FREGEAN LOGICS WITH THE MULTITERM DEDUCTION THEOREM AND THEIR ALGEBRAIZATION

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ABSTRACT. A deductive system $\mathcal{S}$ (in the sense of Tarski) is *Fregean* if the relation of interderivability, relative to any given theory T, i.e., the binary relation between formulas

$$\{\, \langle \alpha, \beta \rangle : T, \alpha \vdash_{\mathcal{S}} \beta \text{ and } T, \beta \vdash_{\mathcal{S}} \alpha \,\},$$

is a congruence relation on the formula algebra. The *multiterm deduction-detachment theorem* is a natural generalization of the deductive theorem of the classical and intuitionistic propositional calculi (IPC) in which a finite system of possibly compound formulas collectively plays the role of the implication connective of IPC. We investigate the deductive structure of Fregean deductive systems with the multiterm deduction-detachment theorem within the framework of abstract algebraic logic. It is shown that each deductive system of this kind has a deductive structure very close to that of the implicational fragment of IPC. Moreover, it is algebraizable and the algebraic structure of its equivalent quasivariety is very close to that of the variety of Hilbert algebras. The equivalent quasivariety is however not in general a variety. This gives an example of a relatively point-regular, congruence-orderable, and congruence-distributive quasivariety that fails to be a variety, and provides what apparently is the first evidence of a significant difference between the multiterm deduction-detachment theorem and the more familiar form of the theorem where there is a single implication connective.

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1. Introduction

A Fregean logic is one in which the *Fregean principle* holds. According to this principle, so called because of its connection with the ontological framework that underlies Frege's logical system, in any given interpretation of the logic the denotation of a sentence coincides with its truth value. Consequently, the relation that holds between two sentences if they have the same truth value in every interpretation is *compositional*; roughly speaking, this says that all connectives of the logic are truth-functional. The first formalized distinction between Fregean and non-Fregean logics appears to have been made by R. Suszko and his collaborators. In addition to all the standard truth-functional connectives, Suszko's logical system contains a non-truth-functional connective that is intended to express the identity of (the denotations of) sentences. The present paper is the second of two papers by the present authors in which the Fregean principle is examined in the wider context of *abstract algebraic logic*. This is a developing area of algebraic logic that focuses on the process of algebraization itself rather than on properties of the classes of algebras that arise when specific logics are algebraized. It also focuses on the connection between metalogical and algebraic properties and how this can be used to establish a particular property of a logic by establishing the corresponding property of its algebraic counterpart, and vice versa.

We consider deductive systems of a very general kind that do not include a priori connectives dedicated to reifying any particular ontological principle. Let $\mathcal{S}$ be such a deductive system. Following Tarski we take $\mathcal{S}$ to be an algebraic closure operation on the set Fm_Λ of all formulas over a fixed but arbitrary set of connectives Λ. Let $\vdash_\mathcal{S}$ denote the associated consequence relation. Two formulas α and β are assumed to have the same truth value, in the abstract sense relative to $\mathcal{S}$, if they are interderivable with respect to every set Γ of formulas in the following precise sense.

$$\Gamma, \alpha \vdash_\mathcal{S} \beta \quad \text{and} \quad \Gamma, \beta \vdash_\mathcal{S} \alpha. \tag{1}$$

$\mathcal{S}$ is defined to be *Fregean* if the interderivability relation is compositional, i.e., if (1) implies

$$\Gamma, \vartheta(x/\alpha) \vdash_\mathcal{S} \vartheta(x/\beta) \quad \text{and} \quad \Gamma, \vartheta(x/\beta) \vdash_\mathcal{S} \vartheta(x/\alpha)$$

for every formula ϑ and every variable x occurring in ϑ. The motivation behind this definition is another principle that is implicit in Frege, namely that there are just two truth values, "truth" and "falsity". The entailments $\Gamma, \alpha \vdash_\mathcal{S} \beta$ and $\Gamma, \beta \vdash_\mathcal{S} \alpha$ can be interpreted as asserting that α and β are "true" in exactly the same interpretations of Γ relative to $\mathcal{S}$.

The paradigm for Fregean deductive systems is the intuitionistic propositional calculus (IPC) and its axiomatic extensions, including the classic propositional calculus (CPC). On the other hand, most modal logics fail to be Fregean. [13] contains a detailed analysis of the Fregean principle in arbitrary deductive systems for which a generalized version of the well-known deduction theorem of CPC and IPC holds. The deduction theorem can be generalized in various ways. One important distinction is made between the uniterm and the multiterm forms of the theorem. A deductive system has the *multiterm deduction-detachment theorem* if there exists a finite system of possibly compound formulas $x \Rightarrow_0 y, \dots, x \Rightarrow_{n-1} y$ in two variables, called a *deduction-detachment system*, such that, for every set Γ of formulas and every pair of formulas α and β,

$$\Gamma, \alpha \vdash_\mathcal{S} \beta \quad \text{iff} \quad \Gamma \vdash_\mathcal{S} \alpha \Rightarrow_i \beta \text{ for all } i < n.$$

If the deduction-detachment system can be taken to be a single formula $x \Rightarrow y$ so that

$$\Gamma, \alpha \vdash_\mathcal{S} \beta \quad \text{iff} \quad \Gamma \vdash_\mathcal{S} \alpha \Rightarrow \beta,$$

then $\mathcal{S}$ is said to have the *uniterm deduction-detachment theorem*. There seem to be no examples among the traditional deductive systems of a deduction-detachment theorem that is properly multiterm. In spite of this the multiterm version seems to be the more interesting notion, at least from the point of view of abstract algebraic logic. It can be characterized entirely in terms of the abstract consequence relation, independently of the existence of a system of formulas with some special property. More precisely, a deductive system $\mathcal{S}$ has the multiterm deduction-detachment theorem iff the upper semilattice of finitely generated theories of $\mathcal{S}$ is relatively pseudo-complemented and hence forms a Brouwerian semilattice; see [9]. As far as we know the uniterm version has no similar characterization.

Quite natural examples of a properly multiterm deduction-detachment theorem can be found among the deductive systems of equational logic. For these logics it is known to be closely connected to the algebraic property of having equationally definable principal congruences; see [3, 7]). For other work on the multiterm deduction-detachment theorem and closely related notions see [5, 10, 11, 12, 17].

It is proved in [13] that, in the presence of the uniterm deduction-detachment theorem, the Fregean principle essentially characterizes the implicational fragment of IPC and all its various expansions by additional connectives and additional axioms. As a corollary of this result we get that every Fregean deductive system with the uniterm deduction-detachment theorem is *strongly algebraizable*, i.e., it is algebraizable in the sense of [4][1] and its equivalent algebraic semantics is a variety. More precisely, the equivalent algebraic semantics of a Fregean deductive system with the uniterm deduction-detachment theorem is term wise definitionally equivalent to a variety of Hilbert algebras in the sense of [14], possibly with additional operations that are compatible with the operations of the Hilbert algebra in a natural way.

In the present paper we study the Fregean principle in the presence of the weaker multiterm deduction-detachment theorem. We generalize the result of [13] mentioned above by proving that the Fregean principle together with the multiterm deduction-detachment theorem characterizes a natural generalization of the implicational fragment of IPC and its various expansions. A family of deductive systems is defined (Def. 3.12) whose characteristic property is the existence of a system of binary formulas that collectively behave like the single implication connective of IPC. We prove that every Fregean deductive system with the multiterm deduction-detachment theorem is an axiomatic extension of one of these systems and vice versa (Cor. 3.15). It is not difficult to see that every Fregean deductive

[1]In the current terminology of abstract algebraic logic deductive systems that are algebraizable in this sense are said to be *finitely algebraizable*, and the term "algebraizable" is reserved for the notion introduced in [19, 20]. Since in this paper we deal only with algebraizability in the sense of [4] we will omit the qualifying "finitely".

system with the multiterm deduction-detachment theorem is algebraizable, in fact regularly algebraizable ([13]). Not surprisingly, its equivalent algebraic semantics can be shown to be a natural "multiterm" generalization of Hilbert algebras (Thm. 4.2). However, in contrast to the uniterm case, they are not in general strongly algebraizable (Cor. 5.5). This gives an example of a relatively point-regular, relatively congruence-orderable (Def. 2.9), and relatively congruence-distributive quasivariety that is not a variety. The existence of such a quasivariety had been an open problem. These results are interesting because they provide the first evidence of a significant difference, at the abstract algebraic logic level, between the uniterm and the multiterm deduction-detachment theorems.

1.1. **Outline of the paper.** In the second section we review the elements of the theory of multidimensional deductive systems that allow us to investigate, within a common framework, both the deduction-detachment theorem (in its generalized form) and the Fregean property in both classical 1-dimensional deductive systems and the 2-dimensional systems of equational logic. Elements of the theory of algebraizable and regularly algebraizable deductive systems are also reviewed.

Section 3 is devoted to obtaining a characterization of multidimensional systems with the multiterm deduction-detachment theorem (Thm. 3.6). This is applied to 2-deductive systems of equational logic to obtain a new characterization of quasivarieties with equational definable principal relative congruences (EDPRC) (Cor. 3.7).

At this point we restrict ourselves to 1-dimensional systems in order to obtain more definitive results. The main result here is that all Fregean 1-dimensional systems with the multiterm deduction-detachment theorem can be uniformly formalized in a way that generalizes naturally the standard axiomatization of IPC and its axiomatic extensions (Cor. 3.15). This gives rise to a family of deductive systems called the *multiterm intuitionistic propositional calculi* (IPC_P).

Each of the IPC_P is regularly algebraizable and hence has a unique relatively point-regular quasivariety associated with it, its so-called equivalent quasivariety. Applying a general result from [4] we show that each of these quasivarieties has an axiomatization that generalizes in a natural way one of the standard axiomatizations of the variety of Hilbert algebras and its subvarieties (Thm. 4.2). This gives rise to the *multiterm Hilbert algebras* (HI_P). The main result of Section 4 is that each of the quasivarieties HI_P is relatively congruence-orderable and has EDPRC (Thm. 4.6).

The results of the previous sections are collected and refined in the last section in order to formalize the precise connections between the deductive systems IPC_P, the quasivarieties HI_P, the metalogical properties of being Fregean and having the multiterm deduction-detachment theorem, and the algebraic properties of being relatively congruence-orderable and having EDPRC (Thm. 5.2). We end the paper by showing that HI_P is not in general a variety (Thm. 5.4) and that IPC_P is not in general strongly algebraizable (Cor. 5.5).

1.2. **Connection with other work.** The recent monograph [16] by J. M. Font and R. Jansana contains a number of deep results on Fregean deductive systems of a very general kind. (An extended abstract of this work appears in [15].) A companion paper [13] to this one by the present authors contains a systematic study of Fregean deductive systems, with particular emphasis on protoalgebraic systems. Several of the key results of [13] will be used in the present paper, but otherwise it is self-contained and can be read independently of [13]. Other work on Fregean logic within the context of abstract algebraic logic, some of it more algebraic than metalogical in nature, can be found in the works [1, 2, 21, 23].

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2. Preliminaries

k-dimensional deductive systems, for any nonzero natural number k, were first considered in [6] and then systematically used in [7] as a vehicle for studying algebraizability and the deduction theorem in the context of abstract algebraic logic. They are a general class of logical systems that generalize both deductive systems in the classical sense of Tarski, and also deductive systems of equational logic, and thus provide an ideal framework for a comparative study of classical and equational systems in the context of abstract algebraic logic.

By a *language type* we will mean a set Λ of connectives, when viewed from a logical context, or a set of operation symbols, in an algebraic context. Each connective has associated with it a natural number, its *rank* or *arity.* A fixed, denumerable set of *variable symbols* is assumed, and the set of Λ-*formulas* (or Λ-*terms* in an algebraic context) is formed in the usual way. The set of all Λ-formulas is denoted by Fm_Λ and the corresponding algebra of Λ-formulas by $\mathbf{Fm}_\Lambda$. The subscript is usually omitted when only one language type is under consideration. The operation of simultaneously substituting fixed but arbitrary formulas for variables is identified with the unique endomorphism of the formula algebra it determines.

Let $1 \leq k < \omega$; let Λ be a language type. Let

$$\mathrm{Fm}^k_\Lambda = \{\, \langle \alpha_0, \ldots, \alpha_{k-1} \rangle : \alpha_i \in \mathrm{Fm}_\Lambda, i < k \,\}.$$

Elements of Fm^k are called *k-formulas*. The k-formula $\langle \alpha_0, \dots, \alpha_{k-1}\rangle$ will be denoted by $\boldsymbol{\alpha}$, and in general k-formulas and sets of k-formulas will be represented by boldface lower and upper case Greek letters, respectively. 1-formulas are identified with ordinary formulas, and we drop the ordered 1-tuple representation and boldface notation in this case. By a *k-variable* we will mean a k-formula of the form $\boldsymbol{x} = \langle x_0, \dots, x_{k-1}\rangle$, where $x_0, \dots, x_{k-1}$ are distinct variables. If $h : \mathbf{Fm} \to \mathbf{Fm}$ is a substitution, and $\boldsymbol{\alpha} = \langle \alpha_0, \dots, \alpha_{k-1}\rangle$ is a k-formula, then $h\,\boldsymbol{\alpha} = \langle h\,\alpha_0, \dots, h\,\alpha_{k-1}\rangle$. For any set $\boldsymbol{\Gamma}$ of k-formulas, $h(\boldsymbol{\Gamma}) = \{\, h\,\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \boldsymbol{\Gamma}\,\}$.

Definition 2.1. A *k-dimensional deductive system* $\mathcal{S}$ (*over* Λ) is an ordered pair $\langle \Lambda, \vdash_{\mathcal{S}}\rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}^k) \times \mathrm{Fm}^k$ (here $\mathrm{Fm} = \mathrm{Fm}_\Lambda$) and such that the following conditions hold for all $\boldsymbol{\Gamma}, \boldsymbol{\Delta} \subseteq \mathrm{Fm}^k$ and $\boldsymbol{\alpha} \in \mathrm{Fm}^k$.

(i) $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}$ implies $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$;
(ii) $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ and $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Delta}$ implies $\boldsymbol{\Delta} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$;
(iii) $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ and $\boldsymbol{\Delta} \vdash_{\mathcal{S}} \boldsymbol{\beta}$ for every $\boldsymbol{\beta} \in \boldsymbol{\Gamma}$ implies $\boldsymbol{\Delta} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$;
(iv) $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ implies $\boldsymbol{\Gamma}' \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ for some finite $\boldsymbol{\Gamma}' \subseteq \boldsymbol{\Gamma}$;
(v) $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ implies $h(\boldsymbol{\Gamma}) \vdash_{\mathcal{S}} h\,\boldsymbol{\alpha}$ for every substitution h.

We often refer to $\mathcal{S}$ simply as a *k-deductive system*. 1-deductive systems are identified with classical deductive systems in the sense of Tarski. Condition 2.1(ii) is called *substitution-invariance*.

By a *k-dimensional sequent*, or simply a *k-sequent*, over Λ we mean a pair $\langle \boldsymbol{\Gamma}, \boldsymbol{\alpha}\rangle$ where $\boldsymbol{\Gamma}$ is a set of k-formulas and $\boldsymbol{\alpha}$ is a single k-formula; the k-sequent is *finite* if $\boldsymbol{\Gamma}$ is finite and *proper* if $\boldsymbol{\Gamma}$ is nonempty. The k--sequent $\langle \boldsymbol{\Gamma}, \boldsymbol{\alpha}\rangle$ is often written in the vertical form $\dfrac{\boldsymbol{\Gamma}}{\boldsymbol{\alpha}}$. A k-formula $\boldsymbol{\alpha}$ is a *theorem* of $\mathcal{S}$ if $\vdash_{\mathcal{S}} \boldsymbol{\alpha}$ (i.e., $\emptyset \vdash_{\mathcal{S}} \boldsymbol{\alpha}$). The k-sequent $\dfrac{\boldsymbol{\Gamma}}{\boldsymbol{\alpha}}$ is called a *rule* of $\mathcal{S}$ if $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$

A k-formula $\boldsymbol{\beta}$ is *directly derivable* from a set $\boldsymbol{\Delta}$ of k-formulas by the k-sequent $\dfrac{\boldsymbol{\Gamma}}{\boldsymbol{\alpha}}$ if there is a substitution $h\colon \mathbf{Fm} \to \mathbf{Fm}$ such that $h(\boldsymbol{\Gamma}) \subseteq \boldsymbol{\Delta}$ and $h\,\boldsymbol{\alpha} = \boldsymbol{\beta}$. $\boldsymbol{\beta}$ is *derivable* from $\boldsymbol{\Delta}$ with respect to a set Ax of k-formulas and a set Ru of k-sequents if it is contained in the smallest set of k-formulas that (1) includes $\boldsymbol{\Delta}$ together with all substitution instances of the k-formulas in Ax and (2) is closed under direct derivability by each k-sequent in Ru. The pair Ax and Ru is called a *presentation* of a k-dimensional deductive system $\mathcal{S}$ if, for every $\boldsymbol{\Delta} \subseteq \mathrm{Fm}^k$ and every $\boldsymbol{\alpha} \in \mathrm{Fm}^k$, $\boldsymbol{\Delta} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ iff $\boldsymbol{\alpha}$ derivable from $\boldsymbol{\Delta}$ with respect to Ax and Ru. The k-formulas of Ax and the k-sequents of Ru are called respectively the *axioms* and *rules of inference*, or *primitive rules*, of the presentation. We assume every k-deductive system is defined by a presentation, and thus we speak freely of the axioms and primitive rules of the system; the presentation however is not normally not explicitly given.

A deductive system $\mathcal{S}'$ is said to an *axiomatic extension* of a second system $\mathcal{S}$ if it has the same language type and a presentation that can be obtained from a presentation of $\mathcal{S}$ by adjoining any number of new axioms but no new inference rules.

Note that, in the recursive definition of derivability, all substitution instances of an axiom are automatically admitted at the base step, and any substitution instance of an inference rule can be used at the recursive step. For this reason axioms and inference rules are more properly viewed as schemes rather than individual k-formulas and k-sequents.

A set $\boldsymbol{T}$ of k-formulas is called a *theory of* $\mathcal{S}$ (a $\mathcal{S}$*-theory* for short) if $\boldsymbol{T} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ implies $\boldsymbol{\alpha} \in \boldsymbol{T}$, for each $\boldsymbol{\alpha} \in \mathrm{Fm}^k$. The set of $\mathcal{S}$-theories is closed under arbitrary intersection and thus forms a complete lattice under the partial ordering of set-theoretical inclusion. The set of all $\mathcal{S}$-theories is denoted $\mathrm{Th}\,\mathcal{S}$ and the lattice of $\mathcal{S}$-theories by $\mathbf{Th}\,\mathcal{S} = \langle \mathrm{Th}\,\mathcal{S}, \cap, \vee \rangle$.

A large number of different logical systems are either k-deductive systems or can be reformalized as k-deductive systems. All the familiar sentential logics together with their various fragments and refinements are naturally formalized as 1-deductive systems—for example, the classical and intuitionistic sentential (CPC and IPC) and all intermediate logics, the various modal logics (including Lewis's S_4 and S_5 when formulated as deductive systems rather than as sets of theorems), and the multiple-valued logics of Łukasiewicz and Post when formulated as 1-deductive systems. Of particular interest from our perspective is the following refinement of a fragment of the intuitionistic propositional calculus.

$\mathrm{IPC}_{\Lambda}^{\to,\top}$*: The* $\{\to, \top\}$*-fragment of* IPC *with additional connectives* Λ*.* Let Λ be an arbitrary language type and let $\to$ and $\top$ be new connectives of rank 2 and 0, respectively. The axioms and rules of inference of $\mathrm{IPC}_{\Lambda}^{\to,\top}$ are:

$$x \to (y \to x),$$
$$(x \to (y \to z)) \to ((x \to y) \to (x \to z)),$$

and, for each $\lambda \in \Lambda$ with n the rank of λ,

$$(x_0 \to y_0) \to ((y_0 \to x_0) \to \ldots \to ((x_{n-1} \to y_{n-1}) \to ((y_{n-1} \to x_{n-1}) \to (\lambda x_0 \ldots x_{n-1} \to \lambda y_0 \ldots y_{n-1}))) \ldots),$$

and the single inference rule

$$\frac{x, x \to y}{y}.$$

The most important example of a 2-deductive system is an equational (or quasi-equational) deductive system. In this context a 2-formula $\langle\alpha,\beta\rangle$ is to be interpreted as the equation $\alpha\approx\beta$.

Free equational logic.[2] Let Λ be any language type. The axioms and rules of inference of the system of free equational logic over Λ are:

(A1) $\langle x,x\rangle$;

(R1) $\dfrac{\langle x,y\rangle}{\langle y,x\rangle}$;

(R2) $\dfrac{\langle x,y\rangle,\langle y,z\rangle}{\langle x,z\rangle}$;

($\mathrm{R3}_\lambda$) $\dfrac{\langle x_0,y_0\rangle,\dots,\langle x_{n-1},y_{n-1}\rangle}{\langle \lambda x_0\dots x_{n-1},\ \lambda y_0\dots y_{n-1}\rangle}$, for each $\lambda\in\Lambda$, n the rank of λ.

The theories of free equational logic are exactly the congruences on the formula algebra $\mathbf{Fm}_\Lambda$.

Applied equational logic. This refers to any one of the extensions of free equational logic by additional axioms and rules of inference. An applied equational logic $\mathcal{S}^{\mathrm{EQL}}\,\mathsf{Q}$ is associated with a each quasivariety Q over the language type Λ. The axioms and rules of inference of $\mathcal{S}^{\mathrm{EQL}}\,\mathsf{Q}$ include (A1), (R1), (R2), and ($\mathrm{R3}_\lambda$), for $\lambda\in\Lambda$, and, if Q is defined by a set Id of identities and a set Qd of quasi-identities, we adjoin the axiom

$$\langle\alpha,\beta\rangle,\quad \text{for every identity } \alpha\approx\beta\in\mathrm{Id},$$

and the inference rule

$$\frac{\langle\gamma_0,\delta_0\rangle,\dots,\langle\gamma_{n-1},\delta_{n-1}\rangle}{\langle\alpha,\beta\rangle},\quad \text{for every quasi-identity}$$

$$\Big(\bigwedge_{i<n}\gamma_i\approx\delta_i\Big)\ \to\ \alpha\approx\beta\in\mathrm{Qd}\,.$$

It is easy to see that any extension of the free equational logic over Λ by new axioms and rules is of the form $\mathcal{S}^{\mathrm{EQL}}\,\mathsf{Q}$ for some quasivariety Q. We define $\mathcal{S}^{\mathrm{EQL}}\,\mathsf{Q}$ to be the *equational logic* of Q.

The equation $\alpha\approx\beta$ will be identified with the 2-formula $\langle\alpha,\beta\rangle$, and the quasi-equation $\gamma_0\approx\delta_0\wedge\dots\wedge\gamma_{n-1}\approx\delta_{n-1}\to\alpha\approx\beta$ will be identified with

[2]Equational logic in this sense differs from the equational logic of identities as it is commonly understood in universal algebra. The latter normally refers to those special applied equational logics (see below) in which Q is a variety. In addition, one is only interested in the identities that can be derived, i.e., the theorems, as opposed to arbitrary quasi-identities.

the 2-sequent

$$\frac{\gamma_0 \approx \delta_0, \dots, \gamma_{n-1} \approx \delta_{n-1}}{\alpha \approx \beta}.$$

In particular, the identities and quasi-identities of Q are identified respectively with the theorems and rules of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$.

Let Q be a quasivariety and let $\boldsymbol{A}$ be a Λ-algebra (not necessarily in Q). A congruence Θ on $\boldsymbol{A}$ is called a Q-*congruence* if $\boldsymbol{A}/\Theta \in \mathsf{Q}$. The set of all Q-congruences on $\boldsymbol{A}$ is denoted by $\mathrm{Co}_{\mathsf{Q}}\,\boldsymbol{A}$. The theories of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$ are just the Q-congruences on the formula algebra $\mathbf{Fm}_\Lambda$.

$\mathrm{Co}_{\mathsf{Q}}\,\boldsymbol{A}$ is always closed under arbitrary intersections and the unions of directed sets. Thus it forms an algebraic lattice with set-theoretical intersection as the infinite meet operation. Given any $R \subseteq A^2$, the intersection of all Q-congruences on $\boldsymbol{A}$ that include R is denoted by $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}\,R$, and is called the Q-congruence *generated* by R.

Quasivarieties with the following property, especially those that are varieties, have been extensively studied; see [7] and the references given there.

Definition 2.2. A quasivariety Q has *equationally definable principal relative congruences* (EDPRC) if there is a finite system of equations $\alpha_i(x_0, x_1, y_0, y_1) \approx \beta_i(x_0, x_1, y_0, y_1)$, $i < n$, in four variables, such that for every $\boldsymbol{A} \in \mathsf{Q}$ and all $a, b, c, d \in A$,

$$c \equiv d \pmod{\mathrm{Cg}_{\mathsf{Q}}(a,b)} \quad \text{iff} \quad \alpha_i^{\boldsymbol{A}}(a,b,c,d) = \beta_i^{\boldsymbol{A}}(a,b,c,d) \text{ for } i < n.$$

2.1. **Regularly algebraizable 1-deductive systems and their equivalent quasivarieties.** It is often the case that a 1-deductive system that has at least one theorem has a canonical one in the sense that Λ contains a constant symbol $\top$ with the property that $\top$ is a theorem. In the context of this paper there is only a slight loss of generality in restricting ourselves to 1-deductive systems of this kind because almost all the 1-deductive systems we deal with have at least one theorem and also satisfy the so-called G-rule, which implies that theorems are all equivalent in a natural sense described below. If such a system does not have a canonical theorem we can adjoin a new constant symbol $\top$ to the language and add the axiom $\top$. Since all theorems are equivalent, this turns out to have little effect on metalogical properties of the system. So, in the sequel we assume all 1-deductive systems have a constant symbol $\top$ in their language type and that $\top$ is a theorem of the system; language types and 1-deductive systems of this kind are called *pointed*. Analogously, all quasivarieties are assumed to be pointed, with the distinguished element represented by the constant symbol $\top$.

Let Q be a pointed quasivariety. By the *assertional logic* of Q, as opposed to its equational logic, we mean the 1-deductive system determined by the class of all matrices of the form $\langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\}\rangle$, where $\boldsymbol{A} \in \mathsf{Q}$. The assertional logic of Q is denoted by $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$. It is easy to see that $\alpha_0, \dots, \alpha_{n-1} \vdash_{\boldsymbol{\mathcal{S}}^{\text{ASL}}\mathsf{Q}} \beta$ iff

$$\frac{\alpha_0 \approx \top, \dots, \alpha_{n-1} \approx \top}{\beta \approx \top}$$

is a quasi-identity of Q. The connection between the assertional and equational logics of Q is given by the equivalence.

$$\alpha_0, \dots, \alpha_{n-1} \vdash_{\boldsymbol{\mathcal{S}}^{\text{ASL}}\mathsf{Q}} \beta \quad \text{iff} \quad \alpha_0 \approx \top, \dots, \alpha_{n-1} \approx \top \vdash_{\boldsymbol{\mathcal{S}}^{\text{EQL}}\mathsf{Q}} \beta \approx \top.$$

Definition 2.3. Let Λ be an arbitrary language type and let

(2) $$E(x, y) = \{\varepsilon_0(x, y), \dots, \varepsilon_{n-1}(x, y)\}$$

be a finite nonempty set of formulas in two variables over Λ. A 1-deductive system $\mathcal{S}$ over Λ is *E-regularly algebraizable* if the following formulas and sequents are respectively theorems and rules of $\mathcal{S}$.

(i) $E(x, x)$ (i.e., $\varepsilon_i(x, x)$ is a theorem for each $i < n$);

(ii) $\dfrac{E(x, y)}{E(y, x)}$ (i.e., $\dfrac{E(x, y)}{\varepsilon_i(x, y)}$ is a rule for each $i < n$);

(iii) $\dfrac{E(x, y)\ ,\ E(y, z)}{E(x, z)}$;

(iv) $\dfrac{E(x_0, y_0)\ , \dots,\ E(x_{n-1}, y_{n-1})}{E(\lambda x_0 \dots x_{n-1},\ \lambda y_0 \dots y_{n-1})}$, for each $\lambda \in \Lambda$ (n is the rank of λ);

(v) $\dfrac{x, E(x, y)}{y}$ (E-modus ponens);

(vi) $\dfrac{x, y}{E(x, y)}$ (E-G-rule).

We say that $\mathcal{S}$ is *regularly algebraizable* if $\mathcal{S}$ is E-regularly algebraizable for some finite nonempty set $E(x, y)$ of binary formulas. E is called a *(finite) equivalence system* for $\mathcal{S}$.

Note that every axiomatic extension of a regularly algebraizable 1-deductive system is regularly algebraizable with the same equivalence system.

Every regularly algebraizable 1-deductive system is algebraizable in the sense of [4, Def. 2.10]; see [4, Corollary 4.8] or [13, Theorem 1.28].

A pointed quasivariety Q is said to be *relatively point-regular* if, for every Λ-algebra $\boldsymbol{A}$, each Q-congruence of $\boldsymbol{A}$ is uniquely determined by its $\top^{\boldsymbol{A}}$-equivalence class, i.e., for each Λ-algebra $\boldsymbol{A}$ and all $\Theta, \Phi \in \mathrm{Co}_{\mathsf{Q}}\,\boldsymbol{A}$, $\top^{\boldsymbol{A}}/\Theta = \top^{\boldsymbol{A}}/\Phi$ implies $\Theta = \Phi$.

The following characterization of relatively point-regular quasivarieties is the analog of a well-known result for pointed varieties (see [18]).

Theorem 2.4 ([13, Theorem 1.36]). *A pointed quasivariety* Q *is relatively point-regular iff there is a finite nonempty set of formulas* (2) *such that*

$$\varepsilon_i(x,x) \approx \top, \quad \textit{for each } i < n,$$

is an identity of Q*, and*

$$\frac{\varepsilon_0(x,y) \approx \top, \dots, \varepsilon_{n-1}(x,y) \approx \top}{x \approx y}$$

is a quasi-identity of Q.

We say that Q is E*-relatively-point-regular* in this case.

Theorem 2.5 ([13, Theorem 1.34]). *A deductive system is regularly algebraizable iff it is the assertional logic of a relatively point-regular quasivariety.*

By this theorem we see that, for a fixed but arbitrary pointed language type, there is a one-one correspondence between relatively point-regular quasivarieties and regularly algebraizable deductive systems. Every relatively point-regular quasivariety determines a unique regularly algebraizable 1-deductive system, its assertional logic. Conversely, every regularly algebraizable deductive system $\mathcal{S}$ is the assertional logic of a unique relatively point-regular quasivariety Q, which is the equivalent quasivariety of $\mathcal{S}$ in the sense of [4].

From the fact that $\mathcal{S}$ is the assertional logic of Q we have that α is a theorem of $\mathcal{S}$ iff $\alpha \approx \top$ is an identity of Q, and $\dfrac{\alpha_0, \dots, \alpha_{n-1}}{\beta}$ is a rule of $\mathcal{S}$ iff $\dfrac{\alpha_0 \approx \top, \dots, \alpha_{n-1} \approx \top}{\beta \approx \top}$ is a quasi-identity of Q. And from the fact that Q is the equivalent quasivariety of $\mathcal{S}$ we have that an equation $\alpha \approx \beta$ is an identity of Q iff each of the formulas of $E(\alpha,\beta)$, where $E(x,y)$ is an equivalence system for $\mathcal{S}$, is a theorem of $\mathcal{S}$, and the quasi-equation $\dfrac{\gamma_0 \approx \delta_0, \dots, \gamma_{n-1} \approx \delta_{n-1}}{\alpha \approx \beta}$ is a quasi-identity of Q iff each sequent of

$$\frac{E(\gamma_0,\delta_0)\,, \dots,\; E(\gamma_{n-1},\delta_{n-1})}{E(\alpha,\beta)}$$

is a rule of $\mathcal{S}$.

In [4, Theorem 2.17] an algorithm is given for constructing an equational definition of the equivalent quasivariety of an algebraizable 1-deductive system $\mathcal{S}$ from a given axiomatization of $\mathcal{S}$; a minor improvement can be found in [13, Theorem 1.13]. When the algorithm is applied to regularly algebraizable 1-deductive systems we get the following result.

Theorem 2.6 ([13, Theorem 1.30]). *Let $\mathcal{S}$ be a 1-deductive system presented by a set* Ax *of axioms and a set* Ru *of proper inference rules. Assume $\mathcal{S}$ is regularly algebraizable with equivalence system $E(x,y) = \{\varepsilon_0(x,y),\dots,\varepsilon_{n-1}(x,y)\}$. Then the unique equivalent relatively point-regular quasivariety of $\mathcal{S}$ is defined by the identities*

(i) $\varphi \approx \top$, *for each* $\varphi \in \mathrm{Ax}$

together with the following quasi-identities

(ii) $\dfrac{\alpha_0 \approx \top,\dots,\alpha_{m-1} \approx \top}{\beta \approx \top}$, *for each rule* $\dfrac{\alpha_0,\dots,\alpha_{m-1}}{\beta}$ *in* Ru;

(iii) $\dfrac{E(x,y) \approx \top}{x \approx y}$.

2.2. **Fregean 1-deductive systems and quasivarieties.** Let $\mathcal{S}$ be a 1-deductive system over an arbitrary language type Λ. For each $\Gamma \subseteq \mathrm{Fm}$, let $\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\,\Gamma$ be the binary relation between formulas defined by

$$\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\,\Gamma := \{\, \langle \alpha,\beta\rangle \in \mathrm{Fm}^2 : \Gamma,\alpha \vdash_{\mathcal{S}} \beta \text{ and } \Gamma,\beta \vdash_{\mathcal{S}} \alpha \,\}.$$

Definition 2.7. Let $\mathcal{S}$ be a deductive system over an arbitrary language type Λ. $\mathcal{S}$ is *Fregean* if $\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\,\Gamma$ is a congruence relation on the formula algebra $\mathbf{Fm}$ for every $\Gamma \subseteq \mathrm{Fm}$.

A 1-deductive system $\mathcal{S}$ is *protoalgebraic* iff there is a finite set of binary formulas $\Delta(x,y) = \{\delta_0(x,y),\dots,\delta_{n-1}(x,y)\}$ such that $\delta_i(x,x)$ is a theorem of $\mathcal{S}$, for each $i < n$, and $\dfrac{x,\Delta(x,y)}{y}$ is a rule of $\mathcal{S}$. (See [6, Theorem 13.2].) $\Delta(x,y)$ is called a *protoequivalence system* for $\mathcal{S}$. Protoalgebraic Fregean 1-deductive systems are investigated in some detail in [13].

Theorem 2.8 ([13, Theorem 2.18]). *Every protoalgebraic, Fregean 1-deductive system with at least one theorem is regularly algebraizable.*

In fact, if $\Delta(x,y)$ is a protoequivalence system for a protoalgebraic, Fregean deductive system, then $\Delta(x,y) \cup \Delta(y,x)$ is a finite equivalence system and $\mathcal{S}$ is $(\Delta(x,y) \cup \Delta(y,x))$-regularly algebraizable.

Definition 2.9. A pointed quasivariety Q is *relatively congruence-orderable* if, for any $\boldsymbol{A} \in \mathsf{Q}$ and all $a,b \in A$, $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(a,\top^{\boldsymbol{A}}) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(b,\top^{\boldsymbol{A}})$ implies $a = b$.

The term "congruence-orderable" was first used in [21], where it is applied to varieties. It comes from the fact that Q is congruence-orderable iff the relation that holds between elements a and b of $\boldsymbol{A}$ when $b \in \mathrm{Cg}_{\mathsf{Q}}(a,\top^{\boldsymbol{A}})$, which in general is a quasi-ordering, is in fact a partial ordering of A. In [8] essentially the same notion is called "fission free". Congruence-orderable varieties were introduced in [23] where they were called "Fregean". But in light of the next theorem it seems more appropriate to reserve this latter term for (relatively) point-regular Fregean varieties (and quasivarieties) that are (relatively) congruence-orderable. However, in this paper we will not use the term "Fregean" in reference to varieties or quasivarieties.

Theorem 2.10 ([13, Theorem 3.11])**.** *A relatively point-regular quasivariety is relatively congruence-orderable iff its assertional logic is Fregean.*

3. The Multiterm Deduction-Detachment Theorem in k-Deductive Systems

We investigate a generalized notion of the deduction theorem in k-deductive systems. This gives a uniform treatment of the deduction theorem in a general class of logical systems that includes classical deductive systems in the Tarski sense as well as systems of equational logic.

Let Λ be a language type. Let $P(\boldsymbol{x},\boldsymbol{y})$ be a finite nonempty sequence of k-formulas in the two k-variables $\boldsymbol{x} = \langle x_0,\dots,x_{k-1}\rangle$ and $\boldsymbol{y} = \langle y_0,\dots,y_{k-1}\rangle$. $\boldsymbol{x} \Rightarrow_i \boldsymbol{y}$ will represent the i-th component of $P(\boldsymbol{x},\boldsymbol{y})$, a k-formula, and we write $\boldsymbol{x} \Rightarrow_P \boldsymbol{y}$ in place of $P(\boldsymbol{x},\boldsymbol{y})$. Thus $\boldsymbol{x} \Rightarrow_P \boldsymbol{y} := \langle \boldsymbol{x} \Rightarrow_0 \boldsymbol{y},\dots,\boldsymbol{x} \Rightarrow_{n-1} \boldsymbol{y}\rangle$, where, for $i = 0,\dots,n-1$,

$$\boldsymbol{x} \Rightarrow_i \boldsymbol{y} = \langle \varphi^i_0(x_0,\dots,x_{k-1},y_0,\dots,y_{k-1}),\dots,\\ \varphi^i_{k-1}(x_0,\dots,x_{k-1},y_0,\dots,y_{k-1})\rangle,$$

for some formulas φ^i_j, with $i < n, j < k$, in $2k$-variables. If $\boldsymbol{\alpha},\boldsymbol{\beta}$ are k-formulas, then $\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}$ is the result of substituting $\boldsymbol{\alpha}$ for $\boldsymbol{x}$ and $\boldsymbol{\beta}$ for $\boldsymbol{y}$. More precisely, if $\boldsymbol{\alpha} = \langle \alpha_0,\dots,\alpha_{k-1}\rangle$ and $\boldsymbol{\beta} = \langle \beta_0,\dots,\beta_{k-1}\rangle$, then

$$\boldsymbol{\alpha} \Rightarrow_i \boldsymbol{\beta} = \langle \varphi^i_0(\alpha_0,\dots,\alpha_{k-1},\beta_0,\dots,\beta_{k-1}),\dots,\\ \varphi^i_{k-1}(\alpha_0,\dots,\alpha_{k-1},\beta_0,\dots,\beta_{k-1})\rangle\,.$$

Next we extend the definition of $\Rightarrow_P$ to arbitrary finite nonempty sequences of k-formulas. The set of all finite sequences of k-formulas is denoted by $(\mathrm{Fm}^k)^*$. Elements of $(\mathrm{Fm}^k)^*$ are represented by boldface lower case Greek letters with a bar over them, e.g., $\langle \boldsymbol{\alpha}_0,\dots,\boldsymbol{\alpha}_{n-1}\rangle \in (\mathrm{Fm}^k)^m$ is represented by $\overline{\boldsymbol{\alpha}}$. We write $\operatorname{rng}\overline{\boldsymbol{\alpha}}$ for the range $\{\boldsymbol{\alpha}_0,\dots,\boldsymbol{\alpha}_{n-1}\}$ of $\overline{\boldsymbol{\alpha}}$. Sets of finite sequences of k-formulas are represented by boldface uppercase Greek letters with an over bar, e.g., $\overline{\boldsymbol{\Gamma}}$, $\overline{\boldsymbol{\Delta}}$, etc. If $h\colon \mathbf{Fm} \to \mathbf{Fm}$ is a substitution, then $h\,\overline{\boldsymbol{\alpha}} = \langle h\,\boldsymbol{\alpha}_0,\dots,h\,\boldsymbol{\alpha}_{n-1}\rangle$ and $h(\overline{\boldsymbol{\Gamma}}) = \{\, h\,\overline{\boldsymbol{\alpha}} : \overline{\boldsymbol{\alpha}} \in \overline{\boldsymbol{\Gamma}}\,\}$.

If $\boldsymbol{\alpha} \in \mathrm{Fm}^k$ and $\overline{\boldsymbol{\beta}} = \langle \boldsymbol{\beta}_0,\dots,\boldsymbol{\beta}_{m-1}\rangle \in (\mathrm{Fm}^k)^m$, we define

$$\boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}} := (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}_0) \frown (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}_1) \frown \cdots \frown (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}_{m-1}),$$

i.e., the concatenation of the sequences $\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}_i$ for $i = 0,\dots,m-1$. Finally, given two sequences of k-formulas $\overline{\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}_0,\dots,\boldsymbol{\alpha}_{l-1}\rangle$ and $\overline{\boldsymbol{\beta}} = \langle \boldsymbol{\beta}_0,\dots,\boldsymbol{\beta}_{m-1}\rangle$, we define

$$\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}} := \boldsymbol{\alpha}_0 \Rightarrow_P (\boldsymbol{\alpha}_1 \Rightarrow_P (\cdots \Rightarrow_P (\boldsymbol{\alpha}_{l-1} \Rightarrow_P \overline{\boldsymbol{\beta}})\cdots)),$$

i.e., successive applications of the operations $\boldsymbol{\alpha}_i \Rightarrow_P *$ for $i = l-1, l-2,\dots,0$. Finally, let $\varphi(v_0,\dots,v_{n-1})$ be any formula constructed from the symbol "$\Rightarrow_P$" and the variable symbols $v_0,\dots,v_{n-1}$. Then for all $\overline{\boldsymbol{\alpha}}_0,\dots,\overline{\boldsymbol{\alpha}}_{n-1} \in (\mathrm{Fm}^k)^*$, $\varphi(\overline{\boldsymbol{\alpha}}_0,\dots,\overline{\boldsymbol{\alpha}}_{n-1})$ is defined in the usual way by recursion on the structure of φ. $\varphi(\overline{\boldsymbol{\alpha}}_0, \dots, \overline{\boldsymbol{\alpha}}_{n-1})$ is called an *iteration instance* of $\varphi(v_0, \dots, v_{n-1})$.

We emphasize that in the construction of the iteration instance, the iteration instance $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$ of $\boldsymbol{x} \Rightarrow_P \boldsymbol{y}$ for instance, $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$ play very different roles. If $\overline{\boldsymbol{\beta}}'$ is a permutation of $\overline{\boldsymbol{\beta}}$, then $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}'$ is a permutation of $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$, and with respect to the consequence relation of $\mathcal{S}$, they are for most purposes indistinguishable. But if $\overline{\boldsymbol{\alpha}}'$ is a permutation of $\overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\alpha}}' \Rightarrow_P \overline{\boldsymbol{\beta}}$ is not a permutation of $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$ and their metalogical properties may be very different, unless $\mathcal{S}$ has very special properties.

Lemma 3.1. *For any substitution* $h: \mathbf{Fm} \to \mathbf{Fm}$ *and all* $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}} \in (\mathrm{Fm}^k)^*$, $h(\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}) = h\overline{\boldsymbol{\alpha}} \Rightarrow_P h\overline{\boldsymbol{\beta}}$.

Proof. An easy induction based on the recursive definition of $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$. □

We introduce some special notation for the consequence relation of $\mathcal{S}$ when sets of k-formulas are involved that will prove very useful in the sequel. $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}, \boldsymbol{\beta}$ means $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ and $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\beta}$. $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\Delta}$ means $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\delta}$ for every $\boldsymbol{\delta} \in \boldsymbol{\Delta}$. $\boldsymbol{\Gamma} \dashv\vdash_{\mathcal{S}} \boldsymbol{\Delta}$ means $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\Delta}$ and $\boldsymbol{\Delta} \vdash_{\mathcal{S}} \boldsymbol{\Gamma}$. With regard to the consequence relation, a sequence of k-formulas will be identified with the set k-formulas that forms its range. Thus for example, if $\overline{\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}_0, \dots, \boldsymbol{\alpha}_{n-1} \rangle$ and $\overline{\boldsymbol{\beta}} = \langle \boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_{m-1} \rangle$, then $\overline{\boldsymbol{\alpha}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}$ means $\{\boldsymbol{\alpha}_0, \dots, \boldsymbol{\alpha}_{n-1}\} \vdash_{\mathcal{S}} \{\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_{m-1}\}$. More generally, if $\overline{\boldsymbol{\Gamma}} \cup \overline{\boldsymbol{\Delta}} \subseteq (\mathrm{Fm}^k)^*$, then $\overline{\boldsymbol{\Gamma}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\Delta}}$ means

$$\bigcup \{ \operatorname{rng} \overline{\boldsymbol{\gamma}} : \overline{\boldsymbol{\gamma}} \in \boldsymbol{\Gamma} \} \vdash_{\mathcal{S}} \bigcup \{ \operatorname{rng} \overline{\boldsymbol{\delta}} : \overline{\boldsymbol{\delta}} \in \boldsymbol{\Delta} \}.$$

We will speak of a sequence $\overline{\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}_0, \dots, \boldsymbol{\alpha}_{n-1} \rangle$ of k-formulas as being *a theorem of* $\mathcal{S}$ when $\vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}$. What we mean of course is that each of the components $\boldsymbol{\alpha}_i$ of $\overline{\boldsymbol{\alpha}}$ is a theorem. Similar references that treat a sequence of k-formulas as a single k-formula will be used further warning.

A caveat: the iteration instance of a $(\Rightarrow_P)$-formula φ is not in general equal to a substitution instance of φ, even in the 1-dimensional case, and it is important that the two not be confused. For example, let $P = \langle x \to y \rangle$, a sequence of a single 1-formula, and consider the sequence $\langle z, w \rangle$ of two 1-formulas (variables) z and w. Then

$$\langle z, w \rangle \Rightarrow_P \langle z, w \rangle = \big\langle z \to (w \to z),\ z \to (w \to w) \big\rangle,$$

which does not coincide with any substitution instance of $v \Rightarrow_P v$. The distinction is important because, if a $(\Rightarrow_P)$-formula is a theorem of a deductive system, then all substitution instances of it are also theorems automatically because of the property of substitution-invariance, but this need not be true of its iteration instances. A similar remark applies to the rules of the system. But the axioms and inference rules of the deductive systems we consider presently are such that, collectively, they do imply their iteration instances. This is what makes the notation we have just introduced so useful.

A k-deductive system $\mathcal{S}$ over Λ is said to have *modus ponens* or *detachment with respect to* $\boldsymbol{x} \Rightarrow_P \boldsymbol{y}$ (*P-modus ponens* or *P-detachment* for short) if the k-sequent $\dfrac{\boldsymbol{x},\, \boldsymbol{x} \Rightarrow_P \boldsymbol{y}}{\boldsymbol{y}}$ is a rule of $\mathcal{S}$, i.e., if, for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{Fm}^k$,

$$(\mathrm{MP}_P) \qquad \boldsymbol{\alpha},\, \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta} \vdash_{\mathcal{S}} \boldsymbol{\beta}.$$

P-modus ponens is a rule that implies all of its following iteration instances:

Lemma 3.2. *Assume $\mathcal{S}$ has P-modus ponens, i.e., that P-modus ponens is a rule of $\mathcal{S}$. Then for all $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}} \in (\mathrm{Fm}^k)^*$,*

$$(\mathrm{MP}^{\mathrm{I}}_P) \qquad \overline{\boldsymbol{\alpha}},\, \overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}.$$

Proof. The proof goes by induction on the length of $\overline{\boldsymbol{\alpha}}$ as a sequence of k-formulas. Assume first of all that $\overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$, a single k-formula, and $\overline{\boldsymbol{\beta}} = \langle \boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_{n-1} \rangle$. Then by (MP_P),

$$\boldsymbol{\alpha},\, \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}_0, \dots, \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}_{n-1} \vdash_{\mathcal{S}} \boldsymbol{\beta}_i, \quad \text{for all } i < n,$$

i.e., $\boldsymbol{\alpha},\, \boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}$. Assume now that $\overline{\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}_0, \dots, \boldsymbol{\alpha}_{m-1} \rangle$, with $m > 1$. Let $\overline{\boldsymbol{\alpha}}' = \langle \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{m-1} \rangle$, so that $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}} = \boldsymbol{\alpha}_0 \Rightarrow_P (\overline{\boldsymbol{\alpha}}' \Rightarrow_P \overline{\boldsymbol{\beta}})$. Let $\overline{\boldsymbol{\alpha}}' \Rightarrow_P \overline{\boldsymbol{\beta}} = \langle \boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_{l-1} \rangle$. By (MP_P)

$$\boldsymbol{\alpha}_0, \dots, \boldsymbol{\alpha}_{m-1}, \boldsymbol{\alpha}_0 \Rightarrow_P \boldsymbol{\gamma}_0, \dots, \boldsymbol{\alpha}_0 \Rightarrow_P \boldsymbol{\gamma}_{l-1} \vdash_{\mathcal{S}} \boldsymbol{\gamma}_j, \quad \text{for all } j < l,$$

i.e., $\overline{\boldsymbol{\alpha}},\, \boldsymbol{\alpha}_0 \Rightarrow_P (\overline{\boldsymbol{\alpha}}' \Rightarrow_P \overline{\boldsymbol{\beta}}) \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}' \Rightarrow_P \overline{\boldsymbol{\beta}}$. Trivially, $\overline{\boldsymbol{\alpha}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}'$. Thus by the induction hypothesis $\overline{\boldsymbol{\alpha}},\, \overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}$. □

We say that $\mathcal{S}$ has the *deduction theorem with respect to* $\boldsymbol{x} \Rightarrow_P \boldsymbol{y}$, or the *P-deduction theorem* for short, if, for all $\boldsymbol{\Gamma} \cup \{\boldsymbol{\alpha}, \boldsymbol{\beta}\} \subseteq \mathrm{Fm}^k$, the following relationship between $\mathcal{S}$-entailments holds.

$$(\mathrm{DT}_P) \qquad \boldsymbol{\Gamma}, \boldsymbol{\alpha} \vdash_{\mathcal{S}} \boldsymbol{\beta} \quad \text{implies} \quad \boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}.$$

Note that the converse of the (DT_P) is just (MP_P). $\mathcal{S}$ has the *P-deduction-detachment theorem* if it has both P-modus ponens and the P-deduction theorem. $\mathcal{S}$ is said to have the *multiterm deduction-detachment theorem* if it has the P-deduction-detachment theorem for some P. In this case P is said to be a *multiterm deduction-detachment system* for $\mathcal{S}$. $\mathcal{S}$ has the *uniterm deduction-detachment theorem* if it has the P-deduction-detachment theorem for some P consisting of a single binary k-formula $\langle \boldsymbol{x} \Rightarrow \boldsymbol{y} \rangle$, which is called a *uniterm deduction-detachment formula* for $\mathcal{S}$. All the intermediate logics, i.e., all axiomatic extensions of IPC included in CPC, have the uniterm deduction-detachment theorem with respect to $\langle x \to y \rangle$, and the normal modal logics S_4 and S_5 have it with respect to $\langle \Box x \to y \rangle$.

The properly multiterm deduction-detachment theorem is not uncommon among applied equational logics.

Theorem 3.3 ([7, Theorem 4.4]). *Let* Q *be a quasivariety. For any quasivariety* Q, $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$ *has the multiterm deduction-detachment theorem iff* Q *has equationally definable principal relative congruences* (EDPRC).

For example, let V be the variety of bounded distributive lattices. Then $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{V}$ has multiterm deduction-detachment system $P = \big\langle \langle x_0, x_1\rangle \Rightarrow_0 \langle y_0, y_1\rangle, \langle x_0, x_1\rangle \Rightarrow_1 \langle y_0, y_1\rangle \big\rangle$, with

$$\langle x_0, x_1\rangle \Rightarrow_0 \langle y_0, y_1\rangle = y_0 \wedge x_0 \wedge x_1 \approx y_1 \wedge x_0 \wedge x_1,$$
$$\langle x_0, x_1\rangle \Rightarrow_1 \langle y_0, y_1\rangle = y_0 \vee x_0 \vee x_1 \approx y_1 \vee x_0 \vee x_1.$$

This follows from the above theorem and the well known fact that, in bounded distributive lattices, principal congruences are defined by the equations $y_0 \wedge x_0 \wedge x_1 \approx y_1 \wedge x_0 \wedge x_1$ and $y_0 \vee x_0 \vee x_1 \approx y_1 \vee x_0 \vee x_1$. (See for example [22]).

As in the case of (MP_P) the P-deduction theorem implies all of its iteration instances:

Lemma 3.4. *Assume that the* P*-deduction theorem holds for* $\mathcal{S}$*. Then, for all* $\boldsymbol{\Gamma} \subseteq \mathrm{Fm}^k$ *and* $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}} \in (\mathrm{Fm}^k)^*$,

$$(\mathrm{DT}^{\mathrm{I}}_P) \qquad \boldsymbol{\Gamma}, \overline{\boldsymbol{\alpha}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}} \quad \textit{implies} \quad \boldsymbol{\Gamma} \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}.$$

Proof. The proof is by induction on the length of $\overline{\boldsymbol{\alpha}}$ and is similar to the proof of Lem. 3.2. We omit the details. □

If $\mathcal{S}$ has the P-deduction theorem, then it is easy to see that, for all $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathrm{Fm}^k$ and $\overline{\boldsymbol{\beta}} \in (\mathrm{Fm}^k)^*$,

$$\boldsymbol{\alpha} \Rightarrow_P (\boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}}) \dashv\vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}},$$
$$\boldsymbol{\alpha} \Rightarrow_P (\boldsymbol{\alpha}' \Rightarrow_P \overline{\boldsymbol{\beta}}) \dashv\vdash_{\mathcal{S}} \boldsymbol{\alpha}' \Rightarrow_P (\boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}}),$$

and, for $\boldsymbol{\alpha} \in \mathrm{Fm}^k$ and $\overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}' \in (\mathrm{Fm}^k)^*$,

$$\overline{\boldsymbol{\beta}} \dashv\vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}' \quad \text{implies} \quad \boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}} \dashv\vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \overline{\boldsymbol{\beta}}'.$$

Using these facts a straightforward induction argument shows that, for all $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}', \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}' \in (\mathrm{Fm}^k)^*$ such that $\operatorname{rng} \overline{\boldsymbol{\alpha}} = \operatorname{rng} \overline{\boldsymbol{\alpha}}'$ and $\operatorname{rng} \overline{\boldsymbol{\beta}} = \operatorname{rng} \overline{\boldsymbol{\beta}}'$,

$$\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}} \dashv\vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}' \Rightarrow_P \overline{\boldsymbol{\beta}}'.$$

For this reason, in the presence of the P-deduction theorem, we have the option of writing $\boldsymbol{\Gamma} \Rightarrow_P \boldsymbol{\Delta}$ for finite sets $\boldsymbol{\Gamma}, \boldsymbol{\Delta}$ of k-formulas, with the understanding that we mean $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$, where $\operatorname{rng} \overline{\boldsymbol{\alpha}} = \boldsymbol{\Gamma}$ and $\operatorname{rng} \overline{\boldsymbol{\beta}} = \boldsymbol{\Delta}$.

It follows directly from the definition of $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$ that for arbitrary finite sequences $\overline{\boldsymbol{\alpha}}_0, \dots, \overline{\boldsymbol{\alpha}}_{n-1}$ and $\overline{\boldsymbol{\beta}}$ of k-formulas we have

$$\overline{\boldsymbol{\alpha}}_0 \Rightarrow_P (\overline{\boldsymbol{\alpha}}_1 \Rightarrow_P \dots \Rightarrow_P (\overline{\boldsymbol{\alpha}}_{n-1} \Rightarrow_P \overline{\boldsymbol{\beta}}) \dots) = \overline{\boldsymbol{\alpha}}_0 \frown \overline{\boldsymbol{\alpha}}_1 \frown \dots \frown \overline{\boldsymbol{\alpha}}_{n-1} \Rightarrow_P \overline{\boldsymbol{\beta}},$$

where the equality here means lexical identity, not just logical equivalence. Consequently, $(\mathrm{MP}^{\mathrm{I}}_P)$ can be written in either one of the following two more suggestive forms

$$\overline{\boldsymbol{\alpha}}_0, \overline{\boldsymbol{\alpha}}_1, \dots, \overline{\boldsymbol{\alpha}}_{n-1}, \overline{\boldsymbol{\alpha}}_0 \Rightarrow_P (\overline{\boldsymbol{\alpha}}_1 \Rightarrow_P \dots \Rightarrow_P (\overline{\boldsymbol{\alpha}}_{n-1} \Rightarrow_P \overline{\boldsymbol{\beta}}) \dots) \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}},$$

$$\overline{\boldsymbol{\alpha}}_0, \overline{\boldsymbol{\alpha}}_1, \dots, \overline{\boldsymbol{\alpha}}_{n-1}, \overline{\boldsymbol{\alpha}}_0 \frown \overline{\boldsymbol{\alpha}}_1 \frown \dots \frown \overline{\boldsymbol{\alpha}}_{n-1} \Rightarrow_P \overline{\boldsymbol{\beta}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}},$$

where $\overline{\boldsymbol{\alpha}}_0, \dots, \overline{\boldsymbol{\alpha}}_{n-1}, \overline{\boldsymbol{\beta}}$ are arbitrary finite sequences of k-formulas. There are of course similar forms for $(\mathrm{DT}^{\mathrm{I}}_P)$.

Note that the metalinguistic operation of concatenation has many of the formal properties of a conjunction operation in the object language. We will make use of this fact at several places in the sequel. For example, we introduce the abbreviation

$$\overline{\boldsymbol{\alpha}} \Leftrightarrow_P \overline{\boldsymbol{\beta}} := (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}) \frown (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\alpha}}).$$

Thus, if both (MP_P) and (DT_P) hold in $\mathcal{S}$, then $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}} \Leftrightarrow_P \overline{\boldsymbol{\beta}}$ iff both $\boldsymbol{\Gamma}, \overline{\boldsymbol{\alpha}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}$ and $\boldsymbol{\Gamma}, \overline{\boldsymbol{\beta}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}$, by virtue of the meaning of the notation $\vdash_{\mathcal{S}}$ for sequences discussed earlier in this section. Another caveat is in order here however. Concatenation of sequences of formulas should not be confused with the existence of an actual conjunction connective.

Theorem 3.5. *Let Λ be a language type and $P(\boldsymbol{x}, \boldsymbol{y})$ a finite nonempty sequence of binary k-formulas. Let $\mathcal{S}$ be a k-deductive system that has the P-deduction-detachment theorem, i.e., both P-modus ponens and the P-deduction theorem. Then for all $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\gamma}} \in (\mathrm{Fm}^k)^*$,*

$(\mathrm{D0}^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\alpha}}$*;*
$(\mathrm{D1}^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\alpha}})$*;*
$(\mathrm{D2}^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}})\big) \Rightarrow_P \big((\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}) \Rightarrow_P (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\gamma}})\big)$*.*

Moreover, $\mathcal{S}$ has a presentation such that (MP_P) *is the only proper rule of inference.*

Proof. $(\mathrm{D0}^{\mathrm{I}}_P)$. $\overline{\boldsymbol{\alpha}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}$. Thus $\vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\alpha}}$ by $(\mathrm{DT}^{\mathrm{I}}_P)$.

$(\mathrm{D1}^{\mathrm{I}}_P)$. $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}} \vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}}$. Thus $\vdash_{\mathcal{S}} \overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\alpha}})$ by two iteration instances of the P-deduction theorem.

$(\mathrm{D2}^{\mathrm{I}}_P)$. By two applications of iterated P-modus ponens we have $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}} \Rightarrow \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}}) \vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}}$. Another application gives $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}} \Rightarrow \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}}) \vdash_{\mathcal{S}} \overline{\boldsymbol{\gamma}}$. Two iteration instances of the P-deduction theorem now give $(\mathrm{D2}^{\mathrm{I}}_P)$.

Let $\dfrac{\overline{\boldsymbol{\alpha}}_0, \dots, \overline{\boldsymbol{\alpha}}_{n-1}}{\overline{\boldsymbol{\beta}}}$ be any rule of $\mathcal{S}$. Then, by ($\mathrm{DT}^{\mathrm{I}}_P$), $\overline{\boldsymbol{\alpha}}_0 \frown \dots \frown \overline{\boldsymbol{\alpha}}_{n-1} \Rightarrow_P \overline{\boldsymbol{\beta}}$ is a theorem of $\mathcal{S}$. By adjoining (MP_P) as a primitive rule (if not already one) and replacing each primitive rule of $\mathcal{S}$ (other than (MP_P)) by its corresponding theorem as an axiom we obtain a presentation of $\mathcal{S}$ with (MP_P) as the only primitive rule. □

We consider now three axiom schemes, in the k-variables $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, whose iteration instances are ($\mathrm{D0}^{\mathrm{I}}_P$)–($\mathrm{D2}^{\mathrm{I}}_P$) respectively.

($\mathrm{D0}_P$) $$\boldsymbol{x} \Rightarrow_P \boldsymbol{x},$$

($\mathrm{D1}_P$) $$\boldsymbol{x} \Rightarrow_P (\boldsymbol{y} \Rightarrow_P \boldsymbol{x}),$$

($\mathrm{D2}_P$) $$\big(\boldsymbol{x} \Rightarrow_P (\boldsymbol{y} \Rightarrow_P \boldsymbol{z})\big) \Rightarrow_P \big((\boldsymbol{x} \Rightarrow_P \boldsymbol{y}) \Rightarrow_P (\boldsymbol{x} \Rightarrow_P \boldsymbol{z})\big).$$

We emphasize once again that an axiom scheme, i.e., all substitution instances of an axiom, does not in general include all its iteration instances; in particular, none of the above three axioms automatically implies all of its iteration instances. For example, as already noted in the remarks following Lemma 3.1, the fact that $\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\alpha}$ is a theorem of a particular k-deductive system $\mathcal{S}$ for every $\boldsymbol{\alpha} \in \mathrm{Fm}^k$ does not in itself imply that each k-formula of the sequence $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\alpha}}$ is a theorem of $\mathcal{S}$ for every $\overline{\boldsymbol{\alpha}} \in (\mathrm{Fm}^k)^*$. In particular, for a 1-deductive system $\mathcal{S}$, $\vdash_{\mathcal{S}} x \Rightarrow_P x$ does not in itself imply that $\vdash_{\mathcal{S}} \langle x, y\rangle \Rightarrow_P \langle x, y\rangle$, i.e., $\vdash_{\mathcal{S}} x \Rightarrow_P (y \Rightarrow_P x)$ and $\vdash_{\mathcal{S}} x \Rightarrow_P (y \Rightarrow_P y)$, for 1-variables x and y.

However, we now show that, if ($\mathrm{D0}_P$)–($\mathrm{D2}_P$) are theorems in a k-deductive system $\mathcal{S}$ in which (MP_P) is the only primitive rule of inference, then $\mathcal{S}$ has (DT_P) and hence the iteration instances ($\mathrm{D0}^{\mathrm{I}}_P$)–($\mathrm{D2}^{\mathrm{I}}_P$) of ($\mathrm{D0}_P$)–($\mathrm{D2}_P$) hold by virtue of Thm. 3.5.

The next theorem, which combines Thm. 3.5 with a weaker version of its converse, gives a useful characterization of k-deductive systems with the P-deduction-detachment theorem.

Theorem 3.6. *Let $\mathcal{S}$ be a k-deductive system over Λ, and let $P(\boldsymbol{x}, \boldsymbol{y})$ be a finite nonempty sequence of binary k-formulas over Λ. Then $\mathcal{S}$ has the P-deduction-detachment theorem, i.e., both P-modus ponens and the P-deduction theorem, iff $\mathcal{S}$ has an presentation in which* (MP_P) *is its only proper inference rule and* ($\mathrm{D0}_P$)–($\mathrm{D2}_P$) *are theorems.*

Proof. The implication from left to right is contained in Thm. 3.5. To prove the implication in the reverse direction we assume, without loss of generality, that ($\mathrm{D0}_P$)–($\mathrm{D2}_P$) are actually axioms of $\mathcal{S}$ and that (MP_P) is the only proper inference rule and then prove that $\mathcal{S}$ has (DT_P).

The proof is similar to the proof of the deduction theorem in CPC and IPC. It goes by induction on the length of a derivation in $\mathcal{S}$ of a k-formula from $\boldsymbol{\Gamma} \cup \{\boldsymbol{\alpha}\} \subseteq \mathrm{Fm}^k$. Assume $\boldsymbol{\Gamma}, \boldsymbol{\alpha} \vdash_{\mathcal{S}} \boldsymbol{\beta}$ and let $\boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_m$ be a derivation of $\boldsymbol{\beta}$ from $\boldsymbol{\Gamma} \cup \{\boldsymbol{\alpha}\}$, so that $\boldsymbol{\gamma}_m = \boldsymbol{\beta}$. If $m = 0$, then $\boldsymbol{\beta} \in \boldsymbol{\Gamma} \cup \{\boldsymbol{\alpha}\}$. If $\boldsymbol{\beta} \in \boldsymbol{\Gamma}$, then trivially $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\beta}$. Since $\vdash_{\mathcal{S}} \boldsymbol{\beta} \Rightarrow_P (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta})$ by ($\mathrm{D1}_P$), we have $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}$ by (MP_P). If $\boldsymbol{\beta} = \boldsymbol{\alpha}$, then $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}$ by ($\mathrm{D0}_P$).

Assume $m > 0$. If $\boldsymbol{\beta} \in \boldsymbol{\Gamma} \cup \{\boldsymbol{\alpha}\}$, then we have $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}$ as above. Assume $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_{m-1}$ by an application of (MP_P). In particular, for some $i < m$ we have $\boldsymbol{\gamma}_i \Rightarrow_0 \boldsymbol{\beta}, \dots, \boldsymbol{\gamma}_i \Rightarrow_{n-1} \boldsymbol{\beta} \in \{\boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_{m-1}\}$, where $\boldsymbol{x} \Rightarrow_P \boldsymbol{y} = \langle \boldsymbol{x} \Rightarrow_0 \boldsymbol{y}, \dots, \boldsymbol{x} \Rightarrow_{n-1} \boldsymbol{y} \rangle$. By the induction hypothesis

$$\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\gamma}_i \quad \text{and} \quad \boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P (\boldsymbol{\gamma}_i \Rightarrow_j \boldsymbol{\beta}) \text{ for all } j < n. \tag{3}$$

Thus $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \langle \boldsymbol{\gamma}_i \Rightarrow_0 \boldsymbol{\beta}, \dots, \boldsymbol{\gamma}_i \Rightarrow_{n-1} \boldsymbol{\beta} \rangle$, i.e.,

$$\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P (\boldsymbol{\gamma}_i \Rightarrow_P \boldsymbol{\beta}). \tag{4}$$

By ($\mathrm{D2}_P$), $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \big(\boldsymbol{\alpha} \Rightarrow_P (\boldsymbol{\gamma}_i \Rightarrow_P \boldsymbol{\beta})\big) \Rightarrow_P \big((\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\gamma}_i) \Rightarrow_P (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta})\big)$. Thus by (4) and ($\mathrm{MP}_P$) we get $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\gamma}_i) \Rightarrow_P (\boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta})$, and by (3) and another application of (MP_P), $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}$. □

When specialized to the 2-deductive systems of applied equational logic this theorem provides a new characterization of quasivarieties with equationally definable principal relative congruences (EDPRC). Recall that by Theorem 3.3 a quasivariety Q has EDPRC iff its associated equational logic $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ (a 2-deductive system) has the P-deduction-detachment theorem for with respect to some finite sequence $P(\boldsymbol{x}, \boldsymbol{y})$ of 2-formulas in two 2-variables.

Corollary 3.7. *A quasivariety* Q *has* EDPRC *iff, for some finite sequence* $P(\boldsymbol{x}, \boldsymbol{y})$ *of 2-formulas in two 2-variables* $\langle x_0, x_1 \rangle$ *and* $\langle y_0, y_1 \rangle$, $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ *has a presentation in which the only proper quasi-identity is* (MP_P)*, i.e.,*

$$\frac{x_0 \approx x_1,\ \langle x_0, x_1 \rangle \Rightarrow_P \langle y_0, y_1 \rangle}{y_0 \approx y_1},$$

and in which the only identities are ($\mathrm{D0}_P$)–($\mathrm{D2}_P$)*, i.e.,*

$$\langle x_0, x_1 \rangle \Rightarrow_P \langle x_0, x_1 \rangle,$$
$$\langle x_0, x_1 \rangle \Rightarrow_P (\langle y_0, y_1 \rangle \Rightarrow_P \langle x_0, x_1 \rangle,$$
$$\big(\langle x_0, x_1 \rangle \Rightarrow_P (\langle y_0, y_1 \rangle \Rightarrow_P \langle z_0, z_1 \rangle)\big) \Rightarrow_P \big((\langle x_0, x_1 \rangle \Rightarrow_P \langle y_0, y_1 \rangle) \Rightarrow_P (\langle x_0, x_1 \rangle \Rightarrow_P \langle z_0, z_1 \rangle)\big).$$

A different characterization of quasivarieties with EDPRC is obtained in [11, p. 397]. Although we do not formulate it, one can also obtain from Thm. 3.6 a characterization of varieties with EDPC that is similar to Cor. 3.7, but is expressed entirely in terms of identities.[3] A characterization of this kind is obtained in [3] by a different method.

The following technical lemma will be used in the next section.

Lemma 3.8. *Let Λ be an arbitrary language type and $P(\boldsymbol{x},\boldsymbol{y})$ a finite nonempty sequence of binary k-formulas over Λ. Let $\mathcal{S}$ be a k-deductive system over Λ such that* $(\mathrm{D}0_P)$–$(\mathrm{D}2_P)$ *are theorems and* (MP_P) *is a rule of $\mathcal{S}$. Then*

$$(\mathrm{D}0^{\mathrm{I}}_P),\ (\mathrm{D}1^{\mathrm{I}}_P),\ \textit{and}\ (\mathrm{D}2^{\mathrm{I}}_P),\ \textit{are theorems of}\ \mathcal{S}.$$

Moreover, the following entailments (in iteration form) hold in $\mathcal{S}$ for all $\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\beta}},\overline{\boldsymbol{\gamma}},\overline{\boldsymbol{\delta}} \in (\mathrm{Fm}^k)^*$.

$(\mathrm{D}3^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}) \Rightarrow_P \big((\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}}) \Rightarrow_P (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\gamma}})\big)$;
$(\mathrm{D}4^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} \overline{\boldsymbol{\beta}} \Rightarrow_P \big((\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\delta}})) \Rightarrow_P (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\delta}})\big)$;
$(\mathrm{D}5^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\delta}})\big) \Rightarrow_P \big((\overline{\boldsymbol{\gamma}} \Rightarrow_P \overline{\boldsymbol{\alpha}}) \Rightarrow_P ((\overline{\boldsymbol{\gamma}} \Rightarrow_P \overline{\boldsymbol{\beta}}) \Rightarrow_P (\overline{\boldsymbol{\gamma}} \Rightarrow_P \overline{\boldsymbol{\delta}}))\big)$;
$(\mathrm{D}6^{\mathrm{I}}_P)$**:** $\vdash_{\mathcal{S}} \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Leftrightarrow_P \top)\big) \Leftrightarrow_P \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}\big)$.

Proof. Let $\mathcal{T}$ be the k-deductive system with $(\mathrm{D}1_P)$ and $(\mathrm{D}2_P)$ as the only axioms and (MP_P) as the only rule of inference. By Thm. 3.6 $\mathcal{T}$ has (DT_P). So the P-deduction-detachment theorem holds in P. Thus by Lem. 3.4 the iteration instance form of P-deduction theorem, $(\mathrm{DT}^{\mathrm{I}}_P)$, holds in $\mathcal{T}$. The iteration instance form of P-modus ponens, $(\mathrm{MP}^{\mathrm{I}}_P)$, also holds in $\mathcal{T}$ by Lem. 3.2. Finally, we note that, by hypothesis, $\vdash_{\mathcal{T}} \subseteq \vdash_{\mathcal{S}}$.

$(\mathrm{D}1^{\mathrm{I}}_P)$–$(\mathrm{D}2^{\mathrm{I}}_P)$ are theorems of $\mathcal{T}$ and hence of $\mathcal{S}$ by Thm. 3.5.

$(\mathrm{D}3^{\mathrm{I}}_P)$. Two applications of $(\mathrm{MP}^{\mathrm{I}}_P)$ give $\overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}$, $\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}} \vdash_{\mathcal{T}} \overline{\boldsymbol{\gamma}}$. Then three applications of $(\mathrm{DT}^{\mathrm{I}}_P)$ give $\vdash_{\mathcal{T}} (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}) \Rightarrow_P \big((\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\gamma}}) \Rightarrow_P (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\gamma}})\big)$. $(\mathrm{D}3^{\mathrm{I}}_P)$ holds because $\vdash_{\mathcal{T}} \subseteq \vdash_{\mathcal{S}}$.

$(\mathrm{D}4^{\mathrm{I}}_P)$. By two applications of $(\mathrm{MP}^{\mathrm{I}}_P)$ we have $\overline{\boldsymbol{\beta}}$, $\overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\delta}}) \vdash_{\mathcal{T}} \overline{\boldsymbol{\delta}}$, and then three applications of $(\mathrm{DT}^{\mathrm{I}}_P)$ give $(\mathrm{D}4^{\mathrm{I}}_P)$ with $\vdash_{\mathcal{T}}$ in place of $\vdash_{\mathcal{S}}$. $(\mathrm{D}4^{\mathrm{I}}_P)$ now follows.

$(\mathrm{D}5^{\mathrm{I}}_P)$ is proved similarly. By multiple applications of $(\mathrm{MP}^{\mathrm{I}}_P)$ we have $\overline{\boldsymbol{\gamma}}$, $\overline{\boldsymbol{\gamma}} \Rightarrow_P \overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\gamma}} \Rightarrow_P \overline{\boldsymbol{\beta}}$, $\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\delta}}) \vdash_{\mathcal{T}} \overline{\boldsymbol{\delta}}$. Thus three applications of $(\mathrm{DT}^{\mathrm{I}}_P)$ give $(\mathrm{D}5^{\mathrm{I}}_P)$ with $\vdash_{\mathcal{T}}$ in place of $\vdash_{\mathcal{S}}$ and hence $(\mathrm{D}5^{\mathrm{I}}_P)$ itself.

$(\mathrm{D}6^{\mathrm{I}}_P)$. The entailment $\vdash_{\mathcal{T}} \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Leftrightarrow_P \top)\big) \Leftrightarrow_P \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}\big)$ is equivalent to the two entailments

$$\vdash_{\mathcal{T}} \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Leftrightarrow_P \top)\big) \Rightarrow_P \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}\big) \quad \text{and} \tag{5}$$

$$\vdash_{\mathcal{T}} \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}\big) \Rightarrow_P \big(\overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Leftrightarrow_P \top)\big). \tag{6}$$

[3]The method of obtaining such a characterization from Thm. 3.6 is a familiar one from universal algebra and is based on the fact that satisfaction of a fixed but arbitrary quasi-identity in a variety is a Mal'cev condition.

Two applications of (MP^I_P) give $\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}} \Rightarrow_P (\overline{\boldsymbol{\beta}} \Leftrightarrow_P \top) \vdash_{\mathcal{T}} \overline{\boldsymbol{\beta}}$, and then two applications of (DT^I_P) give (5). (6) is obtained similarly. □

We will also make use of the formula sequences whose iteration instances are $(\mathrm{D3}^{\mathrm{I}}_P)$–$(\mathrm{D6}^{\mathrm{I}}_P)$ respectively:

$(\mathrm{D3}_P)$ $$(\boldsymbol{x} \Rightarrow_P \boldsymbol{y}) \Rightarrow_P \big((\boldsymbol{y} \Rightarrow_P \boldsymbol{z}) \Rightarrow_P (\boldsymbol{x} \Rightarrow_P \boldsymbol{z})\big);$$

$(\mathrm{D4}_P)$ $$\boldsymbol{y} \Rightarrow_P \big((\boldsymbol{x} \Rightarrow_P (\boldsymbol{y} \Rightarrow_P \boldsymbol{w})) \Rightarrow_P (\boldsymbol{x} \Rightarrow_P \boldsymbol{w})\big);$$

$(\mathrm{D5}_P)$
$$\big(\boldsymbol{x} \Rightarrow_P (\boldsymbol{y} \Rightarrow_P \boldsymbol{w})\big) \Rightarrow_P \big((\boldsymbol{z} \Rightarrow_P \boldsymbol{x}) \Rightarrow_P ((\boldsymbol{z} \Rightarrow_P \boldsymbol{y}) \Rightarrow_P (\boldsymbol{z} \Rightarrow_P \boldsymbol{w}))\big);$$

$(\mathrm{D6}_P)$ $$\big(\boldsymbol{x} \Rightarrow_P (\boldsymbol{y} \Leftrightarrow_P \top)\big) \Leftrightarrow_P \big(\boldsymbol{x} \Rightarrow_P \boldsymbol{y}\big).$$

3.1. **Fregean k-deductive systems.** The notion of a Fregean k-deductive system is a straightforward generalization of that of a Fregean 1-deductive system.

Let $\mathcal{S}$ be a k-deductive system over an arbitrary language type Λ. For each $\boldsymbol{\Gamma} \subseteq \mathrm{Fm}^k$, let $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}\, \boldsymbol{\Gamma}$ be the binary relation on k-formulas defined by

$$\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}\, \boldsymbol{\Gamma} := \{\, \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \in (\mathrm{Fm}^k)^2 : \boldsymbol{\Gamma}, \boldsymbol{\alpha} \vdash_{\mathcal{S}} \boldsymbol{\beta} \text{ and } \boldsymbol{\Gamma}, \boldsymbol{\beta} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \,\}.$$

Definition 3.9. Let $\mathcal{S}$ be a k-deductive system over an arbitrary language type Λ. $\mathcal{S}$ is *Fregean* if $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}\, \boldsymbol{\Gamma}$ is a congruence relation on the k-th power $\mathbf{Fm}^k$ of the formula algebra, for every $\boldsymbol{\Gamma} \subseteq \mathrm{Fm}^k$.

Recall that $\overline{\boldsymbol{\alpha}} \Leftrightarrow_P \overline{\boldsymbol{\beta}} := (\overline{\boldsymbol{\alpha}} \Rightarrow_P \overline{\boldsymbol{\beta}}) \frown (\overline{\boldsymbol{\beta}} \Rightarrow_P \overline{\boldsymbol{\alpha}})$. Note that $\overline{\boldsymbol{\beta}} \Leftrightarrow_P \overline{\boldsymbol{\alpha}}$ is just a permutation of $\overline{\boldsymbol{\alpha}} \Leftrightarrow_P \overline{\boldsymbol{\beta}}$.

Theorem 3.10. *Let $\mathcal{S}$ be a k-deductive system over Λ. Let P be a finite sequence of binary k-formulas in the two k-variables $\boldsymbol{x}$ and $\boldsymbol{y}$. Assume $\mathcal{S}$ has the P-deduction-detachment theorem. Then $\mathcal{S}$ is Fregean iff, for each $\lambda \in \Lambda$, where n is the rank of λ, the following is a theorem of $\mathcal{S}$.*

$(\mathrm{D}\lambda_P)$ $$(\boldsymbol{x} \Leftrightarrow_P \boldsymbol{y}) \Rightarrow_P (\lambda \overline{\boldsymbol{z}}_{i-1}\, \boldsymbol{x}\, \overline{\boldsymbol{z}}_{i+1} \Rightarrow_P \lambda \overline{\boldsymbol{z}}_{i-1}\, \boldsymbol{y}\, \overline{\boldsymbol{z}}_{i+1}) \quad \textit{for all } i < n,$$

where

$$\lambda \overline{\boldsymbol{z}}_{i-1}\, \boldsymbol{x}\, \overline{\boldsymbol{z}}_{i+1} = \big\langle \lambda z^0_0 \dots z^{i-1}_0\, x_0\, z^{i+1}_0 \dots z^{n-1}_0, \dots, \lambda z^0_{k-1} \dots z^{i-1}_{k-1}\, x_i\, z^{i+1}_{k-1} \dots z^{n-1}_{k-1} \big\rangle,$$

and similarly for $\lambda \overline{\boldsymbol{z}}_{i-1}\, \boldsymbol{y}\, \overline{\boldsymbol{z}}_{i+1}$.

Proof. Assume $\mathcal{S}$ has both (MP_P) and (DT_P) and note that by Thm. 3.5 $(\mathrm{D0}^{\mathrm{I}}_P)$–$(\mathrm{D2}^{\mathrm{I}}_P)$ hold in $\mathcal{S}$. By (MP_P) and (DT_P) $\boldsymbol{\Gamma}, \boldsymbol{\alpha} \vdash_{\mathcal{S}} \boldsymbol{\beta}$ and $\boldsymbol{\Gamma}, \boldsymbol{\beta} \vdash_{\mathcal{S}} \boldsymbol{\alpha}$ iff $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Rightarrow_P \boldsymbol{\beta}$ and $\boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\beta} \Rightarrow_P \boldsymbol{\alpha}$. Hence

$$\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}\, \boldsymbol{\Gamma} = \{\, \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \in (\mathrm{Fm}^k)^2 : \boldsymbol{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\alpha} \Leftrightarrow_P \boldsymbol{\beta} \,\}.$$

Thus $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}\,\boldsymbol{\Gamma}$ is reflexive by ($\mathrm{D0}_P$) and that it transitive follows immediately from ($\mathrm{D3}_P$), which holds by Lem. 3.8. Since it is symmetric by its definition, it forms an equivalence relation. Thus $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}\,\boldsymbol{\Gamma}$ is a congruence relation on $\mathbf{Fm}^k$ iff ($\mathrm{D}\lambda_P$) holds for all $\lambda \in \Lambda$. □

Thms. 3.6 and 3.10 together immediately give

Theorem 3.11. *Let $\mathcal{S}$ be a k-deductive system over Λ, and let $P(\boldsymbol{x}, \boldsymbol{y})$ be a finite nonempty sequence of binary k-formulas over Λ. Then $\mathcal{S}$ is Fregean and has the P-deduction-detachment theorem iff $\mathcal{S}$ has a presentation in which* (MP_P) *is its only proper inference rule and* ($\mathrm{D0}_P$)–($\mathrm{D2}_P$) *and* ($\mathrm{D}\lambda_P$), *for $\lambda \in \Lambda$, are theorems.*

This theorem is a generalization of [13, Theorem 2.20].

The notion of a Fregean 1-deductive system is well-motivated, and there are many natural and important examples. On the other hand, the practical significance of Fregean multidimensional systems is not apparent at this time. In particular, we know very little about those quasivarieties whose equational logics are Fregean 2-deductive systems. For this reason we shall limit the investigation in the sequel to Fregean 1-deductive systems. In this case the "Fregean" axioms ($\mathrm{D}\lambda_P$), $\lambda \in \Lambda$, take a form that is simpler by virtue of the fact that only 1-variables are involved:

$$(\mathrm{D}\lambda_P) \qquad (x \Leftrightarrow_P y) \Rightarrow_P (\lambda \overline{z}_{i-1}\, x\, \overline{z}_{i+1} \Rightarrow_P \lambda \overline{z}_{i-1}\, y\, \overline{z}_{i+1}) \text{ for all } i < n,$$

where $\overline{z}_{i-1} = z_0 \ldots z_{i-1}$ and $\overline{z}_{i+1} = z_{i+1} \ldots z_{n-1}$.

When P is taken to be the 1-element sequence $\langle x \to y \rangle$, where $\to$ is an actual connective of the language type, the axioms ($\mathrm{D1}_P$) and ($\mathrm{D2}_P$) and the rule (MP_P) specialize to the two axioms and the rule of modus ponens that constitute the standard presentation of the implicational fragment of the intuitionistic propositional calculus IPC. Moreover the axioms ($\mathrm{D0}_P$) and ($\mathrm{D}\lambda_P$) for $\lambda \in \{\wedge, \vee, \neg\}$ specialize in this situation to well-known theorems of IPC. These observations, together with Thm. 3.11 justify taking the 1-deductive system presented by these axioms in their general form and with P-detachment as the only rule of inference as the natural generalization of the intuitionistic propositional calculus to an arbitrary language type and arbitrary deduction-detachment system P:

Definition 3.12. Let Λ be an arbitrary language type and $P(x, y)$ a finite nonempty sequence of binary 1-formulas over Λ. By the *multiterm intuitionistic propositional calculus with deduction-detachment system P (over Λ)*, in symbols IPC_P, we mean the 1-deductive system with the axioms ($\mathrm{D0}_P$)–($\mathrm{D2}_P$), and ($\mathrm{D}\lambda_P$), for all $\lambda \in \Lambda$, and with (MP_P) as the only proper rule of inference.

IPC_P is also called the *P-intuitionistic propositional calculus* for short. When we refer simply to a *multiterm intuitionistic propositional calculus* we mean IPC_P for some P.

Theorem 3.13. *Every multiterm intuitionistic propositional calculus is regularly algebraizable. More specifically, Let $P(x,y)$ a finite nonempty sequence of binary 1-formulas over an arbitrary language type Λ. Let $P(x,y) = \langle x \Rightarrow_0 y, \dots, x \Rightarrow_{n-1} y\rangle$, and set*

$$E(x,y) = \{x \Rightarrow_0 y, y \Rightarrow_0 x, \dots, x \Rightarrow_{n-1} y, y \Rightarrow_{n-1} x\}. \tag{7}$$

Then IPC_P is E-regularly algebraizable.

Proof. The six conditions of Def. 2.3 are easily verified for IPC_P: Conditions 2.3(i),(iii),(iv) are immediate consequences respectively of ($\mathrm{D}0_P$), ($\mathrm{D}3_P$), ($\mathrm{D}\lambda_P$) for $\lambda \in \Lambda$. The symmetry condition, 2.3(ii), is a trivial consequence of the definition of E, and E-modus ponens, 2.3(v), follows immediately from P-modus ponens. Finally, the E-G-rule, 2.3(vi), follows directly from the P-deduction theorem in the following way. Then for all Λ-formulas we have $\alpha, \beta \vdash_{\mathrm{IPC}_P} \beta$ and thus $\beta \vdash_{\mathrm{IPC}_P} \alpha \Rightarrow_P \beta$ by (DT_P). Similarly, $\alpha \vdash_{\mathrm{IPC}_P} \beta \Rightarrow_P \alpha$. Hence $\alpha, \beta \vdash_{\mathrm{IPC}_P} E(\alpha, \beta)$. □

We assume that IPC_P is pointed, i.e., that Λ contains a constant $\top$ such that $\vdash_{\mathrm{IPC}_P} \top$. By the E-G-rule, $\vdash_{\mathrm{IPC}_P} E(\alpha, \top)$ for every theorem α of IPC_P.

Definition 3.14. Let Λ be an arbitrary language type and $P(x,y)$ a finite nonempty sequence of binary 1-formulas over Λ. By a *multiterm intermediate propositional calculus with deduction-detachment system P (over Λ)* we mean any axiomatic extension of IPC_P, i.e., an extension of IPC_P obtained by adjoining any number of new axioms (but no new inference rules).

A multiterm intermediate propositional calculus with deduction-detachment system P is called a *P-intermediate propositional calculus* for short, and when we refer simply to a *multiterm intermediate propositional calculus*, we mean a P-intermediate propositional calculus for some deduction-detachment system P.

In [13, Theorem 2.20] it is shown that every Fregean 1-deductive system over the language type Λ with the uniterm deduction-detachment theorem is formula wise definitionally equivalent to an axiomatic extension of $\mathrm{IPC}_\Lambda^{\to,\top}$, the $\{\to,\top\}$-fragment of IPC with the new connectives of Λ adjoined (a presentation $\mathrm{IPC}_\Lambda^{\to,\top}$ is given in Section 2). Conversely, every axiomatic extension of $\mathrm{IPC}_\Lambda^{\to,\top}$ is Fregean with the uniterm deduction-detachment theorem. The following corollary, which is an immediate consequence of Thm. 3.11, is a natural generalization of this result.

Corollary 3.15. *A* 1*-deductive system is Fregean and has the multiterm deduction-detachment theorem iff it is a multiterm intermediate propositional calculus.*

More precisely, let $P(x, y)$ be any sequence of binary formulas over an arbitrary language type Λ. Then a 1-deductive system over Λ is Fregean and has the P-deduction-detachment theorem iff it is a P-intermediate propositional calculus.

Every multiterm deduction-detachment system is a protoequivalence system, so every deductive system with the multiterm deduction-detachment theorem is protoalgebraic. Hence by Thm. 2.8 every multiterm intermediate propositional calculus is regularly algebraizable. This can also be obtained as a consequence of Thm. 3.13 and the fact that every axiomatic extension of a regularly algebraizable deductive system is regularly algebraizable.

It is easy to see that $(\top \Rightarrow_P x) \Rightarrow_P x$ is a theorem sequence of every multiterm intermediate propositional calculus $\mathcal{S}$ with deduction-detachment system $P = \langle x \Rightarrow_0 y, \ldots, x \Rightarrow_{n-1} y\rangle$. But $(T \Rightarrow_i x) \Rightarrow_P x$ is in general not a theorem sequence of $\mathcal{S}$ for any $i < n$. This follows easily from the following theorem together with Theorem 5.4 below. However, if $P = \{x \to y\}$, a singleton, and hence $\mathcal{S}$ is an axiomatic extension of $\mathrm{IPC}_{\Lambda}^{\to,\top}$ (up to definitional equivalence), then of course $(\top \to x) \to x$ is a theorem of $\mathcal{S}$. This is a characteristic property of the systems $\mathrm{IPC}_{\Lambda}^{\to,\top}$ and their axiomatic extensions:

Theorem 3.16. *Let $\mathcal{S}$ be a P-intermediate propositional calculus over the language type Λ. Assume $(\top \Rightarrow_i x) \Rightarrow_P x$ is a theorem sequence of $\mathcal{S}$ for some $i < n$, where $P = \langle x \Rightarrow_0 y, \ldots, x \Rightarrow_{n-1} y\rangle$. Let $P' = \langle x \Rightarrow_i y\rangle$. Then $\mathcal{S}$ has the P'-deduction-detachment theorem. Hence $\mathcal{S}$ is formula wise definitionally equivalent to an axiomatic extension of* $\mathrm{IPC}_{\Lambda}^{\to,\top}$.

Proof. $(\mathrm{DT}_{P'})$ is a trivial consequence of (DT_P): If $\Gamma, \alpha \vdash_{\mathcal{S}} \beta$, then $\Gamma \vdash_{\mathcal{S}} \alpha \Rightarrow_P \beta$, and hence, a fortiori, $\Gamma \vdash_{\mathcal{S}} \alpha \Rightarrow_i \beta$, by (DT_P).

Any formula α is interderivable with $\top$ relative to α, i.e., $\alpha, \alpha \vdash_{\mathcal{S}} \top$ and $\alpha, \top \vdash_{\mathcal{S}} \alpha$, that is, $\langle \alpha, \top\rangle \in \widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\{\alpha\}$. Thus from the hypothesis that $\mathcal{S}$ is a multiterm intermediate propositional calculus and hence Fregean we get that $\langle \alpha \Rightarrow_i \beta,\, \top \Rightarrow_i \beta\rangle \in \widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\{\alpha\}$. Thus $\alpha, \alpha \Rightarrow_i \beta \vdash_{\mathcal{S}} \top \Rightarrow_i \beta$. By hypothesis $\vdash_{\mathcal{S}} (\top \Rightarrow_i \beta) \Rightarrow_P \beta$. So by (MP_P) we get finally that $\alpha, \alpha \Rightarrow_i \beta \vdash_{\mathcal{S}} \beta$. Thus $(\mathrm{MP}_{P'})$ holds in $\mathcal{S}$.

So $\mathcal{S}$ has the P'-deduction-detachment theorem. Since $\mathcal{S}$ is Fregean by hypothesis it is formula wise definitionally equivalent to an axiomatic extension of $\mathrm{IPC}_{\Lambda}^{\to,\top}$ by [13, Theorem 2.20]. □

4. Multiterm Hilbert Algebras

Since each multiterm intuitionistic propositional calculus is regularly algebraizable, it is the assertional logic of a unique relatively point-regular quasivariety, its equivalent algebraic semantics in the sense of [4]. In the special case of the standard intuitionistic propositional calculus, IPC, this turns out to be a variety, the variety of Heyting algebras. In the special case of the implicational fragment of IPC it is the variety of Hilbert algebras; see Diego [14]. We shall now see that the equivalent quasivariety of an arbitrary multiterm intuitionistic propositional calculus can be characterized in a very similar way.

Let $\overline{\alpha} = \langle \alpha_0, \dots, \alpha_{n-1} \rangle, \overline{\beta} = \langle \beta_0, \dots, \beta_{n-1} \rangle \in \mathrm{Fm}^*$ be sequences of formulas (i.e., 1-formulas) of the same length. Then $\overline{\alpha} \approx \overline{\beta}$ will denote the sequence of equations (i.e., 2-formulas) $\langle \alpha_0 \approx \beta_0, \dots, \alpha_{n-1} \approx \beta_{n-1} \rangle$. If β is a single formula, $\overline{\alpha} \approx \beta$ will denote the equation sequence $\overline{\alpha} \approx \overline{\beta}$, where $\overline{\beta}$ is the constant formula sequence $\langle \beta, \beta, \dots, \beta \rangle$ of the same length as $\overline{\alpha}$. If $\boldsymbol{A}$ is a Λ-algebra, then $\overline{\alpha} \approx \overline{\beta}$ is an *identity*, or more precisely an *identity sequence* of $\boldsymbol{A}$, in symbols $\vDash_{\boldsymbol{A}} \overline{\alpha} \approx \overline{\beta}$, if each $\alpha_i \approx \beta_i$, $i < n$, is an identity of $\boldsymbol{A}$ in the usual sense, i.e., for every $h\colon \mathbf{Fm} \to \boldsymbol{A}$, $h\,\alpha_i = h\,\beta_i$. If K is a class of Λ-algebras, $\vDash_{\mathsf{K}} \overline{\alpha} \approx \overline{\beta}$ means $\vDash_{\boldsymbol{A}} \overline{\alpha} \approx \overline{\beta}$ for all $\boldsymbol{A} \in \mathsf{K}$; equivalently, $\vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}} \overline{\alpha} \approx \overline{\beta}$, where Q is the quasivariety generated by K. Let $\overline{\gamma}_0, \overline{\delta}_0, \dots, \overline{\gamma}_{m-1}, \overline{\delta}_{m-1}, \overline{\alpha}, \overline{\beta}$ be sequences of formulas such that $\overline{\gamma}_i$ and $\overline{\delta}_i$ are of the same length, and $\overline{\alpha}$ and $\overline{\beta}$ are of the same length. Then

$$\frac{\overline{\gamma}_0 \approx \overline{\delta}_0, \dots, \overline{\gamma}_{m-1} \approx \overline{\delta}_{m-1}}{\overline{\alpha} \approx \overline{\beta}}$$

will denote the sequence of quasi-equations (i.e., 2-sequents)

$$\left\langle \frac{\overline{\gamma}_0 \frown \cdots \frown \overline{\gamma}_{m-1} \approx \overline{\delta}_0 \frown \cdots \frown \overline{\delta}_{m-1}}{\alpha_i \approx \beta_i} \colon i < l \right\rangle, \tag{8}$$

where l is the common length of $\overline{\alpha}$ and $\overline{\beta}$. (8) is a *quasi-identity*, or more precisely a *quasi-identity sequence* of $\boldsymbol{A}$ if each quasi-equation in the sequence is a quasi-identity of $\boldsymbol{A}$.

Definition 4.1. Let Λ be an arbitrary pointed language type and $P(x, y)$ a finite sequence of binary 1-formulas over Λ. A Λ-algebra is called a *multiterm Hilbert algebra with implication system P and compatible operations*

Λ if it satisfies the following identities (i.e., identity sequences)

(H0$_P$) $$x \Rightarrow_P x \approx \top,$$

(H1$_P$) $$x \Rightarrow_P (y \Rightarrow_P x) \approx \top,$$

(H2$_P$) $$\big(x \Rightarrow_P (y \Rightarrow_P z)\big) \Rightarrow \big((x \Rightarrow_P y) \Rightarrow_P (x \Rightarrow_P z)\big) \approx \top,$$

(Hλ_P,) $$(x \Rightarrow_P y) \Rightarrow_P \big((y \Rightarrow_P x) \Rightarrow_P (\lambda\overline{z}_{i-1}\, x\, \overline{z}_{i+1} \Rightarrow_P \lambda\overline{z}_{i-1}\, y\, \overline{z}_{i+1})\big) \approx \top, \quad \text{for all } i < n$$

together with the single quasi-identity

(HP$_P$) $$\frac{x \Leftrightarrow_P y}{x \approx y}.$$

Note that (HP$_P$) is the shorthand form of $\dfrac{x \Rightarrow_P y \approx \top,\ y \Rightarrow_P x \approx \top}{x \approx y}$.

The quasivariety of all multiterm Hilbert algebras with implication system P and compatible operations Λ is denoted by $\mathsf{HI}_{P,\Lambda}$.

The subscript $_\Lambda$ on $\mathsf{HI}_{P,\Lambda}$ is normally omitted. The members of HI_P are called simply *P-Hilbert algebras*, and by a *multiterm Hilbert algebra* we mean a P-Hilbert algebra for some P. If $\Lambda = \{\to, \top\}$ and $P(x, y) = \langle x \to y\rangle$, then HI_P is the variety of Hilbert algebras.

Theorem 4.2. *Let Λ be an arbitrary pointed language type and $P(x, y)$ a finite sequence of binary formulas over Λ. Then HI_P is the equivalent quasivariety of* IPC$_P$ *and* IPC$_P$ *is the assertional logic of* HI_P.

Proof. Recall that IPC$_P$ is presented by (D0$_P$)-(D2$_P$), (Dλ_P), for $\lambda \in \Lambda$, and (MP$_P$). Recall also that IPC$_P$ is E-regularly algebraizable with the equivalence system $E(x, y) = \langle x \Rightarrow_0 y, y \Rightarrow_0 x, \dots, x \Rightarrow_{n-1} y, y \Rightarrow_{n-1} x\rangle$ if $P(x, y) = \langle x \Rightarrow_0 y, \dots x \Rightarrow_{n-1}\rangle$. Thus Thm. 2.6 applies and the equivalent quasivariety of IPC$_P$ is presented by the identities

(i) (H0$_P$)–(H2$_P$), and (Hλ_P) for $\lambda \in \Lambda$ (from (D0$_P$)–(D2$_P$), and (Dλ_P) for $\lambda \in \Lambda$);

together with the following quasi-identities

(iii) $\dfrac{x \approx \top,\ x \Rightarrow_P y \approx \top}{y \approx \top}$ (from (MP$_P$));

(iv) (HP$_P$) (from $\dfrac{E(x, y) \approx \top}{x \approx y}$).

Note that the identity sequence $x \Rightarrow_P \top \approx \top$ is an immediate consequence of (H0$_P$) and (H1$_P$). So the quasi-identity (iii) is a consequence of (HP$_P$).

Thus we have that HI_P is the equivalent quasivariety of IPC$_P$, and hence IPC$_P$ is its assertional logic since IPC$_P$ is regularly algebraizable. □

We do not know if the identity sequence ($\mathrm{H0}_P$) is a consequence of the other axioms and rule of HI_P. This clearly is closely related to the question if ($\mathrm{D0}_P$) is a consequence of ($\mathrm{D1}_P$), ($\mathrm{D2}_P$) and the rule (MP_P). The answer is positive of course if P consists of a single binary connective $\to$, because in this case IPC_P can essentially be identified with the $\{\to, \top\}$-fragment of IPC; this is reflected in the fact that the identity $x \to x$ is derivable from the axioms of Hilbert algebras. More generally, ($\mathrm{D0}_P$) can be eliminated from the list of axioms of IPC_P whenever P consists of a single compound formula: It is easy to check that ($\mathrm{D0}_P$) can, in general, be derived from ($\mathrm{D1}^{\mathrm{I}}_P$) and ($\mathrm{D2}^{\mathrm{I}}_P$) using ($\mathrm{MP}_P$). But when P is a singleton every iterative instance of a ($\Rightarrow_P$)-formula is a substitution instance if it. Thus in this case ($\mathrm{D0}_P$) is derivable from ($\mathrm{D1}_P$) and ($\mathrm{D2}_P$) by (MP_P), and hence can be removed as an axiom of IPC_P. It follows that ($\mathrm{H0}_P$) can be removed from the list of axioms of HI_P when P is a singleton.

As a special case of Theorem 4.2 we get the well-known result that the variety of Hilbert algebras is the equivalent quasivariety of the $\{\to, \top\}$-fragment of IPC.

In the remaining part of the section we investigate some of the algebraic properties of HI_P. As has already been observed, the following is a quasi-identity of HI_P.

($\mathrm{MP}^{\mathrm{eq}}_P$)
$$\frac{x \approx \top,\ x \Rightarrow_P y \approx \top}{y \approx \top}.$$

This is the equality form of P-detachment and expresses the fact that (MP_P) holds in IPC_P, the assertional logic of HI_P. More generally, α is a theorem of IPC_P iff $\alpha \approx \top$ is an identity of HI_P, and $\dfrac{\alpha_0, \dots, \alpha_{n-1}}{\beta}$ is a rule of IPC_P iff

$$\frac{\alpha_0 \approx \top, \dots, \alpha_{n-1} \approx \top}{\beta \approx \top},$$

its *equational form*, is a quasi-identity sequence of HI_P. As a consequence we get that the following equational form of ($\mathrm{MP}^{\mathrm{I}}_P$), the iteration instance form of (MP_P), is a quasi-identity sequence of HI_P for all $\overline{\alpha}, \overline{\beta} \in \mathrm{Fm}^*$.

($\mathrm{MP}^{\mathrm{I\,eq}}_P$)
$$\frac{\overline{\alpha} \approx \top,\ \overline{\alpha} \Rightarrow_P \overline{\beta} \approx \top}{\overline{\beta} \approx \top}.$$

From Lem. 3.8 we conclude that the following equation sequences, the equational forms respectively of ($\mathrm{D0}^{\mathrm{I}}_P$)–($\mathrm{D6}^{\mathrm{I}}_P$), are identity sequences of HI_P for all $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta} \in \mathrm{Fm}^*$.

$(\mathrm{H0}^{\mathrm{I}}_P)$ $$\overline{\alpha} \Rightarrow_P \overline{\alpha} \approx \top,$$

$(\mathrm{H1}^{\mathrm{I}}_P)$ $$\overline{\alpha} \Rightarrow_P (\overline{\beta} \Rightarrow_P \overline{\alpha}) \approx \top;$$

$(\mathrm{H2}^{\mathrm{I}}_P)$ $$\big(\overline{\alpha} \Rightarrow_P (\overline{\beta} \Rightarrow_P \overline{\gamma})\big) \Rightarrow \big((\overline{\alpha} \Rightarrow_P \overline{\beta}) \Rightarrow_P (\overline{\alpha} \Rightarrow_P \overline{\gamma})\big) \approx \top;$$

$(\mathrm{H3}^{\mathrm{I}}_P)$ $$(\overline{\alpha} \Rightarrow_P \overline{\beta}) \Rightarrow_P \big((\overline{\beta} \Rightarrow_P \overline{\gamma}) \Rightarrow_P (\overline{\alpha} \Rightarrow_P \overline{\gamma})\big) \approx \top;$$

$(\mathrm{H4}^{\mathrm{I}}_P)$ $$\overline{\beta} \Rightarrow_P \big((\overline{\alpha} \Rightarrow_P (\overline{\beta} \Rightarrow_P \overline{\delta})) \Rightarrow_P (\overline{\alpha} \Rightarrow_P \overline{\delta})\big) \approx \top;$$

$(\mathrm{H5}^{\mathrm{I}}_P)$ $$\big(\overline{\alpha} \Rightarrow_P (\overline{\beta} \Rightarrow_P \overline{\delta})\big) \Rightarrow_P \big((\overline{\gamma} \Rightarrow_P \overline{\alpha}) \Rightarrow_P ((\overline{\gamma} \Rightarrow_P \overline{\beta}) \Rightarrow_P (\overline{\gamma} \Rightarrow_P \overline{\delta}))\big) \approx \top;$$

$(\mathrm{H6}^{\mathrm{I}}_P)$ $$\big(\overline{\alpha} \Rightarrow_P (\overline{\beta} \Leftrightarrow_P \top)\big) \Leftrightarrow_P \big(\overline{\alpha} \Rightarrow_P \overline{\beta}\big) \approx \top.$$

Recall that $\alpha \Leftrightarrow_P \beta = \alpha \Rightarrow_P \beta \frown \beta \Rightarrow_P \alpha$.

The $(\Rightarrow_P)$-formula that is obtained from $(\mathrm{H}n^{I}_P)$ for each $n \leq 6$ by replacing the formula sequences $\overline{\alpha}, \overline{\beta}, \dots$ by the single variables $x, y, \dots$ will be denoted by $(\mathrm{H}n_P)$; $(\mathrm{H}n^{I}_P)$ represents all iteration instances of $(\mathrm{H}n_P)$.

A $(\Rightarrow_P)$-formula $\varphi(x_0, \dots, x_{m-1})$ represents a sequence of Λ-formulas

$$\langle \alpha_0(x_0, \dots, x_{m-1}), \dots, \alpha_{n-1}(x_0, \dots, x_{m-1}) \rangle.$$

The evaluation of φ under the assignment of the elements $a_0, \dots, a_{m-1}$ of an algebra $\boldsymbol{A}$ to the variables $x_0, \dots, x_{m-1}$ is just the sequence of elements of $\boldsymbol{A}$

$$\langle \alpha_0^{\boldsymbol{A}}(a_0, \dots, a_{m-1}), \dots, \alpha_0^{\boldsymbol{A}}(a_{n-1}, \dots, a_{m-1}) \rangle$$

that is obtained when the α_i are evaluated in the normal way. We use the natural extension of our notation for Λ-formula sequences to denote this evaluation.

Lemma 4.3. *For every $\boldsymbol{A} \in \mathsf{HI}_P$ and all $a, b, c \in A$,*

$$b \equiv c \pmod{\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(a, \top^{\boldsymbol{A}})} \quad \textit{iff} \quad a \Rightarrow^{\boldsymbol{A}}_P (b \Leftrightarrow^{\boldsymbol{A}}_P c) = \top^{\boldsymbol{A}}.$$

Proof. Let Φ denote the set of pairs $\langle b, c \rangle$ of elements of A that satisfy the equations $a \Rightarrow_P (b \Rightarrow_P c) = \top$ and $a \Rightarrow_P (c \Rightarrow_P b) = \top$. (To simplify notation in proofs we omit the superscript $\boldsymbol{A}$ in most instances when $\boldsymbol{A}$ is the only algebra around.) We show that $\Phi \in \mathrm{Co}_{\mathsf{HI}_P} \boldsymbol{A}$. Φ is trivially symmetric. $b \Rightarrow_P b = \top$ for all $b \in \boldsymbol{A}$ by $(\mathrm{H0}_P)$. As observed previously, $a \Rightarrow_P \top = \top$ is an immediate consequence of $(\mathrm{H0}_P)$ and $(\mathrm{H1}_P)$. So $a \Rightarrow_P (b \Rightarrow_P b) = \top$, and hence Φ is reflexive.

For transitivity assume $b \equiv_\Phi c$ and $c \equiv_\Phi d$, i.e.,

(9) $$a \Rightarrow_P (b \Rightarrow_P c) = \top,\ a \Rightarrow_P (c \Rightarrow_P b) = \top \quad \text{and}$$

(10) $$a \Rightarrow_P (c \Rightarrow_P d) = \top,\ a \Rightarrow_P (d \Rightarrow_P c) = \top.$$

By ($\mathrm{H3}_P$),

$$(b \Rightarrow_P c) \Rightarrow_P \big((c \Rightarrow_P d) \Rightarrow_P (b \Rightarrow_P d)\big) = \top,$$

and by ($\mathrm{H5}^{\mathrm{I}}_P$)

$$\big((b \Rightarrow_P c) \Rightarrow_P \big((c \Rightarrow_P d) \Rightarrow_P (b \Rightarrow_P d)\big)$$
$$\Rightarrow_P \big(a \Rightarrow_P (b \Rightarrow_P c)\big) \Rightarrow_P \big((a \Rightarrow_P (c \Rightarrow_P d)) \Rightarrow_P (a \Rightarrow_P (b \Rightarrow_P d))\big) = \top$$

So by ($\mathrm{MP}^{\mathrm{I\,eq}}_P$), the iteration instance form of ($\mathrm{MP}^{\mathrm{eq}}_P$),

$$(a \Rightarrow_P (b \Rightarrow_P c)) \Rightarrow_P \big(((a \Rightarrow_P (c \Rightarrow_P d))) \Rightarrow_P (a \Rightarrow_P (b \Rightarrow_P d))\big) = \top.$$

Two more applications of ($\mathrm{MP}^{\mathrm{I\,eq}}_P$) using the first parts of (9) and (10) give $a \Rightarrow_P (b \Rightarrow_P d) = \top$. Similarly, by the second parts of (9) and (10), $a \Rightarrow_P (d \Rightarrow_P b) = \top$, i.e., $b \equiv_\varPhi d$. Thus $\varPhi$ is transitive and hence an equivalence relation.

To verify the replacement property, let $\lambda \in \varLambda$ and let n be its rank. Assume $b \equiv_\varPhi c$ and let $i < n$ and $\bar{d}_{i-1} = d_0, \dots, d_{i-1}$ and $\bar{d}_{i+1} = d_{i+1}, \dots, d_{n-1}$ be arbitrary elements of A. By ($\mathrm{H}\lambda_P$) we have

$$(b \Rightarrow_P c) \Rightarrow_P \big((c \Rightarrow_P b) \Rightarrow_P (\lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, b, \bar{d}_{i+1}) \Rightarrow_P \lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, c, \bar{d}_{i+1})\big) = \top.$$

Thus by ($\mathrm{H5}^{\mathrm{I}}_P$) and ($\mathrm{MP}^{\mathrm{I\,eq}}_P$) we have

$$(11)\quad (a \Rightarrow_P (b \Rightarrow_P c)) \Rightarrow_P \big((a \Rightarrow_P (c \Rightarrow_P b))$$
$$\Rightarrow_P (a \Rightarrow_P ((\lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, b, \bar{d}_{i+1}) \Rightarrow_P (\lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, c, \bar{d}_{i+1}))\big) = \top.$$

Two more applications of ($\mathrm{MP}^{\mathrm{I\,eq}}_P$), using the assumption $b \equiv_\varPhi c$, give finally

$$a \Rightarrow_P \big((\lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, b, \bar{d}_{i+1}\big) \Rightarrow_P (\lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, c, \bar{d}_{i+1})\big) = \top.$$

Similarly with b and c interchanged. Thus $\lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, b, \bar{d}_{i+1}) \equiv_\varPhi \lambda^{\boldsymbol{A}}(\bar{d}_{i-1}, c, \bar{d}_{i+1})$. This shows that $\varPhi$ is a congruence on $\boldsymbol{A}$. It remains to show that $\varPhi$ is a HI_P-congruence and, in fact, the smallest such congruence that identifies a and $\top$.

We must show that $\boldsymbol{A}/\varPhi$ satisfies the quasi-identity (HP_P). For this purpose we have to show that, if $b \Rightarrow_P c \equiv_\varPhi \top$ (more precisely, $b \Rightarrow_i c \equiv_\varPhi \top$ for each $i < n$) and $c \Rightarrow_P b \equiv_\varPhi \top$, then $b \equiv_\varPhi c$. The congruence instance $b \Rightarrow_P c \equiv_\varPhi \top$ is equivalent to the sequence of equalities

$$(12)\qquad a \Rightarrow_P \big(\top \Rightarrow_P (b \Rightarrow_P c)\big) = \top,$$

since $a \Rightarrow_P \big((b \Rightarrow_P c) \Rightarrow_P \top\big) = \top$ is always true. Assume $b \Rightarrow_P c \equiv_\varPhi \top$, i.e., that (12) holds. From ($\mathrm{H4}^{\mathrm{I}}_P$) we get

$$\top \Rightarrow_P \big((a \Rightarrow_P (\top \Rightarrow_P (b \Rightarrow_P c))) \Rightarrow_P (a \Rightarrow_P (b \Rightarrow_P c))\big) = \top.$$

Two applications of $(\mathrm{MP}_P^{\mathrm{Ieq}})$ give $a \Rightarrow_P (b \Rightarrow_P c) = \top$, and by symmetry $c \Rightarrow_P b \equiv_\Phi \top$ implies $a \Rightarrow_P (c \Rightarrow_P b) = \top$. Thus $b \Rightarrow_P c \equiv_\Phi \top$ and $c \Rightarrow_P b \equiv_\Phi \top$ imply $b \equiv_\Phi c$. So $\Phi \in \mathrm{Co}_{\mathsf{HI}_P} \boldsymbol{A}$, as was to be shown.

Finally, to show that $\Phi = \mathrm{Cg}_{\mathsf{HI}_P}(a, \top)$ it suffices to show that, for every $\Theta \in \mathrm{Co}_{\mathsf{HI}_P} \boldsymbol{A}$, if $a \equiv_\Theta \top$ and $a \Rightarrow_P (c \Rightarrow_P d) = \top$ and $a \Rightarrow_P (d \Rightarrow_P c) = \top$, then $c \equiv_\Theta d$. Suppose $a \equiv_\Theta \top$ and $a \Rightarrow_P (c \Rightarrow_P d) = \top$. Then $\top \Rightarrow_P (c \Rightarrow_P d) \equiv_\Theta \top$. Thus $\top^{\boldsymbol{A}/\Theta} \Rightarrow_P^{\boldsymbol{A}/\Theta} (c/\Theta \Rightarrow_P^{\boldsymbol{A}/\Theta} d/\Theta) = \top^{\boldsymbol{A}/\Theta}$. Since $\boldsymbol{A}/\Theta \in \mathsf{HI}_P$ by assumption, $c/\Theta \Rightarrow_P^{\boldsymbol{A}/\Theta} d/\Theta = \top^{\boldsymbol{A}/\Theta}$ by $(\mathrm{MP}_P^{\mathrm{Ieq}})$. Similarly, the assumptions $a \equiv_\Theta \top$ and $a \Rightarrow_P (d \Rightarrow_P c) = \top$ give $d/\Theta \Rightarrow_P^{\boldsymbol{A}/\Theta} c/\Theta = \top^{\boldsymbol{A}/\Theta}$. Since $\boldsymbol{A}/\Theta \in \mathsf{HI}_P$, $c/\Theta = d/\Theta$ by (HP_P). I.e., $c \equiv_\Theta d$. So $\Phi = \mathrm{Cg}_{\mathsf{HI}_P}(a, \top)$. □

Corollary 4.4. *For all* $\boldsymbol{A} \in \mathsf{HI}_P$, $\overline{a} = \langle a_0, \dots, a_{n-1} \rangle \in A^*$, *and* $b, c \in A$,

$$b \equiv c \pmod{\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\overline{a}, \top^{\boldsymbol{A}})} \quad \textit{iff} \quad \overline{a} \Rightarrow_P^{\boldsymbol{A}} (b \Leftrightarrow_P^{\boldsymbol{A}} c) = \top^{\boldsymbol{A}},$$

where $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\overline{a}, \top^{\boldsymbol{A}}) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\langle a_0, \top^{\boldsymbol{A}} \rangle, \dots, \langle a_{n-1}, \top^{\boldsymbol{A}} \rangle)$.

Proof. The proof is by induction on n. The basis $n = 1$ is just Lem. 4.3. Assume $n > 1$ and let $\overline{a}' = \langle a_0, \dots, a_{n-2} \rangle$ so that $\overline{a} = \overline{a}' \frown \langle a_{n-1} \rangle$. Let $\Phi = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\overline{a}', \top^{\boldsymbol{A}})$. $\boldsymbol{A}/\Phi \in \mathsf{HI}_P$ and $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\overline{a}, \top^{\boldsymbol{A}}) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(a_{n-1}, \top^{\boldsymbol{A}}) \vee \Phi$, where "$\vee$" denotes the join in the lattice of HI_P-congruences of $\boldsymbol{A}$. Let $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{A}/\Phi$ be the natural homomorphism. By an easy corollary of the correspondence theorem of universal algebra we have

(13)
$$h^{-1}\big(\mathrm{Cg}^{\boldsymbol{A}/\Phi}_{\mathsf{HI}_P}(h\, a_{n-1}, \top^{\boldsymbol{A}/\Phi})\big) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(a_{n-1}, \top^{\boldsymbol{A}}) \vee \Phi = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\overline{a}, \top^{\boldsymbol{A}}).$$

Then
$b \equiv c \pmod{\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(\overline{a}, \top^{\boldsymbol{A}})}$

iff $h\, b \equiv h\, c \pmod{\mathrm{Cg}^{\boldsymbol{A}/\Phi}_{\mathsf{HI}_P}(h\, a_{n-1}, \top^{\boldsymbol{A}/\Phi})}$, by (13)

iff $h\, a_{n-1} \Rightarrow_P^{\boldsymbol{A}/\Phi} (h\, b \Leftrightarrow_P^{\boldsymbol{A}/\Phi} h\, c) = \top^{\boldsymbol{A}/\Phi}$, by Lem. 4.3

iff $h\big(a_{n-1} \Rightarrow_P^{\boldsymbol{A}} (b \Leftrightarrow_P^{\boldsymbol{A}} c)\big) = \top^{\boldsymbol{A}/\Phi}$,

iff $a_{n-1} \Rightarrow_P^{\boldsymbol{A}} (b \Leftrightarrow_P^{\boldsymbol{A}} c) \equiv \top^{\boldsymbol{A}} \pmod{\Phi}$,

iff $\overline{a}' \Rightarrow_P^{\boldsymbol{A}} \big((a_{n-1} \Rightarrow_P^{\boldsymbol{A}} (b \Leftrightarrow_P^{\boldsymbol{A}} c)) \leftrightarrow \top^{\boldsymbol{A}}\big) = \top^{\boldsymbol{A}}$,
by the induction hypothesis

iff $\overline{a}' \Rightarrow_P^{\boldsymbol{A}} \big(a_{n-1} \Rightarrow_P^{\boldsymbol{A}} (b \Leftrightarrow_P^{\boldsymbol{A}} c)\big) = \top^{\boldsymbol{A}}$, by $(\mathrm{H6}_P^{\mathrm{I}})$ and $(\mathrm{MP}_P^{\mathrm{Ieq}})$

iff $\overline{a} \Rightarrow_P^{\boldsymbol{A}} (b \Leftrightarrow_P^{\boldsymbol{A}} c) = \top^{\boldsymbol{A}}$.

□

Corollary 4.5. *For every* $\boldsymbol{A} \in \mathsf{HI}_P$ *and all* $a, b \in A$, $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(a,b) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{HI}_P}(a \Leftrightarrow^{\boldsymbol{A}}_P b, \top^{\boldsymbol{A}})$.

Proof. $\subseteq$: By Cor. 4.4, $a \equiv b \pmod{\mathrm{Cg}_{\mathsf{HI}_P}(a \Leftrightarrow_P b, \top)}$ iff

$$(a \Rightarrow_P b) \Rightarrow_P \big((b \Rightarrow_P a) \Rightarrow_P (a \Rightarrow_P b)\big) \;=\; \top, \text{ and} \tag{14}$$

$$(a \Rightarrow_P b) \Rightarrow_P \big((b \Rightarrow_P a) \Rightarrow_P (b \Rightarrow_P a)\big) \;=\; \top \tag{15}$$

The equality (14) holds by $(\mathrm{H1}^{\mathrm{I}}_P)$, and (15) holds since $(b \Rightarrow_P a) \Rightarrow_P (b \Rightarrow_P a) = \top$ by $(\mathrm{H0}^{\mathrm{I}}_P)$. Thus $\subseteq$ holds.

$\supseteq$: Let $\Phi = \mathrm{Cg}_{\mathsf{HI}_P}(a,b)$. $a{\Rightarrow_P}b \equiv_\Phi a{\Rightarrow_P}a = \top$ and $b{\Rightarrow_P}a \equiv_\Phi b{\Rightarrow_P}b = \top$. So $\mathrm{Cg}_{\mathsf{HI}_P}(a \Leftrightarrow_P b,\, \top) = \mathrm{Cg}_{\mathsf{HI}_P}(a \Rightarrow_P b,\, \top) \vee \mathrm{Cg}_{\mathsf{HI}_P}(b \Rightarrow_P a,\, \top) \subseteq \Phi$. □

Theorem 4.6. *Let* $P(x,y)$ *be any finite sequence of binary 1-formulas over an arbitrary language type* Λ.

(i) HI_P *is a relatively point-regular and relatively congruence-orderable quasivariety.*

(ii) HI_P *has* EDPRC. *Moreover, the equations* $\big((x{\Leftrightarrow_P}y){\Rightarrow_P}(z{\Leftrightarrow_P}w)\big) \approx \top$ *form a system of defining equations for the principal relative congruences of* HI_P.

Proof. (i). By Thm. 4.2 IPC_P is the assertional logic of HI_P. But IPC_P is regularly algebraizable by Thm. 3.13. Thus HI_P is relatively point-regular by Thm. 2.5. Also, IPC_P is Fregean by Cor. 3.15. So HI_P is relatively congruence-orderable by Thm. 2.10.

(ii). IPC_P has the multiterm deduction-detachment theorem (Cor. 3.15). Thus since IPC_P is the assertional logic of HI_P, HI_P has the EDPRC by Thm. 3.3. More precisely, let $\boldsymbol{A} \in \mathsf{HI}_P$ and $a,b,c,d \in A$. Then

$$\begin{aligned}
c \equiv d &\pmod{\mathrm{Cg}_{\mathsf{HI}_P}(a,b)}\\
&\text{iff}\quad c \equiv d \pmod{\mathrm{Cg}_{\mathsf{HI}_P}(a \Leftrightarrow_P b, \top)}, &&\text{by Cor. 4.5}\\
&\text{iff}\quad (a \Leftrightarrow_P b) \Rightarrow_P (c \Leftrightarrow_P d) = \top, &&\text{by Lem. 4.3.}
\end{aligned}$$

□

If $P = \langle x \Rightarrow y\rangle$, a singleton, then the identities $(\mathrm{H1}_P)$ and $(\mathrm{H2}_P)$ and the quasi-identity (HP_P) are just the defining identities and quasi-identity of the variety of Hilbert algebras [14, Def. 1]. (The identity $(\mathrm{H0}_P)$ is known to be derivable in this case). It is not difficult to show that the class of subalgebras of $\{\Rightarrow, \top\}$-reducts of members of HI_P in the case of a single implication formula is termwise definitionally equivalent to the variety of Hilbert algebras, and hence that HI_P itself can in this case be identified with a variety of Hilbert algebras with compatible operations. The variety of Hilbert algebras has no proper subquasivarieties (i.e., every subquasivariety is in fact a subvariety); this follows easily from the properties of Hilbert algebras established in [14]. A similar but more restricted result holds for the variety of Hilbert algebras with compatible operations:

Theorem 4.7 ([13, Corollary 2.25]). *Assume $P(x,y)$ consists of a single binary 1-formula. Then every subquasivariety of HI_P that is relatively congruence-orderable and has* EDPRC *is in fact a variety.*

This fails to be the case in general when P contains two or more terms. In particular, HI_P itself is not a variety in this case. See Theorem 5.4 below.

5. Multiterm Hilbert Quasivarieties and their Relation to Multiterm Intermediate Propositional Calculi

By a *relative subvariety* of a given quasivariety Q we mean a subquasivariety of Q that is defined by adjoining only identities to the identities and quasi-identities of Q.

Definition 5.1. Let Λ be an arbitrary pointed language type and $P(x,y)$ a finite sequence of binary 1-formulas over Λ. A relative subvariety of HI_P is called a *P-Hilbert quasivariety*.

A quasivariety is called a *multiterm Hilbert quasivariety* if it is a P-Hilbert quasivariety for some language type Λ and some implication system P. The multiterm Hilbert quasivarieties are the equivalent quasivarieties of the multiterm intermediate propositional logics. More precisely, if Q is a P-Hilbert quasivariety, then Q is the equivalent quasivariety of some axiomatic extension $\mathcal{S}$ of IPC_P. We will also see that the multiterm Hilbert quasivarieties coincide exactly with those quasivarieties that are relatively point-regular, relatively congruence-orderable, and have EDPRC. It turns out that the most efficient way to prove this is by metalogical means. In the next theorem, which is the main result of the paper, we give this equivalence together with the precise connection between the metalogical property of being Fregean and the algebraic property of being relatively congruence-orderable, in the presence of the multiterm deduction-detachment theorem and of EDPRC.

Theorem 5.2. *Let Λ be an arbitrary pointed language type. Let Q be a pointed quasivariety over Λ and $\mathcal{S}$ its assertional logic. The following four conditions are equivalent.*

- (i) Q *is relatively point-regular, relatively congruence-orderable, and has* EDPRC.
- (ii) $\mathcal{S}$ *is Fregean and has the multiterm deduction-detachment theorem.*

(iii) Q *is a multiterm Hilbert quasivariety, i.e., a relative subvariety of* HI_P *for some implication system* $P(x, y)$.
(iv) $\mathcal{S}$ *is a multiterm intermediate propositional calculus, i.e., an axiomatic extension of* IPC_P *for some deduction-detachment system* $P(x, y)$.

Proof. The equivalence of (ii) and (iv) is just a restatement of Cor. 3.15. We now establish the equivalence of (i) and (ii). Assume (ii) holds, i.e., that $\mathcal{S}$ is Fregean and has P-modus ponens and the P-deduction theorem for some P. $\mathcal{S}$ is an axiomatic extension of IPC_P, by the equivalence of (ii) and (iv), and hence is regularly algebraizable. Thus Q is relatively point-regular by Thm. 2.5 and it is relatively congruence-orderable by Thm. 2.10. It is proved in [7, Thm. 7.3] that an algebraizable deductive system has the multiterm deduction-detachment theorem iff its equivalent quasivariety has EDPRC. It follows that Q has EDPRC, and hence (i) holds. Assume conversely that (i) holds, i.e., Q is relatively point-regular, relatively congruence-orderable, and has EDPRC. Then $\mathcal{S}$ is Fregean by Thm. 2.10 and has the multiterm deduction-detachment theorem by [7, Thm. 7.3]. So (i) and (ii) are equivalent. To complete the proof we show that (iii) implies (i) and that (iv) implies (iii).

Assume (iii) holds, i.e., Q is a relative subvariety of HI_P. HI_P itself is relatively point-regular, and it is relatively congruence-orderable and has EDPRC by Thm. 4.6. We only have to see that these properties are preserved when passing to the relative subvariety Q of HI_P. Let $\boldsymbol{A} \in \mathsf{Q}$ and $a, b \in A$ such that $\mathrm{Cg}_{\mathsf{Q}}(a, \top) = \mathrm{Cg}_{\mathsf{Q}}(b, \top)$. But since Q is a relative subvariety of HI_P, $\mathrm{Cg}_{\mathsf{Q}}(a, \top) = \mathrm{Cg}_{\mathsf{HI}_P}(a, \top)$ and $\mathrm{Cg}_{\mathsf{Q}}(b, \top) = \mathrm{Cg}_{\mathsf{HI}_P}(b, \top)$. So $a = b$ because HI_P is relatively congruence-orderable. Hence Q is also relatively congruence-orderable. Similar argument shows that Q relatively point-regular and inherits the property of having EDPRC from HI_P. So (iii) implies (i).

Finally, assume that $\mathcal{S}$ is an axiomatic extension of IPC_P. Then $\mathcal{S}$ is regularly algebraizable since IPC_P is, and its equivalent quasivariety is a relative subvariety of the equivalent quasivariety IPC_P; but this is HI_P by Thm. 4.2. This establishes the implication from (iv) to (iii) and hence completes the proof of the theorem. □

The three properties of item (i) of the theorem are independent.

- The variety of upper-bounded distributive lattices is pointed, congruence-orderable, and has EDPC. But it is not point-regular: That distributive lattices have EDPC was discussed in the remarks following Thm. 3.3. It follows easily from the equational characterization of principal congruences that, for any distributive lattice $\boldsymbol{A}$ and all $a, b \in A$, $b \equiv \top \pmod{\mathrm{Cg}(a, \top)}$ iff $a \leq b$. Congruence-orderability follows at once. Let $\boldsymbol{A}$ be the 4-element chain; $\bot < a < b < \top$. Let Θ be the congruence that identifies just b and $\top$ and Φ the extension of Θ that identifies $\bot$ and a in addition to b and $\top$. $\top/\Theta = \top/\Phi$. So upper-bounded distributive lattices are not point-regular.

- The variety of classical equivalential algebras, i.e., the variety of Boolean groups, is point-regular and congruence-orderable, but it does not have EDPC: The variety of groups is point-regular and hence, trivially, so is the variety of Boolean groups. If $\boldsymbol{A}$ is a Boolean group, then $b \equiv 0 \pmod{\mathrm{Cg}(a, 0)}$ iff $b \in \{a, 0\}$. So Boolean groups are congruence-orderable. It follows also from this that Boolean groups do not have EDPC, since it is easy to see that the relation $y \in \{x, 0\}$ cannot be expressed by any conjunction of equations in x and y. Boolean groups are not congruence-distributive; this is another way to see that EDPC fails.
- The variety of interior algebras, i.e., the equivalent quasivariety of the normal modal logic S4, is point-regular and has EDPC, but it is not congruence-orderable: Interior algebras inherit point-regularity from Boolean algebras, and they have EDPC because S4 has the (uniterm) deduction-detachment theorem. They are not congruence-orderable because there are interior algebras of arbitrarily large order that are simple, i.e., without any proper congruences. To see this let $\boldsymbol{B}$ be any Boolean algebra and define $\Box$ on B by setting $\Box\top = \top$ and $\Box b = \bot$ for every $b \in B \setminus \{\top\}$. It is easy to check that $\langle \boldsymbol{B}, \Box \rangle$ is a simple interior algebra.

By Theorem 4.7, if P is a singleton, then every relatively congruence-orderable subquasivariety of HI_P with EDPRC is in fact a variety. In metalogical terms this says that every 1-deductive system that is Fregean and has the uniterm deduction-detachment theorem is strongly algebraizable. Equivalently, by Cor. 3.15, every P-intermediate propositional calculus with P a singleton is strongly algebraizable. This result is obtained in [16, Prop. 4.49] as a consequence of a more general result. A simple, direct proof can be found in [13, Theorem 2.24]. Note that it suffices to show only that IPC_P is strongly algebraizable when $|P| = 1$, since by Cor. 3.15 every P-intermediate propositional calculus is an axiomatic extension of IPC_P. This result does not hold in general for multiterm intermediate propositional calculi. The following is a particularly simple example of this kind.

5.1. **Multiterm intermediate propositional calculi are not strongly algebraizable.** Let $\boldsymbol{A} = \langle \{0, 1\}, \to, 1 \rangle$ be the 2-element Hilbert algebra, i.e., the $\{\to, \top\}$-reduct of the 2-element Boolean algebra. Let Λ be an arbitrary pointed language type with two distinguished binary operation symbols $\Rightarrow_0$ and $\Rightarrow_1$; $\Lambda = \{\Rightarrow_0, \Rightarrow_1, \top, \lambda_j\}_{j\in J}$. Set $P(x, y) = \langle x \Rightarrow_0 y, x \Rightarrow_1 y\rangle$. Let $\multimap$ be the binary operation on $\{0,1\}$ that is constant 1. Let $\boldsymbol{B} = \langle \{0,1\}, \to, \multimap, 1, \lambda_j^{\boldsymbol{B}}\rangle_{j\in J}$, where $\Rightarrow_0^{\boldsymbol{B}} = \to$, $\Rightarrow_1^{\boldsymbol{B}} = \multimap$, $\top^{\boldsymbol{B}} = 1$, and the $\lambda_j^{\boldsymbol{B}}$ are all constant operations on $\{0,1\}$ of appropriate rank taking the value 1. Finally, let $\boldsymbol{C}$ be like $\boldsymbol{B}$ but with the two implications interchanged, i.e., $\boldsymbol{C} = \langle \{0,1\}, \multimap, \to, 1, \lambda_j^{\boldsymbol{B}}\rangle_{j\in J}$.

Lemma 5.3. (i) *$\boldsymbol{B} \times \boldsymbol{C}$ is a P-Hilbert algebra.*
(ii) *The variety generated by $\boldsymbol{B} \times \boldsymbol{C}$ fails to be congruence-distributive.*

Proof. (i). It is straightforward but time consuming to verify that $\boldsymbol{B}$ and $\boldsymbol{C}$ both satisfy the identities (H0$_P$)–(H2$_P$), (Hλ_P) for $\lambda \in \{\Rightarrow_0, \Rightarrow_1\}$, and the quasi-identity (HP$_P$). (the identity (Hλ_P) is trivial satisfied for the operation of Λ different from $\Rightarrow_0$ and $\Rightarrow_1$.) As an example we outline the verification that (H2$_P$) is an identity of $\boldsymbol{B}$.

We first make a useful observation about the evaluation of formulas in $\boldsymbol{B}$ in general. In an expression of the form

$$\alpha_0 \Rightarrow_P \alpha_1 \Rightarrow_P \alpha_2 \Rightarrow_P \cdots \Rightarrow_P \alpha_n, \tag{16}$$

it is assumed that association is to the right. Assume that all of the $\alpha_0, \dots, \alpha_n$ are formulas (as opposed to sequences of formulas). The sequence (16) consists of all formulas of the form

$$\alpha_0 \Rightarrow_{\pi 0} \alpha_1 \Rightarrow_{\pi 1} \alpha_2 \Rightarrow_{\pi 2} \cdots \Rightarrow_{\pi(n-1)} \alpha_n, \tag{17}$$

where π ranges over all functions from $\{0, \dots, n-1\}$ to $\{0,1\}$. Recall that $\Rightarrow_0^{\boldsymbol{B}} = \to$ and $\Rightarrow_1^{\boldsymbol{B}} = \multimap$. Consider any evaluation $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{B}$. (17) evaluates to

$$h(\alpha_0) \Rightarrow_{\pi 0}^{\boldsymbol{B}} h(\alpha_1) \Rightarrow_{\pi 1}^{\boldsymbol{B}} h(\alpha_2) \Rightarrow_{\pi 2}^{\boldsymbol{B}} \cdots \Rightarrow_{\pi(n-1)}^{\boldsymbol{B}} h(\alpha_n). \tag{18}$$

There are exactly three situations in which (18) takes the value 0: (I) $\pi i = 0$ (i.e., $\Rightarrow_{\pi i}^{\boldsymbol{B}} = \to$) for all $i < n$, (II) $h(\alpha_i) = 1$ for all $i < n$, and (III) $h(\alpha_n) = 0$.

We first check that

$$x \Rightarrow_P (y \Rightarrow_P z) = \langle x \to (y \to z),\, x \multimap (y \to z),\, x \to (y \multimap z),\, x \multimap (y \multimap z)\rangle,$$

and that

$$\begin{aligned}(x \Rightarrow_P y) \Rightarrow_P (x \Rightarrow_P z) &= \langle x \to y, x \multimap y\rangle \Rightarrow_P (x \Rightarrow_P z)\\ &= (x \to y) \Rightarrow_P (x \multimap y) \Rightarrow_P x \Rightarrow_P z.\end{aligned}$$

Thus $\big(x \Rightarrow_P (y \Rightarrow_P z)\big) \Rightarrow_P \big((x \Rightarrow_P y) \Rightarrow_P (x \Rightarrow_P z)\big)$ equals

$$\begin{aligned}&(x \to (y \to z)) \Rightarrow_P (x \multimap (y \to z)) \Rightarrow_P (x \to (y \multimap z))\\ &\qquad \Rightarrow_P (x \multimap (y \multimap z)) \Rightarrow_P (x \to y) \Rightarrow_P (x \multimap y) \Rightarrow_P x \Rightarrow_P z.\end{aligned}$$

By our previous observations we know that the only member of this sequence of formulas that might possibly have a 0-evaluation is

$$(x \to (y \to z)) \to (x \multimap (y \to z)) \to (x \to (y \multimap z)) \\ \to (x \multimap (y \multimap z)) \to (x \to y) \to (x \multimap y) \to x \to z.$$

Since the second, third, fourth, and sixth antecedents in this formula (namely, $x \multimap (y \to z)$, $x \to (y \multimap z)$, $x \multimap (y \multimap z)$, and $(x \multimap y)))$ always evaluate to 1, the formula will always have the same evaluation as

$$\big(x \to (y \to z)\big) \to \big((x \to y) \to (x \to z)\big).$$

Finally, since $\to$ is the classical implication, this term always evaluates to 1.

Thus (H2$_P$) is an identity of $\boldsymbol{B}$, and essentially the same argument shows that it is also an identity of $\boldsymbol{C}$ and thus of $\boldsymbol{B} \times \boldsymbol{C}$. The other axioms of HI_P and (HP$_P$) can be verified for $\boldsymbol{B} \times \boldsymbol{C}$ in a similar way. So $\boldsymbol{B} \times \boldsymbol{C}$ is a P-Hilbert algebra.

(ii). Let $\alpha(x, y_0, \ldots, y_{n-1})$ be any Λ-term that contains x and possibly other variables. Then α is either (I) the variable x itself, (II) of the form $\beta(x, \bar{y}) \Rightarrow_i \gamma(x, \bar{y})$ with $i \in \{0, 1\}$, or (III) of the form $\lambda_j \beta_0(x, \bar{y}), \ldots, \beta_{m_j - 1}(x, \bar{y})$ for some $j \in J$ (m_j is the rank of λ_j). Let

$$\langle a, b \rangle = \alpha^{\boldsymbol{B} \times \boldsymbol{C}}(\langle 0,0 \rangle, \langle 0,0 \rangle, \ldots, \langle 0,0 \rangle).$$

If (II) holds then $a = 1$ if $i = 1$ and $b = 1$ if $i = 0$. If (III) holds then $a = b = 1$. Thus $\alpha(x, \bar{y}) \approx x$ is an identity of $\boldsymbol{B} \times \boldsymbol{C}$ only if $\alpha(x, \bar{y}) = x$. This implies that the variety generated by $\boldsymbol{B} \times \boldsymbol{C}$ can satisfy no nontrivial Mal'cev condition. In particular it cannot have Jońsson terms that characterize (absolute) congruence-distributivity and hence is not congruence-distributive. □

Theorem 5.4. *Let Λ be an arbitrary pointed language type with two distinguished binary operation symbols $\Rightarrow_0$ and $\Rightarrow_1$; $\Lambda = \{\Rightarrow_0, \Rightarrow_1, \top, \lambda_j\}_{j \in J}$. Set $P(x, y) = \langle x \Rightarrow_0 y, x \Rightarrow_1 y \rangle$. Then HI_P is not a variety.*

Proof. Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be the Λ-algebras defined as above. HI_P is relatively congruence-distributive because it has EDPRC ([4, Theorem 5.5]). If HI_P were a variety it would be (absolutely) congruence-distributive. But it is not because by Lem. 5.3 it includes a subvariety (the subvariety generated by $\boldsymbol{B} \times \boldsymbol{C}$) that is not congruence-distributive. □

We note that HI_P provides an example of a relatively point-regular, relatively congruence-orderable, and relatively congruence-distributive quasivariety that is not a variety.

Corollary 5.5. *Let Λ and $P(x, y)$ be as in the theorem. Then* IPC$_P$ *is fails to be strongly algebraizable.*

HI_P satisfies the identity sequence $(\top \Rightarrow_P x) \Leftrightarrow_P x \approx \top$. In general, however, if $P = \langle x \Rightarrow_0 y, \dots, x \Rightarrow_{n-1} y \rangle$, then $(\top \Rightarrow_i x) \Rightarrow_P x \approx x$ is not an identity sequence of HI_P for any $i < n$; equivalently, $\top \Rightarrow_i x \approx x$ is not an identity for any $i < n$. Of course, if $P = \{\to\}$, a singleton, then $\top \to x \approx x$ is an identity of HI_P. In this case HI_P is termwise definitionally equivalent to a variety of Hilbert algebras with compatible operations. The so-called subtractive identity $\top \to x \approx x$ is characteristic of Hilbert algebras.

Theorem 5.6. *Let* Q *be a multiterm Hilbert quasivariety, i.e., a relative subvariety of* HI_P *for some implication system* $P(x,y) = \langle x \Rightarrow_0 y, \dots, x \Rightarrow_{n-1} y \rangle$. *If the subtractive identity* $\top \Rightarrow_i x \approx x$ *holds in* Q *for some* $i < n$, *then* Q *is termwise definitionally equivalent to a variety of Hilbert algebras with compatible operations.*

Proof. By Thm. 5.2 Q is the equivalent quasivariety of a P-intermediate propositional calculus $\mathcal{S}$. By hypothesis $(\top \Rightarrow_i x) \Leftrightarrow_P x$, and hence a fortiori $(\top \Rightarrow_i x) \Rightarrow_P x$ are theorem sequences of S. Thus by Thm. 3.16 $\mathcal{S}$ is formula wise definitionally equivalent to an axiomatic extension of $\mathrm{IPC}_{\Lambda}^{\to,\top}$. So Q is termwise definitionally equivalent to the equivalent quasivariety of an axiomatic extension of $\mathrm{IPC}_{\Lambda}^{\to,\top}$. This gives the desired result at once. □

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