HOMOLOGICAL LOCALIZATIONS PRESERVE 1-CONNECTIVITY

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Abstract. Every generalized homology theory E yields a localization functor $\mathbf{L}_E$ that sends the E-equivalences to homotopy equivalences. We prove that if X is any 1-connected space, then $\mathbf{L}_E X$ is also 1-connected, for every generalized homology theory E. This is deduced from a result by Hopkins and Smith stating that if $K(\mathbb{Z},2)$ is E-acyclic then E is trivial.

Introduction

A number of results in the literature suggest that idempotent functors in the homotopy category of spaces preserve 1-connectivity, although no proof of this fact has so far been given. One of the earliest examples is localization with respect to ordinary homology, which in fact preserves n-connectivity for all n; see [1].

In the same article [1], Bousfield proved the existence of localization with respect to any generalized homology theory E; that is, a functor $\mathbf{L}_E$ which assigns to every space X a space $\mathbf{L}_E X$ together with a natural map $X \to \mathbf{L}_E X$ which is terminal in the homotopy category among E-equivalences with source X. (An E-equivalence is a map $X \to Y$ inducing isomorphisms $E_n(X) \cong E_n(Y)$ for all n.)

In [9], Mislin showed that K-theory localization does not preserve n-connectivity in general, since for example $\pi_3(\mathbf{L}_K S^{2p+2}; \mathbb{Z}/p) \neq 0$ for every odd prime p. However, Mislin also proved in [9] that the K-llocalization of every 1-connected space is 1-connected. Further evidence of the fact that 1-connectivity could be preserved by arbitrary idempotent functors in the homotopy category was given by Neisendorfer in [10] and by Tai in his detailed study of the problem in [11].

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It is therefore natural to address the question of whether or not localizations with respect to generalized homology theories preserve 1-connectivity. Such localizations were thoroughly discussed by Bousfield in [2], where a description was given of their effect on abelian Eilenberg–Mac Lane spaces. The main tool was an arithmetic square, already exploited by Mislin in [9], allowing one to determine the E-localization of a space (with some restrictions on the fundamental group) from its $E\mathbb{Z}/p$-localizations and rational coherence data.

Our main result is that $\mathbf{L}_E X$ is 1-connected if X is 1-connected, for any generalized homology theory E. This follows by combining the methods of Bousfield in [2] with a result proved by Hopkins and Smith in [8], according to which a $K(\mathbb{Z},2)$ is never E-acyclic if E is nontrivial. We note, however, that $K(\mathbb{Z},3)$ is $K\mathbb{Z}/p$-acyclic for all p, by [9, Corollary 2.3]. It is known that, if L is any homotopy idempotent functor, then $LK(\mathbb{Z},n)$ is necessarily a $K(A,n)$ where A is either zero or a commutative ring with 1, for all n; see [5]. If $L = \mathbf{L}_E$ for some nontrivial homology theory E, then the possibility that $A = 0$ has been discarded for $n = 2$ in [8], and this opens the way to substantial improvements of earlier results or to new results as in this article.

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1. Torsion homology theories

Throughout the paper we denote by E a spectrum or the associated homology theory. For an abelian group R, the corresponding spectrum with coefficients in R is defined as $ER = E \wedge SR$ where SR is the Moore spectrum of type $(R,0)$. The only cases of interest in this article are $R = \mathbb{Z}/p$ and R a subring of $\mathbb{Q}$. A spectrum E is called *torsion* if $E\mathbb{Q}$ is contractible. The ordinary Eilenberg–Mac Lane spectrum with coefficients in R is denoted by HR. We denote by $\mathbb{Z}_p^\wedge$ the p-adics, by $\mathbb{Z}(p^\infty)$ the Prüfer group $\bigcup_{n=1}^\infty \mathbb{Z}/p^n$ and, for a set of primes P, we denote by $\mathbb{Z}_P$ the integers localized at P.

In this first section we concentrate on mod p homology theories, where p is any prime. Using the Atiyah–Hirzebruch spectral sequence, one sees that if E is any homology theory, then every $H\mathbb{Z}/p$-equivalence is an $E\mathbb{Z}/p$-equivalence; details are given in [9, § 1]. Hence, all $E\mathbb{Z}/p$-local spaces are

$H\mathbb{Z}/p$-local and there is a natural transformation of functors $\mu\colon \mathbf{L}_{H\mathbb{Z}/p} \to \mathbf{L}_{E\mathbb{Z}/p}$.

We next prove that, if X is connected, then the induced homomorphism

$$\mu_*\colon \pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$$

is surjective. This result is essentially contained in the proof of Proposition 7.1 in [2], as we next recall for the sake of completeness. The argument is based on Bousfield's version of the Whitehead theorem (cf. [2, Theorem 5.2]), stating that if R is $\mathbb{Z}/p$ or a subring of $\mathbb{Q}$, and $f\colon X \to Y$ is a map inducing isomorphisms $H_i(X;R) \cong H_i(Y;R)$ for $i < n$ and an epimorphism $H_n(X;R) \twoheadrightarrow H_n(Y;R)$, where $n \geq 1$, then f also induces isomorphisms $\pi_i(\mathbf{L}_{HR}X) \cong \pi_i(\mathbf{L}_{HR}Y)$ for $i < n$ and an epimorphism $\pi_n(\mathbf{L}_{HR}X) \twoheadrightarrow \pi_n(\mathbf{L}_{HR}Y)$.

Theorem 1.1. *Let E be any homology theory and p any prime. Then, for every connected space X, the natural homomorphism $\mu_*\colon \pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$ is surjective.*

Proof. The claim is obvious if $E\mathbb{Z}/p$ is trivial. If $E\mathbb{Z}/p$ is not trivial, then $K(\mathbb{Z}/p, 1)$ is not $E\mathbb{Z}/p$-acyclic, as shown in [2, Proposition 2.2]. Since the natural map $\mu\colon \mathbf{L}_{H\mathbb{Z}/p}X \to \mathbf{L}_{E\mathbb{Z}/p}X$ is an $E\mathbb{Z}/p$-equivalence, we obtain an isomorphism

$$\mu_*\colon H_1(\mathbf{L}_{H\mathbb{Z}/p}X;\mathbb{Z}/p) \cong H_1(\mathbf{L}_{E\mathbb{Z}/p}X;\mathbb{Z}/p) \tag{1.1}$$

using [2, Proposition 2.1] or [5, Theorem 1.3], according to which $K(\mathbb{Z}/p, 1)$ is $E\mathbb{Z}/p$-local. By the generalized Whitehead theorem stated above, μ induces then an epimorphism $\pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \twoheadrightarrow \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$, since $\mathbf{L}_{E\mathbb{Z}/p}X$ is $H\mathbb{Z}/p$-local. □

Corollary 1.2. *If E is any torsion homology theory and X is 1-connected, then $\mathbf{L}_EX$ is also 1-connected.*

Proof. As in [2], we denote by $\mathcal{P}E$ the set of primes p such that $\pi_*(E)$ is not uniquely p-divisible. By [2, Proposition 7.1], for each torsion homology theory E and every 1-connected space X, we have a homotopy equivalence

$$\mathbf{L}_EX \simeq \prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X.$$

Now recall from [1] that $\mathbf{L}_{H\mathbb{Z}/p}X$ is 1-connected if X is 1-connected. Therefore, Theorem 1.1 tells us that $\mathbf{L}_EX$ is 1-connected. □

Before discussing non-torsion homology theories, we need to study the second homotopy group $\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ when X is 1-connected. The following result is the main input in our discussion.

Theorem 1.3. *Let E be a homology theory and p any prime. Suppose that $E\mathbb{Z}/p$ is nontrivial. Then either $K(\mathbb{Z}/p,2)$ or $K(\mathbb{Z}_p^\wedge,2)$ is $E\mathbb{Z}/p$-local.*

Proof. The classification of acyclicity patterns for Eilenberg–Mac Lane spaces given by Bousfield in [2, § 4] implies that $\mathbf{L}_{E\mathbb{Z}/p}K(\mathbb{Z},n) = K(A,n)$ for each $n \geq 1$, where the group A can be $\mathbb{Z}_p^\wedge$, or $\mathbb{Z}/p^i$ for some $i \geq 1$, or zero. In [8], it is shown that if a reduced homology theory vanishes on $K(\mathbb{Z},2)$, then it is trivial. (Thus, nontrivial mod p homology theories of type IV-1 as defined in [2, § 4] do not exist.) Therefore, if $\mathbf{L}_{E\mathbb{Z}/p}$ is nontrivial, then the localization $\mathbf{L}_{E\mathbb{Z}/p}K(\mathbb{Z},2)$ is necessarily $K(\mathbb{Z}_p^\wedge,2)$ or $K(\mathbb{Z}/p^i,2)$ for some $i \geq 1$. In the latter case, $K(\mathbb{Z}/p,2)$ cannot be $E\mathbb{Z}/p$-acyclic, as one sees by induction using the fibre sequences

$$K(\mathbb{Z}/p,2) \to K(\mathbb{Z}/p^i,2) \to K(\mathbb{Z}/p^{i-1},2).$$

Hence, $K(\mathbb{Z}/p,2)$ is $E\mathbb{Z}/p$-local, by [2, Proposition 2.1] or [5, Lemma 1.4]. □

If $K(\mathbb{Z}/p,2)$ is $E\mathbb{Z}/p$-local and X is 1-connected, then, using the fact that $\mu\colon \mathbf{L}_{H\mathbb{Z}/p}X \to \mathbf{L}_{E\mathbb{Z}/p}X$ is an $E\mathbb{Z}/p$-equivalence, we obtain as in (1.1) an isomorphism

$$(1.2) \qquad \mu_*\colon H_2(\mathbf{L}_{H\mathbb{Z}/p}X;\mathbb{Z}/p) \cong H_2(\mathbf{L}_{E\mathbb{Z}/p}X;\mathbb{Z}/p).$$

Thus, the homomorphism $\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ induced by μ is surjective, by the generalized Whitehead theorem.

Now suppose that $K(\mathbb{Z}_p^\wedge,2)$ is $E\mathbb{Z}/p$-local and X is 1-connected. Similarly as in the previous case, since μ is an $E\mathbb{Z}/p$-equivalence, we have an isomorphism

$$(1.3) \qquad \mu^*\colon \mathrm{Hom}(\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X),\mathbb{Z}_p^\wedge) \cong \mathrm{Hom}(\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X),\mathbb{Z}_p^\wedge).$$

In order to use this information, we recall the following concept from [4, VI.3] and [7]. An abelian group A is called Ext-*p-complete* if the natural homomorphism $A \to \mathrm{Ext}(\mathbb{Z}(p^\infty),A)$ derived from the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}(p^\infty) \to 0$$

is an isomorphism. Equivalently, an abelian group A is Ext-p-complete if and only if both $\mathrm{Hom}(\mathbb{Z}[1/p],A) = 0$ and $\mathrm{Ext}(\mathbb{Z}[1/p],A) = 0$. As explained in [4, VI.4], Ext-p-complete abelian groups are uniquely q-divisible for primes $q \neq p$, and they admit a canonical $\mathbb{Z}_p^\wedge$-module structure.

An Ext-p-complete abelian group A is called *adjusted* if the quotient A/TA of A by its torsion subgroup TA is p-divisible (hence divisible). Thus, A is adjusted if and only if A does not admit any torsion-free Ext-p-complete quotients other than zero. Since $TA \otimes \mathbb{Z}(p^\infty) = 0$, it also follows that an Ext-p-complete abelian group A is adjusted if and only if $A \otimes \mathbb{Z}(p^\infty) = 0$.

Theorem 1.4. *Let E be a homology theory and p a prime. Suppose that $E\mathbb{Z}/p$ is nontrivial. Then, for every 1-connected space X, the cokernel of the natural homomorphism $\mu_*\colon \pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ is an adjusted* Ext*-p-complete abelian group, which is zero if $K(\mathbb{Z}/p,2)$ is $E\mathbb{Z}/p$-local.*

Proof. The spaces $\mathbf{L}_{H\mathbb{Z}/p}X$ and $\mathbf{L}_{E\mathbb{Z}/p}X$ are $H\mathbb{Z}/p$-local. The abelian groups $\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X)$ and $\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$ are thus Ext-p-complete, by [1, Theorem 5.5]. Hence, $\operatorname{Coker}\mu_*$ is Ext-p-complete, since the cokernel of any homomorphism between Ext-p-complete abelian groups is Ext-p-complete. If $K(\mathbb{Z}/p,2)$ is $E\mathbb{Z}/p$-local, then we already proved, by means of (1.2), that $\operatorname{Coker}\mu_*$ is zero. Thus, we assume that $K(\mathbb{Z}_p^\wedge,2)$ is $E\mathbb{Z}/p$-local. In this case, the isomorphism displayed in (1.3) shows that $\operatorname{Hom}(\operatorname{Coker}\mu_*,\, \mathbb{Z}_p^\wedge) = 0$. For an abelian group A, if $\operatorname{Hom}(A,\mathbb{Z}_p^\wedge) = 0$ then we have $\operatorname{Hom}(A\otimes \mathbb{Z}(p^\infty),\mathbb{Z}(p^\infty)) = 0$ by adjunction. Since $A\otimes\mathbb{Z}(p^\infty)$ is a p-torsion divisible abelian group, we may infer that $A\otimes\mathbb{Z}(p^\infty) = 0$ and this implies that A/TA is p-divisible, as we needed. (In fact, an Ext-p-complete abelian group A is adjusted if and only if the condition $\operatorname{Hom}(A,\mathbb{Z}_p^\wedge) = 0$ holds. This has also been pointed out in [3, Lemma 7.7].) □

2. Non-torsion homology theories

In this section we deal with non-torsion homology theories. In this case, there is an arithmetic square allowing one to compute E-localizations of 1-connected spaces by combining mod p data and rational data. Specifically, the following diagram is a homotopy pull-back square if X is 1-connected (and also under less restrictive conditions; see [2, Proposition 7.2]). Recall that $\mathcal{P}E$ denotes the set of primes p such that $\pi_*(E)$ is not uniquely p-divisible.

$$\begin{array}{ccc}
\mathbf{L}_E X & \longrightarrow & \prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X \\
\downarrow & & \downarrow \\
\mathbf{L}_{H\mathbb{Q}} X & \longrightarrow & \mathbf{L}_{H\mathbb{Q}}\left(\prod_{p\in\mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X\right).
\end{array}$$

We also need the following remark.

Lemma 2.1. *Suppose given a set of primes P and an adjusted* Ext*-p-complete abelian group A_p for all $p\in P$. The rationalization $\prod_{p\in P} A_p \to \left(\prod_{p\in P} A_p\right)\otimes\mathbb{Q}$ is then an epimorphism.*

Proof. Fix any prime $q \in P$. Then we have $A_q \otimes \mathbb{Z}(q^\infty) = 0$ since A_q is adjusted, and $\left(\prod_{p \neq q} A_p\right) \otimes \mathbb{Z}(q^\infty) = 0$ as well, since $\prod_{p \neq q} A_p$ is uniquely q-divisible. Therefore, $\left(\prod_{p \in P} A_p\right) \otimes \mathbb{Z}(q^\infty) = 0$. This shows that $\left(\prod_{p \in P} A_p\right) \otimes \mathbb{Q}/\mathbb{Z} = 0$, which proves our claim. □

Our main result is the following.

Theorem 2.2. *Let E be any homology theory and let X be 1-connected. Then $\mathbf{L}_E X$ is also 1-connected.*

Proof. By Corollary 1.2, we may assume that E is not torsion. Our strategy is to compare the arithmetic squares for E and ordinary homology $H\mathbb{Z}_{\mathcal{P}E}$. The natural maps $\mu\colon \mathbf{L}_{H\mathbb{Z}/p}X \to \mathbf{L}_{E\mathbb{Z}/p}X$ yield a commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & F' & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
F'' & \longrightarrow & \prod_{p \in \mathcal{P}E} \mathbf{L}_{H\mathbb{Z}/p}X & \xrightarrow{\ \mu\ } & \prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{L}_{H\mathbb{Q}}F'' & \longrightarrow & \mathbf{L}_{H\mathbb{Q}}\left(\prod_{p \in \mathcal{P}E} \mathbf{L}_{H\mathbb{Z}/p}X\right) & \longrightarrow & \mathbf{L}_{H\mathbb{Q}}\left(\prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p}X\right)
\end{array}$$

where each row and each column is a fibre sequence. The four spaces in the lower right square are 1-connected by Corollary 1.2. Therefore, all the fibres except perhaps Y are connected. The group $\pi_1(F'')$ is the product of the cokernels of the homomorphisms $\mu_*\colon \pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \to \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$, so it is a product of adjusted Ext-p-complete groups, by Theorem 1.4. Hence, Lemma 2.1 tells us that the induced homomorphism $\pi_1(F'') \to \pi_1(\mathbf{L}_{H\mathbb{Q}}F'')$ is surjective. This implies that Y is connected as well, so the homomorphism $\pi_1(F') \to \pi_1(F)$ is surjective.

From the arithmetic square for E we see that $\mathbf{L}_E X$ is 1-connected if and only if the boundary homomorphism $\pi_2(\mathbf{L}_{H\mathbb{Q}}X) \to \pi_1(F)$ is surjective. Consider now the fibre sequence $F' \to \mathbf{L}_{H\mathbb{Z}_{\mathcal{P}E}}X \to \mathbf{L}_{H\mathbb{Q}}X$ appearing in the arithmetic square for $H\mathbb{Z}_{\mathcal{P}E}$. Since we know that $\mathbf{L}_{H\mathbb{Z}_{\mathcal{P}E}}X$ is 1-connected, the homomorphism $\pi_2(\mathbf{L}_{H\mathbb{Q}}X) \to \pi_1(F')$ is surjective. The composite $\pi_2(\mathbf{L}_{H\mathbb{Q}}X) \to \pi_1(F') \to \pi_1(F)$ is thus also surjective, as we needed. □

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