dt} = e^{-kt}\frac{df}{dt} - kfe^{-kt} \equiv 0,$$

that is, $H(x, y, z, t)$ is a first integral. Inversely, the proof follows from the above equation. This proves the proposition. ∎

Proof of Proposition 3: Assume that $f_{2k}(x,y,z)$ is a homogeneous polynomial of degree $2k$, which satisfies the equation

$$s(y-x)\frac{\partial f_{2k}}{\partial x}+(rx-y-xz)\frac{\partial f_{2k}}{\partial y}+(-bz+xy)\frac{\partial f_{2k}}{\partial z}=cf_{2k}. \tag{13}$$

Identifying the terms of degrees $2k+1$ and $2k$ in both sides of the last equality, we get

$$-xz\frac{\partial f_{2k}}{\partial y}+xy\frac{\partial f_{2k}}{\partial z}=0, \tag{14}$$

$$s(y-x)\frac{\partial f_{2k}}{\partial x}+(rx-y)\frac{\partial f_{2k}}{\partial y}-bz\frac{\partial f_{2k}}{\partial z}=cf_{2k}. \tag{15}$$

From the proof of Proposition 1 we know that the homogeneous polynomial solution of degree $2k$ for equation (14) is of the form

$$f_{2k}(x,y,z)=\sum_{i=0}^{k}a_{2k-2i}x^{2k-2i}(y^2+z^2)^i. \tag{16}$$

Substituting (16) into equation (15) and doing some computations, we have

$$\begin{aligned} y\sum_{i=0}^{k-1}&\left[2s(k-i)a_{2(k-i)}+2r(i+1)a_{2(k-i-1)}\right]x^{2k-2i-1}(y^2+z^2)^i\\ &+2(1-b)\sum_{i=1}^{k}ia_{2(k-i)}x^{2(k-i)}(y^2+z^2)^{i-1}z^2\\ &-\sum_{i=0}^{k}\left[c+2s(k-i)+2i\right]a_{2(k-i)}x^{2(k-i)}(y^2+z^2)^i=0. \end{aligned}$$

Comparing the coefficients of the terms with same $x^iy^jz^k$ we obtain

$$2s(k-i)a_{2(k-i)}+2r(i+1)a_{2(k-i-1)}=0, \quad i=0,1,2,\cdots k-1 \tag{17}$$

$$2(1-b)a_{2(k-i)}=0, \quad i=1,2,\cdots,k \tag{18}$$

$$[c+2s(k-i)+2i]a_{2(k-i)}=0, \quad i=0,1,2,\cdots k \tag{19}$$

If $b\neq 1$, then from (18) it follows that $a_{2(k-i)}=0$, $i=1,2,\cdots k$. So $a_{2k}\neq 0$, otherwise $f_{2k}\equiv 0$. Equation (17) with $i=0$ gives $s=0$, which contradicts the assumption. Therefore, we must have $b=1$.

Assume that $r=0$. Since $s\neq 0$, from equation (17) we get that $a_{2(k-i)}=0$, $i=0,1,2,\cdots k-1$. Hence $a_0\neq 0$, otherwise $f_{2k}\equiv 0$. Without loss of generality, we assume that $a_0 = 1$. From equation (19) it follows that $c = -2k$. Therefore, we obtain that

$$f_{2k} = (y^2 + z^2)^k,$$

with cofactor $c = -2k$, and the conditions $b = 1$ and $r = 0$.

Assume that $r \neq 0$. Since $s \neq 0$, we have $a_{2(k-i)} \neq 0$, $i = 0, 1, 2, \cdots k$. Otherwise, $f_{2k} \equiv 0$ by (17). Hence, it follows from equation (19) that $s = 1$ and $c = -2k$.

Equation (17) can be rewritten as

$$a_{2(k-i)} = -r \frac{i+1}{k-i} a_{2(k-i-1)}, \quad i = 0, 1, 2, \cdots, k-1.$$

By recursive calculation we have

$$a_{2i} = (-r)^i \frac{k(k-1)(k-2)\cdots(k+1-i)}{i!} a_0 = (-r)^i \left(\begin{array}{c} k \\ i \end{array} \right) a_0, \quad i = 1, 2, \cdots, k.$$

Without loss of generality, we can assume that $a_0 = 1$. Then

$$f_{2k} = (y^2 + z^2 - rx^2)^k,$$

with cofactor $c = -2k$, and the conditions $b = 1$ and $s = 1$. This completes the proof of the "only if" part of the proposition. The "if" part follows easily by doing direct computations. ∎

4. The proof of Theorem 4.

We first prove statement (a). Assume that the Lorenz system has an invariant algebraic surface of degree n of the form

$$f = \sum_{i=0}^{n} f_i(x, y, z) = 0, \tag{20}$$

with cofactor $k(x, y, z) = c$, where c is a constant, n is odd, and f_i is a homogeneous polynomial of degree i.

Substituting (20) into (1) and comparing the coefficients of the terms with the same degree, we obtain that $f_0 = 0$,

$$-xz \frac{\partial f_n}{\partial y} + xy \frac{\partial f_n}{\partial z} = 0, \tag{21}$$

$$xz\frac{\partial f_i}{\partial y} - xy\frac{\partial f_i}{\partial z} = s(y-x)\frac{\partial f_{i+1}}{\partial x} + (rx-y)\frac{\partial f_{i+1}}{\partial y} - bz\frac{\partial f_{i+1}}{\partial z} - cf_{i+1},$$
$$i = 1, 2, \cdots, n-1. \quad (22)$$

From the proof of Proposition 1, it follows that the homogeneous polynomial solution f_n of (21) is of the form

$$\begin{aligned} f_n(x,y,z) &= a_n x^n + a_{n-2}x^{n-2}(y^2+z^2) + a_{n-4}x^{n-4}(y^2+z^2)^2 \\ &\quad + \cdots + a_3 x^3 (y^2+z^2)^{\frac{n-3}{2}} + a_1 x (y^2+z^2)^{\frac{n-1}{2}}. \end{aligned} \quad (23)$$

Introducing (23) into equation (22) for $i = n-1$, we get that

$$x\left(z\frac{\partial f_{n-1}}{\partial y} - y\frac{\partial f_{n-1}}{\partial z}\right) = sy\sum_{i=0}^{\frac{n-1}{2}}(n-2i)a_{n-2i}x^{n-1-2i}(y^2+z^2)^i + L[f_n], \quad (24)$$

where the operator

$$L = -sx\frac{\partial}{\partial x} + (rx-y)\frac{\partial}{\partial y} - bz\frac{\partial}{\partial z} - c.$$

We remark that the operator L applied to a homogeneous polynomial F in the variables x, y and z, preserves the lowest degree of F thought as a polynomial in x.

Clearly, the variable x divides the right hand side of equation (24). Hence, since x divides $L[f_n]$, it follows that x must divide

$$\sum_{i=0}^{\frac{n-1}{2}}(n-2i)a_{n-2i}x^{n-1-2i}(y^2+z^2)^i.$$

Therefore, we must have $a_1 = 0$.

We rewrite equation (24) as

$$z\frac{\partial f_{n-1}}{\partial y} - y\frac{\partial f_{n-1}}{\partial z} = sy\frac{1}{x}\frac{\partial f_n}{\partial x} + \frac{1}{x}L[f_n], \quad (25)$$

where the lowest degree of x in $\frac{1}{x}\frac{\partial f_n}{\partial x}$ and $\frac{1}{x}L[f_n]$ is 1 and 2, respectively. Working in a similar way to the proof of Proposition 1, we take the change (10) and its inverse transformation (11), then equation (25) becomes the ordinary differential equation:

$$-\sqrt{\mu-\nu^2}\frac{d\overline{f}_{n-1}}{d\nu} = s\sqrt{\mu-\nu^2}\frac{1}{x}\frac{\partial \overline{f}_n}{\partial x} + \frac{1}{x}\overline{L}[\overline{f}_n], \quad (26)$$

where $\overline{f}_n(x,\mu) = f_n(x,y,z)$, $\overline{f}_{n-1}(x,\mu,\nu) = f_{n-1}(x,y,z)$,

$$\overline{L} = -sx\frac{\partial}{\partial x} + 2(rx - \sqrt{\mu-\nu^2})\sqrt{\mu-\nu^2}\frac{\partial}{\partial \mu} - b\nu(2\nu\frac{\partial}{\partial \mu} + \frac{\partial}{\partial \nu}) - c,$$

We note that in the following we do not consider the case for $y = -\sqrt{\mu-\nu^2}$ because the proof would be the same.

Since $\overline{f}_n$ does not contain the variable ν, we have $\dfrac{\partial \overline{f}_n}{\partial \nu} = 0$. By using the relation $\nu^2 = \mu - y^2$ we rewrite equation (26) as

$$\begin{aligned}\frac{d\overline{f}_{n-1}}{d\nu} &= -s\frac{1}{x}\frac{\partial \overline{f}_n}{\partial x} - 2r\frac{\partial \overline{f}_n}{\partial \mu} - \frac{2(b-1)}{x}\frac{\partial \overline{f}_n}{\partial \mu}\sqrt{\mu-\nu^2} \\ &+\frac{1}{x}\left[sx\frac{\partial \overline{f}_n}{\partial x} + 2b\mu\frac{\partial \overline{f}_n}{\partial \mu} + c\overline{f}_n\right]\frac{1}{\sqrt{\mu-\nu^2}}.\end{aligned}$$

Making use of the integrating formulas

$$\begin{aligned}\int \sqrt{\mu-\nu^2}d\nu &= \frac{1}{2}\nu\sqrt{\mu-\nu^2} + \frac{\mu}{2}\int\frac{d\nu}{\sqrt{\mu-\nu^2}}, \\ \int\frac{d\nu}{\sqrt{\mu-\nu^2}} &= \arcsin\frac{\nu}{\sqrt{\mu}} + B,\end{aligned} \tag{27}$$

we get the solution of this last equation

$$\begin{aligned}\overline{f}_{n-1}(x,\mu,\nu) &= -\frac{s}{x}\frac{\partial \overline{f}_n}{\partial x}\nu - 2r\frac{\partial \overline{f}_n}{\partial \mu}\nu - \frac{b-1}{x}\frac{\partial \overline{f}_n}{\partial \mu}(\nu\sqrt{\mu-\nu^2}) \\ &+\frac{1}{x}\left[sx\frac{\partial \overline{f}_n}{\partial x} + (b+1)\mu\frac{\partial \overline{f}_n}{\partial \mu} + c\overline{f}_n(x,\mu)\right]\arcsin\frac{\nu}{\sqrt{\mu}} + \overline{f}^*_{n-1}(x,\mu),\end{aligned}$$

where $\overline{f}^*_{n-1}(x,\mu)$ is an arbitrary function in x and μ.

In order that $f_{n-1}(x,y,z) = \overline{f}_{n-1}(x,\mu,\nu)$ is a homogeneous polynomial of degree $n-1$ in x, y and z, we must have

$$sx\frac{\partial \overline{f}_n}{\partial x} + (b+1)\mu\frac{\partial \overline{f}_n}{\partial \mu} + c\overline{f}_n(x,\mu) \equiv 0,$$

and $f^*_{n-1}(x,y,z) = \overline{f}^*_{n-1}(x,\mu)$ should be selected as a homogeneous polynomial of degree $n-1$ of the form

$$f^*_{n-1}(x,y,z) = \sum_{i=0}^{\frac{n-1}{2}} a^{n-1}_{n-1-2i}x^{n-1-2i}(y^2+z^2)^i.$$

Therefore, we have

$$f_{n-1}(x,y,z) = -\frac{s}{x}\frac{\partial f_n}{\partial x}z - 2r\frac{\partial f_n}{\partial \mu}z - \frac{b-1}{x}\frac{\partial f_n}{\partial \mu}yz + f_{n-1}^*(x,y,z).$$

Using equation (22) for $i = n-2$ yields

$$\begin{aligned} x\left(z\frac{\partial f_{n-2}}{\partial y} - y\frac{\partial f_{n-2}}{\partial z}\right) &= \left(sy\frac{\partial}{\partial x} + L\right) f_{n-1} \\ &= -s^2yz\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f_n}{\partial x}\right) + F + \left(sy\frac{\partial}{\partial x} + L\right) f_{n-1}^*(x,y,z) \\ &= -s^2yz\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f_n}{\partial x}\right) + O\left(\frac{1}{x}\frac{\partial f_n}{\partial x}\right) + \left(sy\frac{\partial}{\partial x} + L\right) f_{n-1}^*(x,y,z), \end{aligned} \tag{28}$$

where

$$F = -syz\frac{\partial}{\partial x}\left(2r\frac{\partial f_n}{\partial \mu} + \frac{b-1}{x}y\frac{\partial f_n}{\partial \mu}\right) - zL\left[\frac{s}{x}\frac{\partial f_n}{\partial x} + 2r\frac{\partial f_n}{\partial \mu} + (b-1)\frac{y}{x}\frac{\partial f_n}{\partial \mu}\right].$$

Moreover, if F and G are two homogeneous polynomials in x, y and z, then $O(F) = O(G)$ denotes that both polynomials F and G start with the same lowest degree terms in x when they are thought as polynomials in x with coefficients polynomials in the variables y and z.

Now we need to get the solution f_{n-2}, a homogeneous polynomial of degree $n-2$, of (28). The variable x should divide the right hand side of equation (28). Therefore, since

$$O\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f_n}{\partial x}\right)\right) = O(a_3),$$

and $-s^2yz\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f_n}{\partial x}\right)$ $\left(\text{respectively } \left(sy\frac{\partial}{\partial x} + L\right) f_{n-1}^*\right)$ is a polynomial in z that has only terms of odd (respectively even) degree, it follows from (23) that $a_3 = 0$, and consequently

$$O\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f_n}{\partial x}\right)\right) = O(a_5x^2).$$

In the following for convenience of notation, we denote by

$$Df = \frac{1}{x}\frac{\partial f}{\partial x}, \quad D^2f = \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f}{\partial x}\right), \quad D^kf = \underbrace{\frac{1}{x}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial}{\partial x}\cdots\left(\frac{1}{x}\frac{\partial f}{\partial x}\right)\right)}_{k},$$

We have that

$$\left(sy\frac{\partial}{\partial x}+L\right)f^*_{n-1}(x,y,z) = \left(s\frac{\partial f^*_{n-1}}{\partial x}+2rx\frac{\partial f^*_{n-1}}{\partial \mu}\right)y-2y^2\frac{\partial f^*_{n-1}}{\partial \mu} \\ -sx\frac{\partial f^*_{n-1}}{\partial x}-2bz^2\frac{\partial f^*_{n-1}}{\partial \mu}-cf^*_{n-1}. \tag{29}$$

Then, by using the relation $y^2=\mu-z^2$, and assuming that x divides expression (29) because we are looking for polynomial solutions of (28), we can write (28) as

$$z\frac{\partial f_{n-2}}{\partial y}-y\frac{\partial f_{n-2}}{\partial z}=-s^2yzD^2f_n+O\left(\frac{1}{x}Df_n\right) \\ +F^1_{n-2}(x,z^2,\mu)+F^2_{n-2}(x,z^2,\mu)y, \tag{30}$$

where F^1_{n-2} and F^2_{n-2} are homogeneous polynomials of degree $n-2$ and $n-3$ in x, y and z, respectively.

Using the transformations (10) and (11), equation (30) becomes

$$-y\frac{d\overline{f}_{n-2}}{d\nu}=-s^2y\nu D^2\overline{f}_n+O\left(\frac{1}{x}D\overline{f}_n\right)+\overline{F}^1_{n-2}(x,\nu^2,\mu)+\overline{F}^2_{n-2}(x,\nu^2,\mu)y.$$

Dividing by $-y$ the two sides of this equation, and integrating with respect to ν, we obtain that

$$\overline{f}_{n-2}(x,\mu,\nu)=\frac{1}{2}s^2\nu^2D^2\overline{f}_n+O\left(\frac{1}{x}D\overline{f}_n\right) \\ -\int\overline{F}^1_{n-2}(x,\nu^2,\mu)\frac{d\nu}{\sqrt{\mu-\nu^2}}-\int\overline{F}^2_{n-2}(x,\nu^2,\mu)d\nu+\overline{f}^*_{n-2}(x,\mu),$$

where $\overline{f}^*_{n-2}$ is an arbitrary function in x and μ.

First we use the following integrating formulas

$$\int\sqrt{\mu-\nu^2}\,\nu^{2k}d\nu=-\frac{1}{2k+2}\left(\mu-\nu^2\right)^{3/2}\nu^{2k-1} \\ +\frac{2k-1}{3}\mu\int\sqrt{\mu-\nu^2}\,\nu^{2k-2}d\nu, \tag{31}$$

$$\int\frac{\nu^{2k}}{\sqrt{\mu-\nu^2}}d\nu=-\nu^{2k-1}\sqrt{\mu-\nu^2}+(2k-1)\int\sqrt{\mu-\nu^2}\,\nu^{2k-2}d\nu,$$

and (27), then apply the relation $y^{2k+1}=(\mu-\nu^2)^ky$, and finally we delete the terms containing $\arcsin\frac{\nu}{\sqrt{\mu}}$ because we want to find polynomial solutions, then we get the homogeneous polynomial solution f_{n-2} of degree $n-2$

in x, y and z of the form

$$\begin{aligned} f_{n-2}(x,y,z) &= \overline{f}_{n-2}(x,\mu,\nu) \\ &= \frac{1}{2}s^2z^2D^2f_n + O\left(\frac{1}{x}Df_n\right) \\ &\quad +z[f^1_{n-2}(x,z^2,y^2+z^2)+f^2_{n-2}(x,z^2,y^2+z^2)y] \\ &\quad +f^*_{n-2}(x,y,z), \end{aligned}$$

where f^1_{n-2} and f^2_{n-2} are homogeneous polynomials of degree $n-3$ and $n-4$ in x, y and z, respectively, f^*_{n-2} is selected as a homogeneous polynomial of degree $n-2$ in x, y and z, and has the following form

$$f^*_{n-2}(x,y,z) = \sum_{i=0}^{\frac{n-3}{2}} a^{n-2}_{n-2-2i}x^{n-2-2i}(y^2+z^2)^i.$$

Substituting f_{n-2} into equation (22) with $i=n-3$ we obtain

$$\begin{aligned} & x\left(z\frac{\partial f_{n-3}}{\partial y} - y\frac{\partial f_{n-3}}{\partial z}\right) \\ &\quad = \left(sy\frac{\partial}{\partial x}+L\right)\left(\frac{1}{2}s^2z^2D^2f_n + O\left(\frac{1}{x}Df_n\right)\right) + \left(sy\frac{\partial}{\partial x}+L\right)f^*_{n-2} \\ &\quad + \left(sy\frac{\partial}{\partial x}+L\right)\left(z\left(f^1_{n-2}(x,z^2,y^2+z^2)+f^2_{n-2}(x,z^2,y^2+z^2)y\right)\right). \end{aligned}$$

Using the previous definition of $O(\cdot)$ and doing similar computations to the cases of degrees $n-1$ and $n-2$, we have

$$\begin{aligned} & x\left(z\frac{\partial f_{n-3}}{\partial y} - y\frac{\partial f_{n-3}}{\partial z}\right) \\ &\qquad = \tfrac{1}{2}s^3yz^2\frac{\partial}{\partial x}D^2f_n + O\left(D^2f_n\right) + sy\frac{\partial f^*_{n-2}}{\partial x} + O(f^*_{n-2}) \\ &\qquad +z\left[\widetilde{F}^1_{n-3}(x,z^2,y^2+z^2)+\widetilde{F}^2_{n-3}(x,z^2,y^2+z^2)y\right], \end{aligned} \tag{32}$$

where $\widetilde{F}^1_{n-3}$ and $\widetilde{F}^2_{n-3}$ are homogeneous polynomials of degree $n-3$ and $n-4$ in x, y and z, respectively.

As the variable x divides the right hand side of expression (32), comparing the coefficients of the terms non containing x, we obtain that

$$a^{n-2}_1 = 0, \qquad a_5 = 0.$$

We eliminate all terms non containing x on the right hand side of (32), and then divide by x the two sides of this equation. Using the changes of variables given in (10) and (11), from expression (32) we get

$$\begin{aligned} -y\frac{d\overline{f}_{n-3}}{d\nu} &= \frac{1}{2}s^3 y\nu^2 D^3\overline{f}_n + O\left(\frac{1}{x}D^2\overline{f}_n\right) + syD\overline{f}^*_{n-2} + O\left(\frac{1}{x}\overline{f}^*_{n-2}\right) \\ &\quad + z\left[\overline{F}^1_{n-3}(x,\nu^2,\mu) + \overline{F}^2_{n-3}(x,\nu^2,\mu)y\right], \end{aligned}$$

where $\overline{F}^1_{n-3}$ and $\overline{F}^2_{n-3}$ are homogeneous polynomials of degree $n-4$ and $n-5$, respectively.

From the integrating formulas

$$\begin{aligned} \int \frac{\nu^{2k+1}}{\sqrt{\mu-\nu^2}}d\nu &= -\frac{1}{2}\nu^{2k}\sqrt{\mu-\nu^2} + k\int\sqrt{\mu-\nu^2}\,\nu^{2k-1}d\nu, \\ \int \nu^{2k+1}\sqrt{\mu-\nu^2}d\nu &= -\frac{1}{2k+1}\nu^{2k}\sqrt{\mu-\nu^2}^3 + \frac{2k\mu}{2k+1}\int\sqrt{\mu-\nu^2}\,\nu^{2k-1}d\nu, \\ \int\sqrt{\mu-\nu^2}\,\nu d\nu &= -\frac{1}{2}(\mu-\nu^2)^{\frac{3}{2}}, \end{aligned} \tag{33}$$

and working in a similar way to the proof of f_{n-2}, it follows that

$$\begin{aligned} \overline{f}_{n-3} &= -\frac{1}{3!}s^3\nu^3 D^3\overline{f}_n + O\left(\frac{1}{x}D^2\overline{f}_n\right) - s\nu D\overline{f}^*_{n-2} + O\left(\frac{1}{x}\overline{f}^*_{n-2}\right) \\ &\quad + \overline{f}^1_{n-3}(x,\nu^2,\mu) + \overline{f}^2_{n-3}(x,\nu^2,\mu)y + \overline{f}^*_{n-3}(x,\mu), \end{aligned}$$

where $y = \sqrt{\mu-\nu^2}$, $\overline{f}^1_{n-3}(x,\nu^2,\mu) = f^1_{n-3}(x,y,z)$, and $\overline{f}^2_{n-3}(x,\nu^2,\mu) = f^2_{n-3}(x,y,z)$ are homogeneous polynomials of degree $n-3$ and $n-4$, respectively; $\overline{f}^*_{n-3}(x,\mu) = f^*_{n-3}(x,y,z)$ is selected as a homogeneous polynomial of degree $n-3$ in x, y and z. Therefore,

$$\begin{aligned} f_{n-3} &= -\frac{s^3}{3!}z^3 D^3 f_n + O\left(\frac{1}{x}D^2 f_n\right) - szDf^*_{n-2} + O\left(\frac{1}{x}f^*_{n-2}\right) \\ &\quad + f^1_{n-3}(x,z^2,y^2+z^2) + f^2_{n-3}(x,z^2,y^2+z^2)y + f^*_{n-3}(x,y^2+z^2). \end{aligned}$$

In order to use the induction method, over k with $k>2$ we assume that for $n-(2k-1)$

$$
\begin{aligned}
&f_{n-(2k-1)}(x,y,z)=\overline{f}_{n-(2k-1)}(x,\mu,\nu)\\
&=-\frac{s^{2k-1}}{(2k-1)!}z^{2k-1}D^{2k-1}f_n+O\left(\frac{1}{x}D^{2k-2}f_n\right)\\
&+\sum_{j=1}^{k-1}\left[-\frac{s^{2j-1}}{(2j-1)!}z^{2j-1}D^{2j-1}f^*_{n-2(k-j)}(x,y,z)+O\left(\frac{1}{x}D^{2j-2}f^*_{n-2(k-j)}\right)\right]\\
&+f^1_{n-(2k-1)}(x,z^2,y^2+z^2)+f^2_{n-(2k-1)}(x,z^2,y^2+z^2)y+f^*_{n-(2k-1)}(x,y,z),
\end{aligned}
\tag{34}
$$

where f^*_l for $l\in\{n-(2k-1)\}\cup\{n-2(k-j):\ j=1,2,\cdots,k-1\}$, are homogeneous polynomials of degree l in x, y and z of form

$$
f^*_l(x,y,z)=\sum_{i=0}^{[l/2]}a^l_{l-2i}x^{l-2i}(y^2+z^2)^i, \tag{35}
$$

$f^1_{n-(2k-1)}$ and $f^2_{n-(2k-1)}$ are homogeneous polynomials of degree $n-(2k-1)$ and $n-2k$ in x, y and z, respectively. Correspondingly we have

$$
\begin{aligned}
&a_1=a_3=a_5=a_7=\cdots=a_{4k-11}=a_{4k-9}=a_{4k-7}=a_{4k-5}=a_{4k-3}=0,\\
&a_1^{n-2}=a_3^{n-2}=a_5^{n-2}=a_7^{n-2}=\cdots=a_{4k-11}^{n-2}=a_{4k-9}^{n-2}=a_{4k-7}^{n-2}=0,\\
&a_1^{n-4}=a_3^{n-4}=a_5^{n-4}=a_7^{n-4}=\cdots=a_{4k-11}^{n-4}=0,\\
&\cdots\quad\cdots\quad\cdots\\
&a_1^{n-2(k-3)}=a_3^{n-2(k-3)}=a_5^{n-2(k-3)}=a_7^{n-2(k-3)}=a_9^{n-2(k-3)}=0,\\
&a_1^{n-2(k-2)}=a_3^{n-2(k-2)}=a_5^{n-2(k-2)}=0,\\
&a_1^{n-2(k-1)}=0.
\end{aligned}
$$

Substituting (34) into equation (22) with $i=n-2k$, we get

$$
\begin{aligned}
&x\left(z\frac{\partial f_{n-2k}}{\partial y}-y\frac{\partial f_{n-2k}}{\partial z}\right)\\
&=\left(sy\frac{\partial}{\partial x}+L\right)\left[-\frac{s^{2k-1}}{(2k-1)!}z^{2k-1}D^{2k-1}f_n+O\left(\frac{1}{x}D^{2k-2}f_n\right)\right]\\
&\quad+\left(sy\frac{\partial}{\partial x}+L\right)\sum_{j=1}^{k-1}\left[-\frac{s^{2j-1}}{(2j-1)!}z^{2j-1}D^{2j-1}f^*_{n-2(k-j)}\right.\\
&\qquad\qquad\left.+O\left(\frac{1}{x}D^{2j-2}f^*_{n-2(k-j)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(sy\frac{\partial}{\partial x} + L \right) \Big[f^1_{n-(2k-1)}(x, z^2, y^2+z^2) \\
&\qquad\qquad\qquad + f^2_{n-(2k-1)}(x, z^2, y^2+z^2)y + f^*_{n-(2k-1)} \Big] \\
&= -\frac{s^{2k}}{(2k-1)!} y z^{2k-1} \frac{\partial}{\partial x} D^{2k-1} f_n + O\left(D^{2k-1} f_n\right) \\
&+ \sum_{j=1}^{k-1} \left[-\frac{s^{2j}}{(2j-1)!} y z^{2j-1} \frac{\partial}{\partial x} D^{2j-1} f^*_{n-2(k-j)} + O\left(D^{2j-1} f^*_{n-2(k-j)}\right) \right] \\
&+ \widetilde{F}^1_{n-2k}(x, z^2, y^2+z^2) + \widetilde{F}^2_{n-2k}(x, z^2, y^2+z^2)y.
\end{aligned}
$$

As x divides the right hand side of this equation, and $\widetilde{F}^1_{n-2k}$ and $\widetilde{F}^2_{n-2k}$ only contain terms of even degree in z, by comparing the coefficients of the terms non containing x, we obtain that

$$
\begin{aligned}
&a_3^{n-2(k-1)} = 0, \ \ a_7^{n-2(k-2)} = 0, \ \ a_{11}^{n-2(k-3)} = 0, \ \cdots, \\
&a_{4k-9}^{n-4} = 0, \ \ a_{4k-5}^{n-2} = 0, \ \ a_{4k-1} = 0.
\end{aligned}
$$

Now we eliminate the terms non containing x on the right hand side of this last equation, we divide by x the two sides of the equation, then using the changes of variables given in (10) and (11) it becomes

$$
\begin{aligned}
-y\frac{d\overline{f}_{n-2k}}{d\nu} &= -\frac{s^{2k}}{(2k-1)!} y \nu^{2k-1} D^{2k} \overline{f}_n + O\left(\frac{1}{x} D^{2k-1} \overline{f}_n\right) \\
&+ \sum_{j=1}^{k-1} \left[-\frac{s^{2j}}{(2j-1)!} y \nu^{2j-1} D^{2j} \overline{f}^*_{n-2(k-j)} + O\left(\frac{1}{x} D^{2j-1} \overline{f}^*_{n-2(k-j)}\right) \right] \\
&+ \overline{F}^1_{n-2k}(x, \nu^2, \mu) + \overline{F}^2_{n-2k}(x, \nu^2, \mu) y,
\end{aligned}
$$

where $y = \sqrt{\mu - \nu^2}$, $F^1_{n-2k}(x, y, z) = \overline{F}^1_{n-2k}(x, \nu^2, \mu)$ and $F^2_{n-2k}(x, y, z) = \overline{F}^2_{n-2k}(x, \nu^2, \mu)$ are homogeneous polynomials of degree $n-2k$ and $n-2k-1$, respectively. We recall that we look for a polynomial solution f_{n-2k}.

We divide by $-y$ the above equation, and then we integrate with respect to ν. From the integrating formulas (31) and (27), and eliminating the terms containing $\arcsin\frac{\nu}{\sqrt{\mu}}$ we obtain the polynomial solution

$$
\begin{aligned}
f_{n-2k}(x, y, z) &= \frac{s^{2k}}{(2k)!} z^{2k} D^{2k} f_n + O\left(\frac{1}{x} D^{2k-1} f_n\right) \\
&\quad + \sum_{j=1}^{k-1} \left[\frac{s^{2j}}{(2j)!} z^{2j} D^{2j} f^*_{n-2(k-j)} + O\left(\frac{1}{x} D^{2j-1} f^*_{n-2(k-j)}\right) \right]
\end{aligned}
$$

$$+ z\left[f^1_{n-2k}(x,z^2,y^2+z^2) + f^2_{n-2k}(x,z^2,y^2+z^2)y\right] + f^*_{n-2k}(x,y,z),$$

where f^1_{n-2k} and f^2_{n-2k} are homogeneous polynomial in x, y and z of degree $n-2k-1$ and $n-2k-2$, respectively; f^*_{n-2k} is a homogeneous polynomial of degree $n-2k$ of the form (35).

Introducing f_{n-2k} into equation (22) with $i = n-(2k+1)$, and working in a similar way to the proof of f_{n-2k}, we have

$$\begin{aligned} x\left(z\frac{\partial f_{n-(2k+1)}}{\partial y} - y\frac{\partial f_{n-(2k+1)}}{\partial z}\right) &= \frac{s^{2k+1}}{(2k)!} y z^{2k}\frac{\partial}{\partial x}D^{2k}f_n + O\left(D^{2k}f_n\right) \\ &+ \sum_{j=0}^{k-1}\left[\frac{s^{2j+1}}{(2j)!}yz^{2j}\frac{\partial}{\partial x}D^{2j}f^*_{n-2(k-j)} + O\left(D^{2j}f^*_{n-2(k-j)}\right)\right] \\ &+ z\left[\widetilde{F}^1_{n-(2k+1)}(x,z^2,y^2+z^2) + \widetilde{F}^2_{n-(2k+1)}(x,z^2,y^2+z^2)y\right], \end{aligned}$$

where $\widetilde{F}^1_{n-(2k+1)}$ and $\widetilde{F}^2_{n-(2k+1)}$ are homogeneous polynomials in x, y and z of degree $n-(2k+1)$ and $n-(2k+2)$, respectively.

Since we want to find a polynomial solution $f_{n-(2k+1)}$, the variable x must divide the right hand side of the above equation. Comparing the coefficients of the terms containing the factor of even degree of z and non containing x, we obtain that

$$a_1^{n-2k} = 0,\quad a_5^{n-2(k-1)} = 0,\quad a_9^{n-2(k-2)} = 0,\ \cdots,\ a_{4k-3}^{n-2} = 0,\quad a_{4k+1} = 0.$$

Eliminating the terms non containing x on the right hand side of this last equation, and then dividing by x, we get by transformations (10) and (11) that

$$\begin{aligned} -y\frac{d\overline{f}_{n-(2k+1)}}{d\nu} &= \frac{s^{2k+1}}{(2k)!}y\nu^{2k}D^{2k+1}\overline{f}_n + O\left(\frac{1}{x}D^{2k}\overline{f}_n\right) \\ &+ \sum_{j=0}^{k-1}\left[\frac{s^{2j+1}}{(2j)!}y\nu^{2j}D^{2j+1}\overline{f}^*_{n-2(k-j)} + O\left(\frac{1}{x}D^{2j}\overline{f}^*_{n-2(k-j)}\right)\right] \\ &+ \nu\left[\overline{F}^1_{n-(2k+1)}(x,\nu^2,\mu) + \overline{F}^2_{n-(2k+1)}(x,\nu^2,\mu)y\right], \end{aligned}$$

where $y = \sqrt{\mu-\nu^2}$.

Dividing by $-y$ the above equation, and then integrating with respect to ν, we obtain from integrating formulas (33) that

$$\overline{f}_{n-(2k+1)} = -\frac{s^{2k+1}}{(2k+1)!}\nu^{2k+1}D^{2k+1}\overline{f}_n + O\left(\frac{1}{x}D^{2k}\overline{f}_n\right)$$

$$
\begin{aligned}
&+\sum_{j=0}^{k-1}\left[-\frac{s^{2j+1}}{(2j+1)!}\nu^{2j+1}D^{2j+1}\overline{f}^*_{n-2(k-j)}+O\left(\frac{1}{x}D^{2j}\overline{f}^*_{n-2(k-j)}\right)\right]\\
&+\overline{f}^1_{n-(2k+1)}(x,\nu^2,\mu)+\overline{f}^2_{n-(2k+1)}(x,\nu^2,\mu)y+\overline{f}^*_{n-(2k+1)}(x,\mu),
\end{aligned}
$$

where $\overline{f}^1_{n-(2k+1)}(x,\nu^2,\mu) = f^1_{n-(2k+1)}(x,y,z)$ and $\overline{f}^2_{n-(2k+1)}(x,\nu^2,\mu) = f^2_{n-(2k+1)}(x,y,z)$ are homogeneous polynomials of degree $n-(2k+1)$ and $n-(2k+2)$, respectively; $\overline{f}^*_{n-(2k+1)}(x,\mu) = f^*_{n-(2k+1)}(x,y,z)$ is a homogeneous polynomial of degree $n-(2k+1)$ with the form (35). Hence, by (10) we get

$$
\begin{aligned}
f_{n-(2k+1)} =& -\frac{s^{2k+1}}{(2k+1)!}z^{2k+1}D^{2k+1}f_n+O\left(\frac{1}{x}D^{2k}f_n\right)\\
&+\sum_{j=0}^{k-1}\left[-\frac{s^{2j+1}}{(2j+1)!}\nu^{2j+1}D^{2j+1}f^*_{n-2(k-j)}+O\left(\frac{1}{x}D^{2j}f^*_{n-2(k-j)}\right)\right]\\
&+f^1_{n-(2k+1)}(x,z^2,y^2+z^2)+f^2_{n-(2k+1)}(x,z^2,y^2+z^2)y\\
&+\overline{f}^*_{n-(2k+1)}(x,y,z).
\end{aligned}
$$

By induction, it follows that

$$a_i = 0, \qquad \text{for } i = 1,3,5,\cdots,n-2,n.$$

This means that $f_n(x,y,z) \equiv 0$, so the Lorenz system has no invariant algebraic surfaces of odd degree. This completes the proof of statement (a).

We now prove statement (b). If $H(x,y,z)$ is a polynomial first integral of odd degree, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R \equiv 0.$$

So, $H(x,y,z) - \text{constant} = 0$ would be an invariant algebraic surface of odd degree for the Lorenz system with cofactor $k(x,y,z) = 0$. It contradicts statement (a). This proves statement (b). ∎

We remark that the homogeneous polynomial $f_n(x,y,z)$ of the form (23) has $(n+1)/2$ coefficients, and that from the proof of Theorem 4 it is sufficient to compute the homogeneous polynomial $f_{\frac{n-1}{2}}$ of degree $(n-1)/2$ in order to obtain that all the coefficients of f_n are equal to zero.

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