Limit theorems for BSDE with local time applications to non–linear PDE

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Abstract

Given a d–dimensional Wiener process W, with its natural filtration $\{\mathcal{F}_t\}$, a $\mathcal{F}_T$–measurable random variable ξ in $\mathbb{R}$, a bounded measure ν on $\mathbb{R}$, and an adapted process $(s, y, z) \to h(s, y, z)$ we consider the following BSDE

$$Y_t \ = \ \xi + \int_t^T h(s, Y_s, Z_s)ds + \int_{\mathbb{R}} (L_T^a(Y) - L_t^a(Y))\, \nu(da) - \int_t^T Z_s dW_s$$

for $0 \le t \le T$. Here $L_t^a(Y)$ stands for the local time of Y at level a. For $h = 0$, we have the existence and the uniqueness of the processes (Y, Z), and if h is continuous with linear growth we have the existence of a solution.

We prove limit theorems for solutions of backward stochastic differential equations of the above form. Those limit theorems will permit us to deduce that any solution of that equation is the limit in a strong sense of a sequence of semi–martingales which are solutions of ordinary BSDE of the form

$$Y_t \ = \ \xi + \int_t^T f(Y_s) Z_s^2 ds - \int_t^T Z_s dW_s.$$

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A comparison theorem for BSDE involving measures is discussed. As an application we obtain, with the help of the connection between BSDE and PDE, some corresponding limit theorems for a class of singular non–linear PDE and a new probabilistic proof of the comparison theorem for PDE.

Keywords: Backward stochastic differential equation; local time; comparison theorem; singular partial differential equations.

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1 Introduction and notations

Backward stochastic differential equations (BSDEs in short) has been invented by Bismut [2] in the linear case. However, the first paper on non–linear BSDE has been introduced by Pardoux and Peng in [20]. Since then many fruitful studies have been produced due to the connections of the theory of BSDEs with mathematical finance (see e.g. [5]), partial differential equation (PDE in short) (see e.g. [18] or [19] and the references therein), stochastic differential games and stochastic control (see e.g. [8], [9] and [7]). More explicitly a BSDE is a stochastic equation of the form:

$$Y_t \ = \ \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \ 0 \le t \le T \tag{1}$$

where $\{W_t \,:\, t \in [0,T]\}$ is a d–dimensional Brownian motion defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, with $(\mathcal{F}_t)_{0\le t\le T}$ the, completed, standard Brownian filtration and ξ is a terminal data (i.e. $Y_T = \xi$ which is $\mathcal{F}_T$–measurable).
It is well–known, [20], that (1) admits a unique adapted solution, that is a couple (Y, Z) of square integrable processes satisfying the equation (1) such that

$$d[(Y,Z),(0,0)]^2 := \mathbb{E} \sup_{0\le t\le T} |Y_t|^2 + \mathbb{E}\int_0^T |Z_s|^2 ds < \infty,$$

when the coefficient $f(t, y, z)$ (called the drift) is Lipschitz continuous in the second and the third argument and ξ is a square integrable.
Many attempts have been suggested to relax the assumptions on the coefficient $f(s, y, z)$ and then to generalize the classical results on existence, uniqueness and comparison theorem for BSDEs with Lipschitz coefficients to more general coefficients as possible. In this direction the results on existence and uniqueness is extended to the case when the drift f is locally Lipschitz, satisfies the linear growth condition on the second argument and the

growth of order $|z|^\alpha$ for α in $[0,2[$ on the third argument in [6]. In the case where the drift f is only continuous and satisfying the linear growth condition, the existence, the uniqueness of a solution and a comparison theorem of the solutions of (1) are obtained in [16] and [17] in the one dimensional case, see also [10] for further consideration. In [13] Kobylanski discusses the existence, uniqueness and the comparison principle for BSDE with continuous coefficient such that f and $\partial f/\partial y$ satisfy the quadratic linear growth in the third argument z and $\partial f/\partial z$ is linear growth in z. He also studies the connections between BSDEs with both viscosity and Sobolev solutions of PDEs when the non–linearity has a quadratic growth in the gradient in [14]. We refer the reader to [1] for a survey on the connections between BSDEs and Sobolev solutions of PDEs. In their recent paper Dermoune *et al.* [4] have extended the existence and uniqueness results for a singular BSDE involving the local time of the unknown process of the form

$$Y_t \;=\; \xi + \int_t^T h(s,Y_s,Z_s)ds + \int_{\mathbb{R}} \left(L_T^a(Y) - L_t^a(Y)\right)\nu(da) - \int_t^T Z_s dW_s \quad (2)$$

where the process $L_t^a(Y)$ is the local time at the level a of the unknown semi–martingale $(Y_t)_{t\geq 0}$. The process h and the measure $\nu(da)$ are given. They have also obtained a probabilistic interpretations for some non–linear PDEs.

In this paper, we are interested in some limit theorems for BSDEs of the form (2) when the process h is identically zero and their corresponding results in PDEs framework.

Let us first consider the BSDE

$$Y_t \;=\; \xi + \int_t^T f(Y_s)Z_s^2 ds - \int_t^T Z_s dW_s, \quad (3)$$

where f is a measurable function from $\mathbb{R}$ to $\mathbb{R}$. The results known, see [3] and the references therein, do not guarantee the existence of a solution for (3). The connection between (3) and the BSDE (2) is made as follows:
¿From the equality $d\langle Y,Y\rangle_t = Z_t^2 dt$ and from occupation time formula, we have, for any bounded measurable function f

$$\int_0^t f(Y_s)Z_s^2 ds \;=\; \int_{-\infty}^{\infty} L_t^a(Y) f(a) da.$$

The process $L_t^a(Y)$ can be expressed by Tanaka's formula as follows

$$L_t^a(Y) \;=\; |Y_t - a| - |Y_0 - a| - \int_0^t \operatorname{sgn}(Y_s - a) dW_s$$

and

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases}$$

So, the BSDE (2) arises naturally. It is proved by Dermoune *et al.* [4] that there exists an adapted couple (Y, Z) solution to equation (2) under the following conditions:

(H1) The random variable ξ is $\mathbb{R}$–valued and belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

(H2) The process $h : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is such that :
 i) For all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $(t, \omega) \mapsto h(t, \omega, y, z)$ is $\mathcal{F}_T$–progressively measurable.
 ii) (t, ω) $dt \otimes d\mathbb{P}$–a.s. the map $(y, z) \mapsto h(t, \omega, y, z)$ is continuous.
 iii) There exists a constant $K > 0$ such that for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$

$$|h(t, \omega, y, z)| \ \leq \ K(1 + |y| + |z|) \quad dt \otimes d\mathbb{P} - a.s.$$

(H3) The signed measure ν is bounded and $|\nu(\{x\})| < 1$, for all x in $\mathbb{R}$.

The purpose of the present paper is to prove some limit theorems for the class of BSDE of the form (2), that are some kind of the stability properties for BSDEs. However we shall restrict our selves to the case where the drift h in the equation (2) is equal to zero that is BSDE of the form:

$$Y_t \ = \ \xi + \int_{\mathbb{R}} (L_T^a(Y) - L_t^a(Y))\, \nu(da) - \int_t^T Z_s dW_s. \tag{4}$$

We use those limit theorems to prove that a solution to (4) can be obtained as a limit of sequence of solution to (3), a much more interesting result is established, that is the relative compactness of the family $\{(Y^\nu, Z^\nu)$ solution to (4) such that the total variation $\|\nu\|$ of ν is less that a constant $C\}$ under the topology induced by the distance $d[\cdot, \cdot]$. The comparison theorem for the above singular BSDE is proved by different methods and in more general and simpler way than [13] and [17]. This comparison result is used to prove some limit theorems in the monotone case. We remark that our results are more general than those obtained in [13] which correspond to the case where the measure ν is absolutely continuous with respect to the Lebesgue measure.
As an application, the probabilistic interpretation obtained in [4] is exploited to prove some limit theorems, and the relative compactness for a class of non–linear PDEs involving the square of the gradient. The comparison theorem is discussed for this PDEs.

The main tool to study the BSDE (2) is the following transformation. Let us set

$$f_\nu(x) \;=\; \exp\left(2\nu^c((-\infty,x])\right)\prod_{y\le x}\left(\frac{1+\nu(\{y\})}{1-\nu(\{y\})}\right)$$

where ν^c is the continuous part of the measure ν.
If f is of bounded variation (increasing in our case), $f(x-)$ will denote the left limit of f at a point x and $f'(dx)$ will be the bounded measure associated with f.
It is well known that the function $f_\nu(\cdot)$ is increasing, right continuous and satisfies

$$0 < m \le f_\nu(x) \le M \quad \forall\, x \in \mathbb{R}$$

for some constants m, M. Moreover f_ν satisfies

$$f_\nu'(dx) - \{f_\nu(x) + f_\nu(x-)\}\,\nu(dx) \;=\; 0.$$

See Le Gall [15] for more details. We set

$$F_\nu(x) \;=\; \int_0^x f_\nu(y)dy \;\text{ and }\; g_\nu(x) \;=\; f_\nu(F_\nu^{-1}(x)).$$

It is well known that F_ν and F_ν^{-1} are Lipschitz functions.
Let us first recall some basic results which will be needed in the sequel, and whose proves can be found in Dermoune *et al.* [4] with minor change.
Let $\mathcal{M}_T^2(\mathbb{R}\times\mathbb{R}^d)$ denote the space of $\mathcal{F}_t$–progressively measurable processes (Y,Z) taking values in $\mathbb{R}\times\mathbb{R}^d$ and which belong to $L^2([0,T]\times\Omega, dt\otimes d\mathbb{P})$.

PROPOSITION 1. *$(Y,Z)\in\mathcal{M}_T^2(\mathbb{R}\times\mathbb{R}^d)$ is a solution of (2) if and only if the process*

$$\left(\tilde{Y},\tilde{Z}\right) \;=\; \left(F_\nu(Y), \frac{Z}{2}\{f_\nu(Y)+f_\nu(Y-)\}\right)$$

is a solution of the BSDE

$$\tilde{Y}_t \;=\; \tilde{\xi} + \int_t^T H(s,\tilde{Y}_s,\tilde{Z}_s)ds - \int_t^T \tilde{Z}_s dW_s, \quad 0\le t\le T, \tag{5}$$

where $\tilde{\xi} = F_\nu(\xi)$ and

$$H(s,y,z) \;=\; \frac{1}{2}\{g_\nu(y)+g_\nu(y-)\}\; h\left(s, F_\nu^{-1}(y), \frac{2z}{g_\nu(y)+g_\nu(y-)}\right).$$

Proof. The proof follows the same line as in [4] but applying Tanaka's formula to $F_\nu(Y_t)$ with the symmetric derivative of the convex function F_ν instead of its left derivative.

REMARK 1. *Stroock and Yor [22], Rutkowski [21] have already used the transformation F_ν to study the SDE*

$$X_t \;=\; x + \int_0^t \sigma(X_s)dW_s + \int_{\mathbb{R}} L_t^a(X)\nu(da).$$

THEOREM 1. *Under the assumptions* (H1), (H2), (H3), *it holds*
1) If $h = 0$ then there exists a unique solution (Y^ν, Z^ν) belonging to $\mathcal{M}_T^2(\mathbb{R}\times \mathbb{R}^d)$ for the equation (2). Moreover

$$Y_t^\nu \;=\; F_\nu^{-1}\left(\mathbb{E}[F_\nu(\xi)\;/\;\mathcal{F}_t]\right), \quad 0 \le t \le T.$$

Assuming moreover that the measure ν is diffuse, then the following assertions hold.
2) There exists a process (Y, Z) in $\mathcal{M}_T^2(\mathbb{R}\times\mathbb{R}^d)$ solution of the BSDE (2).
3) If $H(\cdot, 0, 0) \in L^2(\Omega\times[0,T])$ and there exists $c > 0$ such that

$$|H(\omega,t,y,z) - H(\omega,t,y',z')| \;\le\; c\,(|y-y'| + |z-z'|)$$

for almost all t and ω. Then there exists a unique pair (Y, Z) of processes in $L^2(\Omega\times[0,T])\times L^2(\Omega\times[0,T])$.

EXAMPLE 1. *1) Let $h = 0$ and $\nu = A\delta$, where $|A| < 1$. Then $f_\nu(x) = 1$ for $x < 0$ and $f_\nu(x) = \frac{1+A}{1-A}$ for $x \ge 0$. The function $F_\nu(x) = x$ for $x < 0$ and $F_\nu(x) = \frac{1+A}{1-A}x$ for $x \ge 0$. The solution of the BSDE*

$$Y_t \;=\; \xi + AL_T^0(Y) - AL_t^0(Y) - \int_t^T Z_s dW_s,$$

where $\xi \in]-\infty, 0[$ or $\xi \in [0, \infty[$ is given by

$$Y_t \;=\; \mathbb{E}[\xi\,/\mathcal{F}_t],$$

and $L_t^0(Y) = 0$ for all $0 \le t \le T$.
2) If $h(s, y, z) = h_1(s, y)z$ with h_1 bounded and uniformly Lipschitz with respect to y then the BSDE (2) has a unique solution.

REMARK 2. *In the case where ν is a non necessary bounded measure on $\mathbb{R}$ which is diffuse and σ–finite, the associated function $f_\nu(x)$=$\exp(2\nu((-\infty,x]))$ is positive, continuous and non necessary bounded function. Hence the function $F_\nu(x)$ is only locally Lipschitz, however if ξ and $F_\nu(\xi)$ are square integrable random variables then the BSDE (4) has a unique solution which is given by*

$$Y_t^\nu \;=\; F_\nu^{-1}\left(\mathbb{E}[F_\nu(\xi)\;/\;\mathcal{F}_t]\right).$$

2 Limit theorems for BSDEs

In this section we prove limit theorems for solutions of backward stochastic differential equations of the form (4). Those limit theorems will permit us to deduce some kind of stability properties of BSDE of the form (4).
Assumptions required for BSDE with the local time and techniques employed for the proof are closely related to those in Dermoune *et al.* [4] where BSDE will local time has been studied and in Le Gall [15] where stochastic ordinary differential equations involving the local time of the unknown process are discussed. As an application we shall prove that a solution to equation (4) is the strong limit with respect to some distance of a sequence of semi–martingales which are solutions of ordinary BSDE (3). Another interesting application of our main result will be a comparison theorem for BSDE involving measures that improves the previous result obtained in [17].

THEOREM 2. *Let $\nu_n(da)$, $n = 1, 2, \dots$ be a sequence of Radon measures and ξ^n a sequence of random variables in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Suppose that there exist two positive constants ε, M such that :*

$$|\nu_n|(\mathbb{R}) \quad \leq \quad M \qquad \forall\, n \geq 1,$$

$$|\nu_n(\{x\})| \quad \leq \quad \varepsilon \ < \ 1 \qquad \forall\, n \geq 1,\ \forall\, x \in \mathbb{R}.$$

Let (Y^n, Z^n) be the solution of

$$Y_t \ = \ \xi^n + \int_{\mathbb{R}} \left(L_T^a(Y) - L_t^a(Y)\right) \nu_n(da) - \int_t^T Z_s dW_s.$$

Assume that ξ^n converges to ξ in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ as n tends towards infinity. Assume further that there exist a function f such that :

$$\lim_{n\to+\infty} \int_{-L}^{L} |f_{\nu_n} - f|^2(x)\, dx \ = \ 0 \quad \text{for all } L > 0,$$

set

$$\nu(da) \ = \ \frac{f'(da)}{f(a) + f(a-)}.$$

Then

$$\lim_{n\to+\infty} \mathbb{E} \sup_{0\leq t\leq T} |Y_t^n - Y_t^\nu|^2 + \mathbb{E}\int_0^T |Z_s^n - Z_s^\nu|^2\, ds \ = \ 0$$

where (Y^ν, Z^ν) is the unique solution to the BSDE equation :

$$Y_t \ = \ \xi + \int_{\mathbb{R}} \left(L_T^a(Y) - L_t^a(Y)\right) \nu(da) - \int_t^T Z_s dW_s.$$

Proof. We shall use the following notations :

$$f_{\nu_n}(x) \;=\; \exp(2\nu_n^c((-\infty,x]))\prod_{y\le x}\left(\frac{1+\nu_n(\{y\})}{1-\nu_n(\{y\})}\right)$$

$$F_{\nu_n}(y) \;=\; \int_0^y f_{\nu_n}(x)dx \;\;\text{and}\;\; F(y) \;=\; \int_0^y f(x)dx.$$

By Theorem 1, it holds

$$Y_t^n \;=\; F_{\nu_n}^{-1}\left(\mathbb{E}[F_{\nu_n}(\xi^n)/\mathcal{F}_t]\right) \quad 0\le t\le T.$$

$$Y_t^\nu \;=\; F^{-1}\left(\mathbb{E}[F(\xi)/\mathcal{F}_t]\right) \quad 0\le t\le T.$$

The convergence of f_{ν_n} to f in $L^2_{\text{loc}}(\mathbb{R})$ implies that F_{ν_n} converges to F uniformly on compact sets and then, using a truncating argument, $F_{\nu_n}(\xi^n)$ converges to $F(\xi)$. It follows that $\overline{Y_t}^n := \mathbb{E}[F_{\nu_n}(\xi^n)/\mathcal{F}_t]$ converges to $\mathbb{E}[F(\xi)/\mathcal{F}_t] =: \overline{Y_t}^\nu$ in $L^2(\Omega)$. It is, trivial to see that $F_{\nu_n}^{-1}$ converges to F^{-1} uniformly on compact sets and so $Y_t^n = F_{\nu_n}^{-1}(\overline{Y_t}^n)$ converges to $F^{-1}(\overline{Y_t}^\nu) = Y_t^\nu$. Hence $\mathbb{E}\sup_{0\le t\le T}|Y_t^n - Y_t^\nu|$ tends to zero when n goes to infinity, and using the isometry property, one can see that $\mathbb{E}\int_0^T |Z_s^n - Z_s^\nu|^2\, ds$ converges to zero when n tends to infinity. □

Remark 3. *Let $\xi^n = \xi$ for all n, $\nu_n(dx) = f_n(x)\,dx$ where $f_n(x)\ge 0$; $\int f_n(x)\,dx = 1$ and $\operatorname{supp}(f_n) = [-\frac{1}{n},\frac{1}{n}]$.*
Let us consider the BSDE

$$Y_t^n \;=\; \xi + \int_t^T f_n(Y_s^n)\,(Z_s^n)^2\,ds - \int_t^T Z_s^n dW_s,$$

then the last theorem implies the convergence of $Y_\cdot^n$ to the unique solution of the BSDE

$$Y_t \;=\; \xi + \frac{1-e^2}{1+e^2}\left(L_T^0(Y) - L_t^0(Y)\right) - \int_t^T Z_s dW_s.$$

It is clear from the above Remark that if we consider a sequence of measures ν_n which converges to a measure ν, then in general Y^{ν_n} does not converges to Y^ν. We have presented an affirmative answer to this problem in Theorem 2 by replacing the convergence of measures ν_n by the convergence of its associated function f_{ν_n}.

In the sequel $\mathcal{M}(\mathbb{R})$ will denote the space of all bounded measure on $\mathbb{R}$ such that:

$$|\nu(\{x\})| \quad < 1 \quad \forall\, x \in \mathbb{R}.$$

Let ν be in $\mathcal{M}(\mathbb{R})$. We define

$$\|\nu\| \;=\; |\nu^c(\mathbb{R})| + \frac{1}{2}\sum_y \left|\frac{1+\nu\{y\}}{1-\nu\{y\}}\right|.$$

Note that

$$\|\nu\| \;=\; \mathrm{var}\left(\frac{1}{2}\log\left(f_\nu\right)\right)$$

where, var, denotes the total variation.

In the space of adapted processes (Y, Z), such that :

$$\mathbb{E}\sup_{0\le t\le T}|Y_t|^2 + \mathbb{E}\int_0^T |Z_s|^2\, ds \;<\; +\infty,$$

we consider the associate distance $d[.,.]$ given by :

$$d\left[(Y,Z),(Y',Z')\right] \;=\; \left(\mathbb{E}\sup_{0\le t\le T}|Y_t - Y_t'|^2 + \mathbb{E}\int_0^T |Z_s - Z_s'|^2\, ds\right)^{\dfrac{1}{2}}.$$

With these definitions, we can state the following result :

THEOREM 3. *Let C be a fixed constant. Then, $\mathcal{K} = \{(Y^\nu, Z^\nu) : \|\nu\| \le C\}$ is a compact set for the topology induced by $d[\cdot,\cdot]$.*
The set of all (Y^ν, Z^ν) belonging to $\mathcal{K}$ such that ν is absolutely continuous with respect to Lebesgue measure is dense in $\mathcal{K}$.

Proof. Let ν_n be a sequence in $\mathcal{M}(\mathbb{R})$ such that $\|\nu_n\| \le C$.
Since the total variation of the f_{ν_n}'s are uniformly bounded, we can find a function f of bounded variation and a subsequence $(f_{\nu_{n_k}})$ such that :

$$f_{\nu_{n_k}}(x) \longrightarrow f(x) \quad \text{as } k \longrightarrow +\infty, \;\; \text{for all } x \in \mathbb{R}\backslash D_f$$

where, D_f, is at most countable.
Set

$$\nu(da) \;=\; \frac{f'(da)}{f(a)+f(a-)}.$$

Then the limit Theorem 2 implies that :

$$d\left[(Y^{\nu_{n_k}}, Z^{\nu_{n_k}}), (Z^\nu, Z^\nu)\right] \longrightarrow 0 \;\; \text{when} \;\; k \longrightarrow +\infty.$$

It remains to prove that $\|\nu\| \leq C$.
Note that f satisfies the same equation as f_ν, then, there exist $\lambda > 0$ such that $f = \lambda f_\nu$.
Hence

$$\begin{aligned} \|\nu\| &= \operatorname{var}\left(\frac{1}{2}\log\left(f_\nu\right)\right) &=& \operatorname{var}\left(\frac{1}{2}\log\left(f\right)\right) \\ && \leq& \limsup_{n\to+\infty} \operatorname{var}\left(\frac{1}{2}\log\left(f_{\nu_n}\right)\right) \leq C. \end{aligned}$$

Let us prove the second point; Let $\nu \in \mathcal{M}(\mathbb{R})$ and θ_n an approximation of the identity.
Set

$$f_n = f_\nu * \theta_n \quad \text{and} \quad g_n = \frac{f_n'}{2f_n}.$$

Let (Y^n, Z^n) be the unique solution of the following BSDE

$$Y_t^n = \xi + \int_t^T g_n(Y_s^n)\,(Z_s^n)^2\,ds - \int_t^T Z_s^n dW_s.$$

Using Theorem 2, it is easy to see that :

$$d\left[(Y^n, Z^n), (Y^\nu, Z^\nu)\right] \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

□

3 Comparison theorems for BSDEs

Now we introduce a new result on comparison of solutions of BSDEs involving measures.
In [17], Lepeltier and San Martin consider BSDEs with terminal data $\xi \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$, and gave a comparison theorem for BSDE with parameter (f, ξ) i.e.

$$Y_t = \xi + \int_t^T f(Y_s)Z_s^2 ds - \int_t^T Z_s dW_s.$$

In the following theorem, we give an elegant proof for a general comparison theorem, without boundedness of the terminal value of the BSDE. As a byproduct, we obtain the comparison theorem, for the standard BSDE considered in [17], under fairly weak conditions on the coefficients.

THEOREM 4. *Let ν, μ be in $\mathcal{M}(\mathbb{R})$. Let (Y^ν, Z^ν), (Y^μ, Z^μ) be two processes such that :*

$$\begin{aligned} Y_t^\nu &= \xi + \int_{\mathbb{R}} (L_T^a(Y^\nu) - L_t^a(Y^\nu))\,\nu(da) - \int_t^T Z_s^\nu dW_s, \\ Y_t^\mu &= \xi' + \int_{\mathbb{R}} (L_T^a(Y^\mu) - L_t^a(Y^\mu))\,\mu(da) - \int_t^T Z_s^\mu dW_s. \end{aligned}$$

Assume that $\xi \geq \xi'$ a.s. and the measure $\nu - \mu$ is positive.
Then $Y_t^\nu \geq Y_t^\mu$ for all t $\mathbb{P}$–a.s.

Proof. Let us first recall Tanaka's formula. Since F_μ is a convex function, then

$$\begin{aligned} F_\mu(Y_T^\nu) \;=\; & F_\mu(Y_t^\nu) + \int_t^T \frac{1}{2}\,(f_\mu(Y_s^\nu) + f_\mu(Y_s^\nu -))\,dY_s^\nu \\ & + \frac{1}{2}\int_{\mathbb{R}} (L_T^a(Y^\nu) - L_t^a(Y^\nu))\,f_\mu^{'}(da) \end{aligned}$$

hence

$$\begin{aligned} F_\mu(\xi) \;=\; & F_\mu(Y_t^\nu) + (M_T - M_t) \\ & -\frac{1}{2}\int_{\mathbb{R}} \{f_\mu(a) + f_\mu(a-)\}\,(L_T^a(Y^\nu) - L_t^a(Y^\nu))\,(\nu - \mu)\,(da) \end{aligned}$$

where $M.$ is a square integrable martingale.
Since the function $a \mapsto [f_\mu(a) + f_\mu(a-)](L_T^a(Y^\nu) - L_t^a(Y^\nu))$ is positive, and F_μ is an increasing function, then

$$F_\mu\,(Y_t^\nu) \;\geq\; \mathbb{E}[F_\mu(\xi')\,/\mathcal{F}_t]$$

and

$$Y_t^\nu \;\geq\; F_\mu^{-1}\,(\mathbb{E}[F_\mu\,(\xi')\,/\mathcal{F}_t]) \;=\; Y_t^\mu.$$

□

An immediate consequence of the above comparison result is the

COROLLARY 1. *Let $(\nu_n)_{n\geq 1}$ be an sequence of measures such that $\sup_{n\geq 1} \|\nu_n\| < +\infty$ and f_{ν_n} increases to f almost everywhere. If ξ^n increases to $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ as n tends towards infinity. Then*

$$\lim_{n\to+\infty} \mathbb{E} \sup_{0\leq t\leq T} |Y_t^\nu - Y_t^n|^2 + \int_0^T |Z_s^\nu - Z_s^n|^2\,ds \;=\; 0$$

where

$$\nu(da) \;=\; \frac{f'(da)}{f(a)+f(a-)}.$$

and (Y^n, Z^n), (Y^ν, Z^ν) *are respectively solution to*

$$\begin{aligned} Y_t^n &= \xi^n + \int_{\mathbb{R}} \left(L_T^a(Y^n) - L_t^a(Y^n)\right) \nu_n(da) - \int_t^T Z_s^n dW_s \\ Y_t^\nu &= \xi + \int_{\mathbb{R}} \left(L_T^a(Y^\nu) - L_t^a(Y^\nu)\right) \nu(da) - \int_t^T Z_s^\nu dW_s. \end{aligned}$$

REMARK 4. *An alternative proof of the above comparison theorem, can be given by regularizing the measures* ν *and* μ *and employs the comparison theorem in Lepeltier and San Martin [17] and the above limit theorems.*

COROLLARY 2. *Let* (f^1, ξ^1) *and* (f^2, ξ^2) *be two parameters of BSDE, and let* $\left(Y^1, Z^1\right)$ *and* $\left(Y^2, Z^2\right)$ *be associated solution.*
Suppose that :
$\xi^1 \leq \xi^2$ *a.s. and* $f^1(y) \leq f^2(y)$ *for almost all* y.
Then for all $t \in [0, T]$, *we have* $Y_t^1 \leq Y_t^2$ *a.s.*

As a consequence of the above results, we have obtained an interesting limit theorem for generalized BSDE in monotonic case. We consider a sequence of BSDE

$$Y_t^n \;=\; \xi^n + \int_{\mathbb{R}} \left(L_T^a(Y^n) - L_t^a(Y^n)\right) \nu_n(da) - \int_t^T Z_s^n dW_s,$$

where ξ^n is a sequence of random variables $\mathcal{F}_T$–measurable belonging to $L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$.

THEOREM 5. *Let* $(\nu_n)_{n\geq 1}$ *be an increasing sequence of measures such that* $\sup_{n\geq 1} \|\nu_n\| < +\infty$, *assume* ξ^n *increases to* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ *as* n *tends towards infinity. Then*

$$\lim_{n\to+\infty} \mathbb{E} \sup_{0\leq t\leq T} |Y_t^\nu - Y_t^n|^2 + \int_0^T |Z_s^\nu - Z_s^n|^2 \, ds \;=\; 0,$$

where (Y^ν, Z^ν) *is the unique solution of the BSDE*

$$Y_t^\nu \;=\; \xi + \int_{\mathbb{R}} \left(L_T^a(Y^\nu) - L_t^a(Y^\nu)\right) \nu(da) - \int_t^T Z_s^\nu dW_s,$$

and $\nu = \sup_{n\geq 1} \nu_n$.

Proof. For any measurable set, we have $\nu(A) = \sup_{n\geq 1}\nu_n(A)$, it follows from the bound $\sup_{n\geq 1}\|\nu_n\| < +\infty$, that ν is a bounded measure.
Set

$$F_n(y) := F_{\nu_n}(y) \quad \text{and} \quad F(y) := F_\nu(y).$$

Then $F_n(\cdot)$ is increasing and converges to the continuous function $F(\cdot)$, hence by Dini theorem this convergence is uniform. By the comparison Theorem 4, the sequence Y_t^n is increasing. Set

$$Y_t^\nu \;=\; \lim_{n\to+\infty} Y_t^n,$$

hence $F_n(Y_t^n)$ converges to $F(Y_t^\nu)$. But

$$F_n\left(Y_t^n\right) \;=\; \mathbb{E}[F_n(\xi^n)\,/\mathcal{F}_t] \quad 0\leq t\leq T.$$

and

$$\left|F_n\left(\xi^n\right)\right| \;\leq\; \left(\left|\xi^1\right| + |\xi|\right)\exp\left(2|\nu|(\mathbb{R})\right).$$

Then passing to the limit, using dominated convergence theorem for conditional expectation, it holds that

$$Y_t^\nu \;=\; F^{-1}\left(\mathbb{E}[F(\xi)\,/\mathcal{F}_t]\right) \quad 0\leq t\leq T.$$

By Theorem 1, (Y^ν, Z^ν) is the unique solution of the BSDE

$$Y_t^\nu \;=\; \xi + \int_{\mathbb{R}} \left(L_T^a(Y^\nu) - L_t^a(Y^\nu)\right)\nu(da) - \int_t^T Z_s^\nu dW_s.$$

We deduce from Burkholder–Davis–Gundy inequality, that

$$\lim_{n\to+\infty} \mathbb{E}\sup_{0\leq t\leq T}\left|Y_t^\nu - Y_t^n\right|^2 \;=\; 0,$$

using the transformation F_ν and the isometry property, we get

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T \left|Z_s^\nu - Z_s^n\right|^2 ds \;=\; 0.$$

□

4 Applications to non–linear partial differential equations

This section is devoted to some limit theorems for PDE that can be deduced from the above limit theorems for BSDE using the connection between these

different kind of equations.
Let $\{X_s^{x,t} : 0 \leq t \leq s \leq T\}$ be the unique solution of the stochastic differential equation

$$X_s^{x,t} \;=\; x + \int_t^s b(X_r^{x,t})dr + \int_t^s \sigma(X_r^{x,t})dW_r,$$

where the coefficients b and σ are globally Lipschitz.
Let ν be a measure on $\mathbb{R}$ and satisfy the assumption (H3), with the same notations as in section 1, page 4, we consider the singular non–linear Cauchy problem

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = Lu(t,x) - \frac{1}{2}\sigma^2(x)\left(\frac{\partial F_\nu(u(t,x))}{\partial x}\right)^2 F_\nu(u(t,x))^* \left(\frac{d^2F_\nu^{-1}}{d^2x}\right)^2 \\ &u(0,x) = g(x),\; x \in \mathbb{R}, \end{aligned}\right\} \tag{6}$$

where g is a continuous real valued function with polynomial growth and L is the infinitesimal generator of the diffusion process $\{X_s^{x,t} : 0 \leq t \leq s \leq T\}$ and $\pi^*(\phi)$ stands for the pullback of the distribution ϕ by π (see e.g. [11]).

Definition 1. *The function u in $\mathcal{C}([0,T] \times \mathbb{R})$ is said to be a viscosity solution of (6) if and only if :*
For all ϕ in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, if $(t_0, x_0) \in [0,T]\times\mathbb{R}$ is a local maximum point of $u - \phi$, one has

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t_0,x_0) \;\geq\; L\phi(t_0,x_0) - \frac{1}{2}\sigma^2(x_0)\left(\frac{\partial F_\nu(\phi(t_0,x_0))}{\partial x}\right)^2 \times \\ F_\nu(\phi((t_0,x_0)))^* \left(\frac{d^2F_\nu^{-1}}{d^2x}\right)^2 \end{aligned} \tag{7}$$

and for all ϕ in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R})$, if $(t_0, x_0) \in [0,T] \times \mathbb{R}$ is a local minimum point of $u - \phi$, one has

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t_0,x_0) \;\leq\; L\phi(t_0,x_0) - \frac{1}{2}\sigma^2(x_0)\left(\frac{\partial F_\nu(\phi(t_0,x_0))}{\partial x}\right)^2 \times \\ F_\nu(\phi((t_0,x_0)))^* \left(\frac{d^2F_\nu^{-1}}{d^2x}\right)^2. \end{aligned} \tag{8}$$

If u satisfies (7), u is said to be a viscosity subsolution of the equation while u is said to be a viscosity supersolution if (8) holds.

In the case where the convex function F_ν is twice continuously differentiable, the equation (6) takes the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= Lu(t,x) + \sigma^2(x)\left(\frac{F''_\nu(u(t,x))}{2F'_\nu(u(t,x))}\right)\left(\frac{\partial u}{\partial x}(t,x)\right)^2 \\ u(0,x) &= g(x),\ x \in \mathbb{R}. \end{aligned}\right\} \qquad (9)$$

This situation corresponds to the case where the measure ν is absolutely continuous with respect to the Lebesgue measure.

Using standard argument one can prove that the stochastic process $\{Y_s^{x,t} : s \in [t,T]\}$ constructed as the unique solution to BSDE

$$Y_s^{x,t} = g(X_T^{x,t}) + \int_s^T \frac{F''_\nu}{2F'_\nu}(Y_r^{x,t})\left(Z_r^{x,t}\right)^2 ds - \int_s^T Z_r^{x,t} dW_r, \text{ for all } t \le s \le T\cdot$$

taking at time $s = t$ is a viscosity solution to the equation (9).

Now, let $\{(Y_s^{x,t}, Z_s^{x,t}) : s \in [t,T]\}$ be the unique solution to the singular BSDE

$$Y_s^{x,t} \;=\; g(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t}\right) - L_s^a\left(Y^{x,t}\right)\right)\nu(da) - \int_s^T Z_r^{x,t} dW_r.$$

With the help of the transformation F_ν one can see that $Y_t^{x,t}$ is a viscosity solution to the equation (6).

Let us now go back to the corresponding limit theorems.

THEOREM 6. *Let $\nu_n(da)$, $n = 1, 2, \ldots$ be a sequence of Radon measures. Suppose that there exist two positive constants ε, M such that :*

$$|\nu_n|(\mathbb{R}) \quad \le M \qquad \forall\, n \ge 1,$$

$$|\nu_n(\{x\})| \quad \le \varepsilon < 1 \quad \forall\, n \ge 1,\ \forall\, x \in \mathbb{R}.$$

and there exist a function f such that :

$$\int_{-L}^{L} |f_{\nu_n} - f|^2(x)\,dx \longrightarrow 0 \ \text{ as } \ n \to +\infty \ \text{ for all } \ L > 0,$$

set

$$\nu(da) \;=\; \frac{f'(da)}{f(a) + f(a-)} \quad \text{and} \quad F(x) := \int_0^x f(y)dy\cdot$$

If $u^n(t,x)$ and $u(t,x)$ denote respectively the unique solution of the PDE (6) with ν_n respectively with ν.

Then $u^n(t,x)$ converges towards $u(t,x)$ as n tends to infinity for any $(t,x) \in [0,T] \times \mathbb{R}$.

Proof. For any $t \in \mathbb{R}_+$, we let $\{Y_s^{x,t,n} : t \leq s \leq T\}$ and $\{Y_s^{x,t} : t \leq s \leq T\}$ be respectively the solution of the BSDE

$$Y_s^{x,t,n} \;=\; g(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t,n}\right) - L_s^a\left(Y^{x,t,n}\right)\right)\nu_n(da) - \int_s^T Z_r^{x,t,n} dW_r$$

and

$$Y_s^{x,t} \;=\; g(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t}\right) - L_s^a\left(Y^{x,t}\right)\right)\nu(da) - \int_s^T Z_r^{x,t} dW_r.$$

If we set $\tilde{u} := F_\nu(u)$, then equation (6) becomes

$$\left.\begin{aligned} \frac{\partial \tilde{u}}{\partial t}(t,x) \;&=\; L\tilde{u}(t,x) \\ \tilde{u}(0,x) \;&=\; F_\nu(g(x)),\; x \in \mathbb{R}. \end{aligned}\right\}$$

Therefore the process $\{\tilde{u}(s, X_s^{x,t}) : t \leq s \leq T\}$ is the unique solution to the BSDE

$$\tilde{Y}_s^{x,t} \;=\; F_\nu\left(g(X_T^{x,t})\right) - \int_s^T \tilde{Z}_r^{x,t} dW_r,$$

hence $Y_s^{x,t} = F_\nu^{-1}(\tilde{Y}_s^{x,t}) = F_\nu^{-1}(\tilde{u}(s, X_s^{x,t})) = u(s, X_s^{x,t})$, in particular $u^n(t,x) = Y_t^{x,t,n}$ and $Y_t^{x,t} := u^\nu(t,x)$, then by virtue of the previous results, $u^n(t,x)$ and $u^\nu(t,x)$ are respectively the unique viscosity solution to (6) with ν_n respectively ν. So Theorem 2 implies that

$$\lim_{n \to +\infty} \mathbb{E} \sup_{t \leq s \leq T} \left|Y_s^{x,t,n} - Y_s^{x,t}\right| \;=\; 0$$

which implies that $u^n(t,x)$ converges towards $u(t,x)$ as n tends to 0. The convergence is uniform on compacts by continuity. □

Let u^{h_n} be the unique solution of the following PDE

$$\left.\begin{aligned} \frac{\partial u}{\partial t}(t,x) \;&=\; Lu(t,x) + \sigma^2(x) h_n(u(t,x)) \left(\frac{\partial u}{\partial x}(t,x)\right)^2 \\ u(0,x) \;&=\; g(x),\; x \in \mathbb{R}. \end{aligned}\right\} \qquad (10)$$

The following theorem gives the relative compactness of the family $\{u^\nu : \|\nu\| \leq C\}$ and states that a solution to equation (6) is a limit of sequence of solution to the equation (10).

THEOREM 7. *Let C be a fixed constant. Then, $\mathcal{K} = \{u^\nu : \|\nu\| \leq C\}$ is a compact set for the topology induced by uniform convergence.*
The set of all u^ν belonging to $\mathcal{K}$ such that ν is absolutely continuous with respect to Lebesgue measure is dense in $\mathcal{K}$.

Proof. The proof of the first part is an immediate consequence of the connection between BSDEs and PDEs and Theorem 3.
Let us prove the second part; Let ν be in $\mathcal{M}(\mathbb{R})$ and θ_n be an approximation of the identity, we set $f_n = f_\nu * \theta_n$ and $g_n = f_n'/2f_n$.
Let u^{g_n} be the unique solution of the PDE (10). Using Theorem 6, it is easy to see that:

$$\lim_{n\to+\infty} \|u^{g_n} - u^\nu\|_\infty \;=\; 0.$$

□

4.1 Comparison theorem for PDEs.

In this subsection we use the connection between BSDEs and PDEs to give a probabilistic proof to a comparison theorem for non–linear PDE.

THEOREM 8. *Let g_1 and g_2 be two function such that $g_1(x) \leq g_2(x)$ for all $x \in \mathbb{R}$.*
Let ν_1 and ν_2 be in $\mathcal{M}(\mathbb{R})$ such that the measure $\nu_2 - \nu_1$ is positive.
If $u^1(t,x)$ and $u^2(t,x)$ are the solution to the PDE (6) corresponding to F_{ν_1} and F_{ν_2}.
Then $u^1(t,x) \leq u^2(t,x)$.

Proof. Following the same notations as in the proof of Theorem 6, we can write $u^1(t,x) = Y_t^{x,t,1}$ and $u^2(t,x) = Y_t^{x,t,2}$, where $\{Y_s^{x,t,i} : t \leq s \leq T\}$ is the unique solution to the BSDE equation

$$Y_s^{x,t,i} = g_i(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t,i}\right) - L_s^a\left(Y^{x,t,i}\right)\right)\nu_i(da) - \int_s^T Z_r^{x,t,i}dW_r \quad \text{for } i = 1,2. \quad (11)$$

Now, from Theorem 4, we have $Y_s^{x,t,1} \leq Y_s^{x,t,2}$ for all $t \leq s \leq T$, in particular $Y_t^{x,t,1} \leq Y_t^{x,t,2}$.

REMARK 5. *Using the same argument as in Corollary 1 and the comparison Theorem 8 one can obtain the corresponding limit theorem in the monotone case for PDE.*

It is clear that Theorem 8 implies the uniqueness property for a class of non–linear PDEs (at least of the form (6)). Some stability theorems are discussed in [14] but by different method and under more restrictive conditions.

In Theorem 6 we are given an affirmative answer, in certain sense, to the conjecture stated in [4], subsection, conclusions and remarks, (2).

EXAMPLE 2. *Let us consider the Cauchy problem for the parabolic non–linear equation*

$$\frac{\partial u}{\partial t}(t,x) \;=\; \frac{1}{2}\Delta u(t,x) - \frac{1}{2}\left(\frac{\partial u}{\partial x}(t,x)\right)^2 + k(x) \;\; \text{and}\; u(0,x) \;=\; g(x), x \in \mathbb{R}, \tag{12}$$

where $k : \mathbb{R} \to [0, +\infty)$ *is a continuous function.*
If $g = 0$*, It is known from Karatzas and Shreve [12] p. 270 that the only solution* $u : [0, +\infty[\times\mathbb{R} \to \mathbb{R}$*, to the equation (12), which is continuous on its domain, of* $C^{1,2}$ *on* $]0, +\infty[\times\mathbb{R}$ *and satisfies the quadratic growth condition for every* $T > 0$*:* $-u(t, x) \le C + ax^2$ *for all* $(t, x) \in [0, T] \times \mathbb{R}$ *where* $T > 0$ *is arbitrary and* $0 < a < 1/2T$*, is given by:*

$$u(t,x) \;=\; -\log \mathbb{E}\left[\exp\left\{-\int_0^t k(x+W_s)ds\right\}\right];\; (t,x) \in [0,+\infty[\times\mathbb{R}.$$

In the case where $k = 0$ *the equation (12) is a particular equation of (10) which corresponds to the case where* $\sigma = 1$*,* $b = 0$ *and* $h = -\frac{1}{2}$*, hence we can use BSDE to construct a solution to the equation (12), more precisely, let* $\{Y_s^{x,t} : t \le s \le T\}$ *be the unique solution to the BSDE*

$$Y_s^{x,t} \;=\; g(X_T^{x,t}) - \int_s^T \frac{1}{2}\left(Z_r^{x,t}\right)^2 dr - \int_s^T Z_r^{x,t} dW_r \tag{13}$$

where $X_T^{x,t} = x + \frac{1}{2}(B_T - B_t)$*.*
It is clear that the hypotheses of the Remark 2 are satisfied since the Brownian motion has exponential finite moment, and hence the equation (13) has a unique solution and $u(t, x) = Y_t^{x,t}$ *is the viscosity solution to the equation (12) with* $u(0,\cdot) = g(\cdot)$*.*
In the case where $k \neq 0$*, the BSDE associated to the equation (12) is the following*

$$Y_s^{x,t} \;=\; g(X_T^{x,t}) + \int_s^T k(X_r^{x,t})dr - \int_s^T \frac{1}{2}\left(Z_r^{x,t}\right)^2 dr - \int_s^T Z_r^{x,t} dW_r,$$

consequently the function $u(t,x) = Y_t^{x,t}$ is the unique viscosity solution to the equation (12).

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