On the chaos expansion of some additive functionals of the Brownian motion and applications

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Abstract

This paper deals with some additive functionals based on the Brownian local time. In concrete, we obtain the chaos expansion of the Hilbert transform and the fractional derivative of the Brownian local time. These functionals appear in stochastic calculus and in limit theorems for occupation times of the Wiener process. As an application, we apply those results to deduce some regularity properties of these functionals in Watanabe–Sobolev spaces.

Keywords Brownian motion, Local time, Chaos expansion, Hilbert transform, Fractional derivative, Additives functionals, Watanabe–Sobolev spaces.

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1 Introduction

In the present paper, we are interested in the chaos expansion and the regularity in some Sobolev spaces of a class of continuous Brownian additive functionals of zero energy, in the sense of Fukushima (1979). This contains important as well as interesting examples such as Hilbert transform and fractional derivatives of the Brownian local time. Concerning these additive functionals, their investigations are being pursued from various points of views by many authors. The existence of the principal values of Brownian local time, which is related with its Hilbert transform, has firstly been remarked by Itô and Mckean (1965). Fractional derivatives of local time have been discussed for physical purposes in the paper by Ezawa *et al.* (1975). The local time of the Brownian motion is a particular additive Brownian functional that has been studied by Nualart and Vives (1992a, 1992b and 1994), in the sense of the stochastic calculus of variations. More precisely the authors established a decomposition in Wiener chaos of the local time at zero, and used this decomposition to obtain some regularity property of this process in some Sobolev spaces.

In this article, we would like to see how the ideas developed by Nualart and Vives (1992a, 1992b and 1994), and Imkeller *et al.* (1995) combined with those of Yamada (1984, 1985 and 1986) can be exploited to study the chaos expansion of some additive functionals of the Brownian motion like the Cauchy's principal value and fractional derivative of its local time. Concrete and systematic studies of principal values of local times have been started with Yor (1982), Biane and Yor (1987) and Yamada (1984, 1985). In the early of 80's Yamada showed that principal values of local times can be represented as Hilbert transform or fractional derivatives of local times. Thanks to these analytic tool, we see naturally that some properties such as the Hölder continuity inherit from Brownian local time to their principal values. Fractional derivative of the Brownian local time and related functionals have arisen naturally in some new limit theorems for occupation times of a linear Brownian motion (see e.g. Yamada (1986)). Similar limit theorems in the context of stable processes have been obtained recently by Fitzsimmons and Getoor (1992). Related topics have been discussed also by Pitman and Yor (1989). Since principal values of local times do not belong to the class of semi–martingales, stochastic integrals based on them have to be defined in a different way from those based on semi–martingales. In this direction, Nakao (1985) and Bertoin (1989) contributed new ideas on stochastic integrals and chain rules for functionals of zero energy. These stochastic integrals are closely related with Young–Stieltjes one (see Young, 1936). In connection with their definition of stochastic integrals, p–variation properties of principal values of the local times have been investigated by Bertoin (1990).

Let us note that the chaos expansion of this type on the multi–parameter Wiener process has been discussed by Imkeller and Weisz (1994) but for the sake of completeness in the present paper we present details for the proof of the decomposition of the local time of the real valued Brownian motion at any point x of the real line.

We close this introduction with a few notation and examples of additive functionals. Considering that the class of additive functionals of zero energy forms an important theme in Brownian motion theory it seems to be worth discussing on the application of the Malliavin calculus to obtain the chaos expansion of the above particular additive functionals which can be expressed in explicit formulas.

Let $\{B_t : t \in [0,T]\}$ be a 1–dimensional Brownian motion defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, with $(\mathcal{F}_t)_{0\le t\le T}$ the, completed, standard Brownian filtration. For all $0 \le t \le T$, we define the random measure $\mu_t(\cdot)$ by $\mu_t(A) = \int_0^t \mathbb{1}_A(B_s)ds$, where $A \subset \mathbb{R}$ is a Borel set of $\mathbb{R}$ and $\mathbb{1}_A(\cdot)$ is its characteristic function. The measure $\mu_t(A)$ is called the occupation measure of B on the Borel set A. It is well known by Boylan (1964), Blumental and Getoor (1968) and Barlow (1988) that the measure $\mu_t(A)$ has a density, denoted by $L(t,x)$, with respect to the Lebesgue measure and $\{L(t,x) : t \ge 0, x \in \mathbb{R}\}$ is called the family of local times associated with B. Moreover, $L(t,x)$ has a version which is almost surely Hölder continuous of order β in each variable as well as uniformly continuous on any compact set, where $0 \le \beta < 1/2$. On the other $L(t,x)$ satisfies the following occupation density formula:

$$\int_0^t f(B_s)ds \quad = \quad \int_{\mathbb{R}} f(x)L(t,x)dx,$$

for all bounded Borel function f, and the scaling identity:

$$\{L(\lambda t, \lambda^{\frac{1}{2}}x) : t \ge 0\} \quad \stackrel{\mathcal{L}}{=} \quad \{\lambda^{\frac{1}{2}}L(t,x) : t \ge 0\},$$

for all $\lambda > 0$.

The process $L(t,x)$ can be expressed, using Tanaka's formula, as follows

$$L(t,x) \;=\; |B_t - x| - |B_0 - x| - \int_0^t \operatorname{sgn}(B_s - x)dB_s$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases}$$

One can also write

$$L(t,x) = \int_0^t \delta_x(B_s)ds.$$

Let C_t^x be the continuous additive functional of B_t which corresponds to Cauchy's principal value $v.p.(\frac{1}{x})$. More precisely C_t^x is defined, using the generalized Itô formula, as

$$C_t^x = 2\left(F_x(B_t) - F_x(B_0) - \int_0^t F_x'(B_s)dB_s\right),$$

where

$$F_x(y) = (y-x)\log|y-x| - (y-x).$$

Here and in the sequel $\int_0^t F_x'(B_s)dB_s$ is understood as a stochastic Itô integral.

Next, we will define the additive functional which corresponds to the Hadamard finite part $p.f.(x_+^{-1-\alpha})$ for $0 < \alpha < \frac{1}{2}$, by

$$H_t^x(-1-\alpha) = 2\left(G_x(B_t) - G_x(B_0) - \int_0^t G_x'(B_s)dB_s\right),$$

where $G_x(\cdot)$ is a function given by

$$G_x(y) = \frac{(y-x)_+^{1-\alpha}}{(-\alpha)(1-\alpha)} = \begin{cases} 0 & \text{for} \quad y < x, \\ \frac{(y-x)^{1-\alpha}}{(-\alpha)(1-\alpha)} & \text{for} \quad y \geq x. \end{cases}$$

The processes $L(t,x)$, C_t^x and $H_t^x(-1-\alpha)$ are additive functionals, of zero energy, associated respectively with the distributions δ_x, $v.p.(\frac{1}{x})$ and $p.f.(x_+^{-1-\alpha})$.

Our aim in this paper is to give a chaos expansion for the Hilbert transform of the local time and for its fractional derivative by establishing a decomposition in Wiener chaos of the local time at any x. More explicitly, we want to prove how the functionals $L(t,x)$, C_t^x and $H_t^x(-1-\alpha)$ are regular as functionals of the Brownian motion, and more, which are their chaos development. As an application we deduce some regularity properties for these functionals. Moreover this chaos expansion provides an approximation of these functionals in the uniform and the Hölder norm with respect to the space variable.

This paper is organized as follows: In the next section, we are interested in the chaos expansion of the local time at any point x in $\mathbb{R}$. We give an extension of the result presented by Nualart and Vives (1992b). In section 3, we will use the result of section 2 to give the chaos expansion and to deduce the regularity of C_t^x as a functional of the Wiener process. The Section 4, deals with the chaos expansion and the regularity of the fractional derivative of the local time. Other related functionals are presented in section 5.

2 Chaos expansion of the Brownian local time in x

In previous works Nualart and Vives (1992b, 1994) have obtained the chaos expansion of $L(t,0)$. On the other hand, using other methods, they have proved in Nualart and Vives (1992a) that $L(t,x)$ belongs to $\mathbb{D}^{\alpha,2}$, for all $\alpha < 1/2$ and $L(t,x)$ is not in $\mathbb{D}^{\frac{1}{2},2}$. Remark that $\{\mathbb{D}^{\alpha,2},\, \alpha \in \mathbb{R}\}$ are the Watanabe–Sobolev spaces (see e.g. Watanabe 1984). For $\alpha < 0$ they are big spaces of distributions on the Wiener space. For $\alpha = 0$ we have $L^2(\Omega)$. And for $\alpha > 0$, they are more and more little spaces of smooth functionals. Remember also that $\mathbb{D}^{1,2}$ is the domain of the Malliavin's derivative.

Finally, we recall

$$F \in \mathbb{D}^{\alpha,2} \quad \Longleftrightarrow \quad \sum_{n=0}^{\infty} (1+n)^{\alpha} \left\| I_n\left(f_n\right)\right\|_2^2 \quad < \quad \infty,$$

where $\sum_{n=0}^{\infty} I_n\left(f_n\right)$ is the chaos expansion of the random variable F, (see, e.g. Nualart (1995) for a complete survey).

Let $p_\varepsilon(\cdot)$ be the centered Gaussian kernel with variance ε. It is not difficult to prove that $\int_0^t p_\varepsilon(B_s - x)ds$ converges to $L(t,x)$ as ε tends to 0 in $L^p(\Omega)$ for all $p \geq 1$.

For each $n \geq 0$, we denote by H_n, the n–*th* Hermite polynomial defined by

$$H_n(x) \quad = \quad \frac{(-1)^n}{\sqrt{n!}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right), \ \ x \in \mathbb{R}.$$

The result of this section, which is the main tool in this paper, is the following

Proposition 2.1. *The chaos expansion of the local time at any x is given by the following formula:*

$$L(t,x) \ = \ \sum_{n=0}^{\infty} I_n\left(f_n(\cdot;t,x)\right),$$

where

$$f_n(t_1,\dots,t_n;t,x) \;=\; \frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t} \frac{e^{-\frac{x^2}{2s}}}{s^{\frac{n+1}{2}}} H_n\left(\frac{x}{\sqrt{s}}\right) ds.$$

Consequently, $L(t,x)$ *belongs to* $\mathbb{D}^{\alpha,2}$ *for all* $0<\alpha<1/2$.

Remark 2.1. *This proposition yields an alternative proof of the result obtained by Nualart and Vives (1992a).*

Remark 2.2. *The Brownian local time coincides almost–surely, with its Wiener chaos expansion, this means that the series* $\sum_{n=0}^{\infty} I_n(f_n(\cdot;t,x))$ *converges to* $L(t,x)$ *a.s.*
This result is a consequence of the Rademacher–Menchov lemma (cf. Stout (1984)):
Let $\{J_n : n \geq 1\}$ *be an orthogonal sequence of random variables such that* $\sum_{n=0}^{\infty} \log(n)^2 \mathbb{E}[J_n]^2$ *is finite, then the series* $\sum_{n=0}^{\infty} J_n$ *converges almost surely.*

Following Nualart and Vives (1992b, 1994), we can apply the following lemma to obtain the chaos expansion of $L(t,x)$.

Lemma 2.1. *Let* $\{F_\varepsilon,\ \varepsilon>0\}$ *be a family of random variables of* $L^2(\Omega)$, *with chaos expansion*

$$F_\varepsilon \;=\; \sum_{n=0}^{\infty} I_n\left(f_n^\varepsilon\right), \quad f_n^\varepsilon \in L_s^2([0,\infty)^n),$$

where $L_s^2([0,\infty)^n)$ *stands for the* $L^2([0,\infty)^n)$ *symmetric kernels.*
Assume that :

i) f_n^ε *converges to* $f_n \in L_s^2([0,\infty)^n)$ *as* ε *tends to* 0 *in* $L^2([0,\infty)^n)$.

ii) $\sum_{n=0}^{\infty} \sup_{\varepsilon>0} n! ||f_n^\varepsilon||_2^2 < \infty$.

Then F_ε *converges to* $F := \sum_{n=0}^{\infty} I_n\left(f_n\right)$ *as* ε *tends to* 0 *in* $L^2(\Omega)$.

Proof of Proposition 2.1: We define

$$F_\varepsilon \;=\; L_\varepsilon(t,x) \;:=\; \int_0^t p_\varepsilon(B_s-x)ds.$$

Using the Stroock formula and the properties of the orthogonal Hermite polynomials it is easy to compute (see Nualart and Vives (1992b, 1994)) that

$$p_\varepsilon(B_s-x) \;=\; \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi n!}} \frac{e^{-\frac{x^2}{2(s+\varepsilon)}}}{(s+\varepsilon)^{\frac{n+1}{2}}} H_n\left(\frac{x}{\sqrt{s+\varepsilon}}\right) I_n\left(\mathbb{1}_{[0,s]}^{\otimes n}\right),$$

and

$$L_\varepsilon(t,x) \;=\; \sum_{n=0}^{\infty} I_n\left(f_n^\varepsilon(\cdot;t,x)\right).$$

where

$$f_n^\varepsilon(t_1,\ldots,t_n;t,x) \;=\; \frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t} \frac{e^{-\frac{x^2}{2(s+\varepsilon)}}}{(s+\varepsilon)^{\frac{n+1}{2}}} H_n\left(\frac{x}{\sqrt{s+\varepsilon}}\right) ds.$$

It is easy to prove that f_n^ε converges pointwise to f_n given by

$$f_n(t_1,\ldots,t_n;t,x) \;=\; \frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t} \frac{e^{-\frac{x^2}{2s}}}{s^{\frac{n+1}{2}}} H_n\left(\frac{x}{\sqrt{s}}\right) ds. \tag{2.1}$$

Observe that

$$|f_n^\varepsilon(t_1,\ldots,t_n;t,x)| \;\le\; \frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t} \frac{e^{-\frac{x^2}{2(s+\varepsilon)}}}{(s+\varepsilon)^{\frac{n+1}{2}}} \left|H_n\left(\frac{x}{\sqrt{s+\varepsilon}}\right)\right| ds.$$

But, we know from Szegö (1939) that

$$H_n(y)e^{-\frac{y^2}{2}} \;=\; \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}+1}}{\sqrt{\pi n!}}\int_0^\infty s^n e^{-s^2} g\left(ys\sqrt{2}\right) ds,\; y\in\mathbb{R} \tag{2.2}$$

where $g(r)=\cos(r)$ if $n\in 2\mathbb{N}$ and $g(r)=\sin(r)$ if $n\notin 2\mathbb{N}$.

Remark that the formula (2.2) is not necessary to compute the chaos expansion of $L(t,0)$, and is the key here.

It is clear from (2.2) that

$$|H_n(y)|\, e^{-\frac{y^2}{2}} \;\le\; \frac{2^{\frac{n}{2}+1}}{\sqrt{\pi n!}}\int_0^\infty s^n e^{-s^2} ds \;=\; \frac{2^{\frac{n}{2}}}{\sqrt{\pi n!}}\Gamma\left(\frac{n+1}{2}\right)$$

and therefore

$$\begin{aligned} |f_n^\varepsilon(t_1,\ldots,t_n;t,x)| \quad &\le \quad \frac{2^{\frac{n-1}{2}}}{\pi n!}\Gamma\left(\frac{n+1}{2}\right)\int_{t_1\vee\cdots\vee t_n}^{t} s^{-\frac{n+1}{2}} ds \\ &=: \quad g_n(t_1,\ldots,t_n;t). \end{aligned}$$

In order to apply the Lemma 2.1 we have to show

$$\sum_{n=0}^{\infty} n!\,\|g_n(t_1,\ldots,t_n;t)\|_2^2 \;<\; \infty.$$

But we have

$$\begin{aligned}
&n!\,\|g_n(t_1,\dots,t_n;t)\|_2^2 \\
&= \frac{2^{n-1}}{\pi^2}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 \int_0^t \int_0^{t_n}\cdots\int_0^{t_2}\left(\int_{t_1\vee\cdots\vee t_n}^t s^{-\frac{n+1}{2}}ds\right)^2 dt_1\cdots dt_n \\
&= \frac{2^{n-1}}{\pi^2}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 \int_0^t \frac{y^{n-1}}{(n-1)!}\left(\int_y^t s^{-\frac{n+1}{2}}ds\right)^2 dy.
\end{aligned}$$

On the other hand, we have

$$\left(\int_y^t s^{-\frac{n+1}{2}}ds\right)^2 = \int_0^t \int_0^t u^{-\frac{n+1}{2}} v^{-\frac{n+1}{2}} 1\!\!1_{[0,u]}(y) 1\!\!1_{[0,v]}(y)dudv,$$

hence

$$\begin{aligned}
&\int_0^t \frac{y^{n-1}}{(n-1)!}\left(\int_y^t s^{-\frac{n+1}{2}}ds\right)^2 dy \\
&\qquad = \int_0^t \int_0^t (uv)^{-\frac{n+1}{2}}\left(\int_0^t \frac{y^{n-1}1\!\!1_{[0,u\wedge v]}(y)}{(n-1)!}dy\right)dudv \\
&\qquad = \int_0^t \int_0^t (uv)^{-\frac{n+1}{2}}\frac{(u\wedge v)^n}{n!}dudv.
\end{aligned}$$

Then

$$\begin{aligned}
&n!\,\|g_n(t_1,\dots,t_n;t)\|_2^2 \\
&\qquad = \frac{2^{n-1}}{\pi^2}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 \int_0^t \int_0^t (uv)^{-\frac{n+1}{2}}\frac{(u\wedge v)^n}{n!}dudv \\
&\qquad = \frac{2^{n}}{\pi^2 n!}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 \int_0^t \int_0^v (uv)^{-\frac{n+1}{2}}u^n dudv \\
&\qquad = \frac{2^{n+1}}{\pi^2 n!}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 \int_0^t v^{-\frac{n+1}{2}}\frac{v^{\frac{n+1}{2}}}{n+1}dv \\
&\qquad = \frac{2^{n+1}}{\pi^2 (n+1)!}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 t \sim C\frac{t}{n^{\frac{3}{2}}},
\end{aligned}$$

and the series $\sum_{n=0}^{\infty} n^{-\frac{3}{2}}$ is convergent, hence the proof is complete. □

3 The chaos expansion of the principal value of the local time

Let the function g belongs to $L^2(\mathbb{R})$, we consider the Hilbert transform of the function g defined by

$$\mathcal{H}(g)(x) \ := \ \frac{1}{\pi}\left(v.p.\left(\frac{1}{x}\right) * g\right)(x)$$

It is well known that $\mathcal{H}(g)(\cdot)$ also belongs $L^2(\mathbb{R})$, moreover $\mathcal{H}$ is a very known operator that acts from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$, which can be rewritten as follows

$$\mathcal{H}(f)(x) \ = \ \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{f(y)}{y-x}\mathbb{1}_{\{|y-x|\geq\varepsilon\}}dy$$

This limit exists x–a.e. and the operator is linear, bounded on $L^p(\mathbb{R})$, hence, $\mathcal{H}$ is a closed operator. Moreover $\|\mathcal{H}(f)\|_2 = \|f\|_2$, $\mathcal{H}^{-1} = -\mathcal{H}$ and $\mathcal{H}^2 = -Id$. This means that the operator $\mathcal{H}$ is an isometry on $L^2(\mathbb{R})$.

If f is Hölder–continuous and with compact support, the limit above is defined for all $x \in \mathbb{R}$. This is the case for $L(t,x)$, the local time of the standard Brownian motion as a function of the space variable. It is known from Yamada's result (1984) that we can write

$$\mathcal{H}(L(t,\cdot))(x) \ = \ \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_0^t \frac{1}{B_s - x}\mathbb{1}_{\{|B_s-x|\geq\varepsilon\}}ds.$$

We can write also

$$\mathcal{H}(L(t,\cdot))(x) \ = \ \int_0^{+\infty}\frac{L(t,x+y)-L(t,x-y)}{\pi y}dy.$$

Hence the functional C_t^x can be represented via Hilbert transform of the Brownian local time as

$$C_t^x \quad = \quad \pi\mathcal{H}(L(t,\cdot))(x)$$

Using that $L(t,x)$ belongs to $L^2(\Omega)$ and has a chaos expansion we shall obtain the chaos expansion of C_t^x.

Proposition 3.1. *The additive functional C_t^x has the following decomposition*

$$C_t^x \quad = \quad \sum_{n=0}^{\infty} I_n\left(\pi\mathcal{H}\left(f_n\left(\cdot;t,\cdot\right)\right)(x)\right) =$$

$$= \sum_{n=0}^{\infty} I_n \left(\int_0^{\infty} \frac{1}{y} \left(f_n \left(\cdot ; t, x+y \right) - f_n \left(\cdot ; t, x-y \right) \right) dy \right),$$

where

$$f_n \left(t_1, \ldots, t_n ; t, x \right) = \frac{1}{\sqrt{2\pi n!}} \int_{t_1 \vee \cdots \vee t_n}^{t} \frac{e^{-\frac{x^2}{2s}}}{s^{\frac{n+1}{2}}} H_n \left(\frac{x}{\sqrt{s}} \right) ds.$$

For proving this Proposition we will use the following two lemmas.

Lemma 3.1. *The functional* $\widetilde{C}_t^x$ *defined as*

$$\widetilde{C}_t^x := \sum_{n=0}^{\infty} I_n \left(\pi \mathcal{H} \left(f_n \left(\cdot ; t, \cdot \right) \right) (x) \right)$$

belongs to $\mathbb{D}^{\alpha,2}$, *for all* $\alpha < \frac{1}{2}$, *uniformly on* x.

Proof:

Recall that

$$\begin{aligned}
&\mathcal{H} \left(f_n \left(t_1, \ldots, t_n ; t, \cdot \right) \right) (x) \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{1}{y} \left[f_n \left(t_1, \ldots, t_n ; t, x+y \right) - f_n \left(t_1, \ldots, t_n ; t, x-y \right) \right] dy \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{1}{y} \left\{ \frac{1}{\sqrt{2\pi n!}} \int_{t_1 \vee \cdots \vee t_n}^{t} \frac{1}{s^{\frac{n+1}{2}}} \right. \\
&\qquad \times \left. \left[e^{-\frac{(x+y)^2}{2s}} H_n \left(\frac{x+y}{\sqrt{s}} \right) - e^{-\frac{(x-y)^2}{2}} H_n \left(\frac{x-y}{\sqrt{s}} \right) \right] ds \right\} dy \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{1}{y} \left\{ \frac{1}{\sqrt{2\pi n!}} \int_{t_1 \vee \cdots \vee t_n}^{t} \frac{(-1)^{[\frac{n}{2}]} 2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}} \frac{2}{\sqrt{\pi n!}} \right. \\
&\qquad \times \left. \left[\int_0^{\infty} r^n e^{-r^2} \left\{ g \left(r\sqrt{2} \frac{x+y}{\sqrt{s}} \right) - g \left(r\sqrt{2} \frac{x-y}{\sqrt{s}} \right) \right\} dr \right] ds \right\} dy.
\end{aligned}$$

Now, if $n \in 2\mathbb{N}$, $g(r) = \cos(r)$ and then

$$\cos \left(r\sqrt{2} \frac{x+y}{\sqrt{s}} \right) - \cos \left(r\sqrt{2} \frac{x-y}{\sqrt{s}} \right) = -2 \sin \left(\frac{x\sqrt{2}}{\sqrt{s}} r \right) \sin \left(\frac{y\sqrt{2}}{\sqrt{s}} r \right).$$

If $n \notin 2\mathbb{N}$, $g(r) = \sin(r)$ and then

$$\sin \left(r\sqrt{2} \frac{x+y}{\sqrt{s}} \right) - \sin \left(r\sqrt{2} \frac{x-y}{\sqrt{s}} \right) = 2 \cos \left(\frac{x\sqrt{2}}{\sqrt{s}} r \right) \sin \left(\frac{y\sqrt{2}}{\sqrt{s}} r \right).$$

Summarizing

$$g\left(r\sqrt{2}\frac{x+y}{\sqrt{s}}\right)-g\left(r\sqrt{2}\frac{x-y}{\sqrt{s}}\right) \;=\; 2h\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)$$

where $h(r)=-\sin(r)$ if $n\in 2\mathbb{N}$ and $h(r)=\cos(r)$ if $n\notin 2\mathbb{N}$.
Then

$$\begin{aligned}
&\mathcal{H}\left(f_n\left(t_1,\ldots,t_n;t,\cdot\right)\right)(x)= \\
&\quad=\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n+1}{2}}}{\pi^2 n!}\int_0^\infty\frac{1}{y}\int_{t_1\vee\cdots\vee t_n}^t\frac{1}{s^{\frac{n+1}{2}}} \\
&\qquad\times\int_0^\infty 2r^n e^{-r^2}h\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)drdsdy.
\end{aligned}$$

Now recall that

$$\int_0^\infty\frac{1}{y}\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)dy \;=\; \frac{\pi}{2}.$$

Therefore

$$\begin{aligned}
&\mathcal{H}\left(f_n\left(t_1,\ldots,t_n;t,\cdot\right)\right)(x) \\
&=\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n+1}{2}}}{\pi n!}\int_{t_1\vee\cdots\vee t_n}^t\frac{1}{s^{\frac{n+1}{2}}}\int_0^\infty r^n e^{-r^2}h\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)drds.
\end{aligned}$$

But

$$\begin{aligned}
\left|\mathcal{H}\left(f_n\left(t_1,\ldots,t_n;t,\cdot\right)\right)(x)\right| &\le \frac{2^{\frac{n+1}{2}}}{\pi n!}\int_{t_1\vee\cdots\vee t_n}^t\frac{1}{s^{\frac{n+1}{2}}}\int_0^\infty r^n e^{-r^2}drds \\
&\le \frac{2^{\frac{n-1}{2}}}{\pi n!}\Gamma\left(\frac{n+1}{2}\right)\int_{t_1\vee\cdots\vee t_n}^t\frac{1}{s^{\frac{n+1}{2}}}ds \\
&=: g_n\left(t_1,\ldots,t_n;t\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^\infty n!\left\|\mathcal{H}\left(f_n\left(t_1,\ldots,t_n;t,\cdot\right)\right)(x)\right\|_2^2\le \\
&\qquad\qquad\le\sum_{n=0}^\infty n!\left\|g_n\left(t_1,\ldots,t_n;t\right)\right\|_2^2<\infty,
\end{aligned}$$

being the last line a consequence of the fact that

$$n!\,\|g_n\,(t_1,\ldots,t_n;t)\|_2^2 \quad \sim \quad C\cdot n^{-3/2}.$$

□

Lemma 3.2. *The operator $\mathcal{H}$ defined on the space $L^2(\mathbb{R},L^2(\Omega))$ is an isometry.*

Proof: Let F be an element of $L^2(\mathbb{R},L^2(\Omega))$ such that for all $x\in\mathbb{R}$ the random variable $F(x)$ belongs to $L^2(\Omega)$.
Since the Hilbert transform is a closed linear operator, in fact an isometry on $L^2(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}}\|\mathcal{H}(F)(x)\|^2_{L^2(\Omega)}\,dx &= \int_{\mathbb{R}}\mathbb{E}\,[\mathcal{H}(F)(x)]^2\,dx\\ &= \mathbb{E}\int_{\mathbb{R}}[\mathcal{H}(F)(x)]^2\,dx\\ &= \mathbb{E}\int_{\mathbb{R}}[F(x)]^2\,dx \;=\; \|F\|^2_{L^2(\mathbb{R},L^2(\Omega))}\,.\end{aligned}$$

Lemma 3.3. *Let us set $L^m(t,x)=\sum_{n=0}^m I_n(f_n(\cdot;t,x))$.
Then $\|L^m(t,\cdot)-L(t,\cdot)\|_{L^2(\mathbb{R},L^2(\Omega))}$ converges to zero as m tends to infinity*

Proof: Let us fix the time t and set

$$B_n(x) \quad := \quad \int_0^t\int_0^t\frac{(s\wedge t)^n}{(st)^{\frac{n+1}{2}}}\exp\left(-\frac{x}{s}\right)\exp\left(-\frac{x}{t}\right)dsdt$$

It has been proved in Imkeller and Weisz (1994) that the local time of the Brownian motion can be rewritten in the following form

$$L(t,x) \;=\; \sum_{n=0}^{\infty}\int_0^t\frac{1}{\sqrt{n!}}I_n\left(\left(\frac{\mathbb{1}_{[0,s]}}{\sqrt{s}}\right)^{\otimes n}\right)H_n\left(\frac{x}{\sqrt{s}}\right)p_s(x)ds \qquad (3.3)$$

which is a technical and useful form.
We have

$$\begin{aligned}&\|L^m(t,\cdot)-L(t,\cdot)\|^2_{L^2(\mathbb{R},L^2(\Omega))} =\\ &\quad=\int_{\mathbb{R}}\sum_{n=m+1}^{\infty}\int_0^t\int_0^t\frac{(u\wedge v)^n}{(uv)^{\frac{n+1}{2}}}H_n\left(\frac{x}{\sqrt{u}}\right)H_n\left(\frac{x}{\sqrt{v}}\right)p_u(x)p_v(x)dudvdx\end{aligned}$$

$$\leq \sum_{n=m+1}^{\infty} \frac{C}{n^{\frac{8\mu-1}{6}}} \int_{\mathbb{R}} B_n\left(\left(\frac{1}{2}-\mu\right)x^2\right) dx \quad \text{for any} \quad 1/8 < \mu < 1/2.$$

But

$$\begin{aligned} \int_{\mathbb{R}} B_n\left(\left(\frac{1}{2}-\beta\right)x^2\right) dx \quad &= \\ = \quad & \int_0^t \int_0^t \frac{(u\wedge v)^n}{(uv)^{\frac{n+1}{2}}} \int_{\mathbb{R}} \exp\left(-\left(\frac{1}{2}-\mu\right)^2 x^2 \left(\frac{1}{u}+\frac{1}{v}\right)\right) dxdudv \\ \leq \quad & \frac{C}{n}. \end{aligned}$$

Therefore the series $\sum_{n=0}^{\infty} \int_0^t \frac{1}{\sqrt{n!}} I_n((\frac{\mathbb{1}_{[0,s]}}{\sqrt{s}})^{\otimes n}) H_n(\frac{x}{\sqrt{s}}) p_s(x) ds$ converges in the space $L^2(\mathbb{R}, L^2(\Omega))$, hence $\|L^m(t,\cdot) - L(t,\cdot)\|^2_{L^2(\mathbb{R},L^2(\Omega))}$ converges to zero as m tends to infinity.

Proof of Proposition 3.1:

It is an immediate consequence of Lemmas 3.1, 3.2 and 3.3. □

Remark 3.1. *In fact the kernels of* $L(t,x)$ *are the same that the kernels of* C_t^x *changing* $\cos(r)$ *by* $-\sin(r)$ *if* $n \in 2\mathbb{N}$ *and* $\sin(r)$ *by* $\cos(r)$ *if* $n \notin 2\mathbb{N}$. *This is* $h = g'$.

Corollary 3.1. C_t^x *does not belong to* $L^2(\mathbb{R}; \mathbb{D}^{\frac{1}{2},2})$.

Proof: We have from the proof of Proposition 3.1

$$\begin{aligned} &\|C_t^x\|^2_{L^2(\mathbb{R};\mathbb{D}^{\frac{1}{2},2})} = \\ &= \sum_{n=0}^{\infty} \pi^2 (1+n)^{\frac{1}{2}} \int_{\mathbb{R}} \|I_n(\mathcal{H}(f_n(\cdot;t,\cdot))(x))\|^2_{L^2(\Omega)} dx \\ &= \sum_{n=0}^{\infty} n!\,(1+n)^{\frac{1}{2}} \int_{\mathbb{R}} \left(\int_0^t \cdots \int_0^t [\mathcal{H}(f_n(t_1,\ldots,t_n;t,\cdot))(x)]^2 dt_1\cdots dt_n\right) dx \\ &= \sum_{n=0}^{\infty} n!\,(1+n)^{\frac{1}{2}} \int_{\mathbb{R}} \left(\int_0^t \cdots \int_0^t [f_n(t_1,\ldots,t_n;t,x)]^2 dt_1\cdots dt_n\right) dx, \end{aligned}$$

where we have used Fubini theorem and the isometry property of the operator $\mathcal{H}$.

Now, since, the right hand side diverges, so is the left hand side term. □

4 Chaos expansion of the fractional derivative of the Brownian local time

Let us recall the definition of the fractional derivative of a real function.

Definition 4.1. *Let α be such that $0 < \alpha < 1$ and $f : \mathbb{R} \to \mathbb{R}$ a bounded locally Hölder function of order λ, $\lambda > \alpha$. The fractional derivative (of Marchaud) of order α of f is defined by*

$$D^{\alpha} f(x) \ = \ \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(x) - f(x-y)}{y^{1+\alpha}} dy. \tag{4.4}$$

For more details on the fractional derivative we refer the reader to Hardy and Littlewood (1928) and Samko *et al.* (1993).

Since the local time of the Brownian motion is a β–Hölder continuous function with $\beta < 1/2$, in the space variable, its fractional derivative is given by

$$D^{\alpha} \left(L(t,\cdot) \right)(x) \ = \ \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{L(t,x) - L(t,x-y)}{y^{1+\alpha}} dy.$$

With the same notation as in the previous section, we have

$$D^{\alpha} \left(L(t,\cdot) \right)(x) \ = \ \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \sum_{n \geq 0} \frac{I_n \left(f_n(\cdot;t,x) - f_n(\cdot;t,x-y) \right)}{y^{1+\alpha}} dy.$$

The main result of this section is the following

Proposition 4.1. *$D^{\alpha} \left(L(t,\cdot) \right)(x)$ has the following chaos expansion*

$$D^{\alpha} \left(L(t,\cdot) \right)(x) \ = \ \sum_{n=0}^{\infty} I_n \left(D^{\alpha} \left(f_n(\cdot;t,\cdot) \right)(x) \right),$$

for all $\alpha \in]0, \frac{1}{2}[$. Moreover $D^{\alpha} \left(L(t,\cdot) \right)(x)$ belongs to $\mathbb{D}^{\beta,2}$, for all $\beta \in]0, \frac{1}{2} - \alpha[$.

In order to prove the Proposition 4.1, we establish the following lemmas.

Lemma 4.1. *The series*

$$\sum_{n \geq 0} I_n \left(D^{\alpha} \left(f_n(\cdot;t,\cdot) \right)(x) \right),$$

where

$$D^{\alpha} \left(f_n(t_1,\ldots,t_n;t,\cdot) \right)(x) =$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1}{y^{1+\alpha}} \left[f_n\left(t_1, \ldots, t_n; t, x\right) - f_n\left(t_1, \ldots, t_n; t, x-y\right) \right] dy$$

is convergent in $L^2(\Omega)$*. Moreover* $\sum_{n \geq 0} I_n\left(D^\alpha\left(f_n(\cdot; t, \cdot)\right)(x)\right)$ *belongs to* $\mathbb{D}^{\beta,2}$*, for all* $\beta \in \left]0, \frac{1}{2} - \alpha\right[$

Proof:

We consider

$$\tilde{D}^\alpha\left(L(t, \cdot)\right)(x) \;:=\; \sum_{n \geq 0} I_n\left(D^\alpha\left(f_n(\cdot; t, \cdot)\right)(x)\right).$$

Thanks to the expression of the kernels f_n we have

$$D^\alpha\left(f_n(t_1, \ldots, t_n; t, \cdot)\right)(x) = $$
$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1}{y^{1+\alpha}} \left\{ \frac{1}{\sqrt{2\pi n!}} \int_{t_1 \vee \cdots \vee t_n}^t \frac{(-1)^{[\frac{n}{2}]} 2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}} \frac{2}{\sqrt{\pi n!}} \right.$$
$$\left. \left[\int_0^\infty r^n e^{-r^2} \left\{ g\left(r\sqrt{2}\frac{x}{\sqrt{s}}\right) - g\left(r\sqrt{2}\frac{x-y}{\sqrt{s}}\right) \right\} dr \right] ds \right\} dy.$$

Now, if $n \in 2\mathbb{N}$, $g(r) = \cos(r)$ and then

$$\cos\left(r\sqrt{2}\frac{x-y}{\sqrt{s}}\right) =$$
$$= \cos\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\cos\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right) + \sin\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right).$$

Then

$$D^\alpha\left(f_n(t_1, \ldots, t_n; t, \cdot)\right)(x) =$$
$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1}{y^{1+\alpha}} \left\{ \frac{1}{\sqrt{2\pi n!}} \int_{t_1 \vee \cdots \vee t_n}^t \frac{(-1)^{[\frac{n}{2}]} 2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}} \frac{2}{\sqrt{\pi n!}} \right.$$
$$\int_0^\infty r^n e^{-r^2} \left[\cos\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right) \left(1 - \cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)\right) \right.$$
$$\left. \left. - \sin\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right) \sin\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right) \right] dr \right\} ds\, dy.$$

Now, by Fubini theorem we get

$$
\begin{aligned}
&D^{\alpha}\left(f_n(t_1,\dots,t_n;t,\cdot)\right)(x) = \\
&= \frac{\alpha}{\Gamma(1-\alpha)}\left\{\frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}}\frac{2}{\sqrt{\pi n!}}\right.\\
&\int_0^\infty r^n e^{-r^2}\cos\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right)\left(\int_0^\infty\frac{1}{y^{1+\alpha}}\left(1-\cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)\right)dy\right)dr\\
&\left.-\int_0^\infty r^n e^{-r^2}\sin\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right)\left(\int_0^\infty\frac{1}{y^{1+\alpha}}\sin\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)dy\right)dr\right\}ds.
\end{aligned}
$$

If $n\notin 2\mathbb{N}$, $g(r)=\sin(r)$ then

$$
\begin{aligned}
\sin\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)-\sin\left(r\sqrt{2}\frac{x-y}{\sqrt{s}}\right) &= \sin\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\left(1-\cos\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)\right)\\
&+ \cos\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right).
\end{aligned}
$$

Summarizing

$$
\begin{aligned}
g\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)-g\left(r\sqrt{2}\frac{x-y}{\sqrt{s}}\right) &= g\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\left(1-\cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)\right)\\
&+ g'\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)
\end{aligned}
$$

where $g'(r)=-\sin(r)$ if $n\in 2\mathbb{N}$ and $g'(r)=\cos(r)$ if $n\notin 2\mathbb{N}$.
Then

$$
\begin{aligned}
&D^{\alpha}\left(f_n\left(t_1,\dots,t_n;t,\cdot\right)\right)(x) = \\
&= \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n+1}{2}}}{\pi^2 n!}\int_0^\infty\frac{dy}{y^{1+\alpha}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{ds}{s^{\frac{n+1}{2}}}\\
&\quad\times\int_0^\infty r^n e^{-r^2} g\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\left(1-\cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)\right)dr\\
&+ \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n+1}{2}}}{\pi^2 n!}\int_0^\infty\frac{dy}{y^{1+\alpha}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{ds}{s^{\frac{n+1}{2}}}
\end{aligned}
$$

$$\times \int_0^\infty r^n e^{-r^2} g'\left(\frac{x\sqrt{2}}{\sqrt{s}} r\right) \sin\left(\frac{y\sqrt{2}}{\sqrt{s}} r\right) dr.$$

But

$$\int_0^\infty \frac{1}{y^{1+\alpha}} \sin\left(\frac{r\sqrt{2}}{\sqrt{s}} y\right) dy \;=\; \frac{r^\alpha 2^{\frac{\alpha}{2}}}{s^{\frac{\alpha}{2}}} \int_0^\infty \frac{\sin(y)}{y^{1+\alpha}} dy.$$

Similarly

$$\int_0^\infty \frac{1}{y^{1+\alpha}} \left(1 - \cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)\right) dy \;=\; \frac{r^\alpha 2^{\frac{\alpha}{2}}}{s^{\frac{\alpha}{2}}} \int_0^\infty \frac{1-\cos(y)}{y^{1+\alpha}} dy.$$

Set

$$C_\alpha \;=\; \max\left(\left|\int_0^\infty \frac{\sin(y)}{y^{1+\alpha}} dy\right|, \left|\int_0^\infty \frac{1-\cos(y)}{y^{1+\alpha}} dy\right|\right).$$

Then

$$\begin{aligned}
&\left|D^\alpha \left(f_n\left(t_1, \ldots, t_n; t, \cdot\right)\right)(x)\right| \\
&\leq \frac{2C_\alpha 2^{\frac{n+1+\alpha}{2}}}{\pi n!} \int_{t_1 \vee \cdots \vee t_n}^t \frac{1}{s^{\frac{n+1+\alpha}{2}}} \int_0^\infty r^{n+\alpha} e^{-r^2} dr ds.
\end{aligned}$$

But

$$\begin{aligned}
&\frac{C_\alpha 2^{\frac{n+1+\alpha}{2}}}{\pi n!} \int_{t_1 \vee \cdots \vee t_n}^t \frac{1}{s^{\frac{n+1+\alpha}{2}}} \int_0^\infty r^{n+\alpha} e^{-r^2} dr ds \\
&\leq \quad \frac{C_\alpha 2^{\frac{n+\alpha+1}{2}}}{\pi n!} \Gamma\left(\frac{n+\alpha+1}{2}\right) \int_{t_1 \vee \cdots \vee t_n}^t \frac{1}{s^{\frac{n+1+\alpha}{2}}} ds \\
&=: \quad g_n^\alpha \left(t_1, \ldots, t_n; t\right).
\end{aligned}$$

After some computations we get

$$\begin{aligned}
&n! \left\|g_n^\alpha \left(t_1, \ldots, t_n; t\right)\right\|_2^2 \\
&= \quad \frac{2C_\alpha^2 2^{n+\alpha+3}}{\pi^2 n!} \left(\Gamma\left(\frac{n+\alpha+1}{2}\right)\right)^2 \frac{1}{(n-\alpha+1)} \cdot \frac{t^{1-\alpha}}{1-\alpha} \\
&\sim \quad C \frac{1}{n^{1+\left(\frac{1}{2}-\alpha\right)}}.
\end{aligned}$$

Which completes the proof of the lemma. □

Lemma 4.2. *Let $\mathcal{E}$ a Banach space endowed with the norm $\|\cdot\|_{\mathcal{E}}$. Let α and γ be real numbers such that $(0<\alpha<\gamma\leq 1)$. D^{α} is a bounded linear operator from $\mathcal{C}^{\gamma}(\mathbb{R};\mathcal{E})$: the space of Hölder continuous $\mathcal{E}$–valued functions endowed with the norm $\|f\|_{\infty,\gamma,\mathcal{E}}:=\|f\|_{\infty,\mathcal{E}}+\|f\|_{\gamma,\mathcal{E}}$ where $\|f\|_{\infty,\mathcal{E}}=\sup_x\|f(x)\|_{\mathcal{E}}$ and $\|f\|_{\gamma,\mathcal{E}}=\sup_{x\neq y}\frac{\|f(x)-f(y)\|_{\mathcal{E}}}{|x-y|^{\gamma}}$ to $\mathcal{C}(\mathbb{R};\mathcal{E})$: the space of continuous functions on $\mathbb{R}$ endowed with the sup norm $\|\cdot\|_{\infty,\mathcal{E}}$.*
Consequently D^{α} is continuous, hence is a closed operator.

Proof: Let f be a function in the space $\mathcal{C}^{\gamma}(\mathbb{R};\mathcal{E})$ and let A be a positive constant, we have

$$\begin{aligned}\|D^{\alpha}f(x)\|_{\mathcal{E}} &= \frac{\alpha}{\Gamma(1-\alpha)}\left\|\int_0^{+\infty}\frac{f(x)-f(x-t)}{t^{1+\alpha}}dt\right\|_{\mathcal{E}}\\ &\leq \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{A}\frac{\|f(x)-f(x-t)\|_{\mathcal{E}}}{t^{1+\alpha}}dt\\ &\quad+\frac{\alpha}{\Gamma(1-\alpha)}\int_A^{+\infty}\frac{\|f(x)-f(x-t)\|_{\mathcal{E}}}{t^{1+\alpha}}dt.\end{aligned}$$

Hence,

$$\begin{aligned}\|D^{\alpha}f\|_{\infty,\mathcal{E}} &\leq C(\alpha)\,\|f\|_{\gamma,\mathcal{E}}\int_0^{A}\frac{dt}{t^{1-(\gamma-\alpha)}}+C(\alpha,A)\,\|f\|_{\infty,\mathcal{E}}\\ &\leq C(\alpha,\gamma)\ \|f\|_{\infty,\gamma,\mathcal{E}}\,.\end{aligned}$$

□

In our case the Banach space $\mathcal{E}$ is the Watanabe–Sobolev space $\mathbb{D}^{\beta,2}$

Lemma 4.3. *Set $L^m(t,x)=\sum_{n=0}^{m}I_n(f_n(\cdot;t,x))$.*
Then $L^m(t,x)$ converges to $L(t,x)$ in $\mathcal{C}^{\gamma}(\mathbb{R},\mathbb{D}^{\beta,2})$ for all $\beta<1/2$ and $\gamma<1/2$.
Moreover, for all fixed $m\in\mathbb{N}$, the mapping $x\mapsto L^m(t,x)$ is γ–Hölder continuous function for any $0<\gamma<1$.

Proof: We know that

$$L(t,x)\ =\ \sum_{n=0}^{\infty}I_n\left(f_n(\cdot;t,x)\right).$$

where

$$f_n(t_1,\dots,t_n;t,x) = \\ = \frac{2^{\frac{n+1}{2}}}{\pi n!}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}}{s^{\frac{n+1}{2}}}\left[\int_0^\infty r^n e^{-r^2} g\left(r\sqrt{2}\frac{x}{\sqrt{s}}\right)dr\right]ds.$$

Moreover if $0<t_1<t_2<\cdots<t_n<t$, we have

$$f_n(t_1,\dots,t_n;t,x) = \frac{2^{\frac{n+1}{2}}}{\pi n!}\int_{t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}}{s^{\frac{n+1}{2}}}\left[\int_0^\infty r^n e^{-r^2} g\left(r\sqrt{2}\frac{x}{\sqrt{s}}\right)dr\right]ds.$$

Let $0<\gamma<1/2$ such that $0<\beta<1/2-\gamma$, then we have,

$$\begin{aligned}\|L^m(t,\cdot)-L(t,\cdot)\|^2_{\mathcal{C}^\gamma(\mathbb{R},\mathbb{D}^{\beta,2})} &= \\ &= \left\|\sum_{n=m+1}^{\infty} I_n(f_n(\cdot;t,\cdot))\right\|^2_{\mathcal{C}^\gamma(\mathbb{R},\mathbb{D}^{\beta,2})} \\ &\leq 2\sup_{x\in\mathbb{R}}\left\|\sum_{n=m+1}^{\infty} I_n(f_n(\cdot;t,x))\right\|^2_{\mathbb{D}^{\beta,2}} \\ &\quad +2\sup_{x\neq y}\frac{\left\|\sum_{n=m+1}^{\infty}\{I_n(f_n(\cdot;t,x))-I_n(f_n(\cdot;t,y))\}\right\|^2_{\mathbb{D}^{\beta,2}}}{|x-y|^{2\gamma}}.\end{aligned}$$

Let us set

$$h_n(s,x) := \frac{1}{s^{\frac{n+1}{2}}}\int_0^\infty r^n e^{-r^2} g\left(r\sqrt{2}\frac{x}{\sqrt{s}}\right)dr$$

and

$$v_n(s,x,y) := h_n(s,x)-h_n(s,y)$$

We have

$$\begin{aligned}|I_n(f_n(\cdot;t,x))-I_n(f_n(\cdot;t,y))| &= \\ &\leq \frac{2^{\frac{n+1}{2}}}{\pi n!}\left|\int_0^t I_n\left(\mathbb{1}_{[0,s]}^{\otimes n}\right)(h_n(s,x)-h_n(s,y))\,ds\right|,\end{aligned}$$

$$\mathbb{E}|I_n(f_n(\cdot;t,x))-I_n(f_n(\cdot;t,y))|^2 =$$

$$= \frac{2^{n+1}}{\pi^2 n!} \int_0^t \int_0^t (u \wedge v)^n v_n(u,x,y) v_n(v,x,y) du dv$$

$$\leq \frac{2^{n+1-\gamma}}{\pi^2 n!} \left(\Gamma \left(\frac{n+\gamma+1}{2} \right) \right)^2 \frac{|x-y|^{2\gamma}}{(n-\gamma+1)} \cdot \frac{t^{1-\gamma}}{1-\gamma}$$

Hence

$$\sup_{x \neq y} \frac{\mathbb{E} | I_n(f_n(\cdot; t, x)) - I_n(f_n(\cdot; t, y))|^2}{|x-y|^{2\gamma}}$$
$$\leq \frac{2^{n+1-\gamma}}{\pi^2 n!} \left(\Gamma \left(\frac{n+\gamma+1}{2} \right) \right)^2 \frac{1}{(n-\gamma+1)} \cdot \frac{t^{1-\gamma}}{1-\gamma}$$
$$\sim \frac{C_\gamma}{n^{1+(\frac{1}{2}-\gamma)}},$$

and

$$\left\| \mathbb{E} | I_n(f_n(\cdot; t, \cdot))|^2 \right\|_\infty \leq \frac{2^{n+1}}{\pi^2 n!} \left(\Gamma \left(\frac{n+1}{2} \right) \right)^2 \frac{t}{(n+1)} \sim \frac{C}{n^{3/2}}.$$

Consequently the series $\sum_{n=0}^{\infty} \| I_n(f_n(\cdot; t, \cdot)\|^2_{\mathcal{C}^\gamma(\mathbb{R}, \mathbb{D}^{\beta,2})}$ is convergent and the proof is complete.

□

Proof of Proposition 4.1: It is a consequence of Lemmas 4.1, 4.2 and 4.3, and the fact that

$$D^\alpha \left(I_n \left(f_n \left(\cdot; t, \cdot \right) \right) \right)(x) = I_n \left(D^\alpha \left(f_n \left(\cdot; t, \cdot \right) \right)(x) \right)$$

and

$$n^\beta n! \left\| g_n^\alpha \left(t_1, \ldots, t_n; t \right) \right\|_2^2 \sim \frac{C_\alpha}{n^{1+(\frac{1}{2}-\alpha)-\beta}}.$$

□

In fact using the same tools we can prove the following more general result:

Proposition 4.2. *Let* $\mathbf{L}$ *be a linear closed operator form a Banach space* $\mathcal{E}_1$ *to a Banach space* $\mathcal{E}_2$. *Let* $\{F_t(x) : x \in \mathbb{R}\}$ *be a family of random fields taking its values in* $\mathcal{E}_1$. *Assume that*

$$F_t(x) = \sum_{n=0}^{\infty} I_n \left(f_n \left(\cdot; t, x \right) \right).$$

and $f_n(t_1,\ldots,t_n;t,x)$ belongs to the domain of $\mathbf{L}$.
Then, $\mathbf{L}(F_t)(x)$ *has the following chaos expansion in the space* $\mathcal{E}_2$

$$\mathbf{L}(F_t)(x) = \sum_{n=0}^{\infty} I_n\left(\mathbf{L}\left(f_n(\cdot;t,\cdot)\right)(x)\right).$$

5 Chaos expansion of the additive functional associated to the Hadamard finite part

Let us first recall the representation of the additive functional $H_t^x(-1-\alpha)$ associated to the Hadamard finite part via local times, Hilbert transform and the fractional derivative. We refer to Yamada (1985) and (1986) for more details.

$H_t^x(-1-\alpha)$ can be written as follows:

$$H_t^x(-1-\alpha) = -\cos(\pi(1+\alpha))D^\alpha L(t,\cdot)(x) - \sin(\pi(1+\alpha))\mathcal{H}(D^\alpha L(t,\cdot))(x). \tag{5.5}$$

Now, from the equation (5.5) we deduce the following chaos expansion

Proposition 5.1. *The additive functional $H_t^x(-1-\alpha)$ has the following decomposition*

$$H_t^x(-1-\alpha) = \sum_{n=0}^{\infty} I_n\left(F_n(\cdot;t,x)\right)$$

where

$$F_n(t_1,\ldots,t_n;t,x) = -\cos(\pi(1+\alpha))D^\alpha\left(f_n(t_1,\ldots,t_n;t,\cdot)\right)(x)$$
$$- \sin(\pi(1+\alpha))\mathcal{H}(D^\alpha\left(f_n(t_1,\ldots,t_n;t,\cdot)\right))(x).$$

Remark 5.1. *Combining Propositions 2.1, 3.1, 4.1 and 5.1 and the representation (5.5) we deduce that $H_t^x(-1-\alpha)$ belongs to $\mathbb{D}^{\beta,2}$, for all $\beta \in \left]0,\frac{1}{2}-\alpha\right[$.*

At the end of this section we treat the continuous additive functional which corresponds to the function $(y-x)_+^{\beta-1}$, $x\in\mathbb{R}$ where, $0<\beta<1$, is defined by

$$H_t^x(-1+\beta) = 2\left(\frac{(B_t-x)_+^{1+\beta}}{\beta(\beta+1)} - \frac{(B_0-x)_+^{1+\beta}}{\beta(\beta+1)} - \int_0^t \frac{(B_s-x)_+^{\beta}}{\beta}dB_s\right).$$

Since the function $y\mapsto(y-x)_+^{\beta}$ belongs to $L^2_{loc}(\mathbb{R})$, the right hand side of the above representation formula is well defined.

It has been proved in Yamada (1986) and Boufoussi *et al.* (1997) that the functional $H_t^x(-1-\alpha)$ given by the representation (5.5) has a Hölder continuous version in time and space, but here we note that we can choose a version of $H_t^x(-1+\beta)$, such that the mapping $(t,x) \mapsto H_t^x(-1+\beta)$ is jointly continuous, a.s.

Since the function $y \mapsto (y-x)_+^{\beta}$ belongs to $L^1_{loc}(\mathbb{R})$, we get

$$H_t^x(-1+\beta) \;=\; \int_0^t (B_s - x)_+^{\beta-1} ds.$$

and hence the functional $H_t^x(-1+\beta)$ belongs to the class of continuous additive functionals of B_t locally of zero energy (see e.g. Oshima and Yamada (1984)).

By the occupation density formula, we can write

$$H_t^x(-1+\beta) \;=\; \int_{\mathbb{R}} (y-x)_+^{\beta-1} L(t,y)dy \;=\; \left((y)_+^{\beta-1} * L(t,\cdot)\right)(x).$$

Let g be a continuous function with compact support. If $\mathrm{I}^\beta(g)$ denotes the β–*th* integral of the function g defined by

$$\mathrm{I}^\beta(g)(x) \;=\; \frac{1}{\Gamma(\beta)} \left((y)_+^{\beta-1} * g\right)(x),$$

then

$$H_t^x(-1+\beta) \;=\; \Gamma(\beta)\mathrm{I}^\beta(L(t,\cdot))(x).$$

Since, I^β is a linear continuous operator from the Banach space $\mathcal{C}(\mathbf{I})$ of continuous functions on the compact set $\mathbf{I}$ to $\mathcal{C}(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$, I^β is then a closed linear operator.

Hence, Proposition 4.2 leads to the following result

Corollary 5.1. *The additive functional $H_t^x(-1+\beta)$, $0<\beta<1$, has the following chaos expansion*

$$\begin{aligned} H_t^x(-1+\beta) \;&=\; \Gamma(\beta)\mathrm{I}^\beta\left(\sum_{n=0}^{\infty} I_n\left(f_n(\cdot;t,\cdot)\right)(x)\right) \\ &=\; \Gamma(\beta)\sum_{n=0}^{\infty} I_n\left(\mathrm{I}^\beta\left(f_n(\cdot;t,\cdot)\right)(x)\right), \;\text{ for all }\; \beta \in]0,1[. \end{aligned}$$

Moreover, the functional $H_t^x(-1+\beta)$ belongs to $\mathbb{D}^{\gamma,2}$, for all $\gamma \in]0, \frac{1}{2}+\beta[$.

Proof: Using the same calculation as in section 4, we have

$$\Gamma(\beta)\mathrm{I}^{\beta}\left(f_n(t_1,\ldots,t_n;t,\cdot)\right)(x) \;=\; \int_0^{\infty}\frac{1}{y^{1-\beta}}f_n\left(t_1,\ldots,t_n;t,x+y\right)dy$$

$$= \int_0^{\infty}\frac{1}{y^{1-\beta}}\left\{\frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}}\right.$$
$$\left.\times\frac{2}{\sqrt{\pi n!}}\left[\int_0^{\infty}r^n e^{-r^2}g\left(r\sqrt{2}\frac{x+y}{\sqrt{s}}\right)dr\right]ds\right\}dy.$$

If $n\in 2\mathbb{N}$, $g(r)=\cos(r)$, then

$$\cos\left(r\sqrt{2}\frac{x+y}{\sqrt{s}}\right) =$$
$$=\cos\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\cos\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)-\sin\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right).$$

Hence

$$\Gamma(\beta)\mathrm{I}^{\beta}\left(f_n(t_1,\ldots,t_n;t,\cdot)\right)(x) \;=$$

$$= \int_0^{\infty}\frac{1}{y^{1-\beta}}\left\{\frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}}\frac{2}{\sqrt{\pi n!}}\int_0^{\infty}r^n e^{-r^2}\right.$$
$$\left.\left[\cos\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\cos\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)\right]dr\right\}ds\;dy$$
$$-\int_0^{\infty}\frac{1}{y^{1-\beta}}\left\{\frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}}\frac{2}{\sqrt{\pi n!}}\int_0^{\infty}r^n e^{-r^2}\right.$$
$$\left.\left[\sin\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)\right]dr\right\}ds\;dy.$$

Fubini theorem yields

$$\Gamma(\beta)\mathrm{I}^{\beta}\left(f_n(t_1,\ldots,t_n;t,\cdot)\right)(x) =$$

$$= \frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}}\frac{2}{\sqrt{\pi n!}}$$

$$\left\{\int_0^\infty r^n e^{-r^2}\cos\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right)\left(\int_0^\infty \frac{1}{y^{1-\beta}}\cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)dy\right)dr\right.$$
$$\left.-\int_0^\infty r^n e^{-r^2}\sin\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right)\left(\int_0^\infty \frac{1}{y^{1-\beta}}\sin\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)dy\right)dr\right\}ds.$$

In the case where $n \notin 2\mathbb{N}$, $g(r)=\sin(r)$, we have

$$\sin\left(r\sqrt{2}\frac{x+y}{\sqrt{s}}\right)=$$
$$=\cos\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\sin\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right)+\sin\left(\frac{x\sqrt{2}}{\sqrt{s}}r\right)\cos\left(\frac{y\sqrt{2}}{\sqrt{s}}r\right).$$

Then

$$\Gamma(\beta)\mathrm{I}^\beta\left(f_n(t_1,\ldots,t_n;t,\cdot)\right)(x)=$$

$$= \frac{1}{\sqrt{2\pi n!}}\int_{t_1\vee\cdots\vee t_n}^t \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}}}{s^{\frac{n+1}{2}}}\frac{2}{\sqrt{\pi n!}}$$
$$\times\left\{\int_0^\infty r^n e^{-r^2}\cos\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right)\left(\int_0^\infty \frac{1}{y^{1-\beta}}\sin\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)dy\right)dr\right.$$
$$\left.+\int_0^\infty r^n e^{-r^2}\sin\left(\frac{xr\sqrt{2}}{\sqrt{s}}\right)\left(\int_0^\infty \frac{1}{y^{1-\beta}}\cos\left(\frac{yr\sqrt{2}}{\sqrt{s}}\right)dy\right)dr\right\}ds.$$

Recall that

$$\int_0^\infty \frac{1}{y^{1-\beta}}\sin\left(\frac{r\sqrt{2}}{\sqrt{s}}y\right)dy = \frac{r^{-\beta}2^{\frac{-\beta}{2}}}{s^{\frac{-\beta}{2}}}\int_0^\infty \frac{\sin(y)}{y^{1-\beta}}dy.$$

and

$$\int_0^\infty \frac{1}{y^{1-\beta}}\cos\left(\frac{r\sqrt{2}}{\sqrt{s}}y\right)dy = \frac{r^{-\beta}2^{\frac{-\beta}{2}}}{s^{\frac{-\beta}{2}}}\int_0^\infty \frac{\cos(y)}{y^{1-\beta}}dy.$$

Set

$$C_\beta = \max\left(\left|\int_0^\infty \frac{\sin(y)}{y^{1-\beta}}dy\right|, \left|\int_0^\infty \frac{1}{y^{1-\beta}}\cos(y)\,dy\right|\right).$$

Then

$$\left|\Gamma(\beta)\mathrm{I}^{\beta}\left(f_n\left(t_1,\ldots,t_n;t,\cdot\right)\right)(x)\right| \leq$$

$$\leq \frac{2C_\beta 2^{\frac{n+1-\beta}{2}}}{\pi n!}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{1}{s^{\frac{n+1-\beta}{2}}}\int_0^\infty r^{n-\beta}e^{-r^2}drds.$$

But

$$\begin{aligned}
&\frac{C_\beta 2^{\frac{n+1-\beta}{2}}}{\pi n!}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{1}{s^{\frac{n+1-\beta}{2}}}\int_0^\infty r^{n-\beta}e^{-r^2}drds \\
&\leq \quad \frac{C_\beta 2^{\frac{n-\beta+1}{2}}}{\pi n!}\Gamma\left(\frac{n-\beta+1}{2}\right)\int_{t_1\vee\cdots\vee t_n}^{t}\frac{1}{s^{\frac{n+1-\beta}{2}}}ds \\
&=: \quad h_n^\beta\left(t_1,\ldots,t_n;t\right).
\end{aligned}$$

Hence

$$n!\left\|h_n^\beta\left(t_1,\ldots,t_n;t\right)\right\|_2^2 =$$

$$\begin{aligned}
&= \quad \frac{2C_\beta^2 2^{n-\beta+3}}{\pi^2 n!}\left(\Gamma\left(\frac{n-\beta+1}{2}\right)\right)^2\frac{1}{(n+\beta+1)}\cdot\frac{t^{1+\beta}}{1+\beta} \\
&\sim \quad C\frac{1}{n^{1+(\frac{1}{2}+\beta)}}.
\end{aligned}$$

The proof of the corollary is done. □

Remark 5.2. *We have stated the last result as a corollary, because we can write formally*

$$\mathrm{I}^{\beta}\left(L\left(t,\cdot\right)\right)(x) \;=\; D^{-\beta}\left(L\left(t,\cdot\right)\right)(x)$$

and then deduce the result related to the fractional integral from the Proposition 4.1.

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