

THE COHOMOLOGY OF THE SYLOW 2-SUBGROUP OF THE HIGMAN-SIMS GROUP

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ABSTRACT. In this paper we compute the mod 2 cohomology of the Sylow 2-subgroup of the Higman–Sims group HS , one of the 26 sporadic simple groups. We obtain its Poincaré series as well as an explicit description of it as a ring with 17 generators and 79 relations.

1. INTRODUCTION

Recently there has been substantial progress towards computing the mod 2 cohomology of low rank sporadic simple groups. In fact the mod 2 cohomology of every sporadic simple group not containing $(\mathbb{Z}/2)^5$ has been computed, with the notable exceptions of HS (the Higman–Sims group) and Co_3 (one of the Conway groups). The reason for these exceptions is that these two groups have large and complicated Sylow 2-subgroups, with many conjugacy classes of maximal elementary abelian subgroups. The Higman–Sims group has order $44,352,000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, and its largest elementary abelian 2-subgroup is of rank equal to four. It is a subgroup of Co_3 of index equal to $11178 = 2 \cdot 5589$, hence $Syl_2(HS)$ is an index 2 subgroup of $Syl_2(Co_3)$.

In this paper we compute the mod 2 cohomology ring of $S = Syl_2(HS)$, obtaining explicit generators and relations. In a sequel we will determine the necessary stability conditions for computing the cohomology of HS itself. This is a step towards obtaining a calculation of the mod 2 cohomology of Co_3 . This latter group is of particular interest because of its relation to a homotopy–theoretic construction due to Dwyer and Wilkerson (see [2]) and it would seem that the most viable way of accessing this group is via HS .

The calculation we present is long and highly technical, involving techniques from topology, representation theory and computer algebra. Our main result is the following

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Theorem 1.1. *The mod 2 cohomology of $S = \text{Syl}_2(HS)$ has Poincaré series equal to*

$$\frac{(1+x)^2(1-x+x^2)(1+2x-x^5)}{(1-x)^2(1-x^4)(1-x^8)}.$$

As a ring, $H^(S, \mathbb{F}_2)$ has seventeen generators, in degrees*

$$1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 6, 7, 8,$$

and a (minimal) set of seventy-nine relations.

Remark 1.2. The complete set of relations and the Steenrod operations on the generators are explicitly described in two appendices at the end of the paper.

We briefly outline our method of proof. The group S can be expressed as a semidirect product $(\mathbb{Z}/4)^3 : U_3$, where $U_3 \cong D_8$ is the group of upper triangular 3×3 matrices over \mathbb{F}_2 . Using a recent result in [7], we calculate the E_2 -term of the Lyndon–Hochschild–Serre spectral sequence associated to this extension. This involves using an explicit decomposition of the symmetric algebra of a module.

The second ingredient is a computer-assisted verification of the cohomology of S through degree ten. Combined with the structure of the E_2 term we obtain the following crucial result

Theorem 1.3. *The mod 2 spectral sequence associated to $S = 4^3 : U_3$ collapses at E_2 , and yields the Poincaré series above.*

A critical aspect of this is that the generators for the E_2 term occur in low degrees, where the computer can provide enough information to ensure a collapse. This method seems to be quite effective for cohomology computations and can be used to yield the mod 2 cohomology of U_4 and U_5 .

The next step consists of computing the restriction map from $H^*(S, \mathbb{F}_2)$ to the cohomology of centralizers of rank two elementary abelian subgroups, where we obtain an image ring having the same Poincaré series as the cohomology. From this we conclude

Theorem 1.4. *The cohomology of $S = \text{Syl}_2(HS)$ is detected by the centralizers of rank two elementary abelian subgroups, and its explicit image is the ring described above, with 17 generators and 79 relations.*

As mentioned above, the results here will be used in a sequel to calculate the cohomology of HS , and from there obtain information on the cohomology of Co_3 , thus bringing us closer to a complete understanding of rank 4 sporadic simple groups.

Throughout this paper coefficients will be in \mathbb{F}_2 , so they are suppressed. Occasionally k will be used to denote this coefficient ring, especially if it appears in a representation-theoretic context. We refer the reader to [1] for background on group cohomology.

2. THE SUBGROUP STRUCTURE OF $S = Syl_2(HS)$

The Sylow 2-subgroup of the Higman–Sims group (denoted S from here on) has a description as a semi-direct product $S = 4^3 : D_8$ with v_1, v_2, v_3 the generators of the 4^3 while $D_8 = \{t, s \mid t^2 = s^4 = (ts)^2 = 1\}$, and

$$v_1^t = v_3^{-1}, v_2^t = v_2^{-1}, v_1^s = v_2, v_2^s = v_3, v_3^s = v_2^{-1}v_1v_3.$$

In this section we will develop the required details about the subgroup structure of S to enable us to understand its cohomology. To start we have

Lemma 2.1. $\langle v_1v_3 \rangle = \mathbb{Z}/4$ is a normal subgroup of S and $S/\langle v_1v_3 \rangle \cong 2 \wr 2 \wr 2$.

Proof. In the quotient we have $v_1^t = v_3^{-1} \sim v_1$, while $v_1^{ts^2} = v_1^{-1}$. Also $v_2^t = v_2^{-1}$ while $v_2^{ts^2} = v_2$. Thus

$$\langle v_1, ts^2 \rangle \cong D_8, \langle v_2, t \rangle \cong D_8$$

and these two copies of D_8 commute with each other, giving a copy of $D_8 \times D_8$ in the quotient. Next, note that ts exchanges t, ts^2 , and also exchanges v_1, v_2^{-1} , hence the two copies of D_8 above, and the extension $(D_8 \times D_8) : \langle ts \rangle \cong 2 \wr 2 \wr 2$. \square

Definition 2.2. The group K_β is the inverse image in S of the index two subgroup $D_8 \times D_8 \subset S/\langle v_1v_3 \rangle$.

K_β is given explicitly as an extension

$$\langle v_1, v_2, v_3 \rangle : \langle t, s^2 \rangle = 4^3 : 2^2.$$

Moreover, S is the split extension $K_\beta : 2 = K_\beta : \langle ts \rangle$.

Next we examine the maximal elementary abelian subgroups in S . We show that there are precisely eight copies of $2^4 \subset S$. To begin note that there are exactly 5 conjugacy classes of maximal 2-elementaries in $2 \wr 2 \wr 2$, first 3 conjugacy classes of 2^4 's:

$$2_I^4 = \langle v_1^2, ts^2, v_2^2, t \rangle, 2_{II}^4 = \langle v_1^2, v_1ts^2, v_2^2, v_2^{-1}t \rangle$$

both normal, and $2_{III}^4 = \langle v_1^2ts^2, v_2^2, v_2^{-1}t \rangle$ with Weyl group 2^2 . Then there are two conjugacy classes of 2^3 's, each with Weyl group D_8 ,

$$2_I^3 = \langle (v_1v_2)^2, s^2, ts \rangle, 2_{II}^3 = \langle (v_1v_2)^2, v_1v_2^{-1}s^2, ts \rangle$$

which together generate a copy of $D_8 \times 2$,

$$\Delta(D_8) \times 2 = \langle v_1 v_2^{-1} s^2, ts \rangle.$$

This is all standard and can be found in many references. Lifting these groups to S we find

Lemma 2.3. *The following list describes the groups above and their lifts:*

$$\begin{aligned} & 2_I^4, D_8 * D_8 * 4 \\ & 2_{II}^4, D_8 * D_8 * 4 \\ & 2_{I,II}^4, (8: \text{Aut}(8)) * 4 \\ & \Delta(D_8) \times 2, D_8 \times D_8 \end{aligned}$$

Proof. We note that the lift of 2_I^4 is given as

$$\langle v_1 v_3, v_1^2, ts^2, v_2^2, t \rangle = \langle v_1 v_3^{-1}, ts^2, t, v_1^2, v_1 v_3^{-1} v_2^2 \rangle$$

On the other hand, note that $v_1 v_3^{-1} v_2^2$ commutes with s^2, t , consequently with each of the remaining four generators, while the first two generators commute with the third and fourth, and $\langle v_1 v_3^{-1}, ts^2 \rangle = D_8$, $\langle t, v_1^2 \rangle = D_8$, and all three have the central element $(v_1 v_3)^2$ in common. The verification for the second group is similar. For the third, note that $(v_1 s^2)^2 = v_1 v_3$ so that $v_1 s^2$ has order 8. Also conjugation with v_2^2 takes the element to its fifth power, while conjugation by t takes it to its inverse. This gives the extension $8: \text{Aut}(8)$. The final generator can again be chosen as $v_1 v_3^{-1} v_2^2$.

It remains to check the lift of $\Delta(D_8) \times 2$. Note, first that $\langle s^2, v_1 v_2^{-1} \rangle = D_8$ as a subgroup of S , while $v_1 v_3, ts$ both commute with the elements of this D_8 . On the other hand $\langle v_1 v_3, ts \rangle = D_8$ as well, and the final statement follows. \square

Now, to determine the structure of the set of 2^4 's in S we check the inverse images of the conjugates of 2_I^3 and 2_{II}^3 . The normalizer of each of these groups has order 2^6 so there are a total of 4 such groups in $2 \wr 2 \wr 2$, with the remaining two given by conjugation with v_1 ,

$$\begin{aligned} (2_I^3)^{v_1} &= \langle v_1 v_2 ts, v_2 v_3^{-1} s^2, (v_1 v_2)^2 \rangle, \\ (2_{II}^3)^{v_1} &= \langle v_1 v_2 ts, v_2 v_3^{-1} s^2, (v_1 v_2)^3 \rangle \end{aligned}$$

The lift of each to S is a copy of $2^2 \times D_8$ which contains exactly two copies of 2^4 , so S has at least eight copies of 2^4 . In fact we have

Corollary 2.4. *There are no 2^5 's contained in S . There are exactly eight copies of 2^4 contained in S , breaking up into three conjugacy classes, two with two copies each and one with four.*

Proof. The only thing that needs to be pointed out is that if there were a 2^4 which did not contain the central element $(v_1v_3)^2$, (and hence a 2^5 obtained by adjoining $(v_1v_3)^2$), then it would project non-trivially to one of the four 2^4 's in the quotient $2 \wr 2 \wr 2$. But we have identified the lifts of these groups as copies of groups having rank three! Hence, every possible 2^4 contains $(v_1v_3)^2$, and hence projects to one of the two conjugacy classes of extremal 2^3 's. Consequently, it must lie in one or the other of the four copies of $2^2 \times D_8 \subset S$ that we have constructed. \square

We now list the eight 2^4 's explicitly.

$$\begin{aligned} 2_A &= \langle (v_1v_3)^2, (v_1v_2)^2, s^2, ts \rangle \\ 2_A^{v_1} &= \langle (v_1v_3)^2, (v_1v_2)^2, v_1^{-1}v_3s^2, v_1v_2ts \rangle \\ 2_B &= \langle (v_1v_3)^2, (v_1v_2)^2, v_1v_2^{-1}s^2, ts \rangle \\ 2_B^{v_1} &= \langle (v_1v_3)^2, (v_1v_2)^2, v_2v_3^{-1}s^2, v_1v_2ts \rangle \\ 2_B^t &= \langle (v_1v_3)^2, (v_1v_2)^2, v_3v_2^{-1}s^2, v_3v_2^{-1}ts \rangle \\ 2_B^{v_1t} &= \langle (v_1v_3)^2, (v_1v_2)^2, v_1v_2^{-1}s^2, v_1v_3ts \rangle \\ 2_C &= \langle (v_1v_3)^2, (v_1v_2)^2, s^2, v_1v_3ts \rangle \\ 2_C^{v_1} &= \langle (v_1v_3)^2, (v_1v_2)^2, v_1v_3^{-1}s^2, v_2v_3^{-1}ts \rangle \end{aligned}$$

Lemma 2.5. *There is an outer automorphism $\alpha : K \rightarrow K$ which exchanges 2_A and 2_C .*

Proof. We use the description of S as $4^3 : D_8$ where we write

$$D_8 = 2^2 : 2 = \langle ts, ts^{-1} \rangle : \langle t \rangle .$$

But we can replace this copy of D_8 by

$$\langle v_2v_3^{-1}ts, v_1v_2^{-1}ts^{-1} \rangle : \langle t \rangle$$

and α is defined as the identity on 4^3 and the correspondence above on D_8 . \square

We show that there are precisely two conjugacy classes of $D_8 \times D_8 \subset S$.

Remark 2.6. The group $D_8 \times D_8$ constructed in the proof above as the lift of $\Delta(D_8) \times 2$ is

$$\langle s^2, v_1v_2^{-1} \rangle \times \langle ts, v_1v_3 \rangle .$$

We can also construct a second copy of

$$D_8 \times D_8 \subset S$$

as

$$\langle ts, v_2v_3^{-1} \rangle \times \langle ts^{-1}, v_1v_2^{-1} \rangle .$$

It is direct to check that the two $(D_8)^2$ above are not conjugate in S . In fact, the intersection of the second copy of $(D_8)^2$ and $\langle v_1 v_3 \rangle$ is just $\langle (v_1 v_3)^2 \rangle$ and its image in $2 \wr 2 \wr 2$ is easily seen to be $D_8 * D_8$. Thus there are at least two conjugacy classes of $D_8 \times D_8$'s contained in S . Shortly we will show that there are exactly two.

Lemma 2.7. *The span $\langle 2_I^{e_1}, 2_J^{e_2} \rangle$ is always one of the three groups $D_8 \times D_8$, $2^2 \times D_8$, or $2^{2+4} = \text{Syl}_2(L_3(4))$.*

(a) *It is 2^{2+4} if and only if $I = J$ and $e_2 = e_1 v_1$. Consequently, there are exactly 3 conjugacy classes of 2^{2+4} 's in S , two normal and one containing two elements.*

(b) *It is $D_8 \times D_8$ in case*

$$\langle 2_A, 2_C^{v_1} \rangle = \langle 2_B, 2_B^t \rangle$$

and their conjugates by v , and also in the cases

$$\langle 2_A, 2_B^{v_1} \rangle = \langle 2_C, 2_B^t \rangle$$

and their four conjugates by $2^2 = \langle t, v_1 \rangle$. Consequently, there are exactly two conjugacy classes of $D_8 \times D_8 \subset S$, one containing four groups and the other containing 2.

(c) *In each of the remaining cases it is $2^2 \times D_8$, and each $2^2 \times D_8$ is contained in a $D_8 \times D_8$.*

Proof. First we check the result for $2_A, 2_A^{v_1}$. We have

$$(s^2 v_1^{-1} v_3 s^2)^2 = (v_1 v_3)^2, \quad (s^2 v_1 v_2 t s)^2 = (v_2 v_3)^2,$$

$$(t s v_1^{-1} v_3 s^2)^2 = (v_2 v_3)^2, \quad (t s v_1 v_2 t s)^2 = (v_1 v_2)^2,$$

and this is a presentation of 2^{2+4} . The same calculations result for 2_C using the automorphism above. Moreover, ts commutes with $v_1 v_2^{-1}$ while s^2 inverts it. So this change cancels out in the squares for the pair $\langle 2_B, 2_B^{v_1} \rangle$, and we have verified that the groups asserted to be 2^{2+4} 's in fact are.

The remaining statements are now easily checked by comparing with the $D_8 \times D_8$'s already constructed above. But this gives a complete list of possible pairs and the result follows. \square

We show there there are exactly two conjugacy classes of $2 \wr 2 \wr 2 \subset S$. Note that t normalizes $\langle 2_A, 2_C^{v_1} \rangle$, exchanging the two copies of D_8 , so

$$\langle 2_A, 2_C^{v_1}, t \rangle \cong 2 \wr 2 \wr 2.$$

However, since $s^2, v_i^{e_1} v_j^{e_2} t s$ are not conjugate in S , it follows that no $D_8 \times D_8$ in the second conjugacy class is contained in a $2 \wr 2 \wr 2$.

Corollary 2.8. *There are exactly four copies of $2 \wr 2 \wr 2$ contained in S forming two conjugacy classes with conjugation by v_1 exchanging the groups in each class.*

Proof. The normalizer of $\langle 2_B, 2_C^{v_1} \rangle$ is obtained by adjoining t, v_2^2 . Thus there are three degree two extensions of $\langle 2_B, 2_C^{v_1} \rangle$ in S . The extension by t is $2 \wr 2 \wr 2$. Clearly, the extension by v_2^2 does not give a $2 \wr 2 \wr 2$.

Finally, consider the extension by tv_2^2 . Replace the second D_8 by

$$\langle (v_2v_3)^2ts^{-1}, v_1v_2^{-1} \rangle.$$

Then these two copies of D_8 commute with each other and their span is $D_8 \times D_8$. Moreover, tv_2^2 exchanges them, giving an isomorphism of this group with a second copy of $2 \wr 2 \wr 2$. \square

Remark 2.9. There is an automorphism of S fixing $\langle v_1, v_2, v_3 \rangle$ and exchanging the two conjugacy classes of $2 \wr 2 \wr 2$'s constructed above. Indeed, such an automorphism can be given by setting

$$t \leftrightarrow tv_2^2, ts \leftrightarrow ts, ts^{-1} \leftrightarrow ts^{-1}(v_2v_3)^2$$

There are exactly six conjugacy classes of maximal $2^3 \subset S$.

A computer analysis of S using MAGMA or alternatively a direct analysis of the two copies of $D_8 * D_8 * 4$ and the $(8: \text{Aut}(8)) * 4$ obtained as the lifts of the three 2^4 's in $2 \wr 2 \wr 2$ shows that the maximal elementary abelian subgroups of S consist of the three 2^4 's discussed above and 6 copies of 2^3 , all contained in the subgroup K_β : one given as $I = \langle v_1^2, v_2^2, v_3^2 \rangle$ with centralizer $CI = 4^3$, and the other five all with centralizer of the form $2^2 \times 4$. These centralizers are

$$CII = \langle v_1v_3^{-1}v_2^2, ts^2, v_3^2 \rangle$$

$$CIII = \langle v_1v_3^{-1}v_2^2, t, s^2 \rangle$$

$$CIV = \langle v_1v_3^{-1}v_2^2, s^2, v_1v_3t \rangle$$

$$CV = \langle v_1v_3^{-1}v_2^2, v_1^2, v_1ts^2 \rangle$$

$$CVI = \langle v_1v_3^{-1}v_2^2, v_2t, v_2v_3^{-1}s^2 \rangle$$

with the 2^3 subgroups denoted II, \dots, VI respectively.

The group-theoretic information which we have described in this section will be used subsequently to establish a detection theorem for $H^*(S)$.

3. PRELIMINARIES ON MODULES

Let $U_3 \cong D_8$ denote the Sylow 2-subgroup of $L_3(2)$. In this section we describe the symmetric algebra of the natural U_3 module M by giving a “factorization” of this algebra. This information will be used to describe the E_2 term of a spectral sequence converging to $H^*(S)$. In order to describe this factorization we must make a number of definitions and recall a few results. We will be using ideas and methods from [7]. We write the symmetric algebra as $k[x, y, z]$, where the action of U_3 on the homogeneous polynomials of degree 1 preserves the flag of subspaces $\langle x \rangle \subset \langle x, y \rangle \subset \langle x, y, z \rangle$. Let's denote by b, c, d the elements of U_3 which send $y \mapsto y + x$, $z \mapsto z + x$, and $z \mapsto z + y$, respectively. (b is supposed to fix z , c to fix y and d to fix x .) We write H_1 for the subgroup $\langle b, c \rangle$ and H_2 for the subgroup $\langle d \rangle$.

Remark 3.1. It is worth noting that the module M is *not* isomorphic to M^* , but that the two differ by an outer automorphism of the group U_3 . This means that the symmetric algebras $S^*(M)$ and $S^*(M^*)$ also differ by an outer automorphism of the group; thus if we are only interested in Poincaré series it doesn't matter which one we study. However, it is *not* true that $S^*(M)^* \approx S^*(M^*)$.

Definition 3.2. $d_1 := x$, $d_2 := y^2 + xy$, $d_4 := z(z + y)(z + x)(z + y + x)$

The following result is well-known:

Proposition 3.3. $k[x, y, z]^{U_3} = k[d_1, d_2, d_4]$

Definition 3.4. $\phi := z^2 + yz$, $\theta := z^2 + xz$. We denote by N, K , the U_3 -submodule of the symmetric algebra generated by ϕ, θ , respectively.

Lemma 3.5. N is isomorphic to the permutation module $k[U_3/H_2]$ and has socle d_1^2 . K is isomorphic to the permutation module $k[U_3/H_1]$ and has socle d_2 .

We will now exhibit three U_3 -submodules of $k[x, y, z]$.

Definition 3.6.

$$A := k[d_4], \quad B := k \oplus K \oplus d_2K \oplus d_2^2K \oplus \dots$$

$$C := k \oplus M \oplus N \oplus d_1N \oplus d_1^2N \oplus d_1^3N \oplus \dots$$

Note that A is a $k[d_4][U_3]$ -submodule of $k[x, y, z]$, while B is a $k[d_2][U_3]$ -submodule and C a $k[d_1][U_3]$ -submodule. (The only point to check is that $d_1M \subset N$, but this can be done by working directly with the generating polynomials.)

Now we can describe the “factorization” mentioned above:

Proposition 3.7. *There is an isomorphism of U_3 -modules*

$$A \otimes B \otimes C \rightarrow k[x, y, z]$$

induced by multiplication in the symmetric algebra.

Proof. For any triple of indecomposable modules A_i, B_j, C_l appearing in the direct sum decompositions of $A, B,$ and C above, we have $\text{soc}(A_i \otimes B_j \otimes C_l) = \text{soc}(A_i) \otimes \text{soc}(B_j) \otimes \text{soc}(C_l)$. Thus $\text{soc}(A \otimes B \otimes C) = \text{soc}(A) \otimes \text{soc}(B) \otimes \text{soc}(C) = k[d_4] \otimes k[d_2] \otimes k[d_1]$. This shows that the map above is an isomorphism on the socle, hence it is injective. The Poincaré series of the two sides are equal, by a calculation, so we have an isomorphism. \square

Remark 3.8. No analogous factorization exists for larger fields or for more variables.

We can rewrite this factorization as a direct sum decomposition

Proposition 3.9. *There is an isomorphism of U_3 -modules:*

$$\begin{aligned} k[x, y, z] = & k[d_4] \otimes (k \oplus M) \\ & \oplus k[d_1, d_4] \otimes N \\ & \oplus k[d_2, d_4] \otimes (K \oplus W) \\ & \oplus k[d_1, d_2, d_4] \otimes F. \end{aligned}$$

Here $F = N \otimes K$ is a free U_3 -module, appearing in degree 4, while $k, M, N, W = M \otimes K$, and K appear in degrees 0, 1, 2, 3, and 2 respectively.

4. THE SYLOW 2-SUBGROUP OF THE HIGMAN-SIMS GROUP: MODULES AND COHOMOLOGY

As we mentioned, the Sylow 2-subgroup can be written as the semidirect product $(\mathbb{Z}_4)^3 \rtimes U_3$, where the action of U_3 on the vector space $H^1((\mathbb{Z}_4)^3)$ gives this cohomology group the structure of the natural U_3 -module M . It follows that as a U_3 -module, $H^*((\mathbb{Z}_4)^3) = \Lambda^*(M) \otimes S^*(M)$.

In this section we compute the Poincaré series of $H^*(U_3; \Lambda^*(M) \otimes S^*(M))$. We will also try to find a bound on the degrees of generating elements. To do this we must compute the Poincaré series of the cohomologies of all the U_3 -modules appearing in the decomposition of $\Lambda^*(M) \otimes S^*(M)$. Since $\Lambda^*(M) = k \oplus M \oplus M^* \oplus k$, it is enough to compute the cohomologies of the modules appearing in the decomposition (3.9) of $S^*(M)$, plus the cohomologies of the tensor products of these modules with M and M^* .

The first thing to understand is how the tensor products of the modules $k, M, N, K, W,$ and F with the modules M and M^* break up into direct sums of indecomposables.

The answers are displayed in the table below. A couple of new modules appear; they are described below the table.

\otimes	M	M^*
k	M	M^*
M	Y_9	$F \oplus k$
N	$F \oplus N$	$F \oplus N$
K	W	W^*
W	$F \oplus X_{10}$	$F^{\oplus 2} \oplus K$
F	$F^{\oplus 3}$	$F^{\oplus 3}$

The module X_{10} fits in an exact sequence $X_{10} \hookrightarrow F \oplus F \rightarrow W$. The module Y_9 is by definition $M \otimes M$, and there is an exact sequence $M \hookrightarrow T \oplus F \rightarrow Y_9$. This second exact sequence is constructed by taking the exact sequence $k \hookrightarrow T \rightarrow M$ and applying $-\otimes_k M$. Here we note that T is the permutation module $U_3 / \langle b \rangle$.

The next step is to compute cohomologies for each of the modules listed in the table; we will need Poincaré series and degrees of generators for the cohomologies $H^*(U_3, X)$ as modules over the cohomology $H^*(U_3, k)$, for each indecomposable module X . We display the results in the form of a table and give the methods of proof afterwards.

Module	Poincaré Series	Bound on Generator Degrees
k	$\frac{1}{(1-x)^2}$	0
M	$\frac{1}{(1-x)^2}$	1
M^*	$\frac{1}{(1-x)^2}$	1
N	$\frac{1}{(1-x)}$	0
K	$\frac{1}{(1-x)^2}$	1
W	$1 + \frac{x}{(1-x)^2}$	2
W^*	$\frac{2-x}{(1-x)^2}$	1
F	1	0
Y_9	$\frac{2-2x-x^2}{(1-x)^2}$	2
X_{10}	$2 + x + \frac{x^2}{(1-x)^2}$	3

Now we present brief arguments for these cohomology computations.

In the case of a permutation module, i.e. the cases k , N , K , and F , the cohomology is just the cohomology of the appropriate subgroup, regarded as a module over the cohomology of U_3 via the restriction map. The degrees of the module generators can be worked out from a knowledge of the restriction image.

Now we turn to the more subtle cases. First let us note that there is an exact sequence $k \hookrightarrow T \twoheadrightarrow M$. In fact, more is true, there is also an exact sequence $M \hookrightarrow N \twoheadrightarrow k$, and by dualizing we can get exact sequences for M^* . We'll just work with the first exact sequence.

Lemma 4.1. *The cohomology of M , i.e. $H^*(U_3, M)$, has Poincaré series $(1-x)^{-2}$ and is generated by elements in degrees 0 and 1.*

Proof. Consider the long exact sequence in cohomology arising from the short exact sequence $k \hookrightarrow T \twoheadrightarrow M$. The maps $H^i(U_3, k) \rightarrow H^i(U_3, T)$ are just the restriction homomorphisms for $\langle b \rangle = \mathbb{Z}_2 \subset U_3$, which can be shown to be surjective in all degrees i . This implies that $H^i(U_3, M)$ is isomorphic to the kernel of the restriction map $H^{i+1}(U_3, k) \rightarrow H^{i+1}(U_3, T)$ for all $i \geq 0$. This means that we can write the Poincaré series of the cohomology of M as

$$x^{-1} \left(\frac{1}{(1-x)^2} - \frac{1}{(1-x)} \right) = \frac{1}{(1-x)^2}.$$

The module generators may be taken to be in degrees 0, 1. \square

Lemma 4.2. *The cohomology of M^* , i.e. $H^*(U_3, M^*)$, has Poincaré series $(1-x)^{-2}$ and is generated by elements in degrees 0 and 1.*

Proof. M and M^* are interchanged by an outer automorphism of U_3 . \square

Lemma 4.3. *$H^*(U_3, W)$, has Poincaré series $1 + x(1-x)^{-2}$ and is generated by elements in degrees ≤ 2 .*

Proof. There is an exact sequence $W \hookrightarrow F \twoheadrightarrow K$. \square

Lemma 4.4. *$H^*(U_3, W^*)$, has Poincaré series $(2-x)(1-x)^{-2}$ and is generated by elements in degrees ≤ 1 .*

Proof. There is an exact sequence $K \hookrightarrow F \twoheadrightarrow W^*$. \square

Lemma 4.5. *$H^*(U_3, Y_9)$, has Poincaré series $(2-2x+x^2)(1-x)^{-2}$ and is generated by elements in degrees ≤ 2 .*

Proof. There is an exact sequence $M \hookrightarrow T \otimes M^* \twoheadrightarrow Y_9$, and we consider the associated long exact sequence in cohomology. Since $T \otimes M^* \approx T \oplus F$, we see that $H^i(U_3, T \oplus F) = H^i(U_3, T)$ is one-dimensional if $i > 0$. Furthermore, the induced maps $H^i(U_3, M) \rightarrow H^i(U_3, T)$ are surjective for all $i > 0$. This, plus the fact that the socle of Y_9 is two-dimensional, determines the Poincaré series. For the information on the degrees of the generators, we must study the kernel of the map $H^*(U_3, M) \rightarrow H^*(U_3, T)$. This kernel is generated by elements in degrees ≤ 2 . \square

Lemma 4.6. $H^*(U_3, X_{10})$, has Poincaré series $2 + x + x^2(1-x)^{-2}$ and is generated by elements in degrees ≤ 3 .

Proof. Use the exact sequence $X_{10} \hookrightarrow F \oplus F \twoheadrightarrow W$. \square

5. THE E_2 -TERM FOR THE SYLOW 2-SUBGROUP OF THE HIGMAN-SIMS GROUP

The algebraic computations in the preceding section will now be assembled to describe the E_2 -term of the Lyndon-Hochschild-Serre spectral sequence associated to the group extension $S = 4^3 : D_8$. It can of course be described precisely as $H^*(U_3, \Lambda^*(M) \otimes S^*(M))$.

To start we make a table of the modules appearing in the E_2 -term, which we regard as $\Lambda^*(M) \otimes S^*(M)$, and the Poincaré series of their cohomologies.

Module	Decomposition	Poincaré Series	Degree Bound
k	k	$\frac{1}{(1-x)^2}$	0
M	M	$\frac{1}{(1-x)^2}$	1
M^*	M^*	$\frac{1}{(1-x)^2}$	1
$M \otimes M$	Y_9	$\frac{2-2x+x^2}{(1-x)^2}$	2
$M \otimes M^*$	$F \oplus k$	$1 + \frac{1}{(1-x)^2}$	0
N	N	$\frac{1}{(1-x)}$	0
$N \otimes M$	$F \oplus N$	$1 + \frac{1}{(1-x)}$	0
$N \otimes M^*$	$F \oplus N$	$1 + \frac{1}{(1-x)}$	0
K	K	$\frac{1}{(1-x)^2}$	1
$K \otimes M$	W	$1 + \frac{x}{(1-x)^2}$	2
$K \otimes M^*$	W^*	$\frac{2-x}{(1-x)^2}$	1
W	W	$1 + \frac{x}{(1-x)^2}$	2
$W \otimes M$	$F \oplus X_{10}$	$3 + x + \frac{x^2}{(1-x)^2}$	3
$W \otimes M^*$	$F \oplus F \oplus K$	$2 + \frac{1}{(1-x)^2}$	1
F	F	1	0
$F \otimes M$	$F^{\oplus 3}$	3	0
$F \otimes M^*$	$F^{\oplus 3}$	3	0

Now we must describe the propagation of these modules in the cohomology of $(\mathbb{Z}_4)^3$. This follows from the description of $S^*(M)$ in section 3. It is important to note, however, that we have doubled all degrees in the symmetric algebra. For this reason, we will refer to the invariants as d_2 , d_4 , and d_8 , by abuse of notation. We want to compute the Poincaré series of the E_2 -term, and what we shall do is compute the Poincaré series of the “tensoring with a trivial” part first, then compute the Poincaré series of

the “tensoring with M ” part, and then compute the Poincaré series of the “tensoring with M^* ” part. Finally we combine these pieces.

We make a table for the “tensoring with a trivial part”. Since there are two trivial modules in $\Lambda^*(M)$, we just consider the one in degree 0 and then multiply our Poincaré series by $1 + x^3$. In the table below, “Degree” means the degree in which the propagated module first appears in the symmetric algebra, and “C-Deg.” means the degree in the E_2 -term after which we know no further generators in cohomology appear.

Module	Propagators	Degree	Poincaré Series	C-Deg.
k	d_8	0	$\frac{1}{1-x^8} \cdot \frac{1}{(1-x)^2}$	$0 + 0 = 0$
M	d_8	2	$\frac{x^2}{1-x^8} \cdot \frac{1}{(1-x)^2}$	$2 + 1 = 3$
N	d_8, d_2	4	$\frac{x^4}{(1-x^8)(1-x^2)} \cdot \frac{1}{1-x}$	$4 + 0 = 4$
K	d_8, d_4	4	$\frac{x^4}{(1-x^8)(1-x^4)} \cdot \frac{1}{(1-x)^2}$	$4 + 1 = 5$
W	d_8, d_4	6	$\frac{x^6}{(1-x^8)(1-x^4)} \cdot \left(1 + \frac{x}{(1-x)^2}\right)$	$6 + 2 = 8$
F	d_8, d_4, d_2	8	$\frac{x^8}{(1-x^8)(1-x^4)(1-x^2)}$	$8 + 0 = 8$

Thus the total Poincaré series for the “tensoring with a trivial” part is:

$$(1 + x^3) \cdot \left[\frac{1}{1-x^8} \cdot \frac{1}{(1-x)^2} + \frac{x^2}{1-x^8} \cdot \frac{1}{(1-x)^2} + \frac{x^4}{(1-x^8)(1-x^2)} \cdot \frac{1}{1-x} + \frac{x^4}{(1-x^8)(1-x^4)} \cdot \frac{1}{(1-x)^2} + \frac{x^6}{(1-x^8)(1-x^4)} \cdot \left(1 + \frac{x}{(1-x)^2}\right) + \frac{x^8}{(1-x^8)(1-x^4)(1-x^2)} \right]$$

Note that the maximum degree of an algebra generator for the E_2 -term is 8. Although we are tensoring with a trivial module in degree 3, this is really just multiplying by an element of the E_2 -term and so no new algebra generators in degrees 11 or greater are produced.

Now let’s move on to the “tensoring with M ” part.

We produce the desired information in the form of a table and a Poincaré series, as above.

Module	Propagators	Degree	Poincaré Series	C-Deg.
M	d_8	1	$\frac{x}{1-x^8} \cdot \frac{1}{(1-x)^2}$	$1 + 1 = 2$
$M \otimes M$	d_8	3	$\frac{x^3}{1-x^8} \cdot \frac{2-2x+x^2}{(1-x)^2}$	$3 + 2 = 5$
$N \otimes M$	d_8, d_2	5	$\frac{x^5}{(1-x^8)(1-x^2)} \cdot (1 + \frac{1}{1-x})$	$5 + 0 = 5$
$K \otimes M$	d_8, d_4	5	$\frac{x^5}{(1-x^8)(1-x^4)} \cdot (1 + \frac{x}{(1-x)^2})$	$5 + 2 = 7$
$W \otimes M$	d_8, d_4	7	$\frac{x^7}{(1-x^8)(1-x^4)} \cdot (3 + x + \frac{x^2}{(1-x)^2})$	$7 + 3 = 10$
$F \otimes M$	d_8, d_4, d_2	9	$\frac{3x^9}{(1-x^8)(1-x^4)(1-x^2)}$	$9 + 0 = 9$

Thus the total Poincaré series for the “tensoring with M ” part is:

$$\begin{aligned} & \frac{x}{1-x^8} \cdot \frac{1}{(1-x)^2} + \frac{x^3}{1-x^8} \cdot \frac{2-2x+x^2}{(1-x)^2} + \\ & \frac{x^5}{(1-x^8)(1-x^2)} \cdot (1 + \frac{1}{1-x}) + \frac{x^5}{(1-x^8)(1-x^4)} \cdot (1 + \frac{x}{(1-x)^2}) + \\ & \frac{x^7}{(1-x^8)(1-x^4)} \cdot (3 + x + \frac{x^2}{(1-x)^2}) + \frac{3x^9}{(1-x^8)(1-x^4)(1-x^2)} \end{aligned}$$

Now we do the “tensoring with M^* ” part:

Module	Propagators	Degree	Poincaré Series	C-Deg.
$k \otimes M^*$	d_8	2	$\frac{x^2}{1-x^8} \cdot \frac{1}{(1-x)^2}$	$2 + 1 = 3$
$M \otimes M^*$	d_8	4	$\frac{x^4}{1-x^8} \cdot (1 + \frac{1}{(1-x)^2})$	$4 + 0 = 5$
$N \otimes M^*$	d_8, d_2	6	$\frac{x^6}{(1-x^8)(1-x^2)} \cdot (1 + \frac{1}{1-x})$	$6 + 0 = 6$
$K \otimes M^*$	d_8, d_4	6	$\frac{x^6}{(1-x^8)(1-x^4)} \cdot \frac{2-x}{(1-x)^2}$	$6 + 1 = 7$
$W \otimes M^*$	d_8, d_4	8	$\frac{x^8}{(1-x^8)(1-x^4)} \cdot (2 + \frac{1}{(1-x)^2})$	$8 + 1 = 9$
$F \otimes M^*$	d_8, d_4, d_2	10	$\frac{3x^{10}}{(1-x^8)(1-x^4)(1-x^2)}$	$10 + 0 = 10$

Thus the total Poincaré series for the “tensoring with M^* ” part is:

$$\begin{aligned} & \frac{x^2}{1-x^8} \cdot \frac{1}{(1-x)^2} + \frac{x^4}{1-x^8} \cdot (1 + \frac{1}{(1-x)^2}) + \\ & \frac{x^6}{(1-x^8)(1-x^2)} \cdot (1 + \frac{1}{1-x}) + \frac{x^6}{(1-x^8)(1-x^4)} \cdot \frac{2-x}{(1-x)^2} + \\ & \frac{x^8}{(1-x^8)(1-x^4)} \cdot (2 + \frac{1}{(1-x)^2}) + \frac{3x^{10}}{(1-x^8)(1-x^4)(1-x^2)} \end{aligned}$$

Adding these series up, we obtain the Poincaré series for the E_2 -term. We summarize our computation in the following

Theorem 5.1. *The Poincaré Series for the E_2 term of the Lyndon-Hochschild-Serre spectral sequence for the cohomology of the group extension $S = \text{Syl}_2(HS) = 4^3 : D_8$ is given by the rational function*

$$p(x) = \frac{(1+x)^2(1-x+x^2)(1+2x^2-x^5)}{(1-x)^2(1-x^4)(1-x^8)}.$$

Moreover a complete set of algebra generators for the E_2 term of the spectral sequence occur in degrees eight and below.

Using Maple, this series can be expanded to yield

$$1 + 3x + 7x^2 + 14x^3 + 23x^4 + 34x^5 + 48x^6 + 65x^7 + 84x^8 + 105x^9 + 131x^{10} + 163x^{11} + 198x^{12} + 236x^{13} + 280x^{14} + 330x^{15} + 383x^{16} + 439x^{17} + 503x^{18} + \dots$$

Our goal is to analyze the behaviour of this spectral sequence. Our description provides us with very good control of the generators, especially given that they arise in low degree. To make effective use of this we require an explicit computer-assisted calculation of $H^*(S)$ in low degrees. This can indeed be implemented; details are provided in a subsequent section. For now we use this information to prove the main result in this paper.

Theorem 5.2. *Let $S = (\mathbb{Z}/4)^3 \rtimes U_3$ denote the Sylow 2-subgroup of HS, the Higman-Sims group. The Lyndon-Hochschild-Serre spectral sequence for this semidirect product collapses at E_2 and hence the polynomial $p(x)$ is the Poincaré series for $H^*(S, \mathbb{F}_2)$.*

Proof. A computer calculation using MAGMA for $H^*(S, \mathbb{F}_2)$ shows that through degree 10 the coefficients of the polynomial $p(x)$ agree with the ranks of the cohomology. Now according to the preceding tables, all algebra generators occur by this degree. Hence using the multiplicative structure of the spectral sequence we infer that it must collapse at E_2 , i.e. $E_2 = E_\infty$. \square

6. COMPUTER CALCULATIONS OF THE COHOMOLOGY OF $S = \text{Syl}_2(HS)$

The cohomology $H^*(S, \mathbb{F}_2)$ was calculated directly through degree 10 using a computer. The calculation included a determination of the Betti numbers, a minimal set of ring generators, and a complete collection of relations among the generators in the first ten degrees. We also obtained the images of the generators under various restrictions. Using the latter information and the analysis of the last section we were able to construct the cohomology ring with all generators and relations using computer technology. We give a summary of the calculations in this section.

The calculation was begun by first obtaining a minimal projective resolution

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k$$

of the trivial module $k = \mathbb{F}_2$. The process is mostly linear algebra. The free module over S is generated as a collection of matrices representing the actions of the generators of the group. At each stage in the construction we get a minimal set of generators for the kernel of the previous boundary map, create the free module with exactly that number of generators, make the matrix for the boundary map from the free module to the kernel of the previous boundary map – this is the new boundary map – and find its kernel. The programs are conservative with both time and memory in that they save only the minimal amount of information necessary to reconstruct the boundary homomorphisms and create module structure only when it is necessary. The method is described in detail in [5], [4].

Because the resolution is minimal, any nonzero homomorphism $\gamma' : P_n \rightarrow k$ is a cocycle representing a nonzero cohomology class γ . Any such cocycle can be lifted to a chain map $\hat{\gamma} : P_* \rightarrow P_*$ of degree n which also represents γ . Again this is an exercise in linear algebra on the computer and details of the implementation can be found in the references given above. The point of this operation is that the cup product of two cohomology elements is the class of the composition of the representing chain maps. Thus we can compute the product structure of the cohomology ring. Note that the program computes the chain maps for only a minimal set of generators for the cohomology. That is, at each stage it constructs the subspace of $\text{Hom}_{kG}(P_n, k)$ that can be obtained by compositions with chain maps of lower degrees.

The same implementations have been used to compute the cohomology rings of all but a few of the groups of order dividing 64. ¹The programs are written in the MAGMA language and run in the MAGMA computer algebra system [3]. The computations of the cohomology of S were all run on a SUN ULTRA2200 (named the sloth) that has approximately 1Gb. of RAM and 14Gb. of hard disk. The computation of the projective resolution out to degree 10 for the trivial module k of S took slightly under 31 hours. The computation of the chain maps of the minimal generators of the cohomology ring took more than 55 hours. The attempt to compute an 11th step in the projective resolution failed for lack of memory. Some of the results of this calculation are given in the following. Note that this is precisely the information that is needed to complete the proof of Theorem 5.2.

Proposition 6.1. *The Betti numbers for the cohomology $H^*(S, k)$ through degree 10 are*

$$1, 3, 7, 14, 23, 34, 48, 65, 84, 105, 131,$$

¹See the second author's web page, for the results and some further discussion of the methods <http://www.math.uga.edu/~jfc/groups2/cohomology2.html> .

and the degrees of the minimal generators of the cohomology ring through degree 10 are

$$1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 6, 7, 8.$$

We also computed the restrictions of the cohomology to the elementary abelian subgroups and to the centralizers of the elementary abelian subgroups. The restriction maps are obtained by first converting the projective resolution of k for the group to a projective resolution of the subgroup and then constructing the chain map, lifting the identity on k , from a minimal projective resolution of k for the subgroup to the converted projective resolution for the group. The cocycles for the cohomology elements of the group are then pulled back along the chain map. The process is described in more detail in [6].

Of particular interest was the restrictions to the centralizers of the elementary abelian groups of order 4. This calculation was made on the assumption (hope) that the depth of the cohomology ring $H^*(S, k)$ is at least two and hence that the cohomology of S is detected on these centralizers. An easy calculation shows the following.

Lemma 6.2. *If E is an elementary abelian subgroups of order four in S then the centralizer of E is contained in the subgroup K_β (defined in Section 2) or in one of the three subgroups*

$$\begin{aligned} M &= \langle v_1, v_2, v_3, ts, s^2 \rangle, \\ D_1 &= \langle (v_1 v_2)^2, v_1 v_3, t, s \rangle, \\ D_2 &= \langle (v_1 v_2)^2, v_1 v_3, v_2^{-1} t, ts, v_1 v_2^{-1} s^2 \rangle. \end{aligned}$$

Here M is the centralizer of $(v_1 v_2)^2$, D_1 is the centralizer of s^2 and D_2 is the centralizer of $v_1 v_2^{-1} s^2$. The orders of M is 256 while the orders of D_1 and D_2 are 64.

The computer was able to calculate the cohomology ring of all four of the groups in the Lemma. In the case of K_β it was known that the cohomology is detected on the centralizers of the maximal elementary abelian subgroups and this aided the computation. In all three of the other cases the minimal set of generators for the cohomology ring contained no element of degree greater than 4 and the computation was possible even though all three of the groups have 2-rank 4 and the Betti numbers of the cohomology grow rapidly. In fact, the cohomology rings of D_1 and D_2 are included in the calculations of the second author (see the web page). These groups have Hall-Senior numbers 170 and 110 respectively.

We were able to compute the restriction maps and then get the kernels of the restriction to each of the subgroups. Then the intersection of the kernel of the restriction maps was computed. Actually the problem of getting the

intersection of the kernels seemed to be too difficult for the computer to attack directly using Gröbner basis machinery. Instead we employed an indirect method of turning the restriction maps into linear transformations on the spaces of monomials in each degree and computing the intersection of the kernels out to degree 17. Thus we had a complete set of generators for the intersection of the kernels out through degree 17.

There is one uncertainty that should be noted. The cohomology of M was only calculated out to degree 10 and that is not far enough to pass our test for completeness of the calculation [4]. Nonetheless, the generators of the cohomology of S are in degrees at most 8 and so their restrictions to M are expressed as polynomials in the calculated generators of the cohomology of M . Any polynomial in those generators which is computed to be in the kernel of restriction to M is, in fact, in that kernel. Thus we get the following result, which is used in the next section.

Proposition 6.3. *The ideal \mathcal{I} of Theorem 7.1 is generated by the elements of degree at most 17 that are in the intersection of the kernels of the restriction maps to K_β , M , D_1 and D_2 . Hence the ideal is in the intersection of the kernels of the restrictions to the centralizers of the elementary abelian subgroups of order four.*

In the next section we argue that this information together with some extra verification is sufficient.

7. EXPLICIT GENERATORS AND RELATIONS FOR THE COHOMOLOGY OF S

Our purpose in this section is to give a proof of the following.

Theorem 7.1. *The cohomology of S has the form*

$$H^*(S, \mathbb{F}_2) \cong \mathbb{F}_2[z, y, x, w, v, u, t, s, r, q, p, n, m, k, j, i, h]/\mathcal{I}$$

where the degree of z, y, x is 1, the degree of w, v, u, t is 2, the degree of s, r, q, p is 3, the degree of n, m is 4 and the degrees of k, j, i, h are 5, 6, 7 and 8 respectively. The ideal \mathcal{I} is generated by the relations in the appendix.

Let $R = \mathbb{F}_2[z, y, x, w, v, u, t, s, r, q, p, n, m, k, j, i, h]$ be the polynomial ring in 17 variables in weighted degrees as in the theorem. For $n \geq 0$ let R^n be the space of homogenous polynomials of degree n . From Proposition 6.1 we know that there is a natural homomorphism $\phi : R \rightarrow H^*(S, \mathbb{F}_2)$ taking each variable to the computed cohomology element. Let \mathcal{J} be the kernel of ϕ . Our task is to establish that $\mathcal{I} \subseteq \mathcal{J}$.

Recall that the following is a complete list of representatives for the conjugacy classes of centralizers of maximal elementary abelian subgroups in S :

1. $CI = 4^3$, a characteristic subgroup of S
2. $CII = 2^2 \times 4$,
3. $CIII = 2^2 \times 4$,
4. $CIV = 2^2 \times 4$,
5. $CV = 2^2 \times 4$,
6. $CVI = 2^2 \times 4$,
7. 2_A^4 ,
8. 2_B^4
9. 2_C^4

The following basic detection result involving these groups, these groups was proved by computer calculation.

Lemma 7.2. *Any homogeneous polynomial of R^n in degree $n \leq 20$ is in \mathcal{I} if and only if it is in the kernel of restriction to all of the centralizers listed above.*

Proof. First it was shown that the elements of \mathcal{I} are in the kernels of restriction by computing the restriction maps on the elements. Then for each degree $n \leq 20$ the restriction maps $R^n/\mathcal{I}^n \rightarrow H^*(C, \mathbb{F}_2)$ for C in the list was written as a linear transformation and the intersection of the null spaces was computed to be zero \square

Proof of Theorem 7.1. From the lemma we see that $\mathcal{J}^n \subseteq \mathcal{I}^n$ for $n \leq 20$. But by the Poincaré series (Theorem 5.2) we have that R^n/\mathcal{I}^n has the same dimension as $R^n/\mathcal{J}^n \cong H^*(S, \mathbb{F}_2)$. On the other hand we know that \mathcal{I} is generated by elements of degree at most 14. So $\mathcal{I} \subset \mathcal{J}$, since all generators are in \mathcal{J} . So ϕ induces a surjective homomorphism $R/\mathcal{I} \rightarrow H^*(S, \mathbb{F}_2)$. But again because the Poincaré series are the same this is an isomorphism. \square

8. APPENDIX I: RELATIONS IN THE COHOMOLOGY OF S

The following is a list of the relations in the cohomology ring $H^*(S, \mathbb{F}_2)$. These elements generate the ideal \mathcal{I} in Theorem 7.1. Note that there are 79 relations generating \mathcal{I} and that the relations are minimal in the sense that no collection of fewer than 79 elements will generate the ideal. This set is not a Gröbner basis for the ideal. The computed Gröbner basis using the grevlex ordering on the variables consisted of 884 elements in degrees up to 73. The elements in the list are ordered by degree. The largest degree of any element in the list is 14.

$$\begin{aligned}
& yx, zx, zy, xv, xw, yw + yv + yu, zt, zw, vt, xs, \\
& yq + xq + w^2 + wu, yq + wv, yr + xr + xq + xp + ut, yr + wt, ys + yq, \\
& y^2u + yq + xq + wu, zp + yq + xq + v^2 + vu, zr, zs + zq + xq, \\
& tq, ts, vr, vs + vq, ws + wq, x^2r + xn + wp + vp + up, \\
& x^2r + xn + wr + ur, yu^2 + yn + wp, yvu + ws, \\
& y^2r + wr, z^2p + zvu + zn + ws + vs + us, \\
& z^3u + z^2p + zvu + zu^2 + zn + zm + x^2r + xu^2 + xn + xm, \\
& rq, sp + qp, sq + q^2, sr, s^2 + sq, xuq + vu^2 + vn + sp, \\
& ytr + x^3p + x^2n + xup + xtp + r^2, ytr + x^3p + xuq + xtr + xk + r^2, \\
& yur + x^3p + xuq + xk + tn + rp, y^2n + yvp + yuq + wn, \\
& z^3p + z^2n + zvq + zuq + y^2m + ytr + ytp + yk + x^3p + xk + vn + \\
& vm + tn + tm + p^2, \\
& z^3p + zvq + zup + yuq + xuq + vu^2 + sq, z^3q + z^3p + zuq + zk, sn + qn, \\
& ytn + xtm + xp^2 + rn, yvn + yqp + vuq + vup + qn, \\
& y^5v + y^4q + y^4p + y^3n + y^3m + y^2vq + y^2tp + y^2k + yum + ytn + yr^2 + \\
& yrp + yj + wk, \\
& zvm + zq^2 + zj + yvm + yum + yr^2 + yrp + x^3u^2 + x^3m + vup + vk + \\
& u^2q + u^2p + uk + sm + rn + rm + pn, \\
& z^2vq + zvm + zq^2 + zqp + zj + yvn + yqp + vuq + u^2s, \\
& z^3u^2 + zu^3 + zum + yvn + yqp + x^3u^2 + x^3n + xu^3 + xum + xtn + \\
& xj + vuq + u^2s, \\
& z^5u + z^3m + zu^3 + zum + x^3n + x^3m + xu^3 + xum + xtn + xj + wk + \\
& vk + u^2s + u^2q + uk,
\end{aligned}$$

$$y^4m + y^3k + y^2vn + y^2vm + y^2p^2 + yt^2p + ytk + yrm + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xu^2p + xtk + xrm + xpn + tr^2 + tj + rk,$$

$$y^5p + y^3k + y^2vn + y^2vm + yrm + ypm + yi + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xt^2p + xqm + wun + wum + wj + tr^2 + sk + qk,$$

$$zpn + ypn + xpn + wun + vun + vp^2 + uqp + n^2,$$

$$z^2q^2 + zuk + zqn + zqm + y^4m + y^3vq + y^3tp + y^3k + y^2vm + y^2tm + y^2p^2 + yqm + ypn + x^2un + xuk + xrm + wun + wj + vum + vp^2 + vj + up^2 + qk,$$

$$z^3k + z^2vm + z^2q^2 + zi + y^3tp + y^2tm + yuk + yt^2p + ytk + yqm + x^3k + x^2u^3 + x^2un + x^2tn + xt^2p + xi + wum + vun + vum + urp + up^2 + tr^2 + tj + sk + rk + qk,$$

$$z^3k + z^2vm + zqn + zpn + zi + y^5q + y^4n + y^2q^2 + yqm + x^2u^3 + xu^2p + xuk + xrm + wj + vq^2 + vp^2 + u^2n + uq^2 + uj + sk,$$

$$z^4u^2 + z^2u^3 + z^2un + z^2um + z^2q^2 + zqn + y^5q + y^4n + y^2q^2 + yqm + wun + vun + vq^2 + vp^2 + vj + uq^2 + sk,$$

$$z^5p + z^3k + z^2un + z^2j + zqn + zi + y^3vq + y^2vn + yuk + yrm + ypn + x^2u^3 + xuk + xqm + wum + vp^2 + u^2n + urp + uj + sk,$$

$$y^5n + y^3q^2 + y^2vk + y^2qm + yvqp + yup^2 + yuj + yt^2n + yrk + yqk + ynm + wuk + wqn + wpm + wi + trm + sj + r^2p + rp^2 + qj,$$

$$z^2pn + zuq^2 + zuj + zn^2 + znm + y^6p + y^5m + y^3vm + y^3tm + y^2qn + y^2qm + y^2i + yvqp + yvj + yt^2n + yt^2m + ytj + yrk + ypk + ynm + wqn + wpm + wi + vpm + u^3q + u^2k + urn + trn + q^3 + pj + nk,$$

$$z^2qn + zuj + zn^2 + znm + yup^2 + yt^2n + yrk + ynm + xpk + wpm + wpm + vuk + vqm + u^3q + u^2k + upn + trm + sj + rp^2 + rj + qj,$$

$$z^3un + z^2qn + z^2pn + zvj + zuq^2 + zuj + znm + sj + qj,$$

$$y^2tk + y^2qm + yup^2 + yt^2m + ytp^2 + ytj + yqk + ynm + x^3tn + xrk + xn^2 + wuk + wqn + wpm + t^3r + trn + tpm + ti + sj + r^2p + rp^2 + qj,$$

$$z^3vn + z^3un + z^2vk + z^2pn + zvj + y^5n + y^4vq + y^3vm + y^2qn + y^2pn + yvq^2 + yvqp + yup^2 + yn^2 + xnm + wuk + vqm + vpm + vi + u^3q + u^3p + u^2k + upn + q^3,$$

$$z^3vn + z^2qn + z^2pn + zvj + y^5n + y^2vk + y^2tk + y^2qn + yvqp + yup^2 + yt^2n + yt^2m + ytp^2 + ytj + yrk + yn^2 + x^3tn + wpn + wpm + wi + vuk + vqm + urn + t^3r + trm + tpm + ti + sj + r^2p,$$

$$y^3t^2p + y^2t^2m + yt^3p + yt^2k + yr^2p + yrp^2 + x^2u^2n + x^2t^2n + xrj + t^3n + t^2rp + t^2j + trk + r^2n + r^2m + rpm + ri,$$

$$z^2uq^2 + z^2qk + zpj + y^5vq + y^4j + y^3vk + y^3t^2p + y^3qn + y^3qm + y^3pm + y^3i + y^2vq^2 + y^2vj + y^2t^2m + y^2tp^2 + y^2tj + y^2pk + y^2h + yvqn + yupn + yt^3p + yt^2k + yr^2p + yqp^2 + yqj + x^2t^2n + xu^3p + xtpn + xrj + xpj + wqk + wpk + wnm + vup^2 + upk + t^3n + t^2p^2 + t^2j + trk + r^2n + r^2m + rpn + rpm + q^2n + p^2n + pi + k^2,$$

$$zuqn + y^7q + y^4q^2 + y^3vk + y^3qm + y^2vq^2 + y^2vj + yvqm + yvqn + yvi + yupn + yqj + x^2u^2n + x^2nm + xu^3p + xpj + wqk + wpk + vuj + upk + un^2 + tn^2 + r^2n + q^2n + qpn + qpm + nj,$$

$$z^3uk + zuqm + zupm + zmk + y^3vk + y^3qn + y^3pm + y^2tp^2 + y^2qk + yvqn + yvi + yupn + yupm + yr^2p + yq^2p + yqj + ypj + x^2u^2n + xpj + wpk + vup^2 + upk + t^2rp + rpn + qpm + p^2n + p^2m + pi,$$

$$z^3uk + z^2u^2n + z^2uj + z^2qk + zupm + zui + zpj + y^3qn + y^3qm + y^3pn + y^2vq^2 + y^2nm + yq^2p + x^3uk + x^2u^4 + x^2nm + xu^3p + xui + xrj + wpk + wnm + vuj + usk + upk + un^2 + tn^2 + si + r^2n + qpn + nj,$$

$$z^4vm + z^4j + z^3vk + zuqn + zuqm + zupm + zpj + y^7q + y^4q^2 + y^3vk + y^3qn + y^3pn + y^2vj + y^2nm + yvqn + yvi + yq^2p + yqj + x^2u^2n + x^2nm + vup^2 + u^3n + usk + un^2 + unm + qpn + qi,$$

$$z^2u^2k + z^2uqm + z^2q^3 + z^2qj + zvk + zu^3n + zu^2j + zupk + zun^2 + y^6vq + y^4qm + y^4pn + y^3tp^2 + y^2qj + yvpk + yvn^2 + yvnm + yupk + yq^2n + yqpm + yqi + x^2u^2k + x^2nk + xunm + xtpk + xtnm + xri + xnj + wupm + wpj + wnk + vq^2p + vpj + urp^2 + unk + t^2rn + t^2pm + trp^2 + tp^3 + tpj + rpk + qpk + qnm + pnm + ni,$$

$$z^2vqm + z^2u^2k + z^2uqm + z^2q^3 + zvk + zu^3n + zu^2j + zun^2 + y^9v + y^6vq + y^6k + y^5vm + y^5tm + y^5q^2 + y^4qn + y^4qm + y^3tp^2 + y^3tj + y^3pk + y^3nm + y^3m^2 + y^3h + y^2vi + y^2qj + y^2nk + yvpk + yvnm + yvm^2 + yvh + yupk + yuh + ytm^2 + x^2mk + xunm + xtnm + xqi + wupm + wpj + wnk + wmk + vmk + urj + uqp^2 + upj + unk + t^2pm + tp^3 + r^2k + rn^2 + qpk + qn^2 + qnm + ni + kj,$$

$$z^2q^2n + zvj + zvmk + zupj + zqn^2 + zni + y^8m + y^7vq + y^7tp + y^7k + y^6tm + y^5vk + y^5qn + y^4pk + y^4nm + y^3p^3 + y^2vnm + y^2vh + yvq^3 +$$

$$yvj + yvjk + yvmk + yvp^3 + yumk + ytrj + ytnk + ytmk + yrpj + yqm^2 + y^2k + y^2pm + y^2ni + y^2kj + x^2mj + x^2qm^2 + x^2kj + wum^2 + wq^2m + wpi + vqpn + vp^2m + u^2qk + u^2pk + u^2n^2 + urpm + uri + uq^2m + up^2n + t^2nm + t^2m^2 + tr^2n + tp^2m + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pnk + j^2,$$

$$z^5pm + z^2q^2n + z^2qi + zvqj + zupj + zpm^2 + zkj + y^8m + y^7vq + y^7k + y^6vm + y^5tk + y^5pm + y^4pk + y^3vqn + y^3t^3p + y^3ti + y^2vnm + y^2vh + y^2th + y^2q^2m + y^2k^2 + yvqj + yvnk + yumk + yt^4p + yt^3k + ytnk + ytmk + yrpj + yrn^2 + yq^2k + yqn^2 + y^2k + y^2pm + y^2ni + y^2kj + x^8u^2 + x^8m + x^7k + x^6u^3 + x^5tk + x^5i + x^4tj + x^2t^2j + x^2mj + xt^2pn + xni + wun^2 + wum^2 + wq^2m + wmj + vum^2 + vqpn + vnj + vmj + u^2qk + uri + uqi + up^2n + upi + uk^2 + t^4n + t^3rp + t^3p^2 + t^3j + t^2nm + t^2m^2 + trpm + tk^2 + r^3p + r^2p^2 + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pmk + ki + j^2,$$

$$z^{11}u + z^9u^2 + z^9m + z^7vn + z^6vk + z^6pm + z^3vnm + z^3un^2 + z^3nj + z^2unk + z^2pm^2 + z^2mi + z^2kj + zumj + y^7p^2 + y^6vk + y^4pj + y^4nk + y^3qi + y^3pi + y^2vnk + y^2tmk + y^2q^2k + y^2qnm + y^2qh + y^2p^2k + yvq^2m + yvqpm + yvqi + yvk^2 + yup^2n + yup^2m + yumj + yuk^2 + ytri + yr^2p^2 + yr^2j + yq^4 + yq^2j + yqpj + yn^2m + x^6uk + x^5u^2m + x^4ui + x^4mk + x^3nj + xu^4n + xu^2nm + xupi + xumj + xtnj + xpnk + wunk + wq^2k + wmi + vupj + vumk + vqnm + u^5q + u^4k + u^2p^3 + u^2nk + usm^2 + uq^2k + uqn^2 + t^3rn + trnm + tpm^2 + tpm^2 + q^3n + qp^2n + qnj + p^3m + pnj + nmk + ji,$$

$$z^{11}p + z^5vi + z^5u^2k + z^4pi + z^3umk + z^2qmk + z^2j^2 + zvqn^2 + zuqnm + zumi + zukj + zq^3n + zm^2k + y^{10}n + y^9tp + y^8q^2 + y^8j + y^7tk + y^7qn + y^7i + y^6vj + y^6tj + y^6nm + y^6h + y^5vi + y^5ti + y^4vn^2 + y^4vm^2 + y^4mj + y^4k^2 + y^3vqj + y^3tmk + y^3p^2k + y^3mi + y^2vqi + y^2vnj + y^2t^2m^2 + y^2q^2j + y^2n^2m + y^2mh + y^2ki + yvq^2k + yvqm^2 + yvqh + yvpm + yvmi + yupn^2 + yuph + yuni + yt^3pm + yt^2pj + ytp^2k + ytmi + yr^2pm + yq^3n + yp^2i + yn^2k + x^8u^3 + x^7uk + x^6u^2m + x^6uj + x^4um^2 + x^3mi + x^3kj + x^2u^4n + x^2umj + x^2j^2 + xu^2mk + xrk^2 + xji + wunj + wqnk + wnm^2 + wnh + vumj + vq^2p^2 + vqmk + vn^2m + vj^2 + u^2qi + u^2p^2m + u^2mj + urpj + urmk + uqnk + upmk + un^3 + unm^2 + uki + t^3nm + t^2r^2m + t^2rpm + t^2ri + t^2p^2n + tr^2j + trp^3 + trnk + tp^4 + skj + r^3k + r^2pk + rpj^2 + rni + qmi + pni + mk^2 + i^2$$

9. APPENDIX II: STEENROD OPERATIONS

In this appendix we describe the Steenrod operations on the cohomology generators listed previously. As before this was done on a computer using MAGMA. Note that the program used puts all of the polynomials in “normal form” relative to the Groebner basis of the ideal of relations.

$$Sq^1w = yv, Sq^1v = zv + yv, Sq^1u = zu + xu, Sq^1t = yt + xt$$

$$Sq^1s = vu + yq + xq, Sq^2s = z^2q + us + zn$$

$$Sq^1r = xr + xp, Sq^2r = ur + tr + xn$$

$$Sq^1q = vu + yq + xq, Sq^2q = xu^2 + z^2q + uq + wp + vp + up + zn + xm$$

$$Sq^1p = zp, Sq^2p = z^2q + z^2p + y^2p + tr + wq + uq + vp + ym + k$$

$$Sq^1n = x^2p + wp + vp, Sq^2n = y^4v + y^3q + y^3p + zvq + yvq + xuq + yvp + zup + yup + ytp + z^2n + y^2n + x^2n + y^2m + x^2m + q^2 + rp + qp + tn + vm + yk + xk + j$$

$$Sq^1m = zu^2 + xu^2 + y^2p + us + vq + up + zm + xm$$

$$Sq^2m = y^4v + y^3q + y^3p + u^3 + zvq + xuq + yvp + xup + xtp + z^2m + y^2m + r^2 + q^2 + rp + vn + un + wm + vm + um + tm + xk + j$$

$$Sq^1k = y^3p + ytr + zvq + yvq + yvp + yup + xtp + z^2n + y^2m + qp + p^2 + vn$$

$$Sq^2k = y^4p + xu^3 + x^3m + u^2s + u^2p + yrp + yp^2 + zun + yun + zvm + yvm + xum + xtm + y^2k + x^2k + rn + qn + pn + sm + rm + vk + xj$$

$$Sq^4k = xu^4 + y^4k + yvq^2 + xt^2n + z^2pn + xu^2m + yt^2m + y^2vk + y^3j + r^2p + qp^2 + p^3 + urn + uqn + tpn + zn^2 + uqm + wpm + upm + znm + xnm + ym^2 + wuk + vuk + u^2k + t^2k + yrk + xrk + zqk + yqk + xpk + xuj + y tj + xtj + z^2i + x^2i + mk + pj + vi + yh$$

$$Sq^1j = y^4q + y^2vq + y^3n + y^3m + x^3m + zq^2 + vup + zqp + yvn + ytn + zvm + ytm + y^2k + qn + pn + vk + zj + yj + xj$$

$$Sq^2j = y^2p^2 + x^2tn + z^2vm + ur^2 + vq^2 + tp^2 + t^2n + zqn + vum + t^2m + zqm + xqm + zpm + zvk + yvk + zvk + xtk + z^2j + n^2 + sk + vj$$

$$Sq^4j = y^2t^2m + y^3pm + z^4j + x^4j + yq^3 + t^2rp + yqp^2 + yp^3 + yvqn + yupn + y^2n^2 + yvqm + yupm + z^2nm + xu^2k + z^2qk + z^2vj + y^2tj + x^2tj + y^3i + qpn + un^2 + q^2m + qpm + wnm + tnm + wm^2 + vm^2 + tm^2 + urk + trk + wqk + uqk + wpk + upk + znk + xnk + ymk + vu j + yrj + zqj + zpj + xpj + xti + y^2h + nj + mj + qi + pi + wh$$

$$Sq^1 i = y^2 q^2 + y^2 p^2 + z^2 v m + y^2 t m + u r^2 + t r^2 + w q^2 + u r p + t r p + v q p + v p^2 + t p^2 + v u n + x r n + y q n + z p n + x p n + v u m + t^2 m + y q m + x q m + z p m + y p m + z u k + x u k + z^2 j + n^2 + s k + r k + w j + v j + t j + y i$$

$$Sq^2 i = y^3 t m + y v q^2 + x t^2 n + y^2 p n + y^2 t k + q^2 p + r p^2 + q p^2 + p^3 + v q n + u q n + z n^2 + x n^2 + u r m + t r m + w q m + u q m + w p m + v p m + u p m + t p m + y n m + x n m + y m^2 + x m^2 + v u k + x r k + z q k + x p k + z v j + x u j + z^2 i + x^2 i + n k + m k + s j + r j + q j + v i + t i$$

$$Sq^4 i = y^4 t k + x u^3 m + y t^3 m + y^3 v j + y^3 t j + z^4 i + y^4 i + x^4 i + v q^3 + v u p n + y q p n + y v n^2 + y u n^2 + t^2 p m + y q p m + x u n m + y u m^2 + x u m^2 + y t m^2 + u^3 k + x u p k + z^2 n k + y^2 n k + z^2 m k + y^2 m k + y t^2 j + x t^2 j + z^2 q j + y^2 q j + z^2 p j + y^2 v i + y^2 t i + r n^2 + p n^2 + r n m + r m^2 + q m^2 + r^2 k + v n k + t n k + v m k + u m k + t m k + y k^2 + u r j + w q j + v q j + u q j + w p j + y n j + x n j + z m j + y m j + x m j + t^2 i + y q i + y p i + x p i + y v h + k j + n i + m i + s h + q h$$

$$Sq^1 h = z^7 v + x^7 u + y^2 q n + z^2 p n + y^2 p n + y t^2 m + y^2 p m + y^2 v k + y^2 t k + x^2 t k + y^3 j + r p^2 + p^3 + v p n + u p n + t p n + x n^2 + u s m + t r m + w q m + v q m + w p m + v p m + x n m + w u k + v u k + y r k + z p k + y p k + x p k + z v j + y v j + z u j + y u j + x u j + x^2 i + n k + p j$$

$$Sq^2 h = z^8 v + x^8 u + u^5 + y^3 p m + y^3 t k + t^2 r p + y r p^2 + y q p^2 + y u p n + y^2 n^2 + u^3 m + y v q m + y t p m + z^2 n m + y^2 n m + x^2 n m + z^2 m^2 + y^2 m^2 + x^2 m^2 + x u^2 k + x t^2 k + z^2 q k + y^2 v j + x^2 u j + y^2 t j + x^2 t j + x^3 i + r^2 n + q^2 n + p^2 n + w n^2 + v n^2 + u n^2 + q^2 m + r p m + q p m + w n m + v n m + t r k + w q k + u q k + w p k + u p k + y n k + x n k + z m k + v u j + z q j + y p j + y v i + x u i + y t i + x t i + y^2 h + p i$$

$$Sq^4 h = z^6 j + x^6 j + u^6 + y^2 t^3 m + y^3 t^2 k + x^5 i + t^3 p^2 + t^4 n + t^4 m + z^2 u m^2 + x t^3 k + y^3 n k + y^3 p j + y^3 v i + x^3 t i + z^4 h + y^4 h + x^4 h + q^4 + p^4 + v q^2 n + t p^2 n + u^2 n^2 + y r n^2 + y q n^2 + y p n^2 + t r^2 m + v q^2 m + v q p m + v p^2 m + t p^2 m + v u n m + u^2 n m + t^2 n m + y r n m + y p n m + w u m^2 + v u m^2 + u^2 m^2 + y q m^2 + x q m^2 + y p m^2 + z q^2 k + y r p k + y q p k + z v m k + z u m k + y u m k + u^3 j + t^3 j + y v p j + x u p j + y t p j + x t p j + y^2 n j + x^2 n j + z^2 m j + x u^2 i + x t^2 i + z^2 q i + y^2 q i + y^2 p i + x^2 p i + y^2 v h + z^2 u h + n m^2 + m^3 + p m k + w k^2 + v k^2 + r^2 j + q^2 j + r p j + v n j + u n j + t n j + w m j + t m j + z k j + y k j + x k j + u s i + u r i + v q i + u q i + w p i + u p i + z n i + y n i + y m i + v u h + t^2 h + y r h + x r h + z p h + k i + n h + m h$$

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