RELATION BETWEEN AREA AND VOLUME FOR λ–CONVEX SETS IN HADAMARD MANIFOLDS

A.A. BORISENKO, E. GALLEGO, AND A. REVENTÓS

ABSTRACT. It is known that for a sequence $\{\Omega_t\}$ of convex sets expanding over the whole hyperbolic space $\mathbb{H}^{n+1}$ the limit of the quotient $\mathrm{vol}(\Omega_t)/\mathrm{vol}(\partial\Omega_t)$ is less or equal than $1/n$, and exactly $1/n$ when the sets considered are convex with respect to horocycles. When convexity is with respect to equidistant lines, i.e. curves with constant geodesic curvature λ less than one, the above limit has λ/n as lower bound. Looking how the boundary bends, in this paper we give bounds of the above quotient for a compact λ-convex domain in a complete simply-connected manifold of negative and bounded sectional curvature, a Hadamard manifold. Then we see that the limit of $\mathrm{vol}(\Omega_t)/\mathrm{vol}(\partial\Omega_t)$ for sequences of λ-convex domains expanding over the whole space lies between the values λ/nk_2^2 and $1/nk_1$.

1. INTRODUCTION

When we consider a circumference passing through a point in the hyperbolic space $\mathbb{H}^{n+1}$ and make the center of it to go to infinity, the resulting curve is called an *horocycle*. This curve is characterized by having geodesic curvature equal ± 1. Given two points in $\mathbb{H}^{n+1}$ there is a family of horocycles joining them. We say that a set is *h-convex* if for every couple of points in it, every horocycle joining them is completely contained in the set.

In 1972 Santaló and Yañez ([SYn72]) proved the following result. Let $\{\Omega(t)\}_{t\in\mathbb{R}}$ be a family of compact h-convex domains in $\mathbb{H}^2$ expanding over the whole space. Then

$$\lim_{t\to\infty}\frac{\mathrm{area}(\Omega(t))}{\mathrm{length}(\partial\Omega(t))}=1. \tag{1}$$

For $\mathbb{H}^{n+1}$ it was proven in [BM99] the generalization of this result. Let $\{\Omega(t)\}_{t\in\mathbb{R}}$ be a family of compact h-convex domains expanding over the

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whole space, then

$$\lim_{t\to\infty}\frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))}=\frac{1}{n}.$$

On the other hand, the following linear isoperimetric inequality holds for a domain Ω in a complete simply-connected manifold with negative least upper bound K of the sectional curvatures (cf. [Yau75])

$$n\sqrt{-K}\mathrm{vol}(\Omega)\leq\mathrm{vol}(\partial\Omega).$$

This give us an upper bound for the quotient of volumes, $\mathrm{vol}(\Omega)/\mathrm{vol}(\partial\Omega)\leq 1/n\sqrt{-K}$.

An *h-convex* domain in a simply connected riemannian space M of non-positive curvature is a domain $\Omega\subset M$ with boundary $\partial\Omega$ such that, for every $p\in\partial\Omega$, there is a horosphere $\mathcal{H}$ of M through p such that Ω is locally contained in the horoball of M bounded by $\mathcal{H}$. When M is a Lobachevsky space, then this definition is equivalent to the above definition.

For simply-connected riemannian manifolds with sectional curvature satisfying $-k_2^2\leq K\leq -k_1^2$ it was proved in [BV99] that

$$\frac{1}{nk_2}\leq\liminf_{t\to\infty}\frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))}\leq\limsup_{t\to\infty}\frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))}\leq\frac{1}{nk_1}\tag{2}$$

where $\Omega(t)$ are h-convex bodies expanding over the whole space.

In [GR85] it was shown that equation (1) is not true for general convex sets. This limit can take, in the hyperbolic plane, any value between 0 and 1. Since horocycles are curves of geodesic curvature ±1 and geodesics are curves of geodesic curvature 0, they can be considered as particular cases of curves of constant geodesic curvature λ, $0\leq|\lambda|\leq 1$.

Thus if convexity is defined with respect to horocycles this limit is 1 and when convexity is defined with respect to geodesics the limit of the quotient $\mathrm{area}(\Omega(t))/\mathrm{length}(\Omega(t))$ is less or equal than 1. In [BM99] it was introduced the notion of λ-convexity and the question of the influence of λ in this limit was posed. When convexity is defined with respect λ-geodesic curves it was proved in [GR99] that for each $\alpha\in[\lambda,1]$, there exists a sequence of λ-convex polygons $\{K_n\}$ expanding over the whole hyperbolic plane such that

$$\lim_{t\to\infty}\frac{\mathrm{area}(\Omega(t))}{\mathrm{length}(\Omega(t))}=\alpha.$$

and if the sequence is formed by λ-convex sets with piecewise C^2 boundary, then the lim sup and lim inf of these ratios lie between λ and 1. For Lobachevsky space $\mathbb{H}^{n+1}$ it was proved in [BV99] that

$$\frac{\lambda}{n}\leq\liminf_{t\to\infty}\frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))}\leq\limsup_{t\to\infty}\frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))}\leq\frac{1}{n}.$$

for a family $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ of λ-convex domains expanding over the whole space.

It is possible to generalize in a natural way the notion of λ-convexity for riemannian manifolds. A domain Ω with regular boundary is λ-convex when all the normal curvatures are bounded below by λ (see section 2 for a precise definition). The main result of this work is

Theorem 2. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \le K \le -k_1^2 \qquad k_1, k_2 > 0.$$

Let Ω be a compact λ-convex domain in M with $\lambda \le k_2$. Then there are functions $\alpha(r)$ of the inradius and $\beta(R)$ of the circumradius such that $\alpha(r) \to 1/(nk_2)$ and $\beta(R) \to 1/(nk_1)$ when r and R grow to infinity and that

$$\alpha(r)\frac{\lambda}{k_2} \le \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \le \beta(R).$$

As a consequence we see that

Theorem 3. *If M is a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that $-k_2^2 \le K \le -k_1^2$ with $k_1, k_2 > 0$*

$$\frac{\lambda}{nk_2^2} \le \liminf_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \le \limsup_{t\to\infty} \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \le \frac{1}{nk_1}.$$

for a family $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ of compact λ-convex domains with $\lambda \le k_2$ expanding over the whole space.

The case $\lambda = k_2$ corresponds to a sequence of h-convex sets.

The main tool for proving these results will be an estimation of the angle between the radial direction from an interior point of Ω and the normal of $\partial\Omega$. This will we proved in section 4. We also prove an interesting formula relating the variation of this angle and the normal curvature in a direction of the boundary.

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2. Definitions and preliminary results

Definition 2.1. A *Hadamard manifold* is a simply-connected complete Riemannian manifold of non-positive sectional curvature.

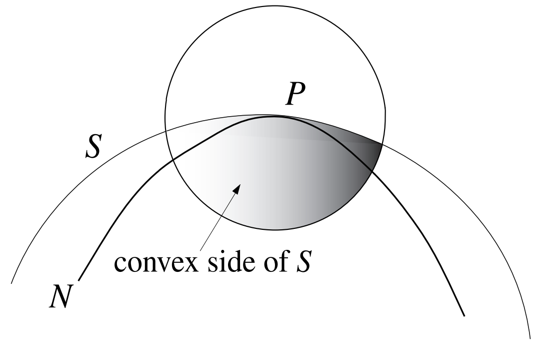

FIGURE 1

In this paper we shall deal with $(n+1)$-dimensional pinched Hadamard manifolds, this means the sectional curvature K satisfies the relation $-k_2^2 \leq K \leq -k_1^2$ with $0 < k_1 \leq k_2$.

Definition 2.2. A C^2 hypersurface $N \subset M$ such that in every point all the normal curvatures are greater or equal than a non-negative λ is said a *regular λ-convex hypersurface.* When N is the boundary of a domain Ω it is said that Ω is a *regular λ-convex domain* when its normal curvature with respect to the inward normal direction is greater than λ.

This definition can be generalized to the non-regular case

Definition 2.3. A *λ-convex hypersurface* is an hypersurface $N \subset M$ such that for every point P there is a regular λ-convex hypersurface S leaving a neighborhood of P in N in the convex side of S. A domain Ω of M is *λ-convex* if its boundary is a λ-convex hypersurface (see figure 1).

Remark. It can be seen that a 0-convex hypersurface is an ordinary locally convex hypersurface and a 0-convex domain is an ordinary convex domain. Also note that λ-convex implies 0-convex.

We shall need the fact, proved for instance in [Pet98], that if (M, g) is a Hadamard manifold with sectional curvature K satisfying $-k_2^2 \leq K \leq -k_1^2$ then the normal curvature k_n in any direction of a geodesic sphere of radius r satisfies

$$k_1 \coth(k_1 r) \leq k_n \leq k_2 \coth(k_2 r). \tag{3}$$

Note that he value $k \coth(kr)$ is the geodesic curvature of a circumference of radius r in Lobachevsky plane of curvature $-k^2$.

Remark. Since $k_1 \leq k_1 \coth(k_1 r) \leq k_n$ we deduce that for every $\lambda \leq k_1$, geodesic spheres are λ-convex hypersurfaces.

Notice also that, if Ω is a λ-convex set with $\lambda > k_2$ then every inscribed ball $B(r)$ must satisfy that $r \leq \frac{1}{k_2}\text{arctanh}\left(\frac{k_2}{\lambda}\right)$. Indeed there are points in $\partial\Omega$ such that the normal curvature is less or equal than the curvature of $\partial B(r)$, therefore $\lambda \leq k_2 \coth(k_2 r)$ and the inequality for r follows. We conclude that λ-convex sets of any radius exists only if $\lambda \leq k_2$.

Definition 2.4. An *horosphere* in a Hadamard manifold is the limit of a geodesic sphere as the radius tends to infinity

Given a point P and a complete geodesic ray γ starting on P, the limit of the sequence of geodesic spheres centered in $\gamma(t)$ an passing by P when t tends to infinity is an horosphere. Using (3) we see that horospheres have normal curvature between k_1 and k_2 when the sectional curvature K of ambient space satisfies $-k_2^2 \leq K \leq -k_1^2$.

Definition 2.5. A locally convex hypersurface N of a Hadamard manifold is said *h-convex* if every point has a locally supporting horosphere.

Remark. This means that for every x in N there is an horosphere H such that x belongs to H and N is locally contained in the convex side defined by H. A convex domain Ω is *h-convex* if its boundary is an h-convex hypersurface. Note also that every λ-convex domain with $\lambda \geq k_2$ is h-convex.

3. Normal curvature on riemannian manifolds

In this section we want to find an estimation of the normal curvature in a point P of N, an hypersurface of a riemannian manifold M. Consider N defined by the equation $t = \rho(\theta)$ of class C^2, the distance to a point O. N can be seen as the 0-level set of the function $F = t - \rho$. Remember that for a function f in M the gradient, $\text{grad} f$, is the unique vector field in M such that $< \text{grad} f, v >= df(v) = v(f)$. ∇ will denote always covariant derivative in M.

With respect to the point O we consider polar coordinates $(t, \theta^1, \ldots, \theta^n)$. The arc element is given by $ds^2 = dt^2 + g_{ij}(t,\theta)d\theta^i d\theta^j$. If we write $\boldsymbol{n} = \text{grad}F/\|\text{grad}F\|$ for the normal unit vector to N and φ for the angle between the radial direction and the unit normal we have that $\cos\varphi =< \boldsymbol{n}, \partial/\partial_t >$. Then $1/\|\text{grad}F\| = \cos\varphi$. Let $f = t$ as a function on M. If $Z \in T_pN$ then $Z(f) =< \partial/\partial_t, Z >$. It follows that $\text{grad}_N\rho$ is the orthogonal projection of ∂/∂_t onto N and the vectors $\boldsymbol{n}$, ∂/∂_t and $Y = \frac{\text{grad}_N\rho}{\|\text{grad}_N\rho\|}$ belong to a 2-dimensional plane (see figure 2). Let denote by X the unit vector in this plane and orthogonal to ∂/∂_t. The normal curvature at $P \in N$ in the direction given by Y is

$$k_n =< \nabla_Y Y, \boldsymbol{n} > .$$

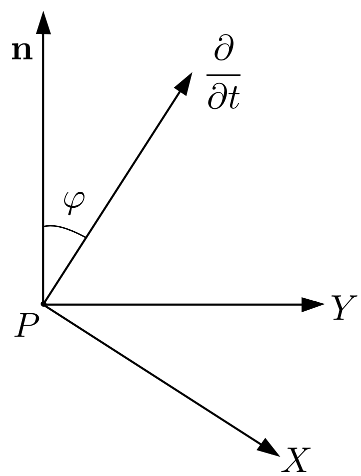

FIGURE 2

Next proposition was announced by A.A. Borisenko who gave a first version of its proof.

Proposition 3.1. *If μ_n is the normal curvature in the direction of X of the sphere centered in O with radius ρ and $\frac{d\varphi}{ds}$ the derivative of φ with respect the arc parameter of the integral curve of Y by P, then*

$$k_n = \mu_n \cos\varphi + \frac{d\varphi}{ds} \tag{4}$$

Remark. This is a kind of Liouville formula. It must be noticed that k_n and μ_n are both negative because the normal vector follows the grad(F) direction which is exterior to the hypersurface.

Proof. We have that

$$\left.\begin{array}{rcl} \boldsymbol{n} & = & \cos\varphi \cdot \partial/\partial_t - \sin\varphi \cdot X \\ Y & = & \cos\varphi \cdot X + \sin\varphi \cdot \partial/\partial_t \end{array}\right\} .$$

Hence

$$k_n = \sin\varphi < \nabla_{\partial/\partial_t} Y, \boldsymbol{n} > + \cos\varphi < \nabla_X Y, \boldsymbol{n} > .$$

A straightforward calculation shows that the first term vanishes. Let us decompose the second term.

$$< \nabla_X Y, \boldsymbol{n} > = \cos\varphi < \nabla_X \cos\varphi\, X, \partial/\partial_t > - \sin\varphi < \nabla_X \cos\varphi\, X, X > +$$
$$\cos\varphi < \nabla_X \sin\varphi\, \partial/\partial_t, \partial/\partial_t > - \sin\varphi < \nabla_X \sin\varphi\, \partial/\partial_t, X > .$$

But

$$< \nabla_X \cos\varphi\, X, \partial/\partial_t >= \cos\varphi < \nabla_X X, \partial/\partial_t >= \mu_n \cos\varphi$$

with μ_n the normal curvature in the direction X of the n-dimensional sphere centered in O with radius ρ.

$$< \nabla_X \cos\varphi \; X, X >= -X(\varphi)\sin\varphi\,,$$

$$< \nabla_X \sin\varphi \; \partial/\partial_t, \partial/\partial_t >= X(\varphi)\cos\varphi\,,$$

and

$$< \nabla_X \sin\varphi \; \partial/\partial_t, X >= -\mu_n \sin\varphi\,.$$

Therefore we obtain

$$k_n = \mu_n \cos\varphi \, + X(\varphi)\cos\varphi\,. \tag{5}$$

Using that $X = Y/\cos\varphi \; + (\tan\varphi)\,\partial/\partial_t$ we obtain

$$k_n = \mu_n \cos\varphi \, + Y(\varphi). \tag{6}$$

But differentiation in direction Y of φ is the derivative with respect the arc parameter of the integral curve of Y by P. This finishes the proof. □

4. Lower bound for $\cos\varphi \; =< \boldsymbol{n}, \partial/\partial_t >$

In this section we shall study the angle φ between the radial direction and the normal direction to the hypersurface. We divide the proof in the regular and the non-regular case.

4.1. **Regular case.** We shall prove the following

Theorem 1. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that $-k_2^2 \le K \le -k_1^2$ with $k_1, k_2 > 0$. Let Ω be a λ-convex domain with C^2 boundary N, $\lambda < k_2$ and O an interior point of Ω. If φ denote the angle of the normal to N an the exterior radial direction, when $d(O,N) \le \frac{1}{k_2}\text{arctanh}(\frac{\lambda}{k_2})$ we have*

$$\cos\varphi \ge \frac{1}{k_2}\sqrt{\lambda^2\cosh^2 k_2 s - k_2^2 \sinh^2 k_2 s}.$$

If $d(O,\partial N) \ge \frac{1}{k_2}\text{arctanh}(\frac{\lambda}{k_2})$ we have

$$\cos\varphi \ge \frac{\lambda}{k_2}.$$

We start studying what happens in the hyperbolic space.

Lemma 4.1 ([BV99])**.** *Let γ be a λ-geodesic line in the Lobachevsky plane of constant curvature $-k^2$. Let O be a point in the convex side of γ. Let r be the distance between γ and O. For each point in γ we define β as the angle between the radial field from O and the outwards normal field of γ. If*

$$r < d := \frac{1}{k}\text{arctanh}\frac{\lambda}{k} \qquad \left(= \log\sqrt{\frac{k+\lambda}{k-\lambda}}\right)$$

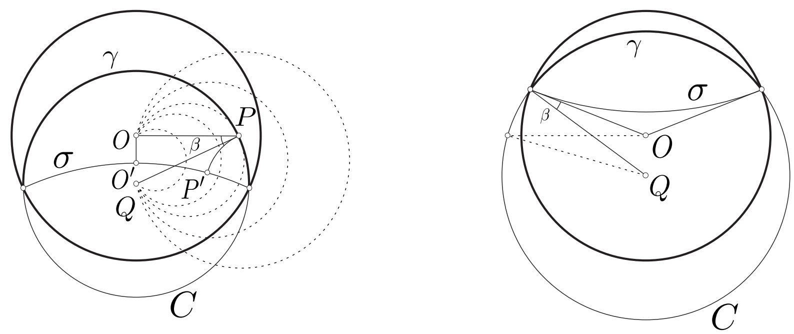

FIGURE 3

then

$$\cos\beta \geq \frac{2\sqrt{\rho(\lambda - k\rho)(k-\lambda\rho)}}{k(1-\rho^2)} \tag{7}$$

where $\rho = \tanh \frac{1}{2}kr$. *Alternatively, if* $r \geq d$ *then*

$$\cos\beta \geq \frac{\lambda}{k}. \tag{8}$$

Remark. The estimate (7) can be given in the following equivalent form

$$\cos\beta \geq \frac{1}{k}\sqrt{\lambda^2\cosh^2 ks - k^2\sinh^2 ks} \tag{9}$$

where $s = d - r$.

We shall see now in a synthetic way a new proof of those expressions. Assume that we are in the conformal Poincaré disk model and that O is the origin. We can also suppose that γ is the intersection with the disk of a circle C centered at $Q = (0, q)$ with $q < 0$. Now, at any point $P \in \gamma$, β is the angle $\widehat{QPO}$. Consider the curves defined as the locus of the point from which OQ is in a given angle. It is known that these level curves are arcs of circles joining O and Q. Two of such arcs are tangent to C. Thus, the maximum of $\widehat{QPO}$ for $P \in C$ is attained when P is one of these tangency points. That is, when $\widehat{POQ} = \pi/2$.

Now, by definition γ is the equidistant curve at distance d to some geodesic σ. If $r < d$ then O is in the region bounded by γ and σ. So, γ meets the boundary of the model at points with negative second coordinate. Thus, the points $P \in C$ where $\widehat{QPO}$ is maximum are in γ. Then, the maximum of β is also attained in P. If O' and P' are the points in σ at minimum distance, respectively, from O and P, then $O'OPP'$ is a quadrilateral with three right angles and an acute angle equal to β. Using a hyperbolic trigonometric formula for quadrilaterals (cf. [Rat94]),

$$\sin\beta = \frac{\cosh k\overline{OO'}}{\cosh k\overline{PP'}}.$$

From this we obtain easily the expression (9). A straightforward computation shows that it is equivalent to (7).

In the case that $r \geq d$, the points $P \in C$ with the greatest angle $\widehat{QPO}$ are outside the disk. Then, at every point of γ, β is less than the angle between the λ-geodesic and the boundary of the disk and this angle has cosine λ/k. □

Proof of theorem 1. Let γ be an integral curve of the field $Y = \mathrm{grad}_N\rho$ through a point P of the boundary. Following γ in the direction that ρ decreases we arrive at a point Q (maybe at infinite time of the parameter). In this point $Y = 0$, hence $\varphi = 0$. Let $d(O,Q) = d\,(\geq d(O,N))$. If $d' = d(O,P)$ we can parametrize the segment of γ between P an Q with the distance $t \in (d,d']$ of O to the corresponding point in the segment. If s is the arc parameter we have by lemma 3.1

$$k_n(\gamma(t)) = \cos\varphi(\gamma(t))\mu_n(\gamma(t)) + \frac{d\varphi}{dt}\frac{dt}{ds}$$

but

$$\frac{dt}{ds} = \frac{Y}{\|Y\|}(\rho) = \frac{< \mathrm{grad}_N\rho, \mathrm{grad}_N\rho >}{\|\mathrm{grad}_N\rho\|} = \sin\varphi.$$

As N is λ-convex and using the comparison formula (3) we have

$$-\lambda \geq -k_2\coth(k_2\cdot t)\cos\varphi + \sin\varphi\frac{d\varphi}{dt}. \tag{10}$$

Now consider in $\mathbb{H}^2(-k_2^2)$ an arbitrary λ-geodesic line $\overline{\gamma}$ and a point $\overline{Q}$ in it. Consider an orthogonal geodesic from $\overline{Q}$ to a point $\overline{O}$ at distance d from $\overline{Q}$. In $\overline{\gamma}$ consider a point $\overline{P}$ at distance $d' = d(O,P)$ from $\overline{O}$. We have the same situation as before, but now in the hyperbolic plane of constant curvature $-k_2^2$. If β is the angle between the normal to $\overline{\gamma}$ in the direction of the ray vector from $\overline{O}$ and this ray vector, we have the exact formula

$$-\lambda = -k_2\coth(k_2\cdot t)\cos\beta + \sin\beta\frac{d\beta}{dt}, \tag{11}$$

where t is again the distance from $\overline{O}$ to the corresponding point in $\overline{\gamma}$ (see figure 4).

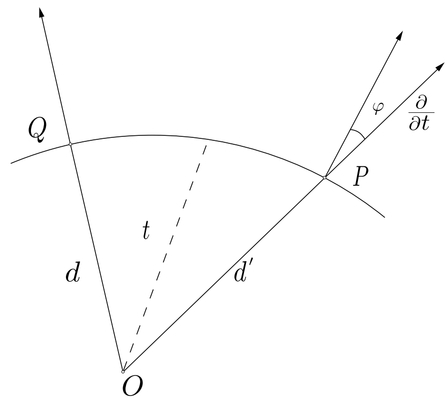

FIGURE 4

Suppose that $\gamma(t) > \beta(t)$. As $\gamma(d) = \beta(d) = 0$ we must have $\gamma' > \beta'$ at some point. From equations (10) and (11) we deduce

$$
\begin{aligned}
&- k_2 \coth(k_2 \cdot t) \cos\beta + \sin\beta \frac{d\beta}{dt} \geq \\
&\qquad - k_2 \coth(k_2 \cdot t) \cos\varphi + \sin\varphi \frac{d\varphi}{dt} > \\
&\qquad\qquad - k_2 \coth(k_2 \cdot t) \cos\beta + \sin\beta \frac{d\beta}{dt}
\end{aligned}
$$

which is a contradiction. Therefore we must have $\varphi \leq \beta$ hence $\cos\varphi(t) \geq \cos\beta(t)$ and the bound follows. □

It is possible to prove in an easier way a less strong result

Proposition 4.1. *Let M be a Hadamard manifold with sectional curvature $-k_2^2 \leq K \leq -k_1^2$. Suppose Ω be a C^2 λ-convex set with $\lambda < k_2$ and $\partial\Omega$ a connected boundary component. Let O a point in the interior of Ω. Then the angle φ between geodesic rays from O and the unit normal to $\partial\Omega$ satisfies the inequality*

$$\cos\varphi \geq \frac{\lambda}{k_2} \tanh(k_2\, r)$$

where r is the minimum distance from O to $\partial\Omega$.

Proof. Note that the field $\mathrm{grad}_N \rho$ is zero if and only if $\cos\varphi = 1$ and in this case $\partial/\partial t = \mathrm{grad} F$.

The angle φ takes its value in the interval $[0, \pi/2]$ then there is a supremum φ_0 of it. Consider any integral curve γ of $Y/\|Y\|$. If at some point

$\gamma(s_0)$ the value φ_0 is achieved we have in this point that $\varphi' = 0$ and so

$$\cos\varphi = \frac{k_n}{\mu_n}$$

concluding that

$$\cos\varphi \geq \frac{\lambda}{k_2 \coth(k_2\rho_o)}. \tag{12}$$

If the maximum value is not achieved we have two different possibilities, there exists a value s_0 such that $\varphi(\gamma(s))$ increases when $s > s_0$ in this case $\varphi' > 0$ and then $(-k_n)\cos\varphi \geq -\mu_n$, it follows (12) again. The other case is that $\varphi(\gamma(s))$ goes to φ_0 in a non monotone way, in this case there is a increasing sequence s_n such that $\varphi'(\gamma(s_n)) = 0$ and $\varphi(\gamma(s_n)) \to \varphi_0$. Again we obtain (12).

□

4.2. **Non regular case.** Now we shall consider a general λ-convex domain Ω. Let N_ϵ be the outer parallel set at distance ϵ to $N = \partial\Omega$. Then it is a general fact that N_ϵ is of class of regularity $C^{1,1}$. When N is λ-convex, N_ϵ is λ_ϵ-convex with $\lambda_\epsilon \geq \lambda - C\epsilon$. It is true also that

$$\lim_{\epsilon\to 0} N_\epsilon = N, \qquad \lim_{\epsilon\to 0} \varphi_\epsilon = \varphi.$$

Here φ corresponds to the angle of the normal of the limit supporting tangent plane with the radial direction $\partial/\partial t$ (see figure 5).

If we found a bound for φ_ϵ then we will obtain an evaluation for φ. Now we consider the gradient of the distance function for N_ϵ, this field has integral curves of class of regularity $C^{1,1}$. In fact in almost all points the class is C^2. Therefore the function $\varphi_\epsilon(t)$ giving the angle is C^1 in those points. Applying proposition 3.1 to φ_ϵ and using that

$$\varphi(s) = \varphi(s_0) + \int_{s_0}^{s} \frac{d\varphi}{ds} dt \tag{13}$$

we obtain that the same evaluation for $\cos\varphi$ as in the regular case is valid now. Taking limits with respect to ϵ we obtain the proof of theorem 1 for the general case.

5. Estimates for the ratio of volumes

First of all we sate the following lemma (see for instance [BZ94]).

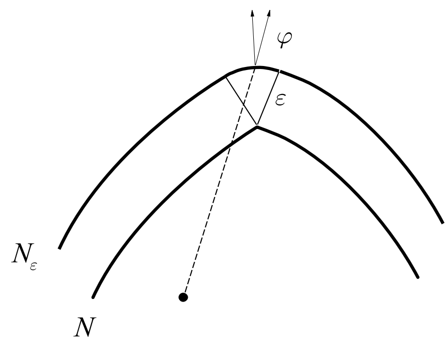

FIGURE 5

Lemma 5.1. *Suppose that on the geodesic line $\gamma : [0, s] \to M$ of a manifold M there are no conjugate points to $\gamma(0)$ and at every point of γ all the sectional curvatures K_σ are bounded by*

$$k_2 \leq K_\sigma \leq k_1.$$

Then, for $t < s$

$$\frac{J_{k_2}(t)}{J_{k_2}(s)} \leq \frac{J(t)}{J(s)} \leq \frac{J_{k_1}(t)}{J_{k_1}(s)}$$

where $J(t)$ and $J_k(t)$ denote the jacobians at the points corresponding to $\gamma(t)$ by the exponential maps of M and of the space with constant curvature k, respectively.

Theorem 2. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \leq K \leq -k_1^2 \qquad k_1, k_2 > 0.$$

Let Ω be a compact λ-convex domain in M. Then if $\lambda < k_2$

$$f(r) \cdot C(r) \frac{\lambda}{k_2} \leq \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \leq h(R)$$

where r is the inradius of Ω, R is the circumradius,

$$f(r) := \frac{1}{(1-e^{-2k_2 r})^n} \left[\frac{1}{k_2 n}(1 - \mathrm{e}^{-k_2 n r}) - \frac{n}{k_2(n-2)}(\mathrm{e}^{-2k_2 r} - \mathrm{e}^{-k_2 n r}) \right]$$

$$h(R) := \frac{1}{k_1 n}(1 - \mathrm{e}^{-k_1 n R})$$

and

$$C(r) := \begin{cases} \frac{1}{k_2}\sqrt{\lambda^2 \cosh^2 k_2 s - k_2^2 \sinh^2 k_2 s} & \text{if} \quad r \leq \frac{1}{k_2}\mathrm{arctanh}\frac{\lambda}{k_2} \\ 1 & \text{if} \quad r > \frac{1}{k_2}\mathrm{arctanh}\frac{\lambda}{k_2}. \end{cases}$$

Proof. Let O be any point interior to Ω. Consider the exponential map in O, $\exp : T_O M \longrightarrow M$. For each unitary vector $u \in T_O M$ we define $l(u)$ as the positive real number such that

$$\exp(l(u)u) \in \partial\Omega.$$

Let r and R be respectively the minimum and the maximum of l. Let $A = \{(u, t \in S^n \times \mathbb{R};\ 0 < t \leq l(u)\}$. Identifying $S^n \times \mathbb{R}$ with $T_O M - \{O\}$ we have $\Omega = \exp(A)$. Hence

$$\mathrm{vol}(\Omega) = \int_\Omega \eta = \int_{esp(A)} \eta = \int_A \exp^* \eta = \int_{S^n} \int_0^{l(u)} J(\exp) t^n \mathrm{d}t \mathrm{d}S.$$

where η and $\mathrm{d}S$ are, respectively, the volume elements of M and S^n.

Analogously, if we define $\phi : S^n \longrightarrow \partial\Omega$ by $\phi(u) = \exp(l(u))u$, then

$$\mathrm{vol}(\partial\Omega) = \int_{\partial\Omega} \mu = \int_{\phi(S^n)} \mu = \int_{Sn} \phi^* \mu = \int_{S^n} \mathrm{Jac}_u(\phi) \mathrm{d}S.$$

where μ is the volume element of $\partial\Omega$. Now, we compute the jacobian of ϕ at a point $u \in S^n$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_u S^n$. By definition, we have

$$\mathrm{Jac}_u(\phi) = \mu(\phi_* e_1, \ldots, \phi_* e_n) = \eta(N, \phi_* e_1, \ldots, \phi_* e_n)$$

where N is orthogonal to $\partial\Omega$. If ∂_t is the radial field from O, we can write

$$\mathrm{Jac}_u(\phi) = \eta(\frac{\partial_t}{< \partial_t, N >}, \phi_* e_1, \ldots, \phi_* e_n).$$

Now, $\phi_*(e_i) = \exp_*(\mathrm{d}l(e_i)u + l(u)e_i)$, so

$$\begin{aligned}\mathrm{Jac}_u(\phi) &= \frac{1}{< \partial_t, N >} \eta(< \partial_t, N >, \exp_*(l(u)e_1), \ldots, \exp_*(l(u)e_n)) = \\ &= \frac{l^n(u)}{< \partial_t, N >} \eta(\exp^*(u), \exp_*(l(u)e_1), \ldots, \exp_*(l(u)e_n)) = \\ &= \frac{l^n(u)}{< \partial_t, N >} \mathrm{Jac}_{l(u)u}(\exp).\end{aligned}$$

Therefore,

$$\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} = \frac{\int_{S^n} \int_0^{l(u)} \mathrm{Jac}_{l(u)u}(\exp) t^n \mathrm{d}t \mathrm{d}S}{\int_{S^n} \frac{l^n(u)}{< \partial_t, N >} \mathrm{Jac}_{l(u)u}(\exp) \mathrm{d}S}.$$

Setting

$$g(u) = \int_0^{l(u)} \frac{\mathrm{Jac}_{tu}(\exp) t^n}{\mathrm{Jac}_{l(u)u}(\exp) l(u)^n} \mathrm{d}t$$

we can write

$$\mathrm{vol}(\Omega) = \int_{S^n} g(u) l(u)^n \mathrm{Jac}_{l(u)u}(\exp) \mathrm{d}S.$$

Now, from lemma 5.1, comparing with the spaces of constant curvature $-k_1^2$ and $-k_2^2$ we can state that

$$\frac{\mathrm{Jac}_{tu}(\exp^{-k_2^2})}{\mathrm{Jac}_{su}(\exp^{-k_2^2})} \leq \frac{\mathrm{Jac}_{tu}(\exp)}{\mathrm{Jac}_{su}(\exp)} \leq \frac{\mathrm{Jac}_{tu}(\exp^{-k_1^2})}{\mathrm{Jac}_{su}(\exp^{-k_1^2})} \qquad \text{for } t < s$$

where $\exp^{-k_i^2}$ denotes the exponential map at any point of the space of curvature $-k_i^2$. It is known that $\mathrm{Jac}_{tu}(\exp^{-k_i^2}) = (\frac{1}{k_i}\sinh k_i t)^n t^{-n}$. Hence

$$\int_0^{l(u)} \frac{(\sinh k_2 t)^n}{(\sinh k_2 s)^n}\mathrm{d}t \leq g(u) \leq \int_0^{l(u)} \frac{(\sinh k_1 t)^n}{(\sinh k_1 s)^n}\mathrm{d}t.$$

We can estimate the first integral by using the fact that $(1-a)^n \geq 1 - na$ for $0 \leq a \leq 1$.

$$\int_0^s \frac{\sinh(k_2 t)^n}{\sinh(k_2 s)^n}\mathrm{d}t = \frac{1}{(1-e^{-2k_2 s})^n}\int_0^s (1-\mathrm{e}^{-2k_2 t})^n \mathrm{e}^{k_2 n(t-s)}\mathrm{d}t \geq$$

$$\geq \frac{1}{(1-e^{-2k_2 s})^n}\int_0^s (1-n\mathrm{e}^{-2k_2 t})\mathrm{e}^{k_2 n(t-s)}\mathrm{d}t =$$

$$= \frac{1}{(1-e^{-2k_2 s})^n}\left[\frac{1}{k_2 n}(1-\mathrm{e}^{-k_2 n s}) - \frac{n}{k_2(n-2)}(\mathrm{e}^{-2k_2 s}-\mathrm{e}^{-k_2 n s})\right] =: f(s)\,.$$

On the other hand,

$$\int_0^s \frac{\sinh(k_1 t)^n}{\sinh(k_1 s)^n}\mathrm{d}t \leq \int_0^s \mathrm{e}^{k_1 n(t-s)}\mathrm{d}t = \frac{1}{k_1 n}(1-\mathrm{e}^{-k_1 n s}) =: h(s)\,.$$

Therefore, since $r \leq l(u) \leq R$ for every $u \in S^n$,

$$f(r)\int_{S^n} l(u)^n \mathrm{Jac}_{l(u)u}(\exp)\mathrm{d}S \leq \mathrm{vol}(\Omega) \leq h(R)\int_{S^n} l(u)^n \mathrm{Jac}_{l(u)u}(\exp)\mathrm{d}S.$$

Finally, using theorem 1, we find that

$$f(r)\cdot C(r)\frac{\lambda}{k_2} \leq \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \leq h(R).$$

Now, choosing O to be the incenter and the circumcenter of Ω, we have proved the two inequalities with r and R the inradius and the circumradius respectively. □

Note that the theorem would be true, with the same proof, if r and R were the radius of any geodesic ball contained and containing, respectively, Ω.

Now, we get the main result of the paper

Theorem 3. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \leq K \leq -k_1^2 \qquad k_1, k_2 > 0.$$

Let $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ be a family of λ-convex compact domains expanding over the whole space. Then, if $\lambda \leq k_2$

$$\frac{\lambda}{nk_2^2} \leq \liminf \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \limsup \frac{\mathrm{vol}(\Omega(t))}{\mathrm{vol}(\partial\Omega(t))} \leq \frac{1}{nk_1}.$$

Proof. Since $\Omega(t)$ expands over the whole hyperbolic space, r and R go to infinity. Then $h(R)$ goes to $1/nk_1$ and $f(r)$ goes to $1/nk_2$. When $\lambda = k_2$ the domains are h-convex and the inequality follows from [BV99]. □

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Geometry Department, Math.-Mech. Faculty, Kharkov National University, Pl. Svobodu 4, 310077–Kharkov, Ukraine

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193–Bellaterra (Barcelona), Spain

E-mail address: `Alexander.A.Borisenko@univer.kharkov.ua`, `egallego@mat.uab.es`, `agusti@mat.uab.es`