

HOMOTOPY EQUIVALENCES OF p -COMPLETED CLASSIFYING SPACES OF FINITE GROUPS

CARLES BROTO, RAN LEVI, AND BOB OLIVER

Throughout this paper, let p denote a fixed prime. For any finite group G , we write BG_p^\wedge to denote the Bousfield-Kan p -completion of BG . The main results in this paper are algebraic conditions for the existence of homotopy equivalences between BG_p^\wedge and BG'_p^\wedge , for any pair of finite groups G and G' , and a description of the space of self homotopy equivalences of BG_p^\wedge .

Spaces of the form BG_p^\wedge for G finite are not generally aspherical. In fact, if G contains a p -perfect subgroup N (i.e. $H_1(BN)$ has order prime to p), whose order is divisible by p , then BG_p^\wedge has non-vanishing higher homotopy groups in arbitrarily high dimensions. For a general discussion of BG_p^\wedge for G finite we refer to [Lev].

This work originated as a study by the first two authors of the space of self equivalences of BG_p^\wedge , which was motivated by the problem of classifying fibrations with BG_p^\wedge as fiber. Some of their work on this subject appears in [BL]. A possibly more general motivation for the project is the analogy with compact Lie groups. For compact connected simple Lie groups G the space of self maps of BG was studied by Jackowski, McClure and Oliver [JMO]. In their study homology decompositions for classifying spaces were used to provide a description of the corresponding mapping spaces.

Homology decompositions are one of the most important tools in the study of classifying spaces and maps between them. Our approach however is different. Central to our study are certain categories $\tilde{\mathcal{L}}_p^c(G)$, defined for any finite group G , which we proceed by describing.

Recall that a p -subgroup $P \leq G$ is p -centric if its center $Z(P)$ is the Sylow p -subgroup of its centralizer $C_G(P)$. Equivalently, the centralizer of such a group naturally splits as a cartesian product $C_G(P) \cong Z(P) \times C'_G(P)$,

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where $C'_G(P)$ is a group of order prime to p . Objects of $\bar{\mathcal{L}}_p^c(G)$ are p -centric subgroups $P \leq G$. For any pair of such subgroups $P, Q \leq G$,

$$\text{Mor}_{\bar{\mathcal{L}}_p^c(G)}(P, Q) = N_G(P, Q)/C'_P(G),$$

where $N_G(P, Q) = \{x \in G \mid xPx^{-1} \leq Q\}$.

Our first theorem is the following:

Theorem A. *For any prime p and a pair G, G' of finite groups, the p -completed classifying spaces BG_p^\wedge and BG'_p^\wedge are homotopy equivalent if and only if the categories $\bar{\mathcal{L}}_p^c(G)$ and $\bar{\mathcal{L}}_p^c(G')$ are equivalent.*

The work in [BL] was motivated in part by the paper [MP], where the authors claim an algebraic condition for determining whether or not two distinct p -completed classifying spaces have the same homotopy type. In the course of trying to compare this claim to their own work, the authors of this article were led to the discovery of an error in [MP], which invalidates the proof of the main theorem there. We still do not know whether or not their theorem is true, but our attempts to fix the error led to Theorem A, which gives a different algebraic criterion for the existence of an equivalence.

We next consider the topological monoid $\text{Aut}(BG_p^\wedge)$ of all self homotopy equivalences of BG_p^\wedge . Let $\text{Out}(BG_p^\wedge)$ denote the group of components $\pi_0(\text{Aut}(BG_p^\wedge))$. For any category \mathcal{C} , let $\text{Aut}(\mathcal{C})$ denote the strict monoidal category whose objects are self equivalences $F : \mathcal{C} \longrightarrow \mathcal{C}$ and whose morphisms are natural isomorphisms of functors. Notice that every morphism in $\text{Aut}(\mathcal{C})$ is invertible making it into a groupoid. Let $\text{Out}(\mathcal{C})$ denote the group of components of $\text{Aut}(\mathcal{C})$, namely the set of equivalence classes obtained by identifying two functors $F', F'' \in \text{Aut}(\mathcal{C})$ if they are naturally isomorphic. These uses of the notation $\text{Out}(-)$ are motivated by the observation that $\text{Out}(BG) \cong \text{Out}(\mathcal{B}(G)) \cong \text{Out}(G)$ for any discrete group G (where $\mathcal{B}(G)$ is the category with one object and morphism group G).

The categories $\bar{\mathcal{L}}_p^c(G)$ come equipped with a natural forgetful functor λ_G from $\bar{\mathcal{L}}_p^c(G)$ to the category of groups. A self equivalence ψ of the category $\bar{\mathcal{L}}_p^c(G)$ is called *isotypical* if the functor λ_G and the composite functor $\lambda_G \psi$ are naturally isomorphic. The full subcategory $\text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$ of $\text{Aut}(\bar{\mathcal{L}}_p^c(G))$ whose objects are all isotypical self equivalences of $\bar{\mathcal{L}}_p^c(G)$ inherits a monoidal structure. We let $\text{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \leq \text{Out}(\bar{\mathcal{L}}_p^c(G))$ denote the subgroup of classes of isotypical self equivalences. Our next theorem gives a description of $\text{Out}(BG_p^\wedge)$.

Theorem B. *For any prime p and a finite group G , there is an isomorphism of groups*

$$\mathrm{Out}_{\mathrm{typ}}(\bar{\mathcal{L}}_p^c(G)) \xrightarrow{\cong} \mathrm{Out}(BG_p^\wedge).$$

The homotopy groups of a small category \mathcal{C} can be defined purely algebraically or equivalently as the homotopy groups of the geometric realization of its nerve. Theorem B claims in fact that $\mathrm{Out}(BG_p^\wedge) = \pi_0(\mathrm{Aut}(BG_p^\wedge))$ is isomorphic as a group to $\pi_0(\mathcal{A}ut_{\mathrm{typ}}(\bar{\mathcal{L}}_p^c(G)))$. The next theorem generalizes Theorem B by showing that in fact the groupoid $\mathcal{A}ut_{\mathrm{typ}}(\bar{\mathcal{L}}_p^c(G))$ completely describes the homotopy type of $\mathrm{Aut}(BG_p^\wedge)$. The proof makes use of the fact, proven in [BL], that $\mathrm{Aut}(BG_p^\wedge)$ is aspherical. For a small category \mathcal{C} let $|\mathcal{C}|$ denote the geometric realization of the nerve of \mathcal{C} .

Theorem C. *For any prime p and a finite group G there is a weak homotopy equivalence*

$$\mathrm{Aut}(BG_p^\wedge) \simeq |\mathcal{A}ut_{\mathrm{typ}}(\bar{\mathcal{L}}_p^c(G))|.$$

In fact these spaces are weakly equivalent as topological monoids, in the sense that their classifying spaces are also weakly homotopy equivalent.

Theorem C is stated in terms of weak equivalences rather than homotopy equivalence because of the usual problem of whether or not a mapping space between two CW complexes has itself the homotopy type of a CW complex. If one is willing to work simplicially, i.e. replace spaces by their singular simplicial sets and mapping spaces by the respective simplicial mapping spaces, then one may replace “weak equivalence” by “equivalence” in the statement of the theorem. Throughout the paper we shall not distinguish between homotopy equivalences and weak homotopy equivalences, but the above comment should be kept in mind.

The identity component of $\mathrm{Aut}(BG_p^\wedge)$ was shown in [BL] to have the homotopy type of $BZ(\bar{G})$, where $\bar{G} = G/O_p G$, the quotient of G by its maximal normal subgroup of order prime to p .

Theorem A will be proven in a more precise form as Theorem 2.9 below. Theorems B and C will be proven together as Theorem 4.5.

Let G and G' be finite groups, with Sylow p -subgroups $S \leq G$ and $S' \leq G'$. An isomorphism $\varphi : S \xrightarrow{\cong} S'$ is called *fusion preserving* if for any $P, Q \leq S$, a group isomorphism $\alpha : P \xrightarrow{\cong} Q$ is induced by conjugation in G if and only if the corresponding isomorphism $\varphi\alpha\varphi^{-1} : \varphi(P) \rightarrow \varphi(Q)$ is induced by conjugation in G' .

The claim of the main theorem in [MP] can be reformulated in those terms. However an error in its proof occurs on [MP, p.129], where a certain

functor fails to be well defined on morphisms. As we have neither a proof nor a counterexample of this statement we restate it as the following

Conjecture D (Martino-Priddy [MP]). *For any prime p and a pair G, G' of finite groups, $BG_p^\wedge \simeq BG'_p^\wedge$ if and only if there is a fusion preserving isomorphism between their Sylow p -subgroups.*

Theorem A gives another purely algebraic condition for BG_p^\wedge and BG'_p^\wedge to be homotopy equivalent. However, the condition in the Martino-Priddy Conjecture is simpler to check in practice. It is easier in general to construct an isomorphism between Sylow subgroups and show that it preserves fusion, than to construct an equivalence between categories $\bar{\mathcal{L}}_p^c(-)$.

In Section 6, we study the relation between Conjecture D and our Theorem A. Let \mathcal{Z} be the contravariant functor on the orbit category for p -centric subgroups of G which sends P to its center $Z(P)$. Let G be a finite group with a Sylow p -subgroup S . Let $\text{Out}_{\text{fus}}(S)$ denote the quotient of the group of fusion preserving automorphisms of S by the normal subgroup of all automorphisms induced by conjugation in G . The following theorem is an algebraic analogue of [BL, Thm. 1.6].

Theorem E. *For any prime p and a finite group G , there is an exact sequence*

$$0 \longrightarrow \varprojlim_{\mathcal{O}_p^c(G)}^1(\mathcal{Z}) \xrightarrow{\lambda_G} \text{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \xrightarrow{\mu_G} \text{Out}_{\text{fus}}(S) \xrightarrow{\omega_G} \varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z}),$$

where λ_G and μ_G are group homomorphisms.

The \varprojlim^1 term in the theorem is known not to vanish in general. The \varprojlim^2 term is the group containing the obstructions for lifting fusion preserving outer automorphisms of S to isotypical outer automorphisms of $\bar{\mathcal{L}}_p^c(G)$. We know of no group G for which the \varprojlim^2 obstruction group is non-vanishing (much less the obstruction itself not vanishing). The best we can show towards proving Conjecture D is the following (see Theorems 5.10 and 6.5 below):

Theorem F. *For any prime p and a pair of finite groups G and G' , Conjecture D is true if either of the following two conditions on G holds:*

(i) $\text{rk}_p(G) < p^2$, or

(ii) *there is no chain $P \not\leq Q \not\leq S$, where $S \leq G$ is a Sylow p -subgroup, and P and Q are both p -centric and p -radical in G .*

Here, a p -subgroup $P \leq G$ is p -radical, or p -stubborn, if $N(P)/P$ contains no nontrivial normal p -subgroups.

The paper is organized as follows. In Section 1, we define the categories $\bar{\mathcal{L}}_p^c(G)$ and discuss some of their basic properties. In Section 2 we construct the categories $\mathcal{L}_p^c(X)$ for an arbitrary space X and prove Theorem A. Sections 3 and 4 are devoted to the proof of Theorems B and C. In section 5, we discuss higher limits obstructions to the existence and uniqueness of a lift of a given fusion preserving isomorphism between Sylow subgroups to a homotopy equivalence between the respective classifying spaces and prove Theorem E and the first half of Theorem F. Section 6 is concerned with the construction of self equivalences of the categories $\bar{\mathcal{L}}_p^c(G)$ by direct methods (i.e. without using higher limits). Finally in Section 7, we give a number of examples to demonstrate the use of our methods.

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1. FROM GROUPS TO CATEGORIES

Fix a finite group G and a prime p . We begin by defining the categories $\mathcal{L}_p(G)$ and $\bar{\mathcal{L}}_p(G)$ which will play a central role throughout this paper. An object in $\mathcal{L}_p(G)$ is a p -subgroup $P \leq G$. For any pair of p -subgroups $P, Q \leq G$, we define

$$\text{Mor}_{\mathcal{L}_p(G)}(P, Q) = N_G(P, Q) \stackrel{\text{def}}{=} \{x \in G \mid xPx^{-1} \leq Q\}.$$

The morphism from P to Q corresponding to $x \in N_G(P, Q)$ will be denoted \hat{x} . Composition is given by $\hat{x} \circ \hat{y} = \widehat{xy}$.

As usual, for any finite group H , $O^p(H)$ denotes the maximal normal p -perfect subgroup of H ; equivalently the minimal normal subgroup of p -power index. We define $\bar{\mathcal{L}}_p(G)$ to be the category with the same objects as $\mathcal{L}_p(G)$, and with morphism sets

$$\text{Mor}_{\bar{\mathcal{L}}_p(G)}(P, Q) = N_G(P, Q) / O^p(C_G(P)).$$

In other words, we have divided out by the action of $O^p(C_G(P))$ on $N_G(P, Q)$ given by right multiplication. In particular, $\bar{\mathcal{L}}_p(G)$ is a quotient category of $\mathcal{L}_p(G)$. The morphism $P \longrightarrow Q$ corresponding to the class of $x \in N_G(P, Q)$ will also be denoted \hat{x} .

Our goal here is to construct a category, depending on G and p , which reflects the structure of BG_p^\wedge in a way which will be made clear later. In

order to do this, we must further restrict the categories $\mathcal{L}_p(G)$ and $\bar{\mathcal{L}}_p(G)$. Following the terminology of Dwyer, we call a subgroup $H \leq G$ *centric* if $C_G(H) \leq H$ (i.e., if $C_G(H) = Z(H)$). A p -subgroup $P \leq G$ is *p -centric* if its centralizer has the form

$$C_G(P) = Z(P) \times C'_G(P),$$

where $C'_G(P)$ has order prime to p . Equivalently, P is p -centric if and only if $|O^p(C_G(P))|$ is prime to p . We let $\mathcal{L}_p^c(G) \subseteq \mathcal{L}_p(G)$ and $\bar{\mathcal{L}}_p^c(G) \subseteq \bar{\mathcal{L}}_p(G)$ denote the full subcategories whose objects are the p -centric subgroups of G . In particular, for any p -centric $P, Q \leq G$,

$$\text{Mor}_{\bar{\mathcal{L}}_p^c(G)}(P, Q) = N_G(P, Q)/C'_G(P).$$

For any finite group G , let $\mathcal{B}(G)$ denote the category with one object o_G and morphism group G . The morphism in $\mathcal{B}(G)$ corresponding to $x \in G$ will be denoted \tilde{x} , and $\tilde{x} \circ \tilde{y} = \widetilde{xy}$. We identify the classifying space BG with the (realization of the) nerve of $\mathcal{B}(G)$. Let

$$\alpha_G: |\mathcal{L}_p(G)| \longrightarrow |\mathcal{B}(G)| = BG$$

be the nerve of the functor $\mathcal{L}_p(G) \longrightarrow \mathcal{B}(G)$ which sends each object of $\mathcal{L}_p(G)$ to o_G , and which sends a morphism $P \xrightarrow{\tilde{x}} Q$ (for $x \in N_G(P, Q)$) to $\tilde{x} \in \text{Aut}_{\mathcal{B}(G)}(o_G)$. The main result of this section is the following:

Proposition 1.1. *For any finite group G , the maps*

$$\alpha_G: |\mathcal{L}_p^c(G)| \longrightarrow BG \quad \text{and} \quad |\pi|: |\mathcal{L}_p^c(G)| \longrightarrow |\bar{\mathcal{L}}_p^c(G)|$$

are both \mathbb{F}_p -homology equivalences. Hence there is a homotopy equivalence

$$\bar{\alpha}_G: |\bar{\mathcal{L}}_p^c(G)|_p^\wedge \xrightarrow{\cong} BG_p^\wedge,$$

unique up to homotopy, such that the following triangle of homotopy equivalences commutes up to homotopy:

$$\begin{array}{ccc} |\mathcal{L}_p^c(G)|_p^\wedge & \xrightarrow[\cong]{|\pi|_p^\wedge} & |\bar{\mathcal{L}}_p^c(G)|_p^\wedge \\ & \searrow \cong & \swarrow \cong \\ & & BG_p^\wedge \end{array}$$

$(\alpha_G)_p^\wedge$ $\bar{\alpha}_G$

Proof. This follows from Lemmas 1.2 and 1.3 below. Lemma 1.2 says that α_G is an \mathbb{F}_p -homology equivalence, and hence induces a homotopy equivalence $(\alpha_G)_p^\wedge$ after p -completion. Lemma 1.3 implies as a special case that

the projection functor $\pi: \mathcal{L}_p^c(G) \longrightarrow \overline{\mathcal{L}}_p^c(G)$ induces an \mathbb{F}_p -homology equivalence between the nerves, and hence a homotopy equivalence between their p -completions. The other statements are then clear. \square

The rest of the section is devoted to the proofs of Lemmas 1.2 and 1.3. Recall [Dw] that a family \mathcal{C} of p -subgroups of G (closed under conjugation) is called *ample* if the projection $EG \times_G |K_{\mathcal{C}}| \longrightarrow BG$ is an \mathbb{F}_p -homology equivalence. Here, $|K_{\mathcal{C}}|$ denotes the realization of the nerve of the poset \mathcal{C} , and the action of G is induced by conjugation. In particular, the family of p -centric subgroups of G is ample by [Dw, §8]. In general we will denote by $\mathcal{L}_p^c(G) \subseteq \mathcal{L}_p(G)$ and $\overline{\mathcal{L}}_p^c(G) \subseteq \overline{\mathcal{L}}_p(G)$ the full subcategories whose objects are in \mathcal{C} .

Lemma 1.2. *For any finite group G , any prime p , and any ample family \mathcal{C} of p -subgroups of G , α_G restricts to an \mathbb{F}_p -homology equivalence*

$$\alpha_G^c: |\mathcal{L}_p^c(G)| \longrightarrow BG.$$

Proof. Let $K_{\mathcal{C}}$ denote the category whose set of objects is \mathcal{C} , and with a unique morphism $P \longrightarrow Q$ whenever $P \leq Q$. Let $\mathcal{E}(G)$ be the category with objects the elements of G and with a unique morphism between each pair of objects (i.e., the category whose nerve is the universal free G -space EG). Let

$$\gamma: \mathcal{E}(G) \times_G K_{\mathcal{C}} \longrightarrow \mathcal{L}_p^c(G)$$

be the functor which sends a pair of objects (g, P) to ${}^gP = gPg^{-1}$, and which sends a pair of morphisms $(g \rightarrow xg, P \leq Q)$ to $\hat{x}: {}^gP \rightarrow {}^{xg}Q$. This clearly induces bijections on objects and on morphisms, and hence a homeomorphism

$$|\gamma|: EG \times_G |K_{\mathcal{C}}| \xrightarrow{\cong} |\mathcal{L}_p^c(G)|.$$

Let $\delta: \mathcal{E}(G) \times_G K_{\mathcal{C}} \longrightarrow \mathcal{B}(G)$ be the functor which sends all objects to o_G , and which sends a morphism $(g \rightarrow xg, P \leq Q)$ to $\tilde{x} \in \text{Aut}_{\mathcal{B}(G)}(o_G)$. The following triangle of categories and functors

$$\begin{array}{ccc} \mathcal{E}(G) \times_G K_{\mathcal{C}} & \xrightarrow{\gamma} & \mathcal{L}_p^c(G) \\ & \searrow \delta & \swarrow \alpha_G^c \\ & & \mathcal{B}(G) \end{array}$$

commutes by definition, and hence induces a commutative triangle of spaces

$$\begin{array}{ccc} EG \times_G |K_{\mathcal{C}}| & \xrightarrow[\cong]{|\gamma|} & |\mathcal{L}_p^{\mathcal{C}}(G)| \\ & \searrow^{|\delta|} & \swarrow_{|\alpha_G^{\mathcal{C}}|} \\ & & BG. \end{array}$$

Since $|\delta|$ is an \mathbb{F}_p -homology equivalence (by definition of “ample”), $|\alpha_G^{\mathcal{C}}|$ is also an \mathbb{F}_p -homology equivalence. \square

The following lemma will be used to show that the projection

$$|\mathcal{L}_p^{\mathcal{C}}(G)| \longrightarrow |\bar{\mathcal{L}}_p^{\mathcal{C}}(G)|$$

is an \mathbb{F}_p -homology equivalence.

Lemma 1.3. *Fix a prime p . Let $\psi: \mathcal{C} \longrightarrow \mathcal{C}'$ be any functor between small categories which has the following properties:*

(i) *ψ is bijective on isomorphism classes of objects and is surjective on morphism sets.*

(ii) *For each object c in \mathcal{C} , the subgroup*

$$K(c) \stackrel{\text{def}}{=} \text{Ker}[\text{Aut}_{\mathcal{C}}(c) \longrightarrow \text{Aut}_{\mathcal{C}'}(\psi(c))]$$

is finite of order prime to p .

(iii) *For each pair of objects c and d , and each $f, g: c \rightarrow d$ in \mathcal{C} , $\psi(f) = \psi(g)$ if and only if there is some $\sigma \in K(c)$ such that $g = f \circ \sigma$ (i.e., $\text{Mor}_{\mathcal{C}'}(\psi(c), \psi(d)) \cong \text{Mor}_{\mathcal{C}}(c, d)/K(c)$).*

Then for any functor $F: \mathcal{C}' \longrightarrow \text{Spaces}$, the induced map

$$\underline{\text{hocolim}}_{\mathcal{C}}(F) \longrightarrow \underline{\text{hocolim}}_{\mathcal{C}'}(F \circ \psi)$$

is an \mathbb{F}_p -homology equivalence, and hence induces a homotopy equivalence between the p -completions. Also, for any functor $T: \mathcal{C}'^{\text{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-mod}$,

$$\underline{\text{lim}}_{\mathcal{C}}^*(T) \cong \underline{\text{lim}}_{\mathcal{C}'}^*(T \circ \psi).$$

Proof. We can assume that ψ is bijective on objects (not just on isomorphism classes). If not, then replace \mathcal{C}' by the equivalent category \mathcal{C}'' whose objects are those of \mathcal{C} , and where $\text{Mor}_{\mathcal{C}''}(c_1, c_2) = \text{Mor}_{\mathcal{C}'}(\psi(c_1), \psi(c_2))$.

Let $\mathcal{C}\text{-mod}$ and $\mathcal{C}'\text{-mod}$ denote the categories of functors $\mathcal{C}^{\text{op}} \rightarrow \text{Ab}$ and $\mathcal{C}'^{\text{op}} \rightarrow \text{Ab}$. We proceed by showing that the functor $\psi^*: \mathcal{C}'\text{-mod} \rightarrow \mathcal{C}\text{-mod}$

given by composition on the right with ψ has a left adjoint, which is a useful observation in proving our claim. Define $(-)_K : \mathcal{C}\text{-mod} \longrightarrow \mathcal{C}'\text{-mod}$ by

$$D_K(c') = D(c)_{K(c)} \stackrel{\text{def}}{=} \mathbb{Z} \otimes_{K(c)} D(c) = H_0(K(c); D(c))$$

for any D in $\mathcal{C}\text{-mod}$, any $c' \in \mathcal{C}'$ and $c = \psi^{-1}(c')$. Note that c is uniquely defined by the assumption that ψ is bijective on objects. To see that D_K is well defined as a (contravariant) functor on \mathcal{C}' , let $f' : \psi(c) \rightarrow \psi(d)$ be any morphism and let $f : c \rightarrow d$ be a lift to \mathcal{C} . Then for any $\tau \in K(d)$ there exists, by (iii) above (applied with $g = \tau \circ f$), some $\sigma \in K(c)$ such that $f \circ \sigma = \tau \circ f$, and any other lift of f' differs from f by an element of $K(c)$.

By construction, for any D in $\mathcal{C}\text{-mod}$ and D' in $\mathcal{C}'\text{-mod}$,

$$\text{Hom}_{\mathcal{C}'\text{-mod}}(D_K, D') \cong \text{Hom}_{\mathcal{C}\text{-mod}}(D, D' \circ \psi), \quad (1)$$

and we have thus shown that $(-)_K$ is left adjoint to ψ^* . In particular, this shows that D_K is $\mathcal{C}'\text{-mod}$ -projective if D is $\mathcal{C}\text{-mod}$ -projective. So if

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$$

is any $\mathcal{C}\text{-mod}$ projective resolution of the constant functor \mathbb{Z} , then

$$\cdots \longrightarrow (P_2)_K \longrightarrow (P_1)_K \longrightarrow (P_0)_K \longrightarrow \underline{\mathbb{Z}} \longrightarrow 0$$

is a chain complex of $\mathcal{C}'\text{-mod}$ -projectives which is exact (a resolution) after localization at p (the groups $K(c)$ all being finite of order prime to p). Since

$$\varprojlim_{\mathcal{C}} (T \circ \psi) \cong \text{Hom}_{\mathcal{C}\text{-mod}}(\underline{\mathbb{Z}}, T \circ \psi),$$

we have

$$\varprojlim_{\mathcal{C}}^* (T \circ \psi) \cong \text{Ext}_{\mathcal{C}\text{-mod}}^*(\underline{\mathbb{Z}}, T \circ \psi) \cong H^*(\text{Hom}_{\mathcal{C}\text{-mod}}(P_*, T \circ \psi)).$$

Similarly, since $T(c)$ is a $\mathbb{Z}_{(p)}$ -module for all c in \mathcal{C}' ,

$$\varprojlim_{\mathcal{C}'}^* (T) \cong \text{Ext}_{\mathcal{C}'\text{-mod}}^*(\underline{\mathbb{Z}}, T) \cong H^*(\text{Hom}_{\mathcal{C}\text{-mod}}((P_*)_K, T)).$$

So $\varprojlim_{\mathcal{C}}^* (T \circ \psi) \cong \varprojlim_{\mathcal{C}'}^* (T)$ by (1).

The \mathbb{F}_p -homology equivalence between homotopy colimits now follows from the isomorphisms

$$\varprojlim_{\mathcal{C}}^* H^*(F \circ \psi(-); \mathbb{F}_p) \cong \varprojlim_{\mathcal{C}'}^* H^*(F(-); \mathbb{F}_p),$$

together with the spectral sequence for the cohomology of a homotopy colimit. \square

Remark 1.4. According to [Dw], any ample family \mathcal{C} of p -subgroups of a finite group G gives rise to an \mathbb{F}_p -homology equivalence

$$\underset{\mathcal{P}_p^{\mathcal{C}}(G)}{\text{hocolim}} EG/C_G(-) \longrightarrow BG$$

where $\mathcal{P}_p^{\mathcal{C}}(G)$ is the Frobenius category whose objects are the subgroups in \mathcal{C} and whose morphisms are homomorphisms induced by conjugation in G . That is,

$$\text{Mor}_{\mathcal{P}_p^{\mathcal{C}}(G)}(P, Q) \cong N_G(P, Q)/C_G(P).$$

By the same methods, we can obtain another \mathbb{F}_p -homology equivalence

$$\varinjlim_{\bar{\mathcal{L}}_p^{\mathcal{C}}(G)} EG/O^p(C_G(-)) \longrightarrow BG.$$

The relationship between these two decompositions can be described as moving information from the functor in the first decomposition to the indexing category in the second. The case corresponding to the discussion above is the family of subgroups P such that $O^p(C_G(P))$ has order prime to p , which holds if and only if P is p -centric. In this case $EG/O^p(C_G(P)) \simeq BO^p(C_G(P))$ is p -acyclic, which explains the existence of the homotopy equivalence $\bar{\alpha}_G: |\bar{\mathcal{L}}_p^{\mathcal{C}}(G)|_p^\wedge \xrightarrow{\cong} BG_p^\wedge$ of Proposition 1.1 in terms of the above \mathbb{F}_p -homology equivalences, and also why we do not expect a similar equivalence to be true for families of subgroups which are not p -centric.

2. FROM SPACES TO DISCRETE CATEGORIES

In this section we introduce a homotopy theoretic analogue of the categories $\mathcal{L}_p(G)$: a category $\mathcal{L}_p(X)$, defined for an arbitrary space X . The remarkable feature of this construction is that $\mathcal{L}_p(BG) \simeq \mathcal{L}_p(G)$ and $\mathcal{L}_p(BG_p^\wedge) \simeq \bar{\mathcal{L}}_p(G)$. The following proposition suggests a way of doing this. For any pair of groups G, H , we set

$$\text{Rep}(G, H) = \text{Hom}(G, H)/\text{Inn}(H) :$$

the set of homomorphisms $G \longrightarrow H$ modulo conjugacy in H .

Proposition 2.1. *For any finite p -group P and any finite group G , the p -completion map $BG \longrightarrow BG_p^\wedge$ induces a (weak) homotopy equivalence*

$$\text{Map}(BP, BG)_p^\wedge \xrightarrow{\cong} \text{Map}(BP, BG_p^\wedge).$$

In particular, the maps

$$\text{Rep}(P, G) \xrightarrow[\cong]{B} [BP, BG] \xrightarrow[\cong]{\kappa_*} [BP, BG_p^\wedge],$$

where B sends a homomorphism $\rho: P \longrightarrow G$ to $B\rho$, are bijections. Also, for each $\rho: P \longrightarrow G$, the induced product map $C_G(\rho) \times P \longrightarrow G$ induces

(after taking adjoints) a homotopy equivalence

$$BC_G(\rho) \longrightarrow \text{Map}(BP, BG)_{B\rho}$$

and a weak homotopy equivalence

$$BC_G(\rho)_p^\wedge \longrightarrow \text{Map}(BP, BG_p^\wedge)_{B\rho}.$$

Proof. The description of $\text{Map}(BP, BG)$ is classical, and in fact holds for any pair of discrete groups. The fact that $\text{Map}(BP, BG_p^\wedge)$ is the p -completion of $\text{Map}(BP, BG)$ (when P is a p -group and G is finite) is shown in [BL, Proposition 2.1]. \square

Proposition 2.1 suggests replacing a homomorphism $P \longrightarrow G$, for a p -group P , by a map $BP \longrightarrow X$. We next state a homotopy theoretic condition which corresponds to a homomorphism being injective. There could be several natural choices for such a condition. The following definition of a *homotopy monomorphism* seems to be the most useful for our purposes here.

Definition 2.2. *A map $f: X \longrightarrow Y$ is said to be a homotopy monomorphism at p if $H^*(X; \mathbb{F}_p)$ is a finitely generated module over $H^*(Y; \mathbb{F}_p)$ via the induced map f^* . For any space X and any prime p , a subgroup of X at p (a p -subgroup of X) is a pair (G, j) where G is a group (a p -group) and $j: BG \longrightarrow X$ is a homotopy monomorphism.*

This definition of homotopy monomorphisms at p is motivated by the following lemma.

Lemma 2.3. *Let G and G' be two finite groups and $f: G \longrightarrow G'$ a homomorphism. Then $Bf: BG \longrightarrow BG'_p^\wedge$ is a homotopy monomorphism at p if and only if $\text{Ker}(f)$ has order prime to p .*

Proof. This statement is implicit in [Sw]. Assume first that $f: G \longrightarrow G'$ is a monomorphism, and fix an embedding $G' \subseteq U(n)$ for some n . The composite $G \longrightarrow G' \longrightarrow U(n)$ induces a fibration

$$U(n)/G \longrightarrow BG \longrightarrow BU(n),$$

and the E^2 -term of the spectral sequence for this fibration is finitely generated as a module over $H^*(BU(n); \mathbb{F}_p)$. Hence, since $H^*(BU(n); \mathbb{F}_p)$ is noetherian, the E^∞ -term is also finitely generated. The composite

$$BG \longrightarrow BG' \longrightarrow BU(n)$$

thus makes $H^*(G; \mathbb{F}_p)$ into a finitely generated $H^*(BU(n); \mathbb{F}_p)$ -module; and hence a finitely generated $H^*(G'; \mathbb{F}_p)$ -module.

If $f: G \longrightarrow G'$ is such that $K = \text{Ker}(f)$ has order prime to p , then $H^*(G; \mathbb{F}_p) \cong H^*(G/K; \mathbb{F}_p)$ by the Serre spectral sequence for the fibration $BK \rightarrow BG \rightarrow B(G/K)$. So $H^*(G; \mathbb{F}_p)$ is finitely generated as an $H^*(G'; \mathbb{F}_p)$ -module in this case also.

Conversely, assume that $H^*(G; \mathbb{F}_p)$ is finitely generated over $H^*(G'; \mathbb{F}_p)$, and again set $K = \text{Ker}(f)$. Since $K \leq G$, $H^*(K; \mathbb{F}_p)$ is a finitely generated $H^*(G; \mathbb{F}_p)$ -module, and thus a finitely generated $H^*(G'; \mathbb{F}_p)$ -module. Since $f|_K: K \longrightarrow G'$ is trivial, $\text{Im}[H^*(G'; \mathbb{F}_p) \longrightarrow H^*(K; \mathbb{F}_p)] = \mathbb{F}_p$; and hence $\dim_{\mathbb{F}_p}(H^*(K; \mathbb{F}_p)) < \infty$. And by [Sw, Corollary 1], this implies that $|K|$ is prime to p . \square

We also need to define the analogue of “ p -centric subgroups” of a space. Recall (Proposition 2.1) that for any finite group G and any p -subgroup $P \leq G$, $BC_G(P) \simeq \text{Map}(BP, BG)_{\text{incl}}$ and $BC_G(P)_p^\wedge \simeq \text{Map}(BP, BG_p^\wedge)_{\text{incl}}$. This motivates the following definition, originally due to Dwyer and Kan [DK].

Definition 2.4. *A map $f: X \longrightarrow Y$ is centric if the induced map*

$$(f \circ -): \text{Map}(X, X)_{\text{id}} \longrightarrow \text{Map}(X, Y)_f$$

is a homotopy equivalence, and is p -centric if $(f \circ -)$ is an \mathbb{F}_p -homology equivalence. A p -subgroup (P, j) of a space X is p -centric if the map $j: BP \longrightarrow X$ is p -centric.

We are now ready to define the categories $\mathcal{L}_p^c(X) \subseteq \mathcal{L}_p(X)$.

Definition 2.5. *Fix a space X . Define $\mathcal{L}_p(X)$ to be the category whose objects are the p -subgroups (P, j) of X ; and where a morphism from (P, j) to (Q, k) is a pair (φ, η) , where $\varphi: P \longrightarrow Q$ is a homomorphism, and where η represents a homotopy class of paths in $\text{Map}(BP, X)$ from j to $k \circ B\varphi$. Let $\mathcal{L}_p^c(X)$ be the full subcategory of $\mathcal{L}_p(X)$ whose objects are the p -centric subgroups of X .*

If $f: X \longrightarrow Y$ is a homotopy monomorphism at p , then there is an induced functor $\mathcal{L}_p(f): \mathcal{L}_p(X) \longrightarrow \mathcal{L}_p(Y)$, and this correspondence satisfies the usual functorial properties. In order to obtain similar functoriality properties for the construction \mathcal{L}_p^c we need maps to preserve p -centricity; that is, for (P, j) a p -centric subgroup of X we need that $(P, f \circ j)$ is also p -centric in Y . This happens only under very restrictive conditions. One relevant case of interest to us is that of the p -completion map $BG \longrightarrow BG_p^\wedge$, for a finite group G . This is clearly a homotopy monomorphism at p and by Proposition 2.1, preserves p -centricity. It therefore induces functors

$$\mathcal{L}_p(\kappa): \mathcal{L}_p(BG) \longrightarrow \mathcal{L}_p(BG_p^\wedge) \quad \text{and} \quad \mathcal{L}_p^c(\kappa): \mathcal{L}_p^c(BG) \longrightarrow \mathcal{L}_p^c(BG_p^\wedge).$$

For any finite group G , any p -subgroups $P, Q \leq G$, and any $x \in N_G(P, Q)$, we let $c_x: P \longrightarrow Q$ denote the homomorphism $c_x(g) = xgx^{-1}$. Also, $\eta_x: BP \times I \longrightarrow BG$ denotes the homotopy (i.e., path in $\text{Map}(BP, BG)$) induced by the natural transformation $o_P \mapsto \tilde{x} \in \text{Aut}_{\mathcal{B}(G)}(o_G)$ between the functors $\mathcal{B}(\text{incl}), \mathcal{B}c_x: \mathcal{B}(P) \longrightarrow \mathcal{B}(G)$.

Proposition 2.6. *For any finite group G and any prime p , let*

$$\beta_G: \mathcal{L}_p(G) \longrightarrow \mathcal{L}_p(BG)$$

be the functor defined as follows. For any p -subgroup $P \leq G$, set $\beta_G(P) = (P, i_P)$, where $i_P: BP \longrightarrow BG$ denotes the inclusion. For any pair of p -subgroups $P, Q \leq G$, and any $x \in N_G(P, Q)$, set

$$\beta_G(\hat{x}) = (c_x, \eta_x): (P, i_P) \longrightarrow (Q, i_Q).$$

Then β_G is an equivalence of categories, and restricts to an equivalence

$$\beta_G^c: \mathcal{L}_p^c(G) \longrightarrow \mathcal{L}_p^c(BG).$$

Proof. By Proposition 2.1, $[BP, BG] \cong \text{Rep}(P, G)$ for any finite group P . By Lemma 2.3, a homomorphism $\rho: P \longrightarrow G$ is injective if and only if $B\rho: BP \longrightarrow BG$ is a homotopy monomorphism. This shows that β_G induces a bijection between isomorphism classes of objects in $\mathcal{L}_p(G)$ and in $\mathcal{L}_p(BG)$.

By Proposition 2.1 again, for any p -subgroup $P \leq G$, (P, i_P) is p -centric in the sense of Definition 2.4 if and only if the inclusion $BZ(P) \rightarrow BC_G(P)$ is an \mathbb{F}_p -homology equivalence. Since $Z(P)$ is central in $C_G(P)$, the inclusion induces an \mathbb{F}_p -homology equivalence if and only if $C_G(P)/Z(P)$ has order prime to p , if and only if P is p -centric in G . This shows that β_G^c also induces a bijection between isomorphism classes of objects.

It remains to show, for all $P, Q \leq G$, that β_G induces a bijection

$$\text{Mor}_{\mathcal{L}_p(G)}(P, Q) \longrightarrow \text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q)).$$

For any $x \in N_G(P, Q)$ (i.e., any morphism $\hat{x}: P \longrightarrow Q$ in $\mathcal{L}_p(G)$), $\beta_G(\hat{x}) = (c_x, \eta_x)$, and by construction the restriction of $\eta_x: BP \times I \longrightarrow BG$ to the basepoint of BP is a loop which represents the element $x \in \pi_1(G) \cong G$. This proves that the map from $\text{Mor}_{\mathcal{L}_p(G)}(P, Q)$ to $\text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$ is injective. So it remains only to show that these morphism sets have the same order.

By definition, $\text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$ consists of pairs (ρ, η) where $\rho: P \longrightarrow Q$ is a homomorphism and η represents a homotopy class of paths

in $\text{Map}(BP, BG)$ from i_P to $i_Q \circ B\rho$. For any fixed ρ , $\pi_1(\text{Map}(BP, BG)_{i_P}) \cong C_G(P)$ acts freely and transitively on the set of homotopy classes of paths of $\text{Map}(BP, BG)$ from i_P to $i_Q \circ B\rho$. Hence $\pi_1(\text{Map}(BP, BG)_{i_P}) \cong C_G(P)$ acts freely on $\text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$ and orbits are in one to one correspondence with all homomorphism $P \longrightarrow Q$ induced by conjugation; that is, $N_G(P, Q)/C_G(P)$. In particular,

$$\begin{aligned} |\text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))| &= |N_G(P, Q)/C_G(P)| \cdot |C_G(P)| = \\ &= |N_G(P, Q)| = |\text{Mor}_{\mathcal{L}_p(G)}(P, Q)|. \quad \square \end{aligned}$$

We next show that the categories $\bar{\mathcal{L}}_p^c(G)$ and $\mathcal{L}_p^c(BG_p^\wedge)$ are equivalent for any finite group G . This uses the description in the previous section of maps $BP \longrightarrow BG_p^\wedge$ for p -groups P .

Proposition 2.7. *The composite functor*

$$\mathcal{L}_p(G) \xrightarrow{\beta_G} \mathcal{L}_p(BG) \xrightarrow{\mathcal{L}_p(\kappa)} \mathcal{L}_p(BG_p^\wedge)$$

factors through an equivalence of categories

$$\bar{\beta}_G: \bar{\mathcal{L}}_p(G) \longrightarrow \mathcal{L}_p(BG_p^\wedge),$$

which restricts to an equivalence of subcategories

$$\bar{\beta}_G^c: \bar{\mathcal{L}}_p^c(G) \longrightarrow \mathcal{L}_p^c(BG_p^\wedge).$$

Proof. Define $\bar{\beta}_G$ on objects by setting $\bar{\beta}_G(P) = (P, i_P)$ for any p -subgroup $P \leq G$ (where i_P now denotes $BP \rightarrow BG_p^\wedge$, the p -completion of the map induced by inclusion). Since $[BP, BG_p^\wedge] \cong [BP, BG]$ for any p -group P (Proposition 2.1), this induces a bijection between isomorphism classes of objects in $\bar{\mathcal{L}}_p(G)$ and those in $\mathcal{L}_p(BG_p^\wedge)$. Also, i_P is p -centric as a map to BG_p^\wedge if and only if it is p -centric as a map to BG (Proposition 2.1 again), and this holds if and only if P is p -centric by Proposition 2.6.

It remains to show, for any pair of p -subgroups $P, Q \leq G$, that there is a (unique) bijection $\bar{\beta}_G(P, Q)$ which makes the following square commute:

$$\begin{array}{ccc} N_G(P, Q) = \text{Mor}_{\mathcal{L}_p(G)}(P, Q) & \xrightarrow[\cong]{\beta_G(P, Q)} & \text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q)) \\ \downarrow & & \downarrow \kappa(P, Q) \\ N_G(P, Q)/C_G(P) = \text{Mor}_{\bar{\mathcal{L}}_p(G)}(P, Q) & \xrightarrow{\bar{\beta}_G(P, Q)} & \text{Mor}_{\mathcal{L}_p(BG_p^\wedge)}((P, i_P), (Q, i_Q)). \end{array}$$

The same argument as that used in the proof of Proposition 2.6 shows that $\pi_1(\text{Map}(BP, BG_p^\wedge)_{Bi_P})$ acts freely on $\text{Mor}_{\mathcal{L}_p(BG_p^\wedge)}((P, i_P), (Q, i_Q))$ with orbit set $N_G(P, Q)/C_G(P)$, the same as the orbit set of the action of

$\pi_1(\text{Map}(BP, BG)_{Bi_P})$ on $\text{Mor}_{\mathcal{L}_p(BG)}((P, i_P), (Q, i_Q))$. Furthermore, the map $\kappa(P, Q)$ is equivariant via the induced map of fundamental groups

$$\pi_1(\text{Map}(BP, BG)_{Bi_P}) \longrightarrow \pi_1(\text{Map}(BP, BG_p^\wedge)_{Bi_P}). \quad (1)$$

By Proposition 2.1,

$$\text{Map}(BP, BG_p^\wedge)_{i_P} \simeq BC_G(P)_p^\wedge \simeq (\text{Map}(BP, BG)_{i_P})_p^\wedge.$$

For any finite group L , $L/O^p(L)$ is a p -group, hence $B(L/O^p(L))$ is p -complete, $BO^p(L)_p^\wedge$ is simply connected by [BK, VII.3.2], and the sequence

$$BO^p(L)_p^\wedge \longrightarrow BL_p^\wedge \longrightarrow B(L/O^p(L))$$

is a fibration sequence by [BK, II.5.2(iv)], hence

$$\pi_1(BL_p^\wedge) \cong L/O^p(L).$$

In particular, the homomorphism of equation (1) is onto, hence so is $\kappa(P, Q)$.

Also, for any $x, y \in N_G(P, Q)$ such that $c_x = c_y \in \text{Hom}(P, Q)$, η_x and η_y are homotopic as paths in $\text{Map}(BP, BG_p^\wedge)$ from i_P to $i_Q \circ Bc_x$ if and only if $\eta_x \eta_y^{-1}$ is trivial in $\pi_1(\text{Map}(BP, BG_p^\wedge)_{i_P})$; that is, if and only if $[xy^{-1}] = 1 \in \pi_1(BC_G(P)_p^\wedge)$, if and only if $xy^{-1} \in O^p(C_G(P))$, if and only if $\hat{x} = \hat{y}$ in $\text{Mor}_{\bar{\mathcal{L}}_p(G)}(P, Q)$. And this shows that $\bar{\beta}_G(P, Q)$ is well defined and a bijection. Since $\bar{\beta}_G$ takes p -centric objects to p -centric objects, its restriction to $\bar{\mathcal{L}}_p^c(G)$ is also an equivalence of categories. \square

Upon combining Propositions 2.6 and 2.7 with Lemma 1.3, we get the following:

Corollary 2.8. *For any finite group G , the functor*

$$\mathcal{L}_p^c(\kappa): \mathcal{L}_p^c(BG) \longrightarrow \mathcal{L}_p^c(BG_p^\wedge)$$

induces an \mathbb{F}_p -homology equivalence on nerves.

We are now ready to prove Theorem A, which we restate as

Theorem 2.9. *For any pair G, G' of finite groups, BG_p^\wedge and BG'_p^\wedge are homotopy equivalent if and only if the categories $\bar{\mathcal{L}}_p^c(G)$ and $\bar{\mathcal{L}}_p^c(G')$ are equivalent.*

Proof. A homotopy equivalence $BG_p^\wedge \longrightarrow BG'_p^\wedge$ induces an equivalence of categories $\mathcal{L}_p^c(BG_p^\wedge) \longrightarrow \mathcal{L}_p^c(BG'_p^\wedge)$, and hence an equivalence of categories $\bar{\mathcal{L}}_p^c(G) \xrightarrow{\cong} \bar{\mathcal{L}}_p^c(G')$ by Proposition 2.7. Conversely, if the categories $\bar{\mathcal{L}}_p^c(G)$ and $\bar{\mathcal{L}}_p^c(G')$ are equivalent, then their nerves are homotopy equivalent, and hence $BG_p^\wedge \simeq BG'_p^\wedge$ by Proposition 1.1. \square

3. SPACES OF EQUIVALENCES

We now turn to the proofs of Theorems B and C. They will first be reduced, in Lemma 3.1, to a problem of comparing two “categories of automorphisms”: one of the space BG_p^\wedge , and the other of the category $\bar{\mathcal{L}}_p^c(G)$. We begin by defining these categories of automorphisms.

For any category \mathcal{C} , we denote by $\mathcal{A}ut(\mathcal{C})$ the category whose objects are the equivalences of categories $\psi : \mathcal{C} \rightarrow \mathcal{C}$, and whose morphisms are the natural isomorphisms $\Psi : \psi \longrightarrow \psi'$ between equivalences. We think of an equivalence between categories either as a functor which is invertible modulo natural isomorphisms of functors, or equivalently as a functor which induces bijections on isomorphism classes of objects and on all morphism sets. Analogously, for any CW complex X , we denote by $\mathcal{A}ut(X)$ the fundamental groupoid of $\text{Aut}(X)$; i.e., the category whose objects are the self homotopy equivalences $\varphi : X \xrightarrow{\simeq} X$, and where $\text{Mor}_{\mathcal{A}ut(X)}(\varphi, \varphi')$ is the set of homotopy classes of paths from φ to φ' in the mapping space. All of these categories $\mathcal{A}ut(-)$ are given the discrete topology. In both cases, we write $\text{Out}(-) = \pi_0(\mathcal{A}ut(-))$. Thus, $\text{Out}(\mathcal{C})$ is the group of self equivalences of \mathcal{C} up to natural isomorphisms of functors, and $\text{Out}(X)$ is the group of homotopy classes of self homotopy equivalences of X .

In general, we write $\mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ to indicate that a functor is an equivalence of categories, and $\psi \cong \psi'$ to denote that two functors ψ and ψ' are naturally isomorphic. When functors $\psi, \psi' : \mathcal{C} \longrightarrow \mathcal{D}$ are already defined, a natural isomorphism $\psi \xrightarrow{\Phi} \psi'$ will be described as a function which sends each object $c \in \text{Ob}(\mathcal{C})$ to an isomorphism Ψ_c in \mathcal{D} , satisfying the obvious naturality properties.

Recall that a natural morphism $\psi \xrightarrow{\Phi} \psi'$ of functors $\mathcal{C} \longrightarrow \mathcal{D}$ is equivalent to a functor $\widehat{\Phi} : \mathcal{C} \times [1] \longrightarrow \mathcal{D}$, where $[1]$ is the category with two objects $0, 1$ and one non-identity morphism $(0 \rightarrow 1)$. For each $c \in \text{Ob}(\mathcal{C})$, $\widehat{\Phi}$ sends $(c, 0)$ to $\psi(c)$ and $(c, 1)$ to $\psi'(c)$, while for each morphism $c \xrightarrow{f} d$,

$$\widehat{\Phi}(f, \text{Id}_0) = \psi(f), \quad \widehat{\Phi}(f, \text{Id}_1) = \psi'(f),$$

and

$$\widehat{\Phi}(f, 0 \rightarrow 1) = \Phi_d \circ \psi(f) = \psi'(f) \circ \Phi_c.$$

Each of these categories $\mathcal{A}ut(-)$ is a strict monoidal category, in the sense that composition defines a strictly associative functor

$$\mathcal{A}ut(-) \times \mathcal{A}ut(-) \longrightarrow \mathcal{A}ut(-)$$

with strict identity. The nerve of each $\mathcal{A}ut(-)$ is thus a simplicial monoid, and $|\mathcal{A}ut(-)|$ is a topological monoid.

For any space X , let $S_\bullet X$ denote its singular simplicial set, and let $\pi(X)$ denote its fundamental groupoid. Let

$$X \xleftarrow{\text{ev}_X} |S_\bullet X| \xrightarrow{\sigma_X} |\pi(X)|$$

denote the evaluation map, and the map which takes each simplex to its homotopy class, respectively. The following lemma shows that $\mathcal{A}ut(BG_p^\wedge) = \pi(\text{Aut}(BG_p^\wedge))$ and $\text{Aut}(BG_p^\wedge)$ have the same homotopy type, and hence that it suffices from now on to work with the former.

Lemma 3.1. *For any finite group G ,*

$$\text{Aut}(BG_p^\wedge) \xleftarrow[\simeq]{\text{ev}} |S_\bullet \text{Aut}(BG_p^\wedge)| \xrightarrow[\simeq]{\sigma} |\mathcal{A}ut(BG_p^\wedge)|$$

are homotopy equivalences of topological monoids.

Proof. Both maps are clearly morphisms of topological monoids. The first is a (weak) homotopy equivalence (cf. [GJ, Theorem I.11.4]), and the second is a homotopy equivalence since by [BL, Theorem 1.1], $\text{Aut}(BG_p^\wedge)$ is aspherical. \square

It now remains to compare $\mathcal{A}ut(BG_p^\wedge)$ with $\mathcal{A}ut(\bar{\mathcal{L}}_p^c(G))$. But in fact, we do not deal with all self equivalences of the category $\bar{\mathcal{L}}_p^c(G)$, but only certain “isotypical” equivalences.

Definition 3.2. *Let \mathcal{C} and \mathcal{D} be two categories equipped with functors $\mathcal{C} \xrightarrow{\gamma} \text{Gr}$ and $\mathcal{D} \xrightarrow{\delta} \text{Gr}$ to the category of groups. A functor $\psi : \mathcal{C} \rightarrow \mathcal{D}$ is isotypical if γ is naturally isomorphic to $\delta \circ \psi$. When γ is understood, $\text{Aut}_{\text{typ}}(\mathcal{C})$ denotes the strict monoidal category of isotypical self-equivalences and natural isomorphisms between them, and $\text{Out}_{\text{typ}}(\mathcal{C})$ denotes the group of isotypical self-equivalences of \mathcal{C} modulo natural isomorphisms of functors.*

We emphasize that the definition of an isotypical functor does not include the natural isomorphism $\gamma \xrightarrow{\cong} \delta \circ \psi$ as part of the data, but only requires that such an isomorphism exists. Thus, if we think of a category together with a functor to groups as a “diagram of groups”, then an equivalence between diagrams of groups defines an isotypical equivalence, but the isotypical equivalence contains less information than the equivalence between diagrams.

The following lemma is immediate, and shows for example that $\text{Out}_{\text{typ}}(\mathcal{C})$ is always a group.

Lemma 3.3. *Let \mathcal{C} and \mathcal{D} be categories equipped with functors $\mathcal{C} \xrightarrow{\gamma} \text{Gr}$ and $\mathcal{D} \xrightarrow{\delta} \text{Gr}$, and let $\psi : \mathcal{C} \longrightarrow \mathcal{D}$ be an isotypical functor. Then any functor $\psi' : \mathcal{C} \longrightarrow \mathcal{D}$ naturally isomorphic to ψ is isotypical, and any right inverse to ψ up to natural isomorphism (i.e., any $\mathcal{D} \xrightarrow{\psi^*} \mathcal{C}$ such that $\psi \circ \psi^* \cong \text{Id}_{\mathcal{D}}$) is isotypical. \square*

The categories $\mathcal{L}_p(G)$, $\bar{\mathcal{L}}_p(G)$, $\mathcal{L}_p(X)$ and all their variations are examples of such diagrams of groups, where the functors

$$\lambda_G : \bar{\mathcal{L}}_p(G) \longrightarrow \text{Gr} \quad \text{and} \quad \lambda_X : \mathcal{L}_p(X) \longrightarrow \text{Gr},$$

are the obvious forgetful functors

$$\lambda_G(P \xrightarrow{\hat{x}} Q) = (P \xrightarrow{c_x} Q)$$

and

$$\lambda_X((P, \alpha) \xrightarrow{(\varphi, \eta)} (Q, \beta)) = (P \xrightarrow{\varphi} Q).$$

(Recall that c_x is the homomorphism $g \mapsto xgx^{-1}$.)

For the categories $\bar{\mathcal{L}}_p^c(G)$, there is an alternative criterion for a functor being isotypical, which is more useful in concrete situations. For each p -centric subgroup $P \leq G$, let $D_P(G)$ denote the “distinguished subgroup” of $\text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(P)$ given by

$$D_G(P) = \{\hat{g} \mid g \in P\} \subseteq \text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(P) \cong N_G(P)/C'_G(P).$$

Since $C'_G(P)$ has order prime to p , $D_G(P) \cong P$, and we can identify these two groups. The next lemma says that an equivalence $\psi : \bar{\mathcal{L}}_p^c(G) \xrightarrow{\cong} \bar{\mathcal{L}}_p^c(G')$ is isotypical if and only if it sends distinguished subgroups isomorphically to distinguished subgroups.

Lemma 3.4. *Fix finite groups G and G' , and an equivalence*

$$\psi : \bar{\mathcal{L}}_p^c(G) \longrightarrow \bar{\mathcal{L}}_p^c(G').$$

Then ψ is isotypical if and only if

$$\psi_{P,P} : \text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(P) \xrightarrow{\cong} \text{Aut}_{\bar{\mathcal{L}}_p^c(G')}(\psi(P))$$

sends $D_G(P)$ isomorphically to $D_{G'}(\psi(P))$ for each p -centric $P \leq G$. In this case, if $P \xrightarrow{\psi_P} \psi(P)$ denotes the restriction of $\psi_{P,P}$ under the identifications $P = D_G(P)$ and $\psi(P) = D_{G'}(\psi(P))$, then $(P \mapsto \psi_P)$ is a natural isomorphism of functors $\lambda_G \xrightarrow{\cong} \lambda_{G'} \circ \psi$.

Proof. To simplify notation, we write $P' = \psi(P)$ for any P in $\bar{\mathcal{L}}_p^c(G)$. Assume first that ψ is isotypical, and let $\Lambda : \lambda_G \xrightarrow{\cong} \lambda_{G'} \circ \psi$ be a natural

isomorphism. Fix P , let $g \in P$ be any element, and set $\hat{x} = \psi_{P,P}(\hat{g})$, where $x \in N_{G'}(P')$. Then Λ sends P to an isomorphism $\Lambda_P \in \text{Iso}(P, P')$ of groups, and

$$c_{\Lambda_P(g)} \circ \Lambda_P = \Lambda_P \circ c_g = c_x \circ \Lambda_P :$$

the first equality holds when Λ_P is replaced by any homomorphism $P \rightarrow P'$, and the second holds by the naturality of Λ with respect to $(P \xrightarrow{\hat{g}} P)$. Thus $c_{\Lambda_P(g)} = c_x$, so $x^{-1}\Lambda_P(g) \in C_{G'}(P')$, and we can assume that $x^{-1}\Lambda_P(g) \in Z(P')$ without changing the class $\hat{x} = x \cdot C'_{G'}(P')$. It follows that $x \in P'$, i.e. that $\psi_{P,P}(\hat{g}) \in D_{G'}(P')$, and thus $\psi_{P,P}(D_G(P)) \subseteq D_{G'}(P')$. And equality now holds since the distinguished subgroups are abstractly isomorphic (and $\psi_{P,P}$ is an isomorphism).

Now assume that $\psi_{P,P}(D_G(P)) = D_{G'}(P')$ for each P , and let $\psi_P: P \xrightarrow{\cong} P'$ be the restriction of $\psi_{P,P}$ under our identifications. We must show that $(P \mapsto \psi_P)$ is natural as an isomorphism of functors

$$\lambda_G \longrightarrow \lambda_{G'} \circ \psi;$$

i.e., that

$$\psi_Q \circ c_x = c_y \circ \psi_P \in \text{Hom}(P, Q') \quad (1)$$

for any morphism $P \xrightarrow{\hat{x}} Q$ in $\bar{\mathcal{L}}_p^c(G)$, where $\hat{y} = \psi_{P,Q}(\hat{x})$. For any $g \in P$, ψ sends $\hat{x} \circ \hat{g} = \hat{x}\hat{g}$ in $\bar{\mathcal{L}}_p^c(G)$ to

$$\hat{y} \circ \widehat{\psi_P(g)} = \widehat{\psi_Q(xg)} \circ \hat{y} \in \text{Mor}_{\bar{\mathcal{L}}_p^c(G')} (P', Q') = N_{G'}(P', Q')/C'_{G'}(P'),$$

where $C_{G'}(P') = Z(P') \times C'_{G'}(P')$ and the second factor has order prime to p . Hence there is $h \in C'_{G'}(P')$ such that $y\psi_P(g) = \psi_Q(xgx^{-1})yh$,

$$yhy^{-1} = \psi_Q(xgx^{-1})^{-1}y\psi_P(g)y^{-1} \in Q'$$

(since $yP'y^{-1} \leq Q'$), and $h = 1$ since it has order prime to p . Thus, $\psi_Q(xgx^{-1}) = y\psi_P(g)y^{-1}$ for all $g \in P$, and this proves (1). \square

From now on, for any isotypical equivalence $\bar{\mathcal{L}}_p^c(G) \xrightarrow[\simeq]{\psi} \bar{\mathcal{L}}_p^c(G')$, and any p -centric $P \leq G$, we let $\psi_P: P \xrightarrow{\cong} \psi(P)$ denote the isomorphism obtained by restricting $\psi_{P,P}$.

We next define functors

$$\text{Aut}(BG_p^\wedge) \xrightleftharpoons[R]{L} \text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)),$$

which will later be seen to be inverses up to natural isomorphism. Very roughly, R is defined to be the realization functor from $\text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$ to

$\mathcal{A}ut(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge)$ followed by an equivalence of categories induced by the homotopy equivalence $|\bar{\mathcal{L}}_p^c(G)|_p^\wedge \simeq BG_p^\wedge$; while L is induced by the functor $\mathcal{L}(-)$ to $\mathcal{A}ut_{\text{typ}}(\mathcal{L}_p^c(BG_p^\wedge))$ followed by an equivalence induced by the homotopy equivalence $\mathcal{L}_p^c(BG_p^\wedge) \simeq \bar{\mathcal{L}}_p^c(G)$.

More precisely, for any category \mathcal{C} , let

$$\mathcal{A}ut(\mathcal{C}) \xrightarrow{|\cdot|} \mathcal{A}ut(|\mathcal{C}|) \quad \text{and} \quad \mathcal{A}ut(\mathcal{C}) \xrightarrow{|\cdot|_p^\wedge} \mathcal{A}ut(|\mathcal{C}|_p^\wedge)$$

denote the functors which take a self equivalence of \mathcal{C} to its geometric realization (before or after p -completion), and which take a natural isomorphism, interpreted as a functor from $\mathcal{C} \times [1]$ to \mathcal{C} , to the homotopy class of its geometric realization. For any finite group G , let $|\bar{\mathcal{L}}_p^c(G)|_p^\wedge \xrightarrow{\bar{\alpha}_G} BG_p^\wedge$ be the homotopy equivalence of Proposition 1.1, and fix a homotopy inverse

$$\alpha^* : BG_p^\wedge \longrightarrow |\bar{\mathcal{L}}_p^c(G)|_p^\wedge.$$

Let

$$c_\alpha : \mathcal{A}ut(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge) \longrightarrow \mathcal{A}ut(BG_p^\wedge)$$

denote the functor ‘‘conjugation by α ’’, defined on objects and morphisms by setting

$$c_\alpha(f) = \alpha^* \circ f \circ \bar{\alpha}_G \quad \text{and} \quad c_\alpha(F) = \alpha^* \circ F \circ (\bar{\alpha}_G \times I)$$

for any $f \in \mathcal{A}ut(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge)$ and any homotopy F . We now define the realization functor

$$R = R_G : \mathcal{A}ut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \xrightarrow{|\cdot|_p^\wedge} \mathcal{A}ut(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge) \xrightarrow[\simeq]{c_\alpha} \mathcal{A}ut(BG_p^\wedge).$$

The first functor in this composite preserves the monoidal structures of the categories, but c_α does not in general.

For any space X , define a functor

$$\mathcal{L}_X : \mathcal{A}ut(X) \longrightarrow \mathcal{A}ut_{\text{typ}}(\mathcal{L}_p^c(X))$$

as follows. On objects, \mathcal{L}_X sends a self equivalence $X \xrightarrow{f} X$ to $\mathcal{L}_p^c(f)$. If $F : X \times I \longrightarrow X$ is a homotopy, representing a morphism in $\mathcal{A}ut(X)$ from f to f' , then $\mathcal{L}_X(F)$ is defined to be the natural isomorphism of functors which sends an object (P, α) to the morphism $(\text{Id}_P, [F \circ (\alpha \times I)])$. (Note that this only depends on the homotopy class of F , as a path in $\mathcal{A}ut(X)$ from f to f' .) One easily checks that \mathcal{L}_X preserves compositions of homotopies and of homotopy equivalences, and is thus a well defined functor of monoidal categories.

Since $\bar{\beta}_G^c : \bar{\mathcal{L}}_p^c(G) \longrightarrow \mathcal{L}_p^c(BG_p^\wedge)$ is an inclusion and an equivalence of categories (Proposition 2.7), it has a left inverse β^* , defined by sending any

object (P, α) in $\mathcal{L}_p^c(BG_p^\wedge)$ not in the image of $\bar{\beta}_G^c$ to some $Q \leq G$ such that $(Q, i_Q) = \bar{\beta}_G^c(Q)$ is isomorphic to (P, α) in $\mathcal{L}_p^c(BG_p^\wedge)$. Also, $\bar{\beta}_G^c$ is isotypical since $\lambda_G = \lambda_{BG_p^\wedge} \circ \bar{\beta}_G^c$, and hence β^* is isotypical by Lemma 3.3. Let

$$\text{Aut}_{\text{typ}}(\mathcal{L}_p^c(BG_p^\wedge)) \begin{array}{c} \xrightarrow{c_\beta} \\ \xleftarrow{c_\beta^*} \end{array} \text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$$

denote the equivalences induced by composition with $\bar{\beta}_G^c$ and β^* . Thus

$$c_\beta(\mathcal{L}_p^c(BG_p^\wedge) \xrightarrow{\rho} \mathcal{L}_p^c(BG_p^\wedge)) = \beta^* \circ \rho \circ \bar{\beta}_G^c$$

and

$$c_\beta^*(\bar{\mathcal{L}}_p^c(G) \xrightarrow{\psi} \bar{\mathcal{L}}_p^c(G)) = \bar{\beta}_G^c \circ \psi \circ \beta^*,$$

and similarly for morphisms. Note that c_β^* preserves the monoidal structures of these categories (since $\beta^* \circ \bar{\beta}_G^c = \text{Id}$ by assumption), while c_β does not in general preserve them. Let $L = L_G$ to be the composite

$$L = L_G: \text{Aut}(BG_p^\wedge) \xrightarrow{\mathcal{L}_{BG_p^\wedge}} \text{Aut}_{\text{typ}}(\mathcal{L}_p^c(BG_p^\wedge)) \xrightarrow[\simeq]{c_\beta} \text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)).$$

It remains to show that the maps induced by R and L on nerves are mutual inverses up to natural isomorphism of functors, thus proving Theorems B and C. We first show that $L \circ R \cong \text{Id}$.

For each p -centric subgroup $P \leq G$, let $t_P: \mathcal{B}(P) \longrightarrow \bar{\mathcal{L}}_p^c(G)$ denote the functor which sends the unique object o_P to P , and sends each morphism \tilde{x} in $\mathcal{B}(P)$ to $\hat{x} \in \text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(P)$. For each pair of p -centric subgroups $P, Q \leq G$ and each $x \in N_G(P, Q)$, let

$$H(x): t_P \longrightarrow t_Q \circ \mathcal{B}(c_x)$$

be the natural morphism of functors $\mathcal{B}(P) \longrightarrow \bar{\mathcal{L}}_p^c(G)$ which sends the object o_P to the morphism $P \xrightarrow{\hat{x}} Q$. This induces a homotopy

$$|H(x)|: BP \times I = |\mathcal{B}(P) \times [1]| \longrightarrow |\bar{\mathcal{L}}_p^c(G)|;$$

i.e., a path in the mapping space $\text{Map}(BP, |\bar{\mathcal{L}}_p^c(G)|)$. Define

$$\bar{\Gamma}_G: \bar{\mathcal{L}}_p^c(G) \longrightarrow \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|)$$

to be the functor which sends an object P to the pair $(P, |t_P|)$, and which sends a morphism $P \xrightarrow{\hat{x}} Q$ in $\bar{\mathcal{L}}_p^c(G)$ (for $x \in N_G(P, Q)$) to $\bar{\Gamma}_G(\hat{x}) = (c_x, |H(x)|)$.

The next two lemmas explore some of the features of $\bar{\Gamma}_G$.

Lemma 3.5. *For any finite group G , the following square*

$$\begin{array}{ccc} \bar{\mathcal{L}}_p^c(G) & \xrightarrow{\bar{\beta}_G^c} & \mathcal{L}_p^c(BG_p^\wedge) \\ \bar{\Gamma}_G \downarrow & & \uparrow \mathcal{L}_p^c(BG_p^\wedge) \\ \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge) \end{array}$$

commutes up to natural isomorphism of functors.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{L}_p^c(BG) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(BG_p^\wedge) \\ & \nearrow \beta_G & \uparrow \mathcal{L}(\alpha_G) & & \simeq \uparrow \mathcal{L}((\alpha_G)_p^\wedge) \\ \mathcal{L}_p^c(G) & \xrightarrow{\Gamma_G} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(|\mathcal{L}_p^c(G)|_p^\wedge) \\ \pi \downarrow & & \downarrow \mathcal{L}(|\pi|) & & \simeq \downarrow \mathcal{L}(|\pi|_p^\wedge) \\ \bar{\mathcal{L}}_p^c(G) & \xrightarrow{\bar{\Gamma}_G} & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|) & \xrightarrow{\mathcal{L}_p^c(\kappa)} & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge) \end{array}$$

of functors between categories. Here Γ_G is defined in a way analogous to $\bar{\Gamma}_G$. Since

$$\mathcal{L}((\alpha_G)_p^\wedge) \cong \mathcal{L}(\bar{\alpha}_G) \circ \mathcal{L}(|\pi|_p^\wedge)$$

by Proposition 1.1, this shows that the two composites

$$\mathcal{L}_p^c(G) \xrightarrow{\pi} \bar{\mathcal{L}}_p^c(G) \xrightarrow[\bar{\beta}_G^c]{\mathcal{L}_p^c(\bar{\alpha}_G) \circ \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G} \mathcal{L}_p^c(BG_p^\wedge)$$

are naturally isomorphic. And since π induces a bijection on objects and a surjection on morphisms, any natural isomorphism of functors after composition with π (when regarded as a map from $\text{Ob}(\mathcal{L}_p^c(G)) = \text{Ob}(\bar{\mathcal{L}}_p^c(G))$ to $\text{Mor}(\mathcal{L}_p^c(BG_p^\wedge))$) is also a natural isomorphism before composition. \square

The second property we need of the functors $\bar{\Gamma}_G$ is their naturality with respect to isotypical equivalences.

Lemma 3.6. *For any isotypical equivalence $\psi: \bar{\mathcal{L}}_p^c(G) \rightarrow \bar{\mathcal{L}}_p^c(G)$, the following square*

$$\begin{array}{ccc} \bar{\mathcal{L}}_p^c(G) & \xrightarrow{\bar{\Gamma}_G} & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|) \\ \psi \downarrow & & \downarrow \mathcal{L}_p^c(|\psi|) \\ \bar{\mathcal{L}}_p^c(G) & \xrightarrow{\bar{\Gamma}_G} & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|); \end{array} \quad (1)$$

commutes up to a natural isomorphism of functors

$$W(\psi) : \mathcal{L}_p^c(|\psi|) \circ \bar{\Gamma}_G \xrightarrow{\cong} \bar{\Gamma}_G \circ \psi,$$

which is itself natural in the sense that the following square commutes for any natural isomorphism $\Psi : \psi \xrightarrow{\cong} \psi'$:

$$\begin{array}{ccc} \mathcal{L}_p^c(|\psi|) \circ \bar{\Gamma}_G & \xrightarrow{W(\psi)} & \bar{\Gamma}_G \circ \psi \\ \mathcal{L}_p^c(|\Psi|) \circ \bar{\Gamma}_G \downarrow & & \bar{\Gamma}_G \circ \Psi \downarrow \\ \mathcal{L}_p^c(|\psi'|) \circ \bar{\Gamma}_G & \xrightarrow{W(\psi')} & \bar{\Gamma}_G \circ \psi'. \end{array} \quad (2)$$

Proof. We first claim that

$$W(\psi) : \mathcal{L}_p^c(|\psi|) \circ \bar{\Gamma}_G \longrightarrow \bar{\Gamma}_G \circ \psi,$$

defined by sending an object P in $\bar{\mathcal{L}}_p^c(G)$ to the morphism

$$(\psi_P, C_P) \in \text{Mor}_{\mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|)}((P, |\psi| \circ |t_P|), (P', |t_{P'}|)),$$

is a natural isomorphism of functors. Here, $P' = \psi(P)$ for short, and C_P denotes the constant homotopy. Note first that the objects are correct: $\mathcal{L}_p^c(|\psi|)(\bar{\Gamma}_G(P)) = (P, |\psi| \circ |t_P|)$ and $\bar{\Gamma}_G(\psi(P)) = (P', |t_{P'}|)$ by definition. Also, (ψ_P, C_P) is a morphism between these objects, since

$$|\psi| \circ |t_P| = |\psi \circ t_P| = |t_{P'} \circ \mathcal{B}(\psi_P)| = |t_{P'}| \circ B\psi_P$$

by definition of ψ_P . To show that $W(\psi)$ is natural, we must check, for each morphism $\hat{x} : P \longrightarrow Q$ in $\bar{\mathcal{L}}_p^c(G)$, with $\hat{y} = \psi(\hat{x})$ and $Q' = \psi(Q)$, that the following square commutes:

$$\begin{array}{ccc} (P, |\psi| \circ |t_P|) & \xrightarrow{(\psi_P, C_P)} & (P', |t_{P'}|) \\ (c_x, |\psi| \circ |H(x)|) \downarrow & & \downarrow (c_y, |H(y)|) \\ (Q, |\psi| \circ |t_Q|) & \xrightarrow{(\psi_Q, C_Q)} & (Q', |t_{Q'}|) \end{array}$$

in $\mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|)$. By Lemma 3.4, $(P \mapsto \psi_P)$ is a natural isomorphism of functors $\lambda_G \xrightarrow{\cong} \lambda_G \circ \psi$, and thus $c_y \circ \psi_P = \psi_Q \circ c_x$. So it remains to show that $|H(y)| \circ (B\psi_P \times I) = |\psi| \circ |H(x)|$; and this follows since both are induced by the natural isomorphism $(o_P \mapsto \hat{y})$ of functors $\mathcal{B}(P) \longrightarrow \bar{\mathcal{L}}_p^c(G)$.

It remains to check that square (2) commutes for all $\Psi : \psi \xrightarrow{\cong} \psi'$. This means showing, for each p -centric $P \leq G$, and z such that $\hat{z} = \Psi_P \in$

$\text{Iso}_{\bar{\mathcal{L}}_p^c(G)}(\psi(P), \psi'(P))$, that the following square commutes in $\mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|)$:

$$\begin{array}{ccc} (P, |\psi| \circ |t_P|) & \xrightarrow{(\psi_P, C_P)} & (\psi(P), |t_{\psi(P)}|) \\ (\text{Id}_P, |\Psi| \circ |t_P|) \downarrow & & \bar{\Gamma}_G(\Psi_P) \downarrow = (c_z, |H(z)|) \\ (P, |\psi'| \circ |t_P|) & \xrightarrow{(\psi'_P, C_P)} & (\psi'(P), |t_{\psi'(P)}|). \end{array}$$

The square of group homomorphisms commutes ($c_z \circ \psi_P = \psi'_P$) since for each $g \in P$, the square

$$\begin{array}{ccc} \psi(P) & \xrightarrow{\widehat{\psi_P(g)} = \psi_{P,P}(\widehat{g})} & \psi(P) \\ \Psi_P = \widehat{\cdot} \downarrow \cong & & \cong \downarrow \Psi_P = \widehat{\cdot} \\ \psi'(P) & \xrightarrow{\widehat{\psi'_P(g)} = \psi'_{P,P}(\widehat{g})} & \psi'(P) \end{array}$$

commutes by naturality of Ψ . Since the C_P are constant homotopies, it remains to check that $|\Psi| \circ |t_P|$ and $|H(\Psi_P)| \circ (B\psi_P \times I)$ are homotopic as paths in the space $\text{Map}(BP, |\bar{\mathcal{L}}_p^c(G)|)$ from $|\psi| \circ |t_P| = |t_{\psi(P)}| \circ B\psi_P$ to $|\psi'| \circ |t_P| = |t_{\psi'(P)}| \circ B\psi'_P$. And this holds since both are induced by the natural isomorphism of functors $B(G) \longrightarrow \bar{\mathcal{L}}_p^c(G)$ which sends the object o_P to the morphism Ψ_P . \square

We are now ready to show:

Proposition 3.7. *For any finite group G , the composite*

$$\text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \xrightarrow{R} \text{Aut}(BG_p^\wedge) \xrightarrow{L} \text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$$

is naturally isomorphic to the identity.

Proof. Fix an isotypical equivalence $\psi : \bar{\mathcal{L}}_p^c(G) \longrightarrow \bar{\mathcal{L}}_p^c(G)$, and consider the following diagram:

$$\begin{array}{ccccc} & \bar{\mathcal{L}}_p^c(G) & \xrightarrow{\psi} & \bar{\mathcal{L}}_p^c(G) & \\ \bar{\Gamma}_G \swarrow & & & & \searrow \bar{\Gamma}_G \\ \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|) & & \mathcal{L}_p^c(BG_p^\wedge) & \xrightarrow{\mathcal{L}_p^c(R(\psi))} & \mathcal{L}_p^c(BG_p^\wedge) & & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|) \\ \bar{\beta}_G^c \searrow \cong & & & & & & \beta^* \nearrow \cong \\ & & & & & & \\ \mathcal{L}_p^c(\kappa) \searrow & & \mathcal{L}_p^c(\alpha^*) & & \mathcal{L}_p^c(\bar{\alpha}_G) & & \mathcal{L}_p^c(\kappa) \\ & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge) & \xrightarrow{\mathcal{L}_p^c(|\psi|_p^\wedge)} & \mathcal{L}_p^c(|\bar{\mathcal{L}}_p^c(G)|_p^\wedge) & \end{array}$$

Here, α^* and β^* are the inverses (up to homotopy or natural isomorphism) of $\bar{\alpha}_G$ and $\bar{\beta}_G^c$ used to define R and L . In particular, $L \circ R(\psi) = \beta^* \circ$

$\mathcal{L}_p^c(R(\psi)) \circ \bar{\beta}_G^c$, and proving the proposition means showing that the upper trapezoid commutes up to natural isomorphism of functors. And this follows since the two squares in the diagram commute up to natural isomorphism by Lemma 3.5, the large hexagon by Lemma 3.6, and the lower trapezoid (commutes precisely) by definition of R .

To make this more precise, fix natural isomorphisms of functors

$$U: \mathcal{L}_p^c(\alpha^*) \circ \bar{\beta}_G^c \longrightarrow \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G$$

and

$$V: \beta^* \circ \mathcal{L}_p^c(\bar{\alpha}_G) \circ \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G \longrightarrow \text{Id}$$

(chosen independently of ψ), and let

$$W_p^\wedge(\psi): \mathcal{L}_p^c(|\psi|_p^\wedge) \circ \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G \longrightarrow \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G \circ \psi$$

be the natural isomorphism of Lemma 3.6 after composing with completion at p . Define $A(\psi): L \circ R(\psi) \longrightarrow \psi$ to be the composite of natural isomorphisms

$$\begin{aligned} L \circ R(\psi) &\stackrel{\text{def}}{=} \beta^* \circ \mathcal{L}_p^c(R\psi) \circ \bar{\beta}_G^c = \beta^* \circ \mathcal{L}_p^c(\bar{\alpha}_G) \circ \mathcal{L}_p^c(|\psi|_p^\wedge) \circ \mathcal{L}_p^c(\alpha^*) \circ \bar{\beta}_G^c \\ &\xrightarrow{(-) \circ U} \beta^* \circ \mathcal{L}_p^c(\bar{\alpha}_G) \circ \mathcal{L}_p^c(|\psi|_p^\wedge) \circ \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G \\ &\xrightarrow{(-) \circ W_p^\wedge(\psi)} \beta^* \circ \mathcal{L}_p^c(\bar{\alpha}_G) \circ \mathcal{L}_p^c(\kappa) \circ \bar{\Gamma}_G \circ \psi \xrightarrow{V \circ \psi} \psi. \end{aligned}$$

To see that $A: L \circ R \longrightarrow \text{Id}$ is an isomorphism of functors, it remains only to check its naturality with respect to morphisms in $\text{Aut}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$. And since U and V are independent of ψ , the naturality of A follows from the naturality of $W_p^\wedge(\psi)$ with respect to isomorphisms $\Psi: \psi \longrightarrow \psi'$, as shown in Lemma 3.6. \square

It remains to show that $R \circ L$ is naturally equivalent to the identity functor. Notice that the map L_G is defined as the composite $c_\beta \circ \mathcal{L}_{BG_p^\wedge}$. Thus, to prove this claim, it suffices to show that the functor $|\mathcal{L}_{BG_p^\wedge}|$ induces a monomorphism on homotopy groups. One way to do this is via a homology decomposition of BG , using standard techniques to construct maps on a homotopy colimit. However, we give a different argument in the next section: one which uses the techniques already developed in this paper.

4. PROOF OF THEOREMS B AND C

We now finish the proof that $\text{Aut}(BG_p^\wedge)$ has the homotopy type of $|\text{Aut}_{\text{typ}}(\tilde{\mathcal{L}}_p^c(G))|$, by showing that $\text{Aut}(BG_p^\wedge) \xrightarrow{L} \text{Aut}_{\text{typ}}(\tilde{\mathcal{L}}_p^c(G))$ is homotopy split injective. This could be done more directly if there were natural “evaluation” maps $\mathcal{L}_p^c(X) \longrightarrow X$, induced by evaluating each $BP \longrightarrow X$ at the basepoint of BP . Since there are no such natural maps, we instead define a simplicial space $\mathbf{M}_\bullet^c(X)$, whose topological realization $|\mathbf{M}_\bullet^c(X)|$ has the mod- p homotopy type of $|\mathcal{L}_p^c(X)|$ when $X = BG$ or BG_p^\wedge , and which is equipped with a natural evaluation map to X . Note that we use $|-|$ to denote both the topological realization of a simplicial space and the topological realization of the nerve of a category.

To help motivate this construction, note the following analogy between $\mathcal{L}_p(X)$ and the fundamental groupoid $\pi(X)$, which can be thought of as the full subcategory of $\mathcal{L}_p(X)$ whose objects are the trivial subgroups ($\text{pt} \rightarrow X$). The realization $|\pi(X)|$ does not have a (natural) evaluation map to X , but there is an obvious map $|S_\bullet(X)| \xrightarrow{\text{ev}} X$ defined on the realization of the singular simplicial set. There is also a natural map $|S_\bullet(X)| \longrightarrow |\pi(X)|$ which sends a singular simplex $(\Delta^n \rightarrow X)$ to the sequence of paths obtained by restriction to the edges $[v_i, v_{i+1}]$, and this map is a homotopy equivalence if X is aspherical. By analogy, $\mathbf{M}_\bullet^c(X)$ should be a simplicial space whose n -simplices are maps on Δ^n which send each point to some mapping space $\text{Map}(BP, X)$ with appropriate continuity conditions. In practice, it is simpler to define $\mathbf{M}_\bullet^c(X)$ using certain spaces $\Delta(\mathbf{P})$ defined as follows.

For any sequence

$$\mathbf{P} = (P_0 \xrightarrow{\varphi_1} P_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} P_n)$$

of finite p -groups and monomorphisms, define the space $\Delta(\mathbf{P})$ inductively as follows. Set $\Delta(P_0) = BP_0$, let $\Delta(P_0 \xrightarrow{\varphi_1} P_1)$ be the mapping cylinder of the map $BP_0 \xrightarrow{B\varphi_1} BP_1$, and in general let $\Delta(\mathbf{P})$ be the mapping cylinder of the map from $\Delta(P_0 \rightarrow \cdots \rightarrow P_{n-1})$ to BP_n induced by the $B\varphi_i$. Upon identifying the n -simplex Δ^n with the mapping cylinder of $(\Delta^{n-1} \rightarrow \text{pt})$, the inclusions of basepoints into the BP_i and the projections to points induce maps

$$\Delta^n \xrightarrow{\iota_{\mathbf{P}}} \Delta(\mathbf{P}) \xrightarrow{\omega_{\mathbf{P}}} \Delta^n.$$

Let $\{v_0, \dots, v_n\}$ denote the vertices of Δ^n , and write $BP_i \times v_i$ ($\cong BP_i$) to denote $\omega_{\mathbf{P}}^{-1}(v_i)$.

Equivalently, for \mathbf{P} as above, we can define

$$\Delta(\mathbf{P}) = \left(\prod_{i=0}^n (BP_i \times \Delta^{n-i}) \right) / \sim,$$

where the equivalence relation is defined by

$$(x, d_0(t)) \sim (B\varphi_i(x), t) \quad \text{for all } x \in BP_{i-1}, t \in \Delta^{n-i}.$$

Here, $d_0: \Delta^{n-i} \longrightarrow \Delta^{n-i+1}$ is the face map which sends the vertex v_j (for $0 \leq j \leq i-1$) to v_{j+1} . Under this identification, $\Delta^n \xrightarrow{\iota_{\mathbf{P}}} \Delta(\mathbf{P})$ is induced by the inclusion of the basepoint in BP_0 , and $\Delta(\mathbf{P}) \xrightarrow{\omega_{\mathbf{P}}} \Delta^n$ is induced by projection to the second factor followed by the face maps $\Delta^{n-i} \longrightarrow \Delta^n$ which send v_j to v_{j+i} .

Alternatively, $\Delta(\mathbf{P})$ can be regarded as the homotopy colimit of the sequence

$$BP_0 \longrightarrow BP_1 \longrightarrow \cdots \longrightarrow BP_n,$$

regarded as a functor on the poset category $[n] = \{0 < 1 < \cdots < n\}$. Or it can be thought of as the realization of the simplicial space $\Delta_{\bullet}(\mathbf{P})$, where

$$\Delta_k(\mathbf{P}) = \prod_{0 \leq i_0 \leq \cdots \leq i_k \leq n} BP_{i_0},$$

with the obvious face and degeneracy maps.

Definition 4.1. For any space X , let $\mathbf{M}_{\bullet}^c(X)$ be the simplicial space defined as follows. The vertices in $\mathbf{M}_{\bullet}^c(X)$ are the p -centric subgroups (P, α) in X . An n -simplex $\eta \in \mathbf{M}_n^c(X)$ is a map

$$\eta: \Delta(\mathbf{P}) \longrightarrow X,$$

for some sequence $\mathbf{P} = (P_0 \rightarrow \cdots \rightarrow P_n)$ of finite p -groups and monomorphisms, such that $\eta|_{(BP_i \times v_i)}$ is a p -centric subgroup of X for each i . This n -simplex spans the vertices $(P_i, \eta|_{(BP_i \times v_i)})$ for $i = 0, \dots, n$. The set of vertices $\mathbf{M}_0^c(X)$ is given the discrete topology. The set of n -simplices spanning a given set of vertices (P_i, α_i) and based on a given sequence of monomorphisms \mathbf{P} has the compact-open topology, and is open in the space of all n -simplices.

The face and degeneracy maps in $\mathbf{M}_{\bullet}^c(X)$ are described as follows. For any $\mathbf{P} = (P_0 \rightarrow \cdots \rightarrow P_n)$ and any morphism $\underline{m} \xrightarrow{\gamma} \underline{n}$ in the simplicial category, let $\gamma^*\mathbf{P}$ denote the sequence $P_{\gamma(0)} \rightarrow \cdots \rightarrow P_{\gamma(m)}$, and let

$$\Delta(\gamma, \mathbf{P}): \Delta(\gamma^*\mathbf{P}) \longrightarrow \Delta(\mathbf{P})$$

denote the map defined by sending $(BP_{\gamma(i)} \times v_i)$ to $(BP_{\gamma(i)} \times v_{\gamma(i)})$ (for $i = 0, \dots, m$) and extending linearly. Then $\mathbf{M}_n^c(X) \xrightarrow{\gamma^*} \mathbf{M}_m^c(X)$ sends an n -simplex $\Delta(\mathbf{P}) \xrightarrow{\eta} X$ to the m -simplex $\eta \circ \Delta(\gamma, \mathbf{P})$.

In fact, wherever $\mathbf{M}_\bullet^c(X)$ is used throughout this section, it will be necessary to replace it by its “levelwise” singular simplicial set $S_\bullet \mathbf{M}_\bullet^c(X)$. In other words, this is the bisimplicial set where each space $\mathbf{M}_n^c(X)$ is replaced by its singular simplicial set $S_\bullet \mathbf{M}_n^c(X)$. In general, this replacement goes through without problems; we make a few comments at key points in the arguments where this can make a difference in statements or definitions.

For each n -simplex $\Delta(\mathbf{P}) = \Delta(P_0 \rightarrow \dots \rightarrow P_n) \xrightarrow{\eta} X$ in $\mathbf{M}_\bullet^c(X)$, the restriction of η to each mapping cylinder $\omega_{\mathbf{P}}^{-1}(\langle v_i, v_{i+1} \rangle) \cong \Delta(P_i \rightarrow P_{i+1})$ determines a morphism in $\mathcal{L}_p^c(X)$. The n -simplex η thus determines an n -simplex in the nerve of $\mathcal{L}_p^c(X)$, regarded as a simplicial space. This induces a map of simplicial spaces

$$\tau_X: \mathbf{M}_\bullet^c(X) \longrightarrow N_\bullet(\mathcal{L}_p^c(X)),$$

where $N_\bullet(-)$ denotes the nerve of the category (as opposed to its topological realization).

In the following lemma, the simplicial space $\mathbf{M}_\bullet^c(-)$ must in fact be replaced by $S_\bullet \mathbf{M}_\bullet^c(-)$ to get a “genuine” (not just weak) homotopy equivalence. This will be important in the proof of Proposition 4.4 (see the explanation after diagram (1) in that proof).

Lemma 4.2. *If $X = BG$ or BG_p^\wedge for a finite group G , then*

$$|\mathbf{M}_\bullet^c(X)| \xrightarrow{|\tau_X|} |\mathcal{L}_p^c(X)|$$

is a homotopy equivalence.

Proof. When $X = BG$ or BG_p^\wedge , then for any p -centric subgroup $BP \xrightarrow{\alpha} X$, $\text{Map}(BP, X)_\alpha$ is aspherical by Proposition 2.1. We claim that $|\tau_X|$ is a homotopy equivalence for any X with this property.

Fix a set of vertices (P_i, α_i) , and a sequence of monomorphisms $\mathbf{P} = (P_0 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_n} P_n)$. Let $\Delta_0(\mathbf{P})$ be the union of the mapping cylinders of the $B\varphi_i$, regarded as a subspace of $\Delta(\mathbf{P})$ (i.e., $\Delta_0(\mathbf{P}) = \bigcup_{i=1}^n (\omega_{\mathbf{P}}^{-1}[v_{i-1}, v_i])$). Let Y denote the space of maps $\Delta(\mathbf{P}) \xrightarrow{\eta} X$ such that $\eta|_{BP_i \times v_i} = \alpha_i$ for each i , and let Y_0 denote the space of maps $\Delta_0(\mathbf{P}) \longrightarrow X$ with the same property. Since $\Delta_0(\mathbf{P})$ is a deformation retract of $\Delta(\mathbf{P})$, the restriction map $Y \longrightarrow Y_0$ is a homotopy equivalence. Also, $\pi_0(Y_0)$ is the set of sequences

of morphisms in $\mathcal{L}_p^c(X)$ of the form

$$(P_0, \alpha_0) \xrightarrow{(\varphi_1, \eta_1)} (P_1, \alpha_1) \xrightarrow{(\varphi_2, \eta_2)} \dots \xrightarrow{(\varphi_n, \eta_n)} (P_n, \alpha_n).$$

In other words, the set of n -simplices in the nerve of $\mathcal{L}_p^c(X)$ is the set of connected components of the space of n -simplices in $\mathbf{M}_\bullet^c(X)$ (this holds for any X), and it remains only to show that each component in Y_0 is contractible. But Y_0 is the product over all i of the spaces of paths in $\text{Map}(BP_i, X)$ from α_i to $\alpha_{i+1} \circ B\varphi_{i+1}$; and the connected components of these spaces are contractible since $\text{Map}(BP_i, X)_{\alpha_i}$ is aspherical by assumption.

This shows that τ_X , as a map of simplicial spaces, is a homotopy equivalence in each degree. So $|\tau_X|$ is a homotopy equivalence by [GJ, Proposition IV.1.7]. \square

For any X , define maps

$$\text{ev}_{X,n}: \mathbf{M}_n^c(X) \times \Delta^n \longrightarrow X,$$

for each $n \geq 0$, by setting

$$\text{ev}_{X,n}(\eta, t) = \eta(\iota_{\mathbf{P}}(t))$$

for each $\Delta(\mathbf{P}) \xrightarrow{\eta} X$ in $\mathbf{M}_n^c(X)$ and each $t \in \Delta^n$. In other words, this is the map defined by restriction to basepoints in the BP_i (recall that $\Delta^n \xrightarrow{\iota_{\mathbf{P}}} \Delta(\mathbf{P})$ is induced by inclusions of basepoints). The $\text{ev}_{X,n}$ are continuous and commute with all face and degeneracy maps, and hence combine to define an evaluation map

$$\text{ev}_X: |\mathbf{M}_\bullet^c(X)| \longrightarrow X.$$

Lemma 4.3. *If $X = BG$ or BG_p^\wedge for some finite group G , then*

$$|\mathbf{M}_\bullet^c(X)| \xrightarrow{\text{ev}_X} X$$

is an \mathbb{F}_p -homology equivalence.

Proof. Assume first that $X = BG$, and consider the following diagram:

$$\begin{array}{ccccc} |\mathcal{L}_p^c(G)| & \xrightarrow[\simeq]{\beta_G^c} & |\mathcal{L}_p^c(BG)| & \xleftarrow[\simeq]{|\tau_{BG}|} & |\mathbf{M}_\bullet^c(BG)| & \xrightarrow{\text{ev}_{BG}} & BG \\ \alpha_G \downarrow \simeq & & \epsilon_1 \downarrow & & \epsilon_2 \downarrow & \nearrow \epsilon_3 & \\ |\mathcal{B}(G)| & \xrightarrow[\simeq]{\gamma_G} & |\pi(BG)| & \xleftarrow[\simeq]{|\tau'|} & |S_\bullet^{\text{top}}(BG)| & \xrightarrow[\simeq]{} & BG \end{array} \quad (1)$$

Here, $S_\bullet^{\text{top}}(BG)$ is the singular simplicial set for BG , but topologized so that the set of vertices is discrete, but the set of n -simplices $\Delta^n \longrightarrow BG$ with

any given set of vertices is given the compact-open topology. The maps ϵ_1 and ϵ_2 are both defined by restricting maps $BP \longrightarrow BG$ to the basepoint (in the case of ϵ_2 via composition with $\iota_{\mathbf{P}}$). And ϵ_3 is the evaluation map for singular simplices in BG . Also, τ' is defined by sending each vertex in $S_{\bullet}^{\text{top}}(BG)$ (i.e., each point of BG) to itself considered as an object in $\pi(BG)$, and by sending each singular simplex $\sigma : \Delta^n \longrightarrow BG$ to its homotopy class regarded as a sequence of paths in BG (i.e., an n -simplex in the nerve of $\pi(BG)$). And γ_G is induced by regarding $\mathcal{B}(G)$ as a subcategory of $\pi(BG)$ — the full subcategory with object the basepoint $*$ of BG . The triangle in (1) commutes by definition of ev_{BG} , and both squares are induced by commutative squares of categories and functors or of simplicial spaces.

Now, $|\tau'|$ is a homotopy equivalence since τ' is a map of simplicial spaces which is a homotopy equivalence at each level (the components of the space of n -simplices in $S_{\bullet}^{\text{top}}(BG)$ are contractible since BG is aspherical). The evaluation map ϵ_3 is also a homotopy equivalence, since $S_n^{\text{top}}(BG)$ is homotopy equivalent to BG for each n . Also, $|\tau_{BG}|$ is a homotopy equivalence by Lemma 4.2, β_G^c and α_G are homotopy equivalences by Propositions 2.6 and 1.1, and γ_G is an equivalence since it is induced by an equivalence of categories. The commutativity of the diagram now proves that ev_{BG} is a homotopy equivalence.

To see that $\text{ev}_{BG_p^\wedge}$ is an \mathbb{F}_p -homology equivalence, consider the following diagram, where the vertical maps are all induced by p -completion:

$$\begin{array}{ccccc} |\mathcal{L}_p^c(BG)| & \xleftarrow{|\tau_{BG}|} & |\mathbf{M}_{\bullet}^c(BG)| & \xrightarrow{\text{ev}_{BG}} & BG \\ \mathcal{L}_p^c(\kappa) \downarrow & & \mathbf{M}_{\bullet}^c(\kappa) \downarrow & & \kappa \downarrow \\ |\mathcal{L}_p^c(BG_p^\wedge)| & \xleftarrow{|\tau_{BG_p^\wedge}|} & |\mathbf{M}_{\bullet}^c(BG_p^\wedge)| & \xrightarrow{\text{ev}_{BG_p^\wedge}} & BG_p^\wedge \end{array}$$

The squares commute by naturality of τ and ev . Also, $|\mathcal{L}_p^c(\kappa)|$ is an \mathbb{F}_p -homology equivalence by Corollary 2.8, and κ is clearly an equivalence. We have just shown that ev_{BG} is a homotopy equivalence, and $|\tau_{BG}|$ and $|\tau_{BG_p^\wedge}|$ are homotopy equivalences by Lemma 4.2. So $\text{ev}_{BG_p^\wedge}$ is also an \mathbb{F}_p -homology equivalence. \square

All of the definitions and results so far in this section were developed to prove the following result:

Proposition 4.4. *For any finite group G , the map*

$$|\mathcal{L}_{BG_p^\wedge}| : |\text{Aut}(BG_p^\wedge)| \longrightarrow |\text{Aut}_{\text{typ}}(\mathcal{L}_p^c(BG_p^\wedge))|$$

induces a split monomorphism on π_i for all $i \geq 0$.

Proof. For any pair of spaces X and Y , let $\mathcal{E}qv(X, Y)$ denote the groupoid whose objects are the mod- p -equivalences $X \longrightarrow Y$, and whose morphisms are homotopy classes of homotopies. Clearly, composition with homotopy equivalences induces homotopy equivalences between the nerves of such categories.

To shorten the notation, we write $\mathbf{M}_\bullet^c = \mathbf{M}_\bullet^c(BG_p^\wedge)$ and $\mathcal{L}_p^c = \mathcal{L}_p^c(BG_p^\wedge)$. We will construct a functor $\xi : \mathcal{A}ut(BG_p^\wedge) \longrightarrow \mathcal{A}ut(\mathbf{M}_\bullet^c)$ which makes the following diagram commute (precisely, not just up to natural isomorphism):

$$\begin{array}{ccc}
 \mathcal{A}ut(BG_p^\wedge) & \xrightarrow{\mathcal{L}_{BG_p^\wedge}} & \mathcal{A}ut(\mathcal{L}_p^c) \\
 \swarrow \begin{array}{c} -\circ\text{ev} \\ \simeq \end{array} & \downarrow \xi & \searrow \begin{array}{c} |-\cdot| \\ \simeq \end{array} \\
 \mathcal{E}qv(|\mathbf{M}_\bullet^c|_p^\wedge, BG_p^\wedge) & & \mathcal{A}ut(|\mathcal{L}_p^c|_p^\wedge) \quad (1) \\
 \swarrow \begin{array}{c} \text{ev}\circ- \\ \simeq \end{array} & \downarrow \begin{array}{c} |\tau| \circ - \\ \simeq \end{array} & \swarrow \begin{array}{c} -\circ|\tau| \\ \simeq \end{array} \\
 \mathcal{A}ut(|\mathbf{M}_\bullet^c|_p^\wedge) & \xrightarrow{\simeq} & \mathcal{E}qv(|\mathbf{M}_\bullet^c|_p^\wedge, |\mathcal{L}_p^c|_p^\wedge)
 \end{array}$$

The lemma then follows immediately, since the maps labeled “ \simeq ” are all induced by composition with homotopy equivalences (and hence themselves induce homotopy equivalences on nerves).

Note that when carrying out this argument, one must actually replace $|\mathbf{M}_\bullet^c(BG_p^\wedge)|$ by $|S_\bullet \mathbf{M}_\bullet^c(BG_p^\wedge)|$ in the above diagram. This is necessary to guarantee that the maps

$$BG_p^\wedge \xleftarrow{|\tau|} |S_\bullet \mathbf{M}_\bullet^c(BG_p^\wedge)|_p^\wedge \xrightarrow{|\text{ev}|} |\mathcal{L}_p^c(BG_p^\wedge)|_p^\wedge$$

are homotopy equivalences (not just weakly), and hence that composition with $|\tau|$ or $|\text{ev}|$ is a homotopy equivalence.

We define ξ as follows. On objects, ξ sends a homotopy equivalence $BG_p^\wedge \xrightarrow{f} BG_p^\wedge$ to the induced functor $|\mathbf{M}_\bullet^c(BG_p^\wedge)| \xrightarrow[=|\mathbf{M}_\bullet^c(f)|]{\xi(f)} |\mathbf{M}_\bullet^c(BG_p^\wedge)|$, defined via composition with f . So it remains to define $\xi(F) : |\mathbf{M}_\bullet^c| \times I \longrightarrow |\mathbf{M}_\bullet^c|$ for each homotopy $F : BG_p^\wedge \times I \longrightarrow BG_p^\wedge$. To do this, we regard I as the realization of the simplicial set $[1]$, where $[1]_n$ is the set of sequences $\sigma = (\sigma(0) \leq \dots \leq \sigma(n))$ taking values in $\{0, 1\}$. For any such σ , we let $\Delta^n \xrightarrow{|\sigma|} I$ denote the affine map which sends v_i to $\sigma(i)$. Let

$$\xi_\bullet(F) : \mathbf{M}_\bullet^c \times [1] \longrightarrow \mathbf{M}_\bullet^c$$

be the map of simplicial spaces which sends the n -simplex

$$(\Delta(\mathbf{P}) \xrightarrow{\eta} BG_p^\wedge, \sigma)$$

in $\mathbf{M}_\bullet^c \times [1]$ to the n -simplex

$$F(\eta, \sigma): \Delta(\mathbf{P}) \xrightarrow{(\text{Id}, |\sigma| \circ \omega_{\mathbf{P}})} \Delta(\mathbf{P}) \times I \xrightarrow{\eta \times I} BG_p^\wedge \times I \xrightarrow{F} BG_p^\wedge,$$

where $\omega_{\mathbf{P}}: \Delta(\mathbf{P}) \longrightarrow \Delta^n$ is the map defined at the beginning of the section. The $F(\eta, \sigma)$ are easily seen to be continuous and to commute with face and degeneracy maps, and hence define a homotopy $\xi(F) = |\xi_\bullet(F)|$.

Note that an obvious modification in the above definition of ξ gives a functor which takes values in the automorphism category $\mathcal{A}ut(|S_\bullet \mathbf{M}_\bullet^c(BG_p^\wedge)|_p^\wedge)$.

The commutativity of (1) follows from the commutativity of the following squares, for any homotopy $F: BG_p^\wedge \times I \longrightarrow BG_p^\wedge$:

$$\begin{array}{ccccc} BG_p^\wedge \times I & \xleftarrow{\text{ev} \times I} & |\mathbf{M}_\bullet^c(BG_p^\wedge)| \times I & \xrightarrow{\tau \times I} & |\mathcal{L}_p^c(BG_p^\wedge)| \times I \\ \downarrow F & & \downarrow \xi_\bullet(F) & & \downarrow \mathcal{L}_{BG_p^\wedge}(F) \\ BG_p^\wedge & \xleftarrow{\text{ev}} & |\mathbf{M}_\bullet^c(BG_p^\wedge)| & \xrightarrow{\tau} & |\mathcal{L}_p^c(BG_p^\wedge)|. \end{array}$$

And this is easily checked. \square

We are now ready to prove Theorems B and C. In particular, we want to show that $\text{Aut}(BG_p^\wedge)$ and $|\mathcal{A}ut(\bar{\mathcal{L}}_p^c(G))|$ are homotopy equivalent, and moreover equivalent as monoids. There is no obvious way to construct a map between these two spaces which is both a homotopy equivalence and a morphism of monoids, so instead we connect them via a sequence of maps. Here $S_\bullet X$, for any space X , denotes the (discrete) singular simplicial set of X .

Theorem 4.5. *For any finite group G and any prime p , $\text{Aut}(BG_p^\wedge)$ has the homotopy type of $|\mathcal{A}ut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))|$. More precisely:*

(a) *R and L induce isomorphisms*

$$\text{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \xleftarrow[\pi_0 R]{\pi_0 L} \text{Out}(BG_p^\wedge)$$

between the groups of components.

(b) *The following maps are morphisms of topological monoids and homotopy equivalences:*

$$\begin{array}{ccccc} \text{Aut}(BG_p^\wedge) & \xleftarrow[\simeq]{\text{ev}} & |S_\bullet \text{Aut}(BG_p^\wedge)| & \xrightarrow[\simeq]{\sigma} & |\mathcal{A}ut(BG_p^\wedge)| \\ & & \xrightarrow[\simeq]{|\mathcal{L}_{BG_p^\wedge}|} & & \xrightarrow[\simeq]{|\mathcal{A}ut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))|} \\ & & |\mathcal{A}ut_{\text{typ}}(\mathcal{L}_p^c(BG_p^\wedge))| & \xleftarrow[\simeq]{|c_\beta^*|} & |\mathcal{A}ut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))|, \end{array}$$

where σ is the map defined in Section 3. In particular,

$$BAut(BG_p^\wedge) \simeq B(|Aut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))|).$$

Proof. Consider the maps

$$|Aut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))| \begin{array}{c} \xleftarrow{|L|} \\ \xrightarrow{|R|} \end{array} |Aut(BG_p^\wedge)|$$

defined in Section 4. By Proposition 3.7, $L \circ R$ is naturally isomorphic to the identity on $Aut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$; while $|L|$ (equivalently $|\mathcal{L}_{BG_p^\wedge}|$) is split injective on homotopy groups by Proposition 4.4. So R and L induce homotopy equivalences between these nerves; and in particular an isomorphism of groups between $Out(BG_p^\wedge) = \pi_0(|Aut(BG_p^\wedge)|)$ and $Out_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) = \pi_0(|Aut_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))|)$.

All four of the above maps in (b) are morphisms of monoids by construction, and ev and σ are homotopy equivalences by Lemma 3.1. Also, $|c_\beta^*|$ is a homotopy equivalence since it is induced by composition with equivalences between the two categories involved. Finally, $|\mathcal{L}_{BG_p^\wedge}|$ induces surjections on all homotopy groups by Proposition 3.7, and injections by Proposition 4.4.

The last statement now follows since a morphism of monoids which is a homotopy equivalence induces a homotopy equivalence between the classifying spaces, using [GJ, Proposition IV.1.7]. \square

5. HIGHER LIMIT OBSTRUCTIONS

We now look more closely at ways of constructing isotypical equivalences between the categories $\bar{\mathcal{L}}_p^c(G)$ and $\bar{\mathcal{L}}_p^c(G')$ when G and G' are finite groups. We construct obstructions to the existence and uniqueness of such equivalences which “extend” given fusion preserving isomorphisms between Sylow p -subgroups. This is then used to prove Conjecture D whenever G has p -rank less than p^2 .

Throughout this section and the next, for any functor $\psi : \mathcal{C} \rightarrow \mathcal{C}'$ and any objects c, d in \mathcal{C} , we let $\psi_{c,d}$ denote the induced map from $\text{Mor}_{\mathcal{C}}(c, d)$ to $\text{Mor}_{\mathcal{C}'}(\psi(c), \psi(d))$.

Let G and G' be finite groups with Sylow p -subgroups S and S' respectively, and assume $S \cong S'$. Recall that an isomorphism $S \xrightarrow{f} S'$ is called *fusion preserving* if given subgroups $P, Q \leq S$ and an isomorphism $P \xrightarrow{\alpha} Q$ between them, α is induced by conjugation in G if and only if

$f(P) \xrightarrow{f \circ \alpha \circ f^{-1}} f(Q)$ is induced by conjugation in G' . The following lemma shows in particular that each equivalence $\psi : \bar{\mathcal{L}}_p^c(G) \longrightarrow \bar{\mathcal{L}}_p^c(G')$ determines a fusion preserving isomorphism f_ψ between Sylow p -subgroups of G and G' .

Lemma 5.1. *Fix finite groups G and G' , and an isotypical equivalence*

$$\psi : \bar{\mathcal{L}}_p^c(G) \longrightarrow \bar{\mathcal{L}}_p^c(G')$$

of categories. Let $S \leq G$ and $S' = \psi(S) \leq G'$ be Sylow p -subgroups. Then

$$N_G(S)/C'_G(S) = \text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(S) \xrightarrow[\cong]{\psi_{S,S}} \text{Aut}_{\bar{\mathcal{L}}_p^c(G')}(S') = N_{G'}(S')/C'_{G'}(S')$$

restricts to a fusion preserving isomorphism $f_\psi : S \xrightarrow{\cong} S'$. Furthermore:

(i) *ψ is naturally isomorphic to an equivalence ψ' which agrees with ψ on S and on $\text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(S)$ (hence $f_{\psi'} = f_\psi$), and which sends inclusions of subgroups of S to inclusions of subgroups of S' . In other words, for any $P \leq S$, $\psi'(P \xrightarrow{\hat{1}} S) = (\psi'(P) \xrightarrow{\hat{1}} S')$.*

(ii) *Assume that ψ sends inclusions of subgroups of S to inclusions of subgroups of S' . Then for any p -centric $P \leq S$, $\psi(P) = f_\psi(P)$, and $\psi_{P,P}$ sends $\hat{g} \in \text{Aut}_{\bar{\mathcal{L}}_p^c(G)}(P)$ to $\widehat{f_\psi(g)} \in \text{Aut}_{\bar{\mathcal{L}}_p^c(G')}(f_\psi(P))$ for all $g \in P$.*

Proof. Since $S \triangleleft N_G(S)/C'_G(S)$ and $S' \triangleleft N_{G'}(S')/C'_{G'}(S')$ are (unique) Sylow p -subgroups, $\psi_{S,S}$ restricts to an isomorphism $f_\psi : S \xrightarrow{\cong} S'$. We prove that f_ψ is fusion preserving as point (iii) below.

(i) For each p -centric $P \leq G$, define a subgroup $\psi'(P) \leq S'$ and an isomorphism $\alpha_P : \psi(P) \xrightarrow{\cong} \psi'(P)$ in $\bar{\mathcal{L}}_p^c(G')$ as follows. If $P \not\leq S$, let $y \in N_{G'}(\psi(P), S')$ be such that $\psi_{P,S}(P \xrightarrow{\hat{1}} S) = \hat{y}$, and set $\psi'(P) = y(\psi(P))y^{-1}$ (the “image” of $\psi_{P,S}(\hat{1})$) and $\alpha_P = \hat{y}$. Otherwise, set $\psi'(P) = \psi(P)$ and $\alpha_P = \text{Id}$.

Now let ψ' be the functor which sends P to $\psi'(P)$, and where

$$\psi'_{P,Q}(P \xrightarrow{\hat{x}} Q) = \left(\psi'(P) \xrightarrow{\alpha(Q) \circ \psi(\hat{x}) \circ \alpha(P)^{-1}} \psi'(Q) \right).$$

Then ψ' is well defined as a functor, α is a natural isomorphism from ψ to ψ' , and ψ' sends inclusions into S to inclusions into S' . Notice that ψ' is automatically an isotypical equivalence.

(ii) Now assume ψ sends inclusions to inclusions. Fix a p -centric subgroup $P \leq S$, and set $P' = \psi(P) \leq S'$ for short. For any element $g \in P$, the

following square commutes in $\overline{\mathcal{L}}_p^c(G')$

$$\begin{array}{ccc} P' & \xrightarrow{\psi(\widehat{g})=\widehat{h}} & P' \\ \widehat{1} \downarrow & & \downarrow \widehat{1} \\ S' & \xrightarrow{\psi(\widehat{g})=\widehat{f_\psi(g)}} & S' \end{array}$$

(for some $h \in N_{G'}(P')$) since it is the image under ψ of a commutative square in $\overline{\mathcal{L}}_p^c(G)$. Thus $h \equiv f_\psi(g) \pmod{C'_{G'}(P')}$, so $f_\psi(g) \in N_{G'}(P')$, and $\psi_{P,P}(\widehat{g}) = \widehat{f_\psi(g)}$. In particular, since ψ is isotypical, this shows that $P' = f_\psi(P)$.

(iii) It remains to show that f_ψ is fusion preserving. This is an easy consequence of Alperin's fusion theorem in the form [Al, Theorem 5.1]; but since our result is much more elementary we give a direct proof.

For each P , let $P \xrightarrow[\cong]{\theta_P} P'$ be the restriction of $\psi_{P,P}$ to $P \leq N_G(P)/C'_G(P)$.

By Lemma 3.4, this defines a natural isomorphism $\theta: \lambda_G \xrightarrow{\cong} \lambda_G \circ \psi$ of functors. Thus, f_ψ may be identified with $\theta_S: S \longrightarrow S'$. By (i), we can assume that ψ sends inclusions of subgroups in S to inclusions of subgroups in S' . It follows that $\theta_P: P \longrightarrow P'$ is given by the restriction of θ_S to P . If $P, Q \leq S$ are p -centric in G and $\alpha = c_x: P \longrightarrow Q$ is a group isomorphism with $x \in N_G(P, Q)$ then

$$f_\psi \alpha f_\psi^{-1} = \theta_Q \alpha \theta_P^{-1} = \psi(\alpha) = c_{x'}$$

for some $x' \in N_{G'}(P', Q')$. Hence fusion is preserved among p -centric subgroups of S and S' .

Fix a subgroup $P \leq S$ which is not p -centric. We may assume inductively that f_ψ preserves fusion among all subgroups of strictly larger order. Let \mathcal{P} be the set of subgroups of S which are G -conjugate to P . We may assume that $N_S(P)$ is a Sylow p -subgroup of $N_G(P)$ (replace P by some other subgroup in \mathcal{P} if necessary).

Let $\overline{P} \leq S$ be a Sylow p -subgroup of $C_G(P) \cdot P$. Then

$$P \not\leq \overline{P} \leq N_G(P) \tag{1}$$

since P is not p -centric, and we claim that

$$N_G(P) = (N_G(P) \cap N_G(\overline{P})) \cdot C_G(P). \tag{2}$$

To see this, notice first that the right hand side is clearly a subgroup of $N_G(P)$. The other inclusion follows because if $x \in N_G(P)$, then $xc \in N_G(\overline{P})$ for some $c \in C_G(P)$. For any $x \in N_G(P)$, $c_x = c_y \in \text{Aut}(P)$ for some

$y \in N_G(P) \cap N_G(\bar{P})$ by (2); f_ψ sends $c_y \in \text{Aut}(\bar{P})$ to $c_{y'} \in \text{Aut}(f_\psi(\bar{P}))$ for some y' by (1) and the induction hypothesis; and hence sends $c_x \in \text{Aut}(P)$ to $c_{y'} \in \text{Aut}(f_\psi(P))$. So fusion is preserved for automorphisms of P .

Now let Q be any other subgroup in \mathcal{P} , and set $R = N_S(Q) (\cong Q)$ for short. Since $N_S(P)$ is a Sylow p -subgroup of $N_G(P)$, there is $x \in N_G(Q, P)$ such that $xRx^{-1} \leq N_S(P)$. By the induction hypothesis, there is $x' \in G'$ such that f_ψ sends $R \xrightarrow[\cong]{c_x} xRx^{-1}$ to $f_\psi(R) \xrightarrow[\cong]{c_{x'}} f_\psi(xRx^{-1})$. But then $x' \in N_G(f_\psi(Q), f_\psi(P))$, and so $f_\psi(Q)$ and $f_\psi(P)$ are conjugate in G' .

This shows that f_ψ sends fusion among subgroups of S to fusion among subgroups of S' . The same argument applied to f_ψ^{-1} shows the converse, and thus that f_ψ is fusion preserving. \square

The isomorphism $f_\psi : S \longrightarrow S'$ of Lemma 5.1 will be referred to as the underlying fusion preserving isomorphism of the equivalence $\psi : \bar{\mathcal{L}}_p^c(G) \longrightarrow \bar{\mathcal{L}}_p^c(G')$.

Martino and Priddy [MP, Corollary 1.2] showed that if $BG_p^\wedge \simeq BG'_p^\wedge$, then there is a fusion preserving isomorphism between Sylow p -subgroups of G and of G' ; this is a consequence of the description of $[BP, BG_p^\wedge]$ (shown here as Proposition 2.1). By Theorem A, an equivalence $BG_p^\wedge \simeq BG'_p^\wedge$ determines an equivalence of categories $\bar{\mathcal{L}}_p^c(G) \simeq \bar{\mathcal{L}}_p^c(G')$, and Lemma 5.1 gives an algebraic explanation of why this determines a fusion preserving isomorphism.

We will need the following property of fusion preserving isomorphisms.

Lemma 5.2. *Let $f : S \xrightarrow{\cong} S'$ be any fusion preserving isomorphism between Sylow subgroups $S \leq G$ and $S' \leq G'$ of finite groups G and G' . Then a subgroup $P \leq S$ is p -centric in G if and only if $f(P)$ is p -centric in G' .*

Proof. Let $P \leq S$ be any subgroup. Then P is G -conjugate to some $Q \leq S$ such that $C_S(Q)$ is a Sylow p -subgroup of $C_G(Q)$. If P is not p -centric in G then the same holds for Q , and $C_S(Q) \not\leq Q$. So $f(Q)$ is not p -centric, and $f(P)$ is G' -conjugate to $f(Q)$ since f is fusion preserving. The converse follows from the same argument applied to f^{-1} . \square

We now want to study the problem of constructing an equivalence of categories $\bar{\mathcal{L}}_p^c(G) \longrightarrow \bar{\mathcal{L}}_p^c(G')$ which has a given isomorphism between Sylow p -subgroups as its underlying fusion preserving isomorphism. The obstructions to doing this lie in certain higher derived functors of inverse limits over orbit categories. The following is a key lemma in the construction.

Lemma 5.3. *Let G and G' be finite groups with Sylow p -subgroup S and S' respectively. Let $f : S \longrightarrow S'$ be a fusion preserving isomorphism. Then for any $P, Q \leq S$ p -centric in G , there is a map*

$$\Phi_{P,Q} : N_G(P, Q) \longrightarrow N_{G'}(f(P), f(Q))$$

which satisfies the following three conditions for all $x \in N_G(P, Q)$ (all x such that $xPx^{-1} \leq Q$):

(a) the following square commutes (in the category of groups):

$$\begin{array}{ccc} P & \xrightarrow{c_x} & Q \\ f \downarrow & & \downarrow f \\ f(P) & \xrightarrow{c_{x'}} & f(Q), \end{array}$$

where $x' = \Phi_{P,Q}(x)$.

(b) $\Phi_{P,Q}(gx) = f(g)\Phi_{P,Q}(x)$ for all $g \in Q$.

(c) $\Phi_{P,Q}$ induces a bijection

$$\hat{\Phi}_{P,Q} : N_G(P, Q)/C'_G(P) \xrightarrow{\cong} N_{G'}(f(P), f(Q))/C'_{G'}(f(P)).$$

Proof. To simplify notation, for any $P \leq S$ we write $P' = f(P) \leq S'$. Conditions (a) and (c) mean that there is a commutative diagram of the following form

$$\begin{array}{ccccccc} N_G(P, Q) & \longrightarrow & N_G(P, Q)/C'_G(P) & \longrightarrow & N_G(P, Q)/C_G(P) & \subseteq & \text{Hom}(P, Q) \\ \Phi_{P,Q} \downarrow & & \Phi_{P,Q} \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ N_{G'}(P', Q') & \longrightarrow & N_{G'}(P', Q')/C'_{G'}(P') & \longrightarrow & N_{G'}(P', Q')/C_{G'}(P') & \subseteq & \text{Hom}(P', Q'), \end{array}$$

where f_* is the bijection between homomorphism sets induced by the isomorphisms $P \cong P'$ and $Q \cong Q'$. The restriction of f_* to $N_G(P, Q)/C_G(P)$ is defined and bijective by the assumption that f is fusion preserving.

Let $X \subseteq N_G(P, Q)$ be a set of orbit representatives for the left action of Q and the right action of $C'_G(P)$. For each $x \in X$, choose an arbitrary element $x' = \Phi_{P,Q}(x)$ such that $c_{x'} = f_*(c_x)$. Extend this to all of $N_G(P, Q)$ by setting $\Phi_{P,Q}(gxa) = f(g)\Phi_{P,Q}(x)$ for all $x \in X$, $g \in Q$, and $a \in C'_G(P)$. Then $\Phi_{P,Q}$ satisfies (a) and (b) above, and induces a map $\hat{\Phi}_{P,Q}$ which makes the above diagram commute. Furthermore, $\hat{\Phi}_{P,Q}$ is bijective, since the third column of the diagram is obtained by dividing out by the free action of $Z(P) \cong Z(P')$ on the second column. \square

For any small category \mathcal{C} and any contravariant functor $\mathcal{C} \rightarrow \mathbf{Ab}$, let $C^*(\mathcal{C}; F)$ denote the chain complex

$$C^n(\mathcal{C}; F) = \prod_{c_0 \rightarrow \cdots \rightarrow c_n} F(c_0),$$

with differentials $\phi \in C^{n-1}(\mathcal{C}; F) \xrightarrow{d} C^n(\mathcal{C}; F)$ defined by

$$d\phi(c_0 \xrightarrow{f} c_1 \rightarrow \cdots \rightarrow c_n) = f^*\phi(c_1 \rightarrow \cdots \rightarrow c_n) + \sum_{k=1}^n (-1)^k \phi(c_0 \rightarrow \cdots \widehat{c}_k \cdots \rightarrow c_n).$$

Then $\varprojlim_{\mathcal{C}}^*(F) \cong H^*(C^*(\mathcal{C}; F), d)$ (cf. [GZ, Appendix II, Proposition 3.3], applied with $\mathcal{M} = \mathbf{Ab}^{\text{op}}$).

Recall the notation used in Section 4 for objects and maps in orbit categories. For any finite G , $\mathcal{O}_p^c(G)$ is the category whose objects are the p -centric subgroups $P \leq G$, and where $\text{Mor}_{\mathcal{O}_p^c(G)}(P, Q) = Q \backslash N_G(P, Q)$ for any objects $P, Q \leq G$. The morphism corresponding to $x \in N_G(P, Q)$ is denoted \hat{x} , and corresponds to the G -map $G/P \rightarrow G/Q$ which sends gP to $gx^{-1}Q$ in the usual definition of $\mathcal{O}_p^c(G)$.

For any element g in a finite group G , we let $g_p \in G$ denote its “ p -part”. This is the unique element of p -power order in the cyclic subgroup $\langle g \rangle$ such that $|g^{-1}g_p|$ is prime to p . Equivalently, it is the unique element of p -power order in G for which there is an element $g' \in G$ of order prime to p such that $[g_p, g'] = 1$ and $g = g_p \cdot g'$. In particular, for any p -centric subgroup $P \leq G$, the projection of $C_G(P) = Z(P) \times C'_G(P)$ to $Z(P)$ sends each $g \in C_G(P)$ to $g_p \in Z(P)$.

Proposition 5.4. *Fix finite groups G and G' , and Sylow p -subgroups $S \leq G$ and $S' \leq G'$. Let $f : S \rightarrow S'$ be any fusion preserving isomorphism. Let $\mathcal{Z} : \mathcal{O}_p^c(G)^{\text{op}} \rightarrow p\text{-Gr}$ denote the functor which sends P to its center $Z(P)$. Then there is an element $\omega(f) \in \varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z})$ with the following property: there is an isotypical equivalence of categories*

$$\psi : \bar{\mathcal{L}}_p^c(G) \xrightarrow{\cong} \bar{\mathcal{L}}_p^c(G')$$

with $f_\psi = f$, if and only if $\omega(f) = 0$.

Proof. Again, for any $P \leq S$ we write $P' = f(P) \leq S'$. Let $\bar{\mathcal{L}}_S^c(G) \subseteq \bar{\mathcal{L}}_p^c(G)$ and $\mathcal{O}_S^c(G) \subseteq \mathcal{O}_p^c(G)$ denote the full subcategories whose objects are the p -centric subgroups of G contained in S , and similarly for $\bar{\mathcal{L}}_{S'}^c(G') \subseteq \bar{\mathcal{L}}_p^c(G')$.

All three of these inclusions are equivalences of categories, since every p -centric subgroup of G is conjugate to a subgroup of S . It follows that $\varprojlim^*_{\mathcal{O}_S^c(G)}(\mathcal{Z}) \cong \varprojlim^*_{\mathcal{O}_p^c(G)}(\mathcal{Z})$. Also, any equivalence $\psi : \bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$ extends to an equivalence $\bar{\mathcal{L}}_p^c(G) \rightarrow \bar{\mathcal{L}}_p^c(G')$. So we can restrict our attention to these subcategories.

We define ψ on objects by setting $\psi(P) = P'$ for all $P \leq S$ p -centric in G . By Lemma 5.2, this defines a bijection between objects in $\bar{\mathcal{L}}_S^c(G)$ and $\bar{\mathcal{L}}_{S'}^c(G')$. The problem is to define ψ on morphisms. Thus, our goal is to construct the obstruction class $\omega(f)$, and show that ψ can be well defined on morphisms if and only if $\omega(f) = 0$.

Given a choice of the $\Phi_{P,Q}$, for all p -centric $P, Q \leq S$, satisfying conditions (a), (b), and (c) of Lemma 5.3, we now define an element

$$\omega_\Phi \in C^2(\mathcal{O}_S^c(G); \mathcal{Z}) = \prod_{P \rightarrow Q \rightarrow R} \mathcal{Z}(P)$$

as follows. For each $x \in N_G(P, Q)$ and $y \in N_G(Q, R)$,

$$f_*(c_y)f_*(c_x) = f_*(c_{yx}),$$

where f_* is as in Lemma 5.3. Hence $\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)\Phi_{P,Q}(x) \in C_{G'}(P')$. Define

$$\omega'_\Phi(y, x) \stackrel{\text{def}}{=} (\Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)\Phi_{P,Q}(x))_p \in Z(P').$$

For each $P \xrightarrow{\dot{x}} Q \xrightarrow{\dot{y}} R$ in $\mathcal{O}_S^c(G)$, set

$$\omega_\Phi(y, x) = \omega_\Phi(P \xrightarrow{\dot{x}} Q \xrightarrow{\dot{y}} R) \stackrel{\text{def}}{=} f^{-1}(\omega'_\Phi(y, x)) \in Z(P).$$

To show that this is well defined we must show that $\omega'_\Phi(hy, gx) = \omega'_\Phi(y, x)$ for any $g \in Q$ and any $h \in R$. Indeed

$$\Phi_{P,R}(hygx) = f(hygy^{-1})\Phi_{P,R}(yx),$$

$$\Phi_{Q,R}(hy)\Phi_{P,Q}(gx) = f(h)\Phi_{Q,R}(y)f(g)\Phi_{P,Q}(x),$$

and hence

$$\begin{aligned} & \Phi_{P,R}(hygx)^{-1}\Phi_{Q,R}(hy)\Phi_{P,Q}(gx) \\ &= \Phi_{P,R}(yx)^{-1}f(yg^{-1}y^{-1})\Phi_{Q,R}(y)f(g)\Phi_{P,Q}(x) \\ &= \Phi_{P,R}(yx)^{-1}(f(yg^{-1}y^{-1})\Phi_{Q,R}(y)f(g)\Phi_{Q,R}(y)^{-1})\Phi_{Q,R}(y)\Phi_{P,Q}(x). \end{aligned}$$

But $f(ygy^{-1}) = \Phi_{Q,R}(y)f(g)\Phi_{Q,R}(y)^{-1}$ by (a). Hence

$$\Phi_{P,R}(hygx)^{-1}\Phi_{Q,R}(hy)\Phi_{P,Q}(gx) = \Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)\Phi_{P,Q}(x),$$

and $\omega'_{\Phi}(hy, gx) = \omega'_{\Phi}(y, x)$ as claimed.

To see that ω_{Φ} is a 2-cocycle, we must show, for each sequence

$$P \xrightarrow{\overset{\circ}{x}} Q \xrightarrow{\overset{\circ}{y}} R \xrightarrow{\overset{\circ}{z}} T$$

in $\mathcal{O}_S^c(G)$, that

$$\begin{aligned} d(\omega_{\Phi})(P \xrightarrow{\overset{\circ}{x}} Q \xrightarrow{\overset{\circ}{y}} R \xrightarrow{\overset{\circ}{z}} T) &\stackrel{\text{def}}{=} \\ (x^{-1}\omega(z, y)x) \cdot \omega(z, yx)^{-1}\omega(z, y)\omega(y, x)^{-1} &= 1. \end{aligned}$$

For any $g \in Z(Q)$,

$$f(x^{-1}gx) \equiv \Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)f(g)\Phi_{Q,R}(y)^{-1}\Phi_{P,Q}(yx)$$

by condition (a) (applied to the squares for y and yx); so it suffices to show that

$$\begin{aligned} \Phi_{P,R}(yx)^{-1}\Phi_{Q,R}(y)f(\omega(z, y))\Phi_{Q,R}(y)^{-1}\Phi_{P,R}(yx) \cdot \\ f(\omega(z, yx)^{-1}\omega(z, y)\omega(y, x)^{-1}) &= 1. \end{aligned}$$

And this follows immediately by substitution and cancellation.

Now set $\omega(f) = [\omega_{\Phi}] \in \varprojlim_{\mathcal{O}_S^c(G)}^2(\mathcal{Z})$. We claim that $\omega(f)$ depends only on

the isomorphism $f : S \rightarrow S'$, and that f extends to an isotypical equivalence $\psi : \bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$ of categories (i.e., $f_{\psi} = f$) if and only if $\omega(f) = 0$. Clearly, the $\Phi_{P,Q}$ induce a functor $\psi : \bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$ by taking the corresponding bijections $\hat{\Phi}_{P,Q}$ if and only if $\omega_{\Phi} = 0$. Also, for any isotypical equivalence $\psi : \bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$ with $f_{\psi} = f$ which sends inclusions in S to inclusions in S' , the maps $\Phi_{P,Q}$ defined by ψ between morphism sets satisfy conditions (a) and (b) above by Lemma 5.1(ii). So it remains to show that

- (1) for any two choices of maps $\Phi_{P,Q}$ and $\Phi'_{P,Q}$, the corresponding 2-cycles ω_{Φ} and $\omega_{\Phi'}$ differ by a coboundary, and
- (2) all coboundaries can be realized in this way.

To prove point 1, fix maps $\Phi_{P,Q}$ and $\Phi'_{P,Q}$ satisfying conditions (a), (b), and (c) of Lemma 5.3. For each morphism $P \xrightarrow{x} Q$ in $\bar{\mathcal{L}}_S^c(G)$, one has $\Phi_{P,Q}(x)^{-1}\Phi'_{P,Q}(x) \in C_{G'}(P')$ by condition (a). Set

$$A(x) \stackrel{\text{def}}{=} \Phi_{P,Q}(x)^{-1}\Phi'_{P,Q}(x); \quad \text{and} \quad \alpha(x) \stackrel{\text{def}}{=} f^{-1}((A(x))_p) \in Z(P). \quad (1)$$

Then $\alpha(x)$ does not depend of the choice of the lifts $\Phi_{P,Q}$ and $\Phi'_{P,Q}$. Also, $\alpha(x)$ depends only on the class $\dot{x} \in \text{Mor}_{\mathcal{O}_S^c(G)}(P, Q)$ by condition (b). So

$$\alpha \in C^1(\mathcal{O}_S^c(G); \mathcal{Z}) = \prod_{P \rightarrow Q} Z(P).$$

Write $a_x = \alpha(x)$ for short. For each sequence $P \xrightarrow{\dot{x}} Q \xrightarrow{\dot{y}} R$ in $\mathcal{O}_S^c(G)$, repeated application of condition (a) gives

$$\begin{aligned} f(\omega_{\Phi'}(y, x)) &= (\Phi'(yx)^{-1}\Phi'(y)\Phi'(x))_p \\ &= (A(yx)^{-1}(\Phi(yx)^{-1}\Phi(y)\Phi(x))(\Phi(x)^{-1}A(y)\Phi(x))A(x))_p \\ &= f(a_{yx})^{-1}\omega'_{\Phi}(y, x)(\Phi(x)^{-1}A(y)\Phi(x))_p f(a_x) \\ &= \omega'_{\Phi}(y, x)[\Phi(x)^{-1}f(a_y)\Phi(x)]f(a_{yx})^{-1}f(a_x) \\ &= \omega'_{\Phi}(y, x)f((x^{-1}a_yx)a_{yx}^{-1}a_x) \\ &= f(\omega_{\Phi}(y, x)) \cdot f(d\alpha(P \xrightarrow{\dot{x}} Q \xrightarrow{\dot{y}} R)). \end{aligned}$$

Thus $\omega_{\Phi'} = \omega_{\Phi} + d\alpha$.

Finally to see 2 above notice that any 1-cochain α can be realized as the difference between Φ and some Φ' , since one can use (1) to define Φ' . This completes the proof of the proposition. \square

Remark 5.5. Let $\mathcal{P}_p^c(G)$ denote the Frobenius category: the category whose objects are the p -centric subgroups of G , and with $\text{Mor}_{\mathcal{P}_p^c(G)}(P, Q) = N_G(P, Q)/C_G(P) (\subseteq \text{Hom}(P, Q))$. Martino and Priddy in [MP] defined an *isotypical equivalence of Frobenius categories* to be an equivalence $\psi : \mathcal{P}_p^c(G) \longrightarrow \mathcal{P}_p^c(G')$ together with isomorphisms $P \xrightarrow{\cong} \psi(P)$ which are natural with respect to all morphisms in $\mathcal{P}_p^c(G)$. They showed [MP, Corollary 1.2] that there is an isotypical equivalence between $\mathcal{P}_p^c(G)$ and $\mathcal{P}_p^c(G')$ if and only if there is a fusion preserving isomorphism between the Sylow p -subgroups of G and G' . Stewart Priddy pointed out to us that the obstruction for lifting an isotypical equivalence of Frobenius categories to an equivalence of categories $\bar{\mathcal{L}}_p^c(-)$ lies in $\varprojlim_{\mathcal{P}_p^c(G)}^2(\mathcal{Z})$, in contrast to the obstruction group of Proposition 5.4. There is a homomorphism

$$\varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z}) \longrightarrow \varprojlim_{\mathcal{P}_p^c(G)}^2(\mathcal{Z})$$

which sends our obstruction to theirs, but it seems quite possible that our obstruction could be nonzero and theirs zero. The difference between them

is that $\omega(f) \in \varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z})$ is the obstruction to constructing an equivalence ψ such that $f_\psi = f$, while its image in $\varprojlim_{\mathcal{P}_p^c(G)}^2(\mathcal{Z})$ is the obstruction to constructing ψ such that $f_\psi \equiv f$ modulo $Z(S)$.

Given a fusion preserving isomorphism f between Sylow p -subgroups of G and G' , one can try directly to construct a homotopy equivalence

$$BG_p^\wedge \simeq \left(\operatorname{hocolim}_{P \in \mathcal{O}_p^c(G)} (EG/P) \right)_p^\wedge \longrightarrow BG'_p{}^\wedge.$$

Maps $EG/P \longrightarrow BG'_p{}^\wedge$ are determined up to homotopy by f , and the obstruction to extending them to a map defined on the homotopy colimit lies precisely in $\varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z})$. It was this observation — that one finds the same

obstruction group to constructing an equivalence of categories $\bar{\mathcal{L}}_p^c(-)$ and to constructing an equivalence of spaces — which first suggested Theorem A to us.

Just as the obstruction to realizing a fusion preserving isomorphism as the underlying isomorphism of an equivalence of categories $\bar{\mathcal{L}}_p^c(-)$ lies in $\varprojlim^2(\mathcal{Z})$, the obstruction to the uniqueness of such a realization lies in $\varprojlim^1(\mathcal{Z})$. To simplify this discussion, we restrict attention to automorphisms of $\bar{\mathcal{L}}_p^c(G)$. Recall that $\operatorname{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$ denotes the group of isotypical self equivalences modulo natural isomorphism. By analogy, we let $\operatorname{Aut}_{\text{fus}}(S)$, for any Sylow p -subgroup $S \leq G$, denote the group of fusion preserving automorphisms of S , and set

$$\operatorname{Out}_{\text{fus}}(S) = \operatorname{Aut}_{\text{fus}}(S) / \{c_x \mid x \in N_G(S)\}.$$

We are now ready to restate and prove Theorem E. As pointed out in the introduction, this is an algebraic version of the exact sequence of [BL, Theorem 1.6], where $\operatorname{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$ has been replaced by $\operatorname{Out}(BG_p^\wedge)$.

Theorem 5.6. *For any finite group G , there is an exact sequence*

$$0 \longrightarrow \varprojlim_{\mathcal{O}_p^c(G)}^1(\mathcal{Z}) \xrightarrow{\lambda_G} \operatorname{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \xrightarrow{\mu_G} \operatorname{Out}_{\text{fus}}(S) \xrightarrow{\omega_G} \varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z}),$$

where λ_G and μ_G are group homomorphisms.

Proof. As in the proof of Proposition 5.4, we replace $\bar{\mathcal{L}}_p^c(G)$ and $\mathcal{O}_p^c(G)$ by their equivalent subcategories $\bar{\mathcal{L}}_S^c(G)$ and $\mathcal{O}_S^c(G)$: the full subcategories whose objects are the subgroups of S which are p -centric in G .

Let ω_G be the obstruction map of Proposition 5.4, and let μ_G be the map which sends $\psi \in \text{Aut}_{\text{typ}}(\widehat{\mathcal{L}}_p^c(G))$ to f_ψ . If ψ and ψ' are naturally isomorphic, by a natural isomorphism which sends S to $\widehat{x} \in \text{Aut}_{\widehat{\mathcal{L}}_p^c(G)}(S)$, then f_ψ and $f_{\psi'}$ differ by conjugation by $x \in N_G(S)$. So μ_G is well defined, and the sequence is exact at $\text{Out}_{\text{fus}}(S)$ by Proposition 5.4.

A 1-cocycle $\alpha \in Z^1(\mathcal{O}_5^c(G); \mathcal{Z})$ is a collection of maps

$$\text{Mor}_{\mathcal{O}_5^c(G)}(P, Q) \xrightarrow{\alpha_{P,Q}} Z(P)$$

such that

$$(x^{-1}\alpha(y)x) \cdot \alpha(yx)^{-1} \cdot \alpha(x) = 1 \quad (1)$$

for all $(P \xrightarrow{\overset{\circ}{x}} Q \xrightarrow{\overset{\circ}{y}} R)$. In particular, $\alpha(\overset{\circ}{g}) = \alpha(\text{Id}_P) = 1$ for all $g \in P$. For any such α , set $\lambda_G(\alpha) = [\psi_\alpha]$, where $\psi_\alpha(P) = P$ for all objects P , and $\psi_\alpha(\widehat{x}) = \widehat{x\alpha(x)}$. Then ψ_α is a functor by (1). And if β is a 0-cochain, then the elements $\beta(P) \in Z(P)$ define a natural isomorphism $\text{Id} \longrightarrow \psi_{d\beta}$. One easily checks that $\psi_{\alpha+\alpha'} = \psi_\alpha \circ \psi_{\alpha'}$; and this shows that λ_G is a well defined homomorphism.

It is straightforward to show that $\text{Im}(\lambda_G) = \text{Ker}(\mu_G)$. To see that λ_G is injective, fix $[\alpha] \in \text{Ker}(\lambda_G)$, and let $\beta : \text{Id} \longrightarrow \psi_\alpha$ be a natural isomorphism. This consists of a choice of $\widehat{\beta(P)} \in \text{Aut}_{\widehat{\mathcal{L}}_p^c(G)}(P)$ for each P , where $\beta(P) \in N(P)$. Naturality with respect to automorphisms $P \xrightarrow{\widehat{g}} P$ for $g \in P$ implies that $\beta(P) \in Z(P)$ for each P (since $\psi_\alpha(\widehat{g}) = \widehat{g}$). Thus $\beta \in C^0(\mathcal{O}_5^c(G); \mathcal{Z})$, and $\alpha = d\beta$. \square

The next proposition deals with the question whether or not the obstruction map is a homomorphism. As we are not aware of an example of a finite group G , where the corresponding $\varprojlim^2(\mathcal{Z})$ is non-zero, we obviously don't have an example where ω_G fails to be a homomorphism. However, the proposition shows that ω_G should be thought of as a 1-cocycle rather than a homomorphism.

Notice that $\text{Out}_{\text{fus}}(S)$ acts on $\varprojlim^2(\mathcal{Z})$ on the right. The action is given as follows. Let $f \in \text{Aut}_{\text{fus}}(S)$ be a fusion preserving automorphism of S . The group $C^2(\mathcal{O}_5^c(G); \mathcal{Z})$ is generated by functions ν from the collection of all 2-chains of morphisms $P \xrightarrow{\overset{\circ}{x}} Q \xrightarrow{\overset{\circ}{y}} R$ in $\mathcal{O}_p^c(G)$ to the product of all centers of p -centric subgroups of S , where $\nu(y, x) \in Z(P)$. Choose maps $\Phi_{P,Q}$ as in Lemma 5.3, and define

$$(\nu \cdot f)(y, x) = f^{-1}(\nu(\Phi_{Q,R}(y), \Phi_{P,Q}(x))).$$

We leave it to the reader to verify that this is well defined and gives the required right action.

Proposition 5.7. *For any finite group G , the obstruction map*

$$\omega_G : \text{Out}_{\text{fus}}(S) \longrightarrow \varprojlim_{\mathcal{O}_p^c(G)}^2(\mathcal{Z})$$

satisfies the relation

$$\omega_G(f_2 \circ f_1) = \omega_G(f_1) + (\omega_G(f_2))f_1.$$

In particular ω_G is a homomorphism if the action of $\text{Out}_{\text{fus}}(S)$ on $\varprojlim^2(\mathcal{Z})$ is trivial.

Proof. For any p -subgroups $P, Q \leq G$, any $x \in N_G(P, Q)$, and any $g \in P$,

$$\Phi_{P,Q}(xg) = f(xgx^{-1}) \cdot \Phi_{P,Q}(x) = \Phi_{P,Q}(x) \cdot f(g) :$$

the first equality follows from (b) and the second from (a). Hence for any $x, x' \in N_G(P, Q)$ such that $x^{-1}x' \in P$,

$$(\Phi_{P,Q}(x))^{-1} \cdot \Phi_{P,Q}(x') = f(x^{-1}x'). \quad (1)$$

Now fix fusion preserving automorphisms f_1, f_2 of S . For each $i = 1, 2$, let $\Phi_{P,Q}^i$ be maps which satisfy conditions (a), (b), and (c) of Lemma 5.3 relative to f_i . To simplify the notation in what follows, we drop the subscripts whenever possible. Fix a sequence of morphisms $P \xrightarrow{\overset{x}{\alpha}} Q \xrightarrow{\overset{y}{\beta}} R$ in $\mathcal{O}_p^c(G)$. Then

$$\begin{aligned} (f_2 \circ f_1)(\omega(f_2 \circ f_1)(y, x)) &= ((\Phi^2 \Phi^1(yx))^{-1} \cdot \Phi^2 \Phi^1(y) \cdot \Phi^2 \Phi^1(x))_p \\ &= ((\Phi^2 \Phi^1(yx))^{-1} \cdot (\Phi^2(\Phi^1(y)\Phi^1(x))))_p \\ &\quad \cdot ((\Phi^2(\Phi^1(y)\Phi^1(x)))^{-1} \cdot \Phi^2 \Phi^1(y) \cdot \Phi^2 \Phi^1(x))_p. \end{aligned}$$

By (1),

$$((\Phi^2 \Phi^1(yx))^{-1} \cdot (\Phi^2(\Phi^1(y)\Phi^1(x))))_p = f_2((\Phi^1(yx))^{-1} \cdot \Phi^1(y)\Phi^1(x))_p;$$

and by definition

$$((\Phi^2(\Phi^1(y)\Phi^1(x)))^{-1} \cdot \Phi^2 \Phi^1(y) \cdot \Phi^2 \Phi^1(x))_p = f_2(\omega(f_2)(\Phi^1(y), \Phi^1(x))).$$

Hence

$$\omega(f_2 \circ f_1)(y, x) = \omega(f_1)(y, x) \cdot f_1^{-1}(\omega(f_2)(\Phi^1(y), \Phi^1(x))),$$

and so

$$\omega(f_2 \circ f_1) = \omega(f_1) + (\omega(f_2))f_1. \quad \square$$

We now discuss vanishing of higher limits groups over orbit categories. Recall that the p -rank $\text{rk}_p(G)$ of a finite group G is the rank of the largest elementary abelian p -subgroup of G . In particular, we show that $\varprojlim_{\mathcal{O}_p^c(G)}^r(\mathcal{Z}) = 0$ whenever $\text{rk}_p(G) < p^r$.

One way to compute higher limits of functors over orbit categories is to filter the functor in a way so that each of the quotient functors vanishes except on one isomorphism class of objects. In particular, if higher limits of the original functor are non-vanishing, then they must also be non-vanishing for at least one of the quotient functors. This motivated the definition of graded abelian groups $\Lambda^*(G; M)$ in [JMO]. For any prime p , any finite group G , and any $\mathbb{Z}_{(p)}[G]$ -module M ,

$$\Lambda^*(G; M) \stackrel{\text{def}}{=} \varprojlim_{\mathcal{O}_p(G)}^*(F_M), \quad \text{where} \quad F_M(P) = \begin{cases} M & \text{if } P = 1 \\ 0 & \text{otherwise.} \end{cases}$$

These groups have the property that if $F : \mathcal{O}_p(G)^{\text{op}} \rightarrow \mathbb{Z}_{(p)}\text{-mod}$ is any functor which vanishes except on subgroups conjugate to P , then $\varprojlim^*(F) \cong \Lambda^*(N(P)/P; F(P))$ (see [JMO, Lemma 5.4]).

Proposition 5.8. *For any finite group G , any finitely generated $\mathbb{F}_p[G]$ -module M , and any $k \geq 1$ such that $\Lambda^k(G; M) \neq 0$, there is an elementary abelian p -subgroup $A \leq G$ of rank k such that $M|_A$ contains the free module $\mathbb{F}_p[A]$ as a direct summand. In particular,*

$$\text{rk}_p(G) \geq k \quad \text{and} \quad \dim_{\mathbb{F}_p}(M) \geq p^k.$$

Proof. Let $\mathcal{S}_p(G)$ denote the nerve of the poset of nontrivial p -subgroups of G , and let $\tilde{C}_*(\mathcal{S}_p(G))$ denote its reduced chain complex. Thus $\tilde{C}_{-1}(\mathcal{S}_p(G)) = \mathbb{Z}$, and $\tilde{C}_k(\mathcal{S}_p(G))$ (for $k \geq 0$) is the free \mathbb{Z} -module with basis the set of sequences $1 \neq P_0 \leq P_1 \leq \dots \leq P_k$. The conjugation action of G on the basis makes this into a chain complex of $\mathbb{Z}[G]$ -modules. By [Gr, Theorem 1.2], for any $\mathbb{Z}_{(p)}[G]$ -module M ,

$$\Lambda^*(G; M) \cong H^{*-1}(\text{Hom}_G(\tilde{C}_*(\mathcal{S}_p(G)), M)).$$

Similarly, let $\mathcal{A}_p(G)$ denote the nerve of the poset of nontrivial elementary abelian p -subgroups of G , and set $C_* = \tilde{C}_*(\mathcal{A}_p(G); \mathbb{F}_p)$. By [Qu], $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ are G -homotopy equivalent, and so the chain complexes $\Lambda^*(G; M) \cong H^{*-1}(\text{Hom}_G(C_*, M))$ for any $\mathbb{F}_p[G]$ -module M . By [Wb2, Theorem 2.7.1], C_* splits as a sum $C_* = D_* \oplus P_*$ of complexes of $\mathbb{F}_p[G]$ -modules, where D_* is exact (in fact, split exact), and where each P_k is projective. In particular, $\Lambda^*(G; M)$ is the cohomology of the complex $\text{Hom}_G(P_*, M)$.

Assume that $\Lambda^k(G; M) \neq 0$. Then $\text{Hom}_G(P_{k-1}, M) \neq 0$ by the above remarks. Since P_{k-1} is $\mathbb{F}_p[G]$ -projective, the module $\text{Hom}_{\mathbb{F}_p}(P_{k-1}, M)$ is also $\mathbb{F}_p[G]$ -projective. Notice that for any projective G -module P , the submodule of G -invariants is given by

$$P^G = \{\nu_G \cdot x \mid x \in P\},$$

where ν_G is the norm element $\sum_{g \in G} g \in \mathbb{F}_p[G]$. The group G acts on $\text{Hom}_{\mathbb{F}_p}(P_{k-1}, M)$ by $(g\phi)(x) = g \cdot \phi(g^{-1}x)$, and

$$0 \neq \text{Hom}_G(P_{k-1}, M) \cong (\text{Hom}_{\mathbb{F}_p}(P_{k-1}, M))^G.$$

Hence there is some $\phi : P_{k-1} \rightarrow M$ such that $\sum_{g \in G} g\phi \neq 0$. By extension, we may assume that ϕ with this property is defined on $C_{k-1} = D_{k-1} \oplus P_{k-1}$.

The complex C_* is generated as an $\mathbb{F}_p[G]$ -module in dimension $k-1$ by the $(k-1)$ -simplices of $\mathcal{A}_p(G)$, namely, by sequences of the form $\tau = (E_1 \not\leq \dots \not\leq E_k)$. Hence there exists such a simplex τ such that

$$\sum_{g \in G} (g\phi)(\tau) = \sum_{g \in G} g \cdot \phi(g^{-1}\tau) \neq 0,$$

where ϕ is as above.

Let b_1, \dots, b_r be a choice of right coset representatives for E_k in G . Notice that E_k is contained in the stabilizer group of τ . Thus

$$0 \neq \sum_{g \in G} (g\phi)(\tau) = \sum_{g \in E_k} \sum_{i=1}^r g b_i \phi(b_i^{-1}\tau).$$

Set $x = \sum_{i=1}^r b_i \phi(b_i^{-1}\tau) \in M$. Then $\sum_{g \in E_k} gx \neq 0$. Notice that $\text{rk}_p(G) \geq \text{rk}_p(E_k) \geq k$.

Let $\iota_x : \mathbb{F}_p[E_k] \longrightarrow M$ denote the E_k -map sending 1 to x . Then $\text{Ker}(\iota_x)$ does not contain the norm element ν_{E_k} and hence is the zero ideal in the group ring (cf. [Se, §8.3, Prop. 26]). Thus ι_x is a monomorphism. Furthermore, since $\mathbb{F}_p[E_k]$ is an injective E_k -module, the module $M|_{E_k}$ contains $\mathbb{F}_p[E_k]$ as a split summand. In particular, $\dim_{\mathbb{F}_p}(M) \geq |E_k| \geq p^k$. This completes the proof. \square

The idea of the last part of the above proof was suggested to us by the proof of [Wb1, 5.3].

The inequalities of Proposition 5.8 are in fact the best possible, as is shown by the following examples. Let V denote the $\mathbb{F}_p[\Sigma_{p+1}]$ -module $V = (\mathbb{F}_p)^{p+1}/(\text{diag})$. Then $\Lambda^1(\Sigma_{p+1}; V) \cong \mathbb{F}_p$ by [JMO, Proposition 6.2(i)]. So for any $k > 0$,

$$\Lambda^k((\Sigma_{p+1})^k, V^{\otimes k}) \neq 0$$

by the Künneth formula in [JMO, Proposition 6.1(v)]; and

$$\Lambda^k(\Sigma_{p+1} \underbrace{\wr C_p \wr \cdots \wr C_p}_{(k-1) \text{ times}}, V^{p^{k-1}}) \neq 0$$

by a general formula $\Lambda^i(G \wr C_p; M^p) \cong \Lambda^{i-1}(G; M)$ when $p \mid |G|$ (unpublished). These are thus two examples of pairs (G, M) for which $\Lambda^k(G; M) \neq 0$, $\text{rk}_p(G) = k$, and $\dim_{\mathbb{F}_p}(M) = p^k$.

When $k = 1$, a similar argument gives a much stronger result. If M is an $\mathbb{F}_p[G]$ -module with $\Lambda^1(G; M) \neq 0$, then for any Sylow p -subgroup $S \leq G$, $M|_S$ contains a summand isomorphic to $\mathbb{F}_p[S]$, and in particular $\dim_{\mathbb{F}_p}(M) \geq |S|$.

As an easy consequence of Proposition 5.8, we get:

Corollary 5.9. *Fix a finite group G , and a functor $F : \mathcal{O}_p^c(G)^{\text{op}} \rightarrow p\text{-Gr}$. Assume, for some $k > 0$, that $\varprojlim^k_{\mathcal{O}_p^c(G)}(F) \neq 0$. Then there is a p -centric subgroup $P \leq G$ such that $\text{rk}_p(F(P)) \geq p^k$.*

Proof. Extend F to a functor on $\mathcal{O}_p(G)$ by setting $F(P) = 0$ whenever P is not p -centric. Since any p -subgroup of G which contains a p -centric subgroup is itself p -centric, the complexes $C^*(\mathcal{O}_p(G); F)$ and $C^*(\mathcal{O}_p^c(G); F)$ are isomorphic. So higher limits of F are the same whether taken over $\mathcal{O}_p(G)$ or over $\mathcal{O}_p^c(G)$.

Now let P_1, P_2, \dots, P_k be conjugacy class representatives for p -centric subgroups of G , arranged such that $|P_1| \leq |P_2| \leq \dots$. For $0 \leq i \leq k$, let F_i be the functor

$$F_i(P) = \begin{cases} 0 & \text{if } P \geq xP_jx^{-1}, \text{ some } x \in G \text{ and } j > i \\ F(P) & \text{otherwise.} \end{cases}$$

Thus, $F_0 = 0$, $F_k = F$, and F_i is a subfunctor of F_{i+1} for each i . By the exact sequences of higher limits for extensions of functors, there is some $1 \leq i \leq k$ such that $\varprojlim^k(F_i/F_{i-1}) \neq 0$. The functor F_i/F_{i-1} vanishes except on subgroups conjugate to P_i , and hence

$$\Lambda^k(N(P_i)/P_i; F(P_i)) \cong \varprojlim^k_{\mathcal{O}_p(G)}(F_i/F_{i-1}) \neq 0$$

by [JMO, Lemma 5.4]. Then $\Lambda^k(N(P)/P; p^j F(P)/p^{j+1} F(P)) \neq 0$ for some $j \geq 0$ (where $P = P_i$), so this module has rank at least p^k by Proposition 5.8, and hence $\text{rk}_p(F(P)) \geq p^k$. \square

We are now ready to prove the first half of Theorem F.

Theorem 5.10. *Fix a prime p , and a finite group G such that $\text{rk}_p(G) < p^2$. Then for any other finite group G' , $BG_p^\wedge \simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism between Sylow p -subgroups of G and of G' .*

Proof. If $BG_p^\wedge \simeq BG'^\wedge_p$, then there is an equivalence of categories

$$\psi : \bar{\mathcal{L}}_p^c(G) \xrightarrow{\cong} \bar{\mathcal{L}}_p^c(G')$$

by Theorem A, and hence a fusion preserving isomorphism between the Sylow p -subgroups by Lemma 5.1.

Conversely, if $\text{rk}_p(G) < p^2$ then $\text{rk}_p(Z(P)) < p^2$ for each p -centric $P \leq G$, and Corollary 5.9 implies that $\varprojlim^2(\mathcal{Z}) = 0$. Here, as usual, $\mathcal{Z} : \mathcal{O}_p^c(G) \rightarrow p\text{-Gr}$ is the functor which sends \bar{P} to $Z(P)$. So if there is a fusion preserving isomorphism between Sylow p -subgroups of G and G' , then the categories $\bar{\mathcal{L}}_p^c(G)$ and $\bar{\mathcal{L}}_p^c(G')$ are equivalent by Proposition 5.4, and so $BG_p^\wedge \simeq BG'^\wedge_p$ by Theorem A. \square

6. A DIRECT CONSTRUCTION OF EQUIVALENCES BETWEEN CATEGORIES

We now describe another, more direct approach to constructing equivalences of categories $\bar{\mathcal{L}}_p^c(-)$. The following lemma will be useful when constructing functors. It follows immediately from the definition of the categories $\bar{\mathcal{L}}_p^c(-)$.

Lemma 6.1. *For any finite group G all morphisms in $\bar{\mathcal{L}}_p^c(G)$ are monomorphism in the categorical sense. In other words, for any p -centric subgroups $P, Q, R \leq G$, and any $\hat{x}, \hat{y} \in \text{Mor}_{\bar{\mathcal{L}}_p^c(G)}(P, Q)$ and $\hat{z} \in \text{Mor}_{\bar{\mathcal{L}}_p^c(G)}(Q, R)$, $\hat{z} \circ \hat{x} = \hat{z} \circ \hat{y}$ implies $\hat{x} = \hat{y}$. \square*

A p -subgroup $P \leq G$ will be called *p -radical* if $O_p(N_G(P)) = P$; i.e., if $N_G(P)/P$ contains no nontrivial normal p -subgroups. In [JMO] such subgroups are called *p -stubborn*. We let $\bar{\mathcal{L}}_p^{cr}(G) \subseteq \bar{\mathcal{L}}_p^c(G)$ denote the full subcategory whose objects are the p -radical p -centric subgroups of G . Also, for any Sylow p -subgroup $S \leq G$, we let

$$\bar{\mathcal{L}}_S(G) \subseteq \bar{\mathcal{L}}_p(G), \quad \bar{\mathcal{L}}_S^c(G) \subseteq \bar{\mathcal{L}}_p^c(G), \quad \text{and} \quad \bar{\mathcal{L}}_S^{cr}(G) \subseteq \bar{\mathcal{L}}_p^{cr}(G)$$

denote the full subcategories whose objects in each case are those subgroups in the larger category which are contained in S . All three of these inclusions are equivalences of categories.

The following lemma shows that equivalences between categories $\bar{\mathcal{L}}_S^{cr}(-)$ are essentially the same as equivalences between the $\bar{\mathcal{L}}_S^c(-)$.

Lemma 6.2. *Fix finite groups G and G' , and Sylow p -subgroups $S \leq G$ and $S' \leq G'$. Let $\psi_0 : \bar{\mathcal{L}}_S^{cr}(G) \longrightarrow \bar{\mathcal{L}}_{S'}^{cr}(G')$ be an isotypical equivalence of categories which sends inclusions to inclusions. Then ψ_0 extends to a unique isotypical equivalence $\psi : \bar{\mathcal{L}}_S^c(G) \longrightarrow \bar{\mathcal{L}}_{S'}^c(G')$ which sends inclusions to inclusions.*

Proof. The equivalence ψ_0 will be extended to an equivalence ψ as above in several steps. To keep the notation simple, these equivalences will all be denoted ψ . So we start by setting $\psi = \psi_0$: an equivalence between the full subcategories of p -centric p -radical subgroups.

For each $P \leq S$, set $\bar{P} = O_p(N_G(P))$ (the intersection of the Sylow p -subgroups of $N_G(P)$) if P is p -centric, and let \bar{P} be a Sylow p -subgroup of $C_G(P) \cdot P$ otherwise. For each $P \leq S$ not in $\bar{\mathcal{L}}_S^{cr}(G)$, \bar{P} satisfies the two conditions: $P \not\leq \bar{P} \leq N_G(P)$, and $N_G(P) = (N_G(P) \cap N_G(\bar{P})) \cdot C_G(P)$. Since S is p -centric and p -radical, ψ is already defined there, and the argument in the proof of Lemma 5.1 now carries through to show that $\psi_{S,S'}$ restricts to a fusion preserving isomorphism

$$f = f_\psi = \psi_S : S \xrightarrow{\cong} S'.$$

Also, for any p -centric p -radical subgroups $P, Q \leq S$, and any $x \in N_G(P, Q)$ and $x' \in N_{G'}(f(P), f(Q))$ such that $\psi_{P,Q}(\hat{x}) = \hat{x}'$, the following square commutes by the same argument as in the proof of Lemma 5.3:

$$\begin{array}{ccc} P & \xrightarrow{c_x} & Q \\ f \downarrow & & \downarrow f \\ f(P) & \xrightarrow{c_{x'}} & f(Q). \end{array} \tag{1}$$

By Lemma 5.2, for any $P \leq S$, P is p -centric in G if and only if $f(P)$ is p -centric in G' . So we can define $\psi : \bar{\mathcal{L}}_S^c(G) \longrightarrow \bar{\mathcal{L}}_{S'}^c(G')$ on objects by setting $\psi(P) = f(P)$ for all p -centric $P \leq S$. It remains to define ψ on morphisms in $\bar{\mathcal{L}}_S^c(G)$.

Since we will be sending inclusions to inclusions (and every morphism is the composite of an isomorphism followed by an inclusion), the problem is to define ψ on isomorphisms. The basic idea is to define $\psi_{P,Q}(\hat{x})$, for any isomorphism $P \xrightarrow{\hat{x}} Q$ in $\bar{\mathcal{L}}_S^c(G)$ for which P and Q are not p -radical, to be the “restriction” of $\psi_{\bar{P},\bar{Q}}(\hat{x})$; i.e., the morphism (unique by Lemma 6.1)

such that $\widehat{1} \circ \psi_{P,Q}(\widehat{x}) = \psi_{\overline{P},\overline{Q}}(\widehat{x}) \circ \widehat{1}$. Here, $\overline{P} = O_p(N_G(P))$ for p -centric $P \leq S$ as above. However, since \overline{P} need not be p -radical (when P is not p -radical), we must combine this with a downward induction on $|P|$. And the procedure is further complicated since for $P \leq S$ not p -radical, \overline{P} need not be contained in S if $N_S(P)$ is not a Sylow p -subgroup of $N_G(P)$.

Set $p^k = |S|$. For each $0 \leq i \leq k$, let $\overline{\mathcal{L}}_i \subseteq \overline{\mathcal{L}}_S^c(G)$ be the full subcategory whose objects are the p -centric subgroups $P \leq S$ which are either p -radical in G , or which satisfy $|P| \geq p^i$. In particular, $\overline{\mathcal{L}}_k = \overline{\mathcal{L}}_S^{cr}(G)$ and $\overline{\mathcal{L}}_0 = \overline{\mathcal{L}}_S^c(G)$. Fix $0 \leq i < k$, and assume inductively that ψ has already been defined on $\overline{\mathcal{L}}_{i+1}$, and that (1) commutes whenever P and Q are in $\overline{\mathcal{L}}_{i+1}$. We must extend this to $\overline{\mathcal{L}}_i$, and so that (1) still commutes when appropriate.

Set $\mathcal{P}_i = \text{Ob}(\overline{\mathcal{L}}_i) \setminus \text{Ob}(\overline{\mathcal{L}}_{i+1})$, the set of p -centric subgroups $P \leq S$ of order p^i which are not p -radical. Let $\mathcal{P}_i^* \subseteq \mathcal{P}_i$ be a set of G -conjugacy class representatives, chosen such that for each $P^* \in \mathcal{P}_i^*$, $N_S(P^*)$ is a Sylow p -subgroup of $N_G(P^*)$. For each $P \in \mathcal{P}_i$, let P^* denote the unique element of \mathcal{P}_i^* which is G -conjugate to P , and fix some element $x_P \in N_G(P, P^*)$ such that $x_P N_S(P) x_P^{-1} \leq N_S(P^*)$.

Fix some $P \in \mathcal{P}_i$ and some $x \in N_G(P^*)$. We claim that there are unique morphisms $\psi_{P,P^*}(\widehat{x}_P)$ and $\psi_{P^*,P^*}(\widehat{x})$ such that the following squares commute:

$$\begin{array}{ccc} f(P) & \xrightarrow{\psi_{P,P^*}(\widehat{x}_P)} & f(P^*) \\ \widehat{1} \downarrow & & \downarrow \widehat{1} \\ f(N_S(P)) & \xrightarrow{\psi_{N_S(P),N_S(P^*)}(\widehat{x}_P)} & f(N_S(P^*)) \end{array} \quad (2) \quad \text{and} \quad \begin{array}{ccc} f(P^*) & \xrightarrow{\psi_{P^*,P^*}(\widehat{x})} & f(P^*) \\ \widehat{1} \downarrow & & \downarrow \widehat{1} \\ f(\overline{P^*}) & \xrightarrow{\psi_{\overline{P^*},\overline{P^*}}(\widehat{x})} & f(\overline{P^*}). \end{array} \quad (3)$$

To see this, set $\widehat{P} = N_S(P)$ and $y = x_P$ if $P \neq P^*$, and $\widehat{P} = \overline{P}$ and $y = x$ if $P = P^*$. In either case, $\widehat{P} \not\cong P$ (so \widehat{P} is in $\overline{\mathcal{L}}_{i+1}$) and $y \in N_G(\widehat{P}, N_S(P^*))$. Choose $y' \in N_{G'}(f(\widehat{P}), f(N_S(P^*)))$ such that $\psi_{\widehat{P},N_S(P^*)}(\widehat{y}) = \widehat{y}'$. Then $y' \in N_{G'}(f(P), f(yPy^{-1}))$ by (1). So if we set $\psi_{P,P^*}(\widehat{y}) = \widehat{y}'$, then the above square (2) or (3) commutes, and the choice of \widehat{y}' to make this commute is unique by Lemma 6.1.

If $P \xrightarrow{\widehat{x}} Q$ is an arbitrary isomorphism between objects of $\overline{\mathcal{L}}_i$ not in $\overline{\mathcal{L}}_{i+1}$, then it factors uniquely as a composite

$$P \xrightarrow{\widehat{x}_P} P^* \xrightarrow{\widehat{y}} P^* = Q^* \xrightarrow{\widehat{x}_Q^{-1}} Q \quad (4)$$

($P^* = Q^*$ since P and Q are conjugate); and we set $\psi_{P,Q}(\hat{x}) \stackrel{\text{def}}{=} \widehat{x'_Q}^{-1} \circ \psi_{P^*,P^*}(\hat{y}) \circ \widehat{x'_P}$. And if $P \xrightarrow{\hat{x}} Q$ is an arbitrary morphism in $\bar{\mathcal{L}}_i$, then it factors as the composite of the isomorphism $P \xrightarrow{\hat{x}} {}^xP$ (recall that ${}^xP = xPx^{-1}$) followed by the inclusion ${}^xP \xrightarrow{\hat{1}} Q$, and we set $\psi_{P,Q}(\hat{x}) \stackrel{\text{def}}{=} \hat{1} \circ \psi_{P,{}^xP}(\hat{x})$.

We have now defined ψ on all morphisms of $\bar{\mathcal{L}}_i$, and it remains to show that it preserves composition of morphisms. This means showing, for each $P \xrightarrow{\hat{x}} Q \xrightarrow{\hat{y}} R$ in $\bar{\mathcal{L}}_i$, that the following diagram commutes:

$$\begin{array}{ccccc}
 f(P) & \xrightarrow{\psi_{P,{}^{yx}P}(\widehat{y\hat{x}})} & f({}^{yx}P) & \xrightarrow{\hat{1}} & f(R) \\
 \searrow \psi_{P,{}^xP}(\hat{x}) & & \nearrow \psi_{{}^xP,{}^{yx}P}(\hat{y}) & & \nearrow \hat{1} \\
 & (5) & & (7) & \\
 & & f({}^xP) & \xrightarrow{\hat{1}} & f({}^yQ) \\
 & & \searrow \hat{1} & & \nearrow \hat{1} \\
 & & & (6) & \\
 & & f(Q) & \xrightarrow{\psi_{Q,{}^yQ}(\hat{y})} & f({}^yQ)
 \end{array}$$

Square (7) clearly commutes. Via the decomposition (4) it suffices to show the commutativity of (5) when $P = {}^xP = {}^{yx}P = P^*$, and this follows from the uniqueness of the morphisms $\psi_{P,P}(-)$ which make square (3) commute. So it remains only to show that (6) commutes. It suffices to show this when either P or Q lies in \mathcal{P}_i , but not both (if $P, Q \in \mathcal{P}_i$ then ${}^xP = Q$ since both have order p^i).

We are thus reduced to showing that the following two squares commute

$$\begin{array}{ccc}
 f(Q) & \xrightarrow{\hat{1}} & f(P) \\
 \psi_{Q,{}^xQ}(\hat{x}) \downarrow \cong & & \downarrow \psi_{P,{}^xP}(\hat{x}) \cong \\
 f({}^xQ) & \xrightarrow{\hat{1}} & f({}^xP)
 \end{array}
 \qquad
 \begin{array}{ccc}
 f(P) & \xrightarrow{\hat{1}} & f(R) \\
 \psi_{P,{}^xP}(\hat{x}) \downarrow \cong & & \downarrow \psi_{R,{}^xR}(\hat{x}) \cong \\
 f({}^xP) & \xrightarrow{\hat{1}} & f({}^xR)
 \end{array}$$

for any $Q \leq P \leq R \leq S$ and any $x \in G$ such that $P \in \mathcal{P}_i$, Q and R are in $\bar{\mathcal{L}}_{i+1}$, and ${}^xR \leq S$. Since $P \xrightarrow{\hat{x}} {}^xP$ factors as in (4), it suffices to do this when ${}^xP = P^*$, and either $P = P^* \in \mathcal{P}_i^*$ or $x = x_P$. If $x = x_P$, then this

follows from the following diagram:

$$\begin{array}{ccccccc}
 f(Q) & \xrightarrow{\hat{1}} & f(P) & \xrightarrow{\hat{1}} & f(R) & \xrightarrow{\hat{1}} & f(T) \\
 \psi_{Q, {}^xQ}(\widehat{x_P}) \downarrow & & \psi_{P, {}^xP}(\widehat{x_P}) \downarrow & & \psi_{R, {}^xR}(\widehat{x_P}) \downarrow & & \psi_{T, {}^xT}(\widehat{x_P}) \downarrow \\
 f({}^xQ) & \xrightarrow{\hat{1}} & f({}^xP) & \xrightarrow{\hat{1}} & f({}^xR) & \xrightarrow{\hat{1}} & f({}^xT),
 \end{array}$$

where $T = R \cdot N_S(P)$. The composite of the two squares on the right commutes by (2). If $Q \not\leq P$, then the composite of the three squares commutes since it is in $\bar{\mathcal{L}}_{i+1}$; otherwise the first square commutes trivially. Similarly, the third square commutes if $P \not\leq R$ and the second square otherwise. With the help of Lemma 6.1, this shows that all three squares commute (in all cases). A similar argument applies when $P = {}^xP \in \mathcal{P}_i^*$, upon using (3) and setting $T = R \cdot \bar{P}$.

We have now constructed a functor $\psi : \bar{\mathcal{L}}_S^c(G) \longrightarrow \bar{\mathcal{L}}_{S'}^c(G')$. From the above procedure (and Lemma 6.1) this is the unique such functor which extends ψ_0 and sends inclusions to inclusions. Upon applying this same procedure to $(\psi_0)^{-1}$, we obtain a functor ψ' in the opposite direction; the two composites $\psi \circ \psi'$ and $\psi' \circ \psi$ are the identity by the uniqueness of such extensions; and this shows that ψ is an isomorphism of categories. \square

The following proposition is our main tool for directly (without reference to higher limits) constructing equivalences between the categories $\bar{\mathcal{L}}_p^c(-)$. To simplify the notation, for a p -centric subgroup $P \leq G$, we write $N'(P) = N_G(P)/C'_G(P)$, where $C'_G(P)$ is the p' -torsion in the centralizer of P . (Note that $N'(P)$ is the automorphism group of P in the category $\bar{\mathcal{L}}_p^c(G)$.) And we let $\text{Equiv}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G), \bar{\mathcal{L}}_p^c(G'))$ denote the set of isotypical equivalences between these categories modulo natural isomorphisms of functors.

Fix finite groups G and G' and Sylow p -subgroups $S \leq G$ and $S' \leq G'$. Let $S = P_0, P_1, \dots, P_m$ denote a choice of G -conjugacy class representatives for all p -centric p -radical subgroups of G contained in S . Let $X(G, G')$ be the set of all $(m+1)$ -tuples $(\theta; \theta_1, \theta_2, \dots, \theta_m)$ such that

$$\theta : N'(S) \xrightarrow{\cong} N'(S') \quad \text{and} \quad \theta_i : N'(P_i) \xrightarrow{\cong} N'(\theta P_i)$$

are isomorphisms, and such that

$$\theta_i|_{N_G(S) \cap N_G(P_i)} = \theta|_{N_G(S) \cap N_G(P_i)}$$

for each i . (Here, by restriction to $N_G(S) \cap N_G(P_i)$, we really mean restriction to its image in $N'(P_i)$). The group $N_{G'}(S')$ act on $X(G, G')$ by

$$x \cdot (\theta; \theta_1, \dots, \theta_m) \stackrel{\text{def}}{=} (c_x \circ \theta; c_x \circ \theta_1, \dots, c_x \circ \theta_m)$$

for $x \in N_{G'}(S')$.

Define a map

$$\Lambda : \text{Equip}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G), \bar{\mathcal{L}}_p^c(G')) \longrightarrow X(G, G')/N_{G'}(S')$$

as follows. For any isotypical equivalence $\psi : \bar{\mathcal{L}}_p^c(G) \rightarrow \bar{\mathcal{L}}_p^c(G')$, let ψ' be an isotypical equivalence which sends inclusions of subgroups of S to inclusions of subgroups of S' and is naturally isomorphic to ψ . Define

$$\Lambda([\psi]) = [\psi_{S,S}; \psi_{P_1,P_1}, \dots, \psi_{P_m,P_m}],$$

where $\psi_{P,P} : N'(P) \longrightarrow N'(\psi(P))$ for $P = S$ or $P = P_i$.

To see that Λ is well defined, replace, as usual, the categories $\bar{\mathcal{L}}_p^c(G)$ and $\bar{\mathcal{L}}_p^c(G')$ by their equivalent subcategories $\bar{\mathcal{L}}_S^c(G)$ and $\bar{\mathcal{L}}_{S'}^c(G')$. By Lemma 5.1(i,ii), any isotypical equivalence $\bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$ is naturally isomorphic to an isotypical equivalence ψ which sends inclusions to inclusions, and $\psi(P) = f_\psi(P)$ for all $P \leq S$. So

$$(\psi_{S,S}; \psi_{P_1,P_1}, \dots, \psi_{P_m,P_m}) \in X(G, G').$$

If ψ' is another isotypical equivalence which sends inclusions to inclusions, and if $\alpha : \psi \longrightarrow \psi'$ is a natural isomorphism, then set $\alpha(S) = \hat{x}$ and $\alpha(P_i) = \hat{x}_i$ for each i . Here, $x \in N_{G'}(S')$, and $x_i \in N_{G'}(\psi(P_i), \psi'(P_i))$. The naturality of α implies that the following squares commute for each $0 \leq i \leq m$ and each $g \in N_G(P_i)$:

$$\begin{array}{ccc} \psi(P_i) & \xrightarrow{\hat{1}} & S' & & \psi(P_i) & \xrightarrow{\psi_{P_i,P_i}(\hat{g})} & \psi(P_i) \\ \hat{x}_i \downarrow & & \hat{x} \downarrow & & \hat{x}_i \downarrow & & \hat{x}_i \downarrow \\ \psi'(P_i) & \xrightarrow{\hat{1}} & S' & & \psi'(P_i) & \xrightarrow{\psi'_{P_i,P_i}(\hat{g})} & \psi'(P_i). \end{array}$$

This shows first that $x_i \equiv x \pmod{C'_{G'}(P'_i)}$ for each i , and that $\psi'_{P_i,P_i} = c_{x_i} \circ \psi_{P_i,P_i} = c_x \circ \psi_{P_i,P_i}$. The $(m+1)$ -tuples $(\psi_{S,S}; \psi_{P_1,P_1}, \dots, \psi_{P_m,P_m})$ and $(\psi'_{S,S}; \psi'_{P_1,P_1}, \dots, \psi'_{P_m,P_m})$ are thus in the same $N_{G'}(S')$ -orbit of $X(G, G')$, and this shows that Λ is well defined.

Proposition 6.3. *The map*

$$\Lambda : \text{Equip}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G), \bar{\mathcal{L}}_p^c(G')) \longrightarrow X(G, G')/N_{G'}(S')$$

is injective. Furthermore, Λ is a bijection if there is no $1 \leq i, j \leq m$ such that P_i is conjugate to a proper subgroup of P_j .

Proof. Proof of injectivity: Fix isotypical equivalences ψ and ψ' such that $\Lambda([\psi]) = \Lambda([\psi'])$. We can assume that ψ and ψ' both send inclusions of subgroups of S to inclusions of subgroups of S' ; and also (since the action of any $x \in N_{G'}(S')$ can be lifted to a natural isomorphism of equivalences) that $\psi_{S,S} = \psi'_{S,S}$ and $\psi_{P_i,P_i} = \psi'_{P_i,P_i}$ for each i .

Since each morphism factors as an isomorphism followed by an inclusion, it suffices to show that $\psi = \psi'$ on isomorphisms. So fix a G -conjugacy class \mathcal{P} of subgroups of S , and assume inductively that $\psi = \psi'$ on all isomorphisms of subgroups of strictly larger order. For any $P, Q \in \mathcal{P}$ such that $N_S(Q)$ is a Sylow p -subgroup of $N_G(Q)$, there is $x \in N_G(P, Q)$ such that $xN_S(P)x^{-1} \leq N_S(Q)$; and since ψ and ψ' agree on the morphism $N_S(P) \xrightarrow{\hat{x}} N_S(Q)$, they must also agree on $P \xrightarrow{\hat{x}} Q$. Thus, by taking composites, we see that for *any* two subgroups $P, Q \in \mathcal{P}$, there is at least one isomorphism $P \xrightarrow{\hat{x}} Q$ such that $\psi_{P,Q}(\hat{x}) = \psi'_{P,Q}(\hat{x})$.

Given an isomorphism $P \xrightarrow{\hat{x}} Q$ between elements of \mathcal{P} , any other isomorphism between these groups can be obtained by composing with an automorphism of P . It thus remains to show that $\psi_{P,P} = \psi'_{P,P}$ for at least one subgroup in \mathcal{P} . If the subgroups in \mathcal{P} are p -radical, then this holds for the representative $P = P_i$ (some i). Otherwise, choose $P \in \mathcal{P}$ such that $N_S(P)$ is a Sylow p -subgroup of $N_G(P)$, and set $\bar{P} = O_p(N_G(P)) \leq S$. Then for any $x \in N_G(P)$, $\psi_{\bar{P},\bar{P}}(\hat{x}) = \psi'_{\bar{P},\bar{P}}(\hat{x})$, and hence $\psi_{P,P}(\hat{x}) = \psi'_{P,P}(\hat{x})$.

Proof of surjectivity: Now assume that no P_i is strictly contained in a subgroup conjugate to any P_j (for $i, j = 1, \dots, m$). Let $\bar{\mathcal{L}} \subseteq \bar{\mathcal{L}}_S^{cr}(G)$ be the full subcategory with objects S, P_1, \dots, P_m . For each i ,

$$N_G(P_i, S) = N_G(S) \cdot N_G(P_i),$$

since if P_i is G -conjugate to $P \leq S$, then by Alperin's theorem again P_i is $N_G(S)$ -conjugate to P . Hence for any $(\theta; \theta_1, \dots, \theta_m) \in X(G, G')$, there is a unique functor $\psi_0 : \bar{\mathcal{L}} \longrightarrow \bar{\mathcal{L}}_S^{cr}(G')$ which sends S to S' and P_i to $\theta(P_i)$, and whose restriction to $\text{Aut}_{\bar{\mathcal{L}}}(S)$ or $\text{Aut}_{\bar{\mathcal{L}}}(P_i)$ is θ or θ_i .

For each P which is G -conjugate to some P_i , a choice of isomorphisms $P \xrightarrow{\hat{x}} P_i$ and $\theta(P) \xrightarrow{\theta(\hat{x})} \theta(P_i)$ (for $x \in N_G(S)$) determines an extension of ψ_0 to the object P . We thus get an isotypical equivalence $\psi_1 : \bar{\mathcal{L}}_S^{cr}(G) \longrightarrow \bar{\mathcal{L}}_S^{cr}(G')$, and this extends to an isotypical equivalence $\psi : \bar{\mathcal{L}}_S^c(G) \longrightarrow \bar{\mathcal{L}}_S^c(G')$ by Lemma 6.2. \square

Proposition 6.3 will now be applied to prove the second part of Theorem F. To do this, we first need to show how to construct isomorphisms

$N'_G(P) \longrightarrow N'_{G'}(P')$ which extend a given fusion preserving isomorphism $S \longrightarrow S'$. This is the goal of the next lemma.

For a finite group G , let $O_{p'}(G)$ denote the maximal normal subgroup of G of order prime to p . We say that G is p' -reduced if $O_{p'}(G)$ is trivial. Notice that for any finite group G , the quotient $G/O_{p'}(G)$ is always p' -reduced.

Lemma 6.4. *Let G and G' be finite groups, containing normal p -centric subgroups $H \triangleleft G$ and $H' \triangleleft G'$. Let $S \leq G$ and $S' \leq G'$ be Sylow p -subgroups. Then any fusion preserving isomorphism $f : S \xrightarrow{\cong} S'$ with $f(H) = H'$ extends to an isomorphism*

$$\Phi : G/C'_G(H) \xrightarrow{\cong} G'/C'_{G'}(H'),$$

and any two such extensions differ by conjugation by an element in $Z(S')$.

Proof. We first claim that

$$O_{p'}(G) = C'_G(H) \quad \text{and} \quad O_{p'}(G') = C'_{G'}(H'). \quad (1)$$

To see this, first notice that both $O_{p'}(G)$ and H are normal subgroups in $H \cdot O_{p'}(G) \leq G$, which is isomorphic to the direct product $H \times O_{p'}(G)$ because of the relatively prime orders. Thus $O_{p'}(G) \leq C'_G(H)$. The opposite inclusion is obvious.

Now let $f : S \xrightarrow{\cong} S'$ be a fusion preserving isomorphism such that $f(H) = H'$. We must construct a homomorphism Φ from $G/C'_G(H)$ to $G'/C'_{G'}(H')$, and show that it is unique up to conjugation by an element of $C_G(S')$. Upon replacing G and G' by $G/O_{p'}(G)$ and $G'/O_{p'}(G')$, respectively, and applying (1), we are reduced to the case where G and G' are p' -reduced, and where $C'_G(H) = C'_{G'}(H') = 1$.

We must show that f extends to an isomorphism $G \xrightarrow{\Phi} G'$. Consider the functors

$$\mathcal{Z} : \mathcal{O}_S^c(G) \longrightarrow p\text{-Gr}, \quad \text{and} \quad \mathcal{Z}_1, \mathcal{Z}_2 : \mathcal{O}_p(G) \longrightarrow p\text{-Gr},$$

where \mathcal{Z} sends each object P to $Z(P)$, and where

$$\mathcal{Z}_1(P) = \begin{cases} Z(P) & \text{if } P \text{ is } p\text{-centric} \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad \mathcal{Z}_2(P) = (Z(H))^P.$$

Then

$$\varprojlim_{\mathcal{O}_S^c(G)}^* (\mathcal{Z}) \cong \varprojlim_{\mathcal{O}_S(G)}^* (\mathcal{Z}_1) \cong \varprojlim_{\mathcal{O}_p(G)}^* (\mathcal{Z}_1) \cong \varprojlim_{\mathcal{O}_p(G)}^* (\mathcal{Z}_2).$$

Here, the first isomorphism holds since $C^*(\mathcal{O}_S^c(G); \mathcal{Z})$ and $C^*(\mathcal{O}_S(G); \mathcal{Z}_1)$ are isomorphic (no p -centric subgroup can be contained in one which is not p -centric). The second isomorphism holds since the two categories are

equivalent. The third holds by [JMO, Corollary 1.8]: $\mathcal{Z}_1(P) \cong \mathcal{Z}_2(P)$ whenever $P \geq H$, and otherwise $N_{PH}(P)/P$ is a nontrivial normal p -subgroup in $N_G(P)/P$ and so P is not p -radical. The functor \mathcal{Z}_2 is acyclic by [JMO, Proposition 5.2] (as is any functor of the form $P \mapsto M^P$); and this proves that $\varprojlim_{\mathcal{O}_S^c(G)}^2(\mathcal{Z}) = 0$.

By Proposition 5.4, there is an isotypical equivalence

$$\psi : \bar{\mathcal{L}}_S^c(G) \xrightarrow{\cong} \bar{\mathcal{L}}_{S'}^c(G')$$

such that $f_\psi = f : S \xrightarrow{\cong} S'$. Notice that $\text{Aut}_{\bar{\mathcal{L}}_S^c(G)}(H) = G/C'_G(H) = G$, since $C'_G(H) = 1$, and similarly for G' and H' . Set

$$\Phi = \psi_{H,H} : G = \text{Aut}_{\bar{\mathcal{L}}_S^c(G)}(H) \xrightarrow{\cong} \text{Aut}_{\bar{\mathcal{L}}_{S'}^c(G')}(H') = G',$$

regarded as an isomorphism $G \xrightarrow{\cong} G'$. For any $g \in S \leq N(H)$, the following square commutes

$$\begin{array}{ccc} H' & \xrightarrow{\widehat{\Phi}(g)} & H' \\ \hat{1} \downarrow & & \downarrow \hat{1} \\ S' & \xrightarrow{\psi(\hat{g}) = \widehat{f}(g)} & S', \end{array}$$

since it is the image under ψ of a commutative square, which shows that $\Phi|_S = f$.

It remains to prove the uniqueness of Φ up to conjugation by an element of $Z(S')$. Let $\Phi_1, \Phi_2 : G \xrightarrow{\cong} G'$ be any two isomorphisms such that $\Phi_i|_S = f$, and set $\Phi = \Phi_1^{-1} \circ \Phi_2 \in \text{Aut}(G)$. Thus $\Phi|_S = \text{Id}_S$, and we must show that $\Phi = c_x$ for some $x \in Z(S)$. Note that $\Phi(g) \equiv g \pmod{Z(H)}$ for all g , since for $h \in H$, $g^{-1}hg = \Phi(g^{-1}hg) = \Phi(g)^{-1}h\Phi(g)$.

Define $\gamma : G/H \rightarrow Z(H)$ by setting $\gamma(gH) = g^{-1}\Phi(g)$. This is a 1-cocycle whose restriction to S/H is trivial. Since the restriction map

$$H^1(G/H; Z(H)) \longrightarrow H^1(S/H; Z(H))$$

is injective (S/H being a Sylow p -subgroup), this implies that γ is a coboundary. In other words, there is some $x \in Z(H)$ such that $\gamma(gH) = (g^{-1}xg)x^{-1}$ for all $g \in G$. Thus, $\Phi(g) = g\gamma(gH) = xgx^{-1}$ for all g , and $\Phi = c_x$. Also, $x \in Z(S)$ since $\Phi|_S = \text{Id}$. \square

We can now prove the second half of Theorem F.

Theorem 6.5. *Fix a prime p and a finite group G . Assume that there is no chain $P \not\leq Q \not\leq S$, where S is a Sylow p -subgroup and P and Q are p -radical and p -centric. Then for any other finite group G' , $BG_p^\wedge \simeq BG'^\wedge_p$ if and only if there is a fusion preserving isomorphism between Sylow p -subgroups of G and of G' .*

Proof. Fix Sylow p -subgroups $S \leq G$ and $S' \leq G'$, and let $X(G, G')$ be as in Proposition 6.3. By Theorem A and Lemma 5.1, it suffices to show, for any fusion preserving isomorphism $f : S \longrightarrow S'$, that $f = f_\psi$ for some isotypical equivalence $\psi : \bar{\mathcal{L}}_p^c(G) \rightarrow \bar{\mathcal{L}}_p^c(G')$ which sends S to S' .

Let $S = P_0, P_1, \dots, P_m$ be G -conjugacy class representatives for the p -centric p -radical subgroups $P \leq S$, chosen such that $N_S(P_i)$ is a Sylow p -subgroup of $N_G(P_i)$. For each i , P_i is normal and p -centric in $N_G(P_i)$, and so Lemma 6.4 applies to show that there is an isomorphism

$$\theta_i : N_G(P_i)/C'_G(P_i) \xrightarrow{\cong} N_{G'}(f(P_i))/C'_{G'}(f(P_i)),$$

whose restriction to $N_S(P_i)$ is f . For $i > 0$, the restrictions of θ_i and $\theta = \theta_0$ to $N(P_i) \cap N(S)$ differ by conjugation by an element of $Z(N_S(P_i))$ by Lemma 6.4 again, and so the θ_i can be chosen to agree with θ on $N(P_i) \cap N(S)$. This defines an element $(\theta; \theta_1, \dots, \theta_m) \in X(G, G')$, and the result now follows from Proposition 6.3. \square

7. AN EXAMPLE: BG_2^\wedge WHEN $G = PSL_2(q)$ AND q IS ODD

Throughout this section, $G = PSL_2(q)$, where $q = p^e$ and p is an odd prime. Set $k = \nu_2(|G|)$, where in general $2^{\nu_2(n)}$ is the highest power of 2 dividing n . Then $k \geq 2$, and $q \equiv 2^k \pm 1 \pmod{2^{k+1}}$. Fix a Sylow 2-subgroup $S \leq G$, and let $\bar{\mathcal{L}}_S^c(G) \subseteq \bar{\mathcal{L}}_2^c(G)$ denote the full subcategory whose objects are the 2-centric subgroups of G which are contained in S . As usual, this inclusion is an equivalence of categories.

For a positive integer n , C_n denotes a cyclic group of order n .

Lemma 7.1. *The Sylow 2-subgroup S in $PSL_2(q)$ is dihedral of order 2^k . If $k = 2$ then $N_G(S) \cong A_4$, while if $k \geq 3$ then $N_G(S) = S$. If $k \geq 3$, and if T_1 and T_2 are representatives for the two conjugacy classes of subgroups $T_i \cong C_2^2$ in S , then T_1 and T_2 also represent the two conjugacy classes of such subgroups in G , and $N_G(T_i) \cong \Sigma_4$. Also, any subgroup of G which strictly contains $N_G(T_i)$ is isomorphic to $PSL_2(u)$, where u is such that $q = u^t$ for some t .*

Proof. See, for example, [Sz, Theorems 6.25–26] or [Hu, Satz 8.27] for a complete description of the subgroups of $PSL_2(q)$. \square

Let q' be another odd prime power and set $G' = PSL_2(q')$. Assume $\nu_2(|G'|) = k$, and let $S' \leq G'$ be a Sylow 2-subgroup (also dihedral). Let $\text{Equiv}_{\text{typ}}(\bar{\mathcal{L}}_S^c(G), \bar{\mathcal{L}}_{S'}^c(G'))$ denote, as before, the set of all isotypical equivalences modulo natural isomorphisms of functors. The next Proposition shows that the map of Proposition 6.3 is a bijection, but does not exclude the possibility that the two sets are empty.

Proposition 7.2. *Assume $k \geq 3$. Let $T_1, T_2 \leq S \cong D_{2^k}$ be conjugacy class representatives for the subgroups isomorphic to C_2^2 . Let $X(G, G')$ denote the set of triples $(\theta; \theta_1, \theta_2)$, where $\theta : S \xrightarrow{\cong} S'$ and the $\theta_i : N_G(T_i) \xrightarrow{\cong} N_{G'}(\theta T_i)$ are all isomorphisms, and where $\theta_i|_{N_S(T_i)} = \theta|_{N_S(T_i)}$ (for $i = 1, 2$). Then there is a bijection*

$$\text{Equiv}_{\text{typ}}(\bar{\mathcal{L}}_S^c(G), \bar{\mathcal{L}}_{S'}^c(G')) \xrightarrow{\cong} X(G, G')/S'.$$

Proof. This follows immediately from Proposition 6.3, since $N_G(S) = S$, and the only proper subgroups of S which are 2-centric and 2-radical in G are those isomorphic to C_2^2 . Furthermore, $C_S(T_i) \leq S$, and so (in the notation of Proposition 6.3) $N'(S) = N_G(S)$ and $N'(T_i) = N_G(T_i)$. \square

As a first consequence, we have:

Corollary 7.3. *Let q and q' be odd prime powers and set $G = PSL_2(q)$ and $G' = PSL_2(q')$. Set $k = \nu_2(|G|)$ and $k' = \nu_2(|G'|)$. Then $BG_2^\wedge \simeq BG'^\wedge_2$ if and only if $k = k'$.*

Proof. If $BG_2^\wedge \simeq BG'^\wedge_2$, then their Sylow 2-subgroups must have the same order, and hence $k = k'$.

Conversely, assume $k = k'$, let S' be any Sylow 2-subgroup of G' , and fix an isomorphism $\theta : S \rightarrow S'$ (both are dihedral of order $2^k = 2^{k'}$). By Theorem A, it suffices to construct an equivalence of categories $\bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$.

If $k = 2$, then $S \cong C_2^2$ is the only object in $\bar{\mathcal{L}}_S^c(G)$, and θ extends to an isomorphism $N_G(S) \xrightarrow{\cong} N_{G'}(S') \cong A_4$, and hence to an equivalence of categories $\bar{\mathcal{L}}_S^c(G) \rightarrow \bar{\mathcal{L}}_{S'}^c(G')$.

If $k \geq 3$, then by Proposition 7.2, it suffices to show that θ extends to a triple $(\theta; \theta_1, \theta_2)$. Furthermore, for $i = 1, 2$, $R_i \stackrel{\text{def}}{=} N_S(T_i) \cong D_8$ is a Sylow 2-subgroup in $N_G(T_i) \cong \Sigma_4$; and if $R'_i = \theta(R_i)$, then $\theta|_{R_i} : R_i \rightarrow R'_i$ is fusion

preserving and hence extends to an isomorphism $\theta_i : N_G(T_i) \longrightarrow N_{G'}(T'_i)$ by Lemma 6.4. Since S is self normalizing in G , $N_G(S) \cap N_G(T_i) = S \cap N_G(T_i) = R_i$, and thus $(\theta; \theta_1, \theta_2) \in X(G, G')$ as defined in Proposition 7.2. \square

We now look at the groups

$$\text{Out}(BG_2^\wedge) \cong \text{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G)) \cong X(G, G)/S,$$

where $S \leq G$ is the Sylow 2-subgroup. The equivalence on the left is a consequence of Theorem B and the one on the right follows from Proposition 7.2. We compare those groups with the outer automorphism group $\text{Out}(G)$. We first recall the well known description of this group.

Recall that $q = p^e$ where p is prime. We regard $\text{Aut}(\mathbb{F}_q) \cong C_e$ as a subgroup of $\text{Aut}(G)$: each $\varphi \in \text{Aut}(\mathbb{F}_q)$ acts on G by sending a matrix $A = (a_{i,j})$ to the matrix $\varphi(A) = (\varphi(a_{i,j}))$.

Lemma 7.4. *Let $\omega \in \text{Aut}(G)$ be any automorphism induced by conjugation by an element $\gamma \in PGL_2(q) \setminus PSL_2(q)$; i.e., $\gamma = [A]$ for some $A \in GL_2(q)$ whose determinant is not a square in $(\mathbb{F}_q)^*$. Then*

$$\text{Out}(G) = \langle \omega \rangle \times \text{Aut}(\mathbb{F}_q) \cong C_2 \times C_e.$$

Proof. See [Ctr, Theorem 12.5.1]. Note that ω is called the “diagonal automorphism”, and the elements coming from $\text{Aut}(\mathbb{F}_q)$ the “field automorphisms”. \square

We next compare the group $\bar{\mathcal{L}}_S^c(G)$ to a certain quotient of $\text{Aut}(S)$. The statement of the following lemma should be compared with [BL, Theorem 1.6(3)].

Lemma 7.5. *Let $\text{Aut}_G(S)$ denote the group of automorphisms of S induced by conjugation in G . Let*

$$\sigma : \text{Out}(BG_2^\wedge) \cong \text{Out}(\bar{\mathcal{L}}_S^c(G)) \longrightarrow \text{Aut}(S)/\text{Aut}_G(S)$$

be the homomorphism defined by restricting an automorphism of the category to its action on $\text{Aut}_{\bar{\mathcal{L}}_S^c(G)}(S) = N_G(S) = S$.

(a) *If $k = 2$ (so $S \cong C_2^2$), then σ is an isomorphism, and*

$$\text{Aut}(S)/\text{Aut}_G(S) \cong \Sigma_3/C_3 \cong C_2.$$

(b) *Assume $k \geq 3$ (so $S \cong D_{2^k}$), and let $z \in S$ be the central element of order 2. Then σ is surjective, and*

$$\text{Aut}(S)/\text{Aut}_G(S) = \text{Out}(S) \cong C_2 \times C_{2^k-3}.$$

Also, $\text{Ker}(\sigma)$ has order 2, and under the identification of Proposition 7.2 is generated by the element $[\text{Id}_S; \text{Id}_{T_1}, \text{conj}(z)_{T_2}]$.

Proof. The surjectivity of σ (for arbitrary k) follows as in the proof of Lemma 7.3, by using Lemma 6.4 to show that each $\theta \in \text{Aut}(S)$ extends to a triple $(\theta; \theta_1, \theta_2)$.

If $k = 2$ (if $q \equiv \pm 3 \pmod{8}$), then $S \cong C_2^2$, and $N_G(S) \cong A_4$ by Lemma 7.1. Then σ is injective since S is the only object in $\bar{\mathcal{L}}_S^c(G)$.

If $k \geq 3$, then fix generators $\alpha, \beta \in S \cong D_{2^k}$, where $|\alpha| = 2^{k-1}$ and $|\beta| = 2$. Then

$$\text{Aut}(D_{2^k}) \cong C_{2^{k-1}} \rtimes \text{Aut}(C_{2^{k-1}}),$$

where the normal cyclic subgroup is generated by the automorphism $\alpha \mapsto \alpha$ and $\beta \mapsto \alpha\beta$; and where the second factor is the subgroup of automorphisms fixing β . Also, $\text{Aut}_G(S) = \text{Inn}(S)$ since $N_G(S) = S$. It now follows easily that

$$\text{Aut}(S)/\text{Aut}_G(S) = \text{Out}(D_{2^k}) \cong C_2 \times \text{Aut}(C_{2^{k-1}})/\{\pm 1\} \cong C_2 \times C_{2^{k-3}},$$

and it remains to calculate the kernel of σ .

Under the identification of $\text{Out}_{\text{typ}}(\bar{\mathcal{L}}_p^c(G))$ with $X(G, G)/S$ given in Proposition 7.2, an element in $\text{Ker}(\sigma)$ is of the form $[\text{Id}_S; \theta_1, \theta_2]$, where $\theta_i \in \text{Aut}(N_G(T_i))$ (recall $T_i \cong C_2^2$ and $N_G(T_i) \cong \Sigma_4$) are such that $\theta_i|_{N_S(T_i)} = \text{Id}$. One easily checks that the only nontrivial automorphism of Σ_4 which is the identity on a subgroup D_8 is conjugation by the central element of that subgroup. Thus, each θ_i is the identity or conjugation by z . And by Lemma 7.2 again, $[\text{Id}_S, \text{conj}(z), \text{conj}(z)] = 0$ in $X(G, G)/S$, while $[\text{Id}_S, \text{Id}_{T_1}, \text{conj}(z)] \neq 0$. \square

By Lemma 7.4, $\text{Out}(G)$ is a finite abelian group. We let $\text{Out}(G)_{(2)}$ denote its Sylow 2-subgroup. We are now ready to determine the group $\text{Out}(BG_2^\wedge)$ and compare it to $\text{Out}(G)_{(2)}$.

Proposition 7.6. *Fix $q = p^e$, where p is an odd prime and $e \geq 1$, and set $G = \text{PSL}_2(q)$. Then the natural homomorphism*

$$B : \text{Out}(G)_{(2)} \longrightarrow \text{Out}(BG_2^\wedge) \cong C_2 \times C_{2^{k-2}}$$

is always injective, and is an isomorphism if and only if $p \equiv \pm 3 \pmod{8}$.

Proof. We first fix Sylow 2-subgroups $S \leq G = \text{PSL}_2(q)$ and $\hat{S} \leq \text{PGL}_2(q)$, such that $S \leq \hat{S}$ has index 2, as follows. If $q \equiv 1 \pmod{4}$, then let S and \hat{S} be the subgroups generated by diagonal matrices, together with the matrix $\beta \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $q \equiv 3 \pmod{4}$, then identify $(\mathbb{F}_q)^2$ with \mathbb{F}_{q^2} , regarded as a

2-dimensional \mathbb{F}_q -vector space, and set $H = \langle (\mathbb{F}_{q^2})^*, \beta \rangle$, where $\beta \in \text{Aut}(\mathbb{F}_{q^2})$ is the automorphism of order 2. Let \widehat{H} be the image of H in $PGL_2(q)$, let \widehat{S} be the Sylow 2-subgroup of \widehat{H} which contains β , and set $S = \widehat{S} \cap PSL_2(q)$.

In either case, $\widehat{S} \cong D_{2^{k+1}}$ and $S \cong D_{2^k}$ are both dihedral groups. Fix $\gamma \in \widehat{S}$ which generates the cyclic subgroup of index 2. By Lemma 7.4, $\text{conj}(\gamma)$ generates the first factor in $\text{Out}(G) \cong C_2 \times \text{Aut}(\mathbb{F}_q)$.

We must compare the following maps,

$$\begin{array}{ccc} \text{Out}(G)_{(2)} & & \\ \downarrow B & \searrow \rho & \\ \text{Out}(BG_2^\wedge) & \xrightarrow{\sigma} & \text{Out}(S), \end{array} \quad (1)$$

where σ is surjective by Lemma 7.5.

Case 1: Assume first that $q \not\equiv 1 \pmod{8}$. Then e is odd (q is not a square), and so $\text{Out}(G)_{(2)} = \langle \text{conj}(\gamma) \rangle$ has order 2. Also, $\text{conj}(\gamma) \notin \text{Ker}(\rho)$ ($\text{conj}(\gamma)|_S \notin \text{Inn}(S)$) by the above description of $\gamma \in \widehat{S}$. So B is injective since ρ is, and B is an isomorphism if and only if $|\text{Out}(BG_2^\wedge)| = 2$, if and only if $k = 2$ ($q \equiv \pm 3 \pmod{8}$) by Lemma 7.5.

Case 2: Now assume that $q \equiv 1 \pmod{8}$. Set $\alpha = \gamma^2 = [\text{diag}(v, v^{-1})]$, where $v \in (\mathbb{F}_q)^*$ has order 2^k . Then $S = \langle \alpha, \beta \rangle \cong D_{2^k}$, and $\text{conj}(\gamma)$ sends α to α and β to $\alpha\beta$. So

$$\text{Out}(S) = \langle \rho(\text{conj}(\gamma)) \rangle \times (\text{Aut}(C_{2^{k-1}})/\{\pm 1\}),$$

where the second factor operates on $\langle \alpha \rangle$ while fixing β .

If q is not a square, then $|\text{Aut}(\mathbb{F}_q)|$ is odd, $\text{Out}(G)_{(2)}$ is generated by $\text{conj}(\gamma)$, and so ρ (and hence B) are injective. So assume that $q = (q_1)^2$ is a square. Since $q \equiv 2^k + 1 \pmod{2^{k+1}}$, we have $q_1 \equiv 2^{k-1} \pm 1 \pmod{2^k}$. Let $\varphi_1 \in \text{Aut}(\mathbb{F}_q) \leq \text{Aut}(G)$ denote the automorphism of order 2: $\varphi_1(a) = a^{q_1}$ for $a \in \mathbb{F}_q$. Since v has order 2^k in $(\mathbb{F}_q)^*$, this shows that $\varphi_1(v) = -v$ or $-v^{-1}$. In particular, $\varphi_1(\alpha) = \alpha$ or α^{-1} (where $\alpha = \text{diag}(v, v^{-1})$), so $\varphi_1|_S \in \text{Inn}(S)$, and $\varphi_1 \in \text{Ker}(\rho)$. Furthermore, $\text{Ker}(\rho)$ is generated by φ_1 , since any automorphism $\varphi_2 \in \text{Aut}(\mathbb{F}_q)$ of order 4 is of the form $(a \mapsto a^{q_2})$ where $q_2 \equiv 2^{k-2} \pm 1 \pmod{2^{k-1}}$.

Define $\Psi \in \text{Aut}(G)$, by setting

$$\Psi = \begin{cases} \varphi_1 & \text{if } q_1 \equiv 1 \pmod{2^{k-1}} \\ \varphi_1 \circ \text{conj}(\beta) & \text{if } q_1 \equiv -1 \pmod{2^{k-1}}. \end{cases}$$

These are chosen so that $[\Psi] = [\varphi_1]$ in $\text{Out}(G)$, and $\Psi|_S = \text{Id}$. Set $z = \text{diag}(i, -i) = \alpha^{2^{k-2}}$, the central element of order two in S , and set $T_1 = \langle z, \beta \rangle$ and $T_2 = \langle z, \alpha\beta \rangle$. By Lemma 7.1, T_1 and T_2 represent the two G -conjugacy classes of subgroups isomorphic to C_2^2 .

Since $\Psi|_{T_1} = \text{Id}$, $\Psi|_{N_G(T_1)}$ must be conjugation by some element of T_1 (just look at the automorphisms of Σ_4). Furthermore, Ψ is the identity on $N_S(T_1) \cong D_8$, and so $\Psi|_{N_G(T_1)}$ must be conjugation by an element of $Z(N_S(T_1)) = \{1, z\}$. After replacing Ψ by $\Psi \circ \text{conj}(z)$ if necessary, we can assume that $\Psi|_{N_G(T_1)} = \text{Id}$. The same argument shows that $\Psi|_{N_G(T_2)}$ is conjugation by 1 or z . By Lemma 7.1, $G' \stackrel{\text{def}}{=} \langle N_G(T_1), N_G(T_2) \rangle \cong PGL_2(q')$ for some q' such that $q = (q')^t$, and t must be odd since $S \leq G'$. Then $\Psi|_{G'} \neq \text{Id}$, since restriction induces an isomorphism $\text{Out}(G)_{(2)} \cong \text{Out}(G')_{(2)}$ by Lemma 7.4. Thus $\Psi|_{N_G(T_2)} \neq \text{Id}$, and hence $\Psi|_{N_G(T_2)} = \text{conj}(z)$. So by Lemma 7.5, $B(\Psi)$ is the nontrivial element in $\text{Ker}[\text{Out}(BG_2^\wedge) \xrightarrow{\sigma} \text{Out}(S)]$, and thus B is injective.

To see when B is an isomorphism, note first that

$$|\text{Out}(BG_2^\wedge)| = 2 \cdot |\text{Out}(S)| = 2^{k-1}$$

and

$$|\text{Out}(G)_{(2)}| = 2^{m+1} \quad (m = \nu_2(e))$$

by Lemmas 7.4 and 7.5. So B is an isomorphism if and only if p has order $2^{k-1} = 2^{m+1}$ in $(\mathbb{Z}/2^{k+1})^*$; and this is the case if and only if $p \equiv \pm 3 \pmod{8}$. In particular, since $BG_2^\wedge \simeq BPSL_2(5^{2^{k-2}})_2^\wedge$ by Proposition 7.3,

$$\text{Out}(BG_2^\wedge) \cong \text{Out}(BPSL_2(5^{2^{k-2}})_2^\wedge) \cong \text{Out}(PSL_2(5^{2^{k-2}})) \cong C_2 \times C_{2^{k-2}}. \quad \square$$

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DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, E-08193 BELLATERRA, SPAIN

E-mail address: `broto@mat.uab.es`

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ABERDEEN, MESTON BUILDING 339, ABERDEEN AB24 3UE, U.K.

E-mail address: `ran@maths.abdn.ac.uk`

LAGA, INSTITUT GALILÉE, AV. J-B CLÉMENT, 93430 VILLETANEUSE, FRANCE

E-mail address: `bob@math.univ-paris13.fr`