On realizability of branched coverings of the sphere

Krzysztof Barański

Institute of Mathematics, Warsaw University,
ul. Banacha 2, 02-097 Warsaw, Poland
e-mail: baranski@mimuw.edu.pl

Abstract

We introduce a simple geometric method of determining which abstract branch data can be realizable by branched coverings of the two-dimensional sphere S^2 (the Hurwitz problem). Using this method we the realizability of some classes of data.

1 Introduction

Let M, N be closed Riemann surfaces and let $f : M \to N$ be a continuous map. We call f *a branched* (*ramified*) *covering* of degree $d > 1$, if there exist a finite number of points $v_1, \dots, v_n \in N$ such that $f|_{M \setminus f^{-1}(\{v_1, \dots, v_n\})}$ is a covering of $N \setminus \{v_1, \dots, v_n\}$ of degree d and for every $i = 1, \dots, n$ there exist $k_i > 0$ points $c_{i,1}, \dots, c_{i,k_i} \in M$ and positive integers $d_{i,1}, \dots, d_{i,k_i}$, such that:

- $d_{i,j} > 1$ for some $j \in \{1, \dots, k_i\}$,
- $\sum_{j=1}^{k_i} d_{i,j} = d$,
- for every $j = 1, \dots, k_i$ the map f on some open neighborhood $U_{i,j} \subset N$ of $c_{i,j}$ is conjugate to $\mathbb{C} \ni z \mapsto z^{d_{i,j}}$ near $z = 0$, such that $f(c_{i,j}) = v_i$.

1991 *Mathematics Subject Classification*: Primary 57M12.

Research supported by Polish KBN Grant No 2 P03A 025 12 and the Centre de Recerca Matemàtica in Bellaterra.

Thus, each branched covering f defines a suitable table of positive integers $\mathcal{D}(f) = [d_{i,j}]_{i,j}$, which will be called *the branch data* of f. Let $\nu(v_i) = \sum_{j=1}^{k_i}(d_{i,j} - 1) = d - k_i$ and define *the total branching* of f as $\nu(f) = \sum_{i=1}^{n} \nu(v_i)$.

Definition 1.1. *An abstract branch data* $\mathcal{D}$ of degree $d > 1$ is a table of positive integers $[d_{i,j}]_{(i,j)\in I}$, where

$$I = \{(i,j) : i = 1, \dots, n,\ j = 1, \dots, k_i\}$$

for some $k_1, \dots, k_n > 0$, such that for every $i = 1, \dots, n$:

- $d_{i,j} > 1$ for some $j \in \{1, \dots, k_i\}$,
- $\sum_{j=1}^{k_i} d_{i,j} = d$.

We assume that for each i the numbers $d_{i,1}, \dots, d_{i,k_i}$ are ordered in a non-increasing way. Let $\nu_i(\mathcal{D}) = \sum_{j=1}^{k_i}(d_{i,j} - 1) = d - k_i$ and define $\nu(\mathcal{D}) = \sum_{i=1}^{n} \nu_i(\mathcal{D})$ to be *the total branching* of $\mathcal{D}$. Note that the definition implies $1 \leq \nu_i(\mathcal{D}) \leq d - 1$.

We consider a question, which abstract branch data $\mathcal{D}$ can be realizable by a branched covering $f : M \to N$ for some Riemann surfaces M, N, i.e. when there exists a branched covering $f : M \to N$, such that $\mathcal{D}(f) = \mathcal{D}$. This problem was studied by Hurwitz in the classical work [Hur]. He proved that realizability is equivalent to finding a set of permutations in the symmetric group Σ_d of d symbols satisfying some algebraic conditions. However, practically it is not easy to check for which abstract branch data such permutations can be found. There are some necessary conditions for realizability, the most important one is the Riemann-Hurwitz formula:

$$\chi(M) = d\chi(N) - \nu(\mathcal{D}), \tag{1}$$

where χ is the Euler characteristic. Moreover, $\nu(\mathcal{D})$ is always even.

If $\chi(N) \leq 0$ then these conditions are also sufficient, i.e. for any closed Riemann surface N with $\chi(N) \leq 0$ and any abstract branch data $\mathcal{D}$ of degree d with even $\nu(\mathcal{D})$ there exist a closed Riemann surface M with Euler characteristic given by (1) and a branched covering $f : M \to N$ of degree d, such that $\mathcal{D}(f) = \mathcal{D}$. This was proved by Husemoller in [Hus] (the orientable case) and by Ezell in [Ez] (the non-orientable case). The complete proof is given also in [EKS] together with a detailed discussion on the subject. Moreover, in [EKS] it is proved that if N is the projective plane, then an abstract branch data $\mathcal{D}$ of degree d is realizable by a branched covering

$f : M \to N$ if and only if $\nu(\mathcal{D})$ is even and $\nu(\mathcal{D}) \geq d-1$. In this way the problem is essentially reduced to the most complicated case when N is the two-dimensional sphere S^2.

From now on, we assume $N = S^2$. In this case the Hurwitz solution of the realizability problem is as follows. An abstract branch data $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ of degree $d > 1$ is realizable if and only if there exist permutations $\alpha_1, \dots, \alpha_n \in \Sigma_d$, such that:

- for every $i = 1, \dots, n$ the action of α_i on $\{1, \dots, d\}$ has exactly k_i orbits of lengths respectively $d_{i,1}, \dots, d_{i,k_i}$,
- $\alpha_1 \cdots \alpha_n = 1$ in Σ_d,
- $\langle \alpha_1, \dots \alpha_n \rangle$ acts transitively on $\{1, \dots, d\}$.

However, it is not easy to determine, which given branch data is realizable by a branched covering of the sphere. By (1), the condition $\nu(\mathcal{D}) \geq 2d-2$ is necessary for realizability. But there exist a lot of non-realizable data with $\nu(\mathcal{D}) \geq 2d-2$, $\nu(\mathcal{D})$ even. In fact, such examples can be found for any non-prime number d. On the other hand, the realizability is proved for some classes of data. Suppose that for an abstract data $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ the necessary conditions $\nu(\mathcal{D}) \geq 2d-2$ and $\nu(\mathcal{D}) \equiv 0 \mod 2$ are satisfied. If there exists i such that $d_{i,1} = d$ (which implies $k_i = 1$), then $\mathcal{D}$ is realizable. If for some i we have $d_{i,1} = d-1$, $d_{i,2} = 1$, then $\mathcal{D}$ is realizable if and only if (up to a permutation of the index i) $\mathcal{D}$ is not of the form

- $[[2,2], \dots, [2,2], [3,1]]$ (provided $d = 4$, $n \geq 3$) or
- $[[2, \dots, 2], [2, \dots, 2], [d-1, 1]]$ (provided d even, $n = 3$).

An exact answer is also known in the case

$$\mathcal{D} = [[d_{1,1}, \dots, d_{1,k_1}], [d_{2,1}, \dots, d_{2,k_2}], [m, 1, \dots, 1]].$$

Moreover, every data with $d \geq 2$, $d \neq 4$, is realizable provided $\nu(\mathcal{D})$ is even and $\nu(\mathcal{D}) \geq 3(d-1)$. For any non-prime number d, if $d = ab$ for some positive integers $a, b > 1$, then $\mathcal{D} = [[a, \dots, a], [b+1, 1 \dots, 1], [a, a(b-1)]]$ satisfies $\nu(\mathcal{D}) = 2d-2$ but is not realizable. All mentioned above facts were proved in [EKS].

It is not known, whether every abstract branch data $\mathcal{D}$ with $\nu(\mathcal{D}) \geq 2d-2$ and $\nu(\mathcal{D}) \equiv 0 \mod 2$ is realizable provided d is prime. (The direct computations show this holds for $d = 2, 3, 5, 7$.) By algebraic techniques from [EKS], it would be sufficient to prove this in the case $n = 3$.

This paper studies the case $M = N = S^2$. Then by (1), every realizable data satisfies $\nu(\mathcal{D}) = 2d-2$. Apart from the results described above, it is known that in this case a data of the form $[[x, d-x], [2, \ldots, 2], [2, \ldots, 2]]$ with $d = 2r$, $x \geq r$ is realizable if and only if $x = r$ (see [EKS]). In [Ge], an equivalent algebraic condition for realizability (different from the Hurwitz one) is given, which leads to some particular examples of non-realizable data. Moreover, a direct checking shows that provided $\nu(\mathcal{D}) = 2d - 2$, all branch data of degree $d = 2, 3, 5, 7$ are realizable and for $d = 4$ there is only one non-realizable data (up to a permutation of the index i), namely $[[3, 1], [2, 2], [2, 2]]$.

From now on, assume $M = N = S^2$ and consider only abstract branch data with $\nu(\mathcal{D}) = 2d-2$. Note that in this case $\mathcal{D} = [d_{i,j}]_{(i,j) \in I}$ is realizable if and only if there exists an (unbranched) covering of degree d from a $\sum_i k_i$-connected domain $U \subset \mathbb{C}$ onto an n-connected domain $V \subset \mathbb{C}$ such that for every component δ_i, $i = 1, \ldots, n$ of ∂V there exist exactly k_i components $\beta_{i,j}$, $j = 1, \ldots, k_i$ of ∂U, such that $f|_{\beta_{i,j}}$ is a covering of δ_i of degree $d_{i,j}$.

By the use of the quasiconformal technique and the measurable Riemann theorem one can easily show that if $\mathcal{D}$ is realizable by a branched covering $f : S^2 \to S^2$ of degree d, then it is also realizable by a rational map of degree d on the Riemann sphere $\widehat{\mathbb{C}}$. Hence, in this case the realizability problem is equivalent to a problem of existence of rational maps with given degrees and combinatorics of critical points and values. Note also that this rational map is a polynomial if and only if there exists i such that $d_{i,1} = d$. Hence, the described above results of [EKS] imply that every abstract "polynomial" branch data can be realized. This was proved also by Thom in [Th].

Definition 1.2. *A rational branch data* of degree $d > 1$ is an abstract branch data $\mathcal{D} = [d_{i,j}]_{(i,j) \in I}$ of degree d, such that $\nu(\mathcal{D}) = 2d - 2$. Define

$$I' = \{(i, j) \in I : d_{i,j} > 1\}, \quad I'_i = \{j : (i, j) \in I'\}.$$

Set also $k'_i = \#I'_i$, $k = \sum_{i=1}^n k_i = \#I$ and $k' = \sum_{i=1}^n k'_i = \#I'$.

Definition 1.3. *A polynomial branch data* of degree d is a rational branch data $\mathcal{D}$ of degree d, such that $d_{i,1} = d$ for some $i \in \{1, \ldots, n\}$ (which implies $k_i = 1$).

Remark. By definition, for a rational branch data of degree d we have $nd = \sum_I d_{i,j} = 2d - 2 + k$, so $k = nd - 2d + 2$.

In this paper (Section 2) we introduce a simple, purely geometric method of determining which rational branch data are realizable. Using this, in Section 3 we present a new proof of the polynomial realizability and indicate that this implies the straightening lemma for "refined" polynomial-like maps. In Section 4 we prove the realizability of some classes of rational branch data. In particular, we show:

- (Theorem 4.3) *Every rational branch data* $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ *of degree* d *with* $n \geq d$ *is realizable.*
- (Corollary 4.6) *Every rational branch data* $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ *of degree* d *with* $d_{i,j} \leq 2$ *and* $\nu_i(\mathcal{D}) \leq \sqrt{d/2}$ *is realizable.*

Notation. The closure and boundary of a set U will be denoted by $\mathrm{cl}\, U$, ∂U respectively. We write $\mathbb{D}$ for the open unit disc in $\mathbb{C}$. A topological disc is a set homeomorphic to $\mathbb{D}$. By a simple curve we mean a curve homeomorphic to a line segment. The number of elements of a set A is denoted by $\#A$. To avoid confusion in notation, we write $\boldsymbol{i}$ for the imaginary unit.

Acknowledgment. The author thanks the Centre de Recerca Matemàtica in Bellaterra for the support and hospitality.

2 Geometric condition for realizability

Let $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ be a rational data of degree d. We develop the following combinatorial method of determining whether $\mathcal{D}$ is realizable. Consider a unit circle $\partial\mathbb{D}$ in $\mathbb{C}$ with points

$$x_{i,l} = \exp(2\pi\boldsymbol{i}(nl+i)/(nd)), \quad i = 1, \ldots, n,\ l = 1, \ldots d.$$

We say that each point $x_{i,l}$ is coloured by colour i.

Lemma 2.1. $\mathcal{D}$ *is realizable if and only if there exist* $2d-2$ *simple curves* $\gamma_{i,m} : [0,1] \to \mathbb{C}$, $i = 1, \ldots, n$, $m = 1, \ldots, d - k_i$, *such that:*

- *The ends of* $\gamma_{i,m}$ *are two distinct points of colour* i,
- $\gamma_{i,m}((0,1)) \cap \partial\mathbb{D} = \emptyset$ *and if* $(i_1, m_1) \neq (i_2, m_2)$ *then* $\gamma_{i_1,m_1} \cap \gamma_{i_2,m_2}$ *is either empty or consists of one point of colour* $i = i_1 = i_2$,
- $\bigcup_m \gamma_{i,m}$ *has exactly* k'_i *components* $\Gamma_{i,j}$, $j \in I'_i$, *such that each* $\Gamma_{i,j}$ *is contractible and consists of* $d_{i,j} - 1$ *curves* $\gamma_{i,m}$.

Proof. Suppose we can construct the curves $\gamma_{i,m}$. It is easy to check that $d-1$ of them is inside the unit circle and the same number is outside. Let $\Gamma = \partial\mathbb{D} \cup \bigcup_{i,m} \gamma_{i,m}$. Then $\widehat{\mathbb{C}} \setminus \Gamma$ has $2d$ disc-like components $D_l^{\pm}$, $l = 1, \ldots, d$, where D_l^+ are inside the unit circle and D_l^- outside. Since each $\Gamma_{i,j}$ is contractible, we can shrink it to one point $c_{i,j}$ of colour i. Then Γ is changed into a curve with branch points at $c_{i,j}$, $(i,j) \in I'$, such that every $c_{i,j}$ lies in the boundary of exactly $d_{i,j}$ domains D_l^+ and $d_{i,j}$ domains D_l^-. Moreover, the boundary of each $D_l^{\pm}$ is a Jordan curve with exactly one point of colour i for $i = 1, \dots, n$, situated in counter-clockwise order for D_l^+ and in clockwise order for D_l^-. Let

$$v_i = \exp(2\pi \boldsymbol{i} i/n), \quad i = 1, \dots, n.$$

Then we can map homeomorphically each $\operatorname{cl} D_l^+$ onto $\operatorname{cl} \mathbb{D}$ and each $\operatorname{cl} D_l^-$ onto $\widehat{\mathbb{C}} \setminus \mathbb{D}$, such that each point of colour i is mapped onto v_i and the maps are consistent on Γ. In this way we construct a branched covering of $\widehat{\mathbb{C}}$ with branch data $\mathcal{D}$.

Suppose now that $\mathcal{D}$ is realizable by a branched covering f of $\widehat{\mathbb{C}}$. Composing f with a suitable homeomorphism of $\widehat{\mathbb{C}}$, we can assume that the branch values v_i are situated as above. Let $\Gamma = f^{-1}(\partial \mathbb{D})$. Then Γ is connected, because its complement consists of preimages of $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \operatorname{cl} \mathbb{D}$, which are topological discs, since all critical values are in $\partial \mathbb{D}$. Let $c_{i,j} \in f^{-1}(v_i), j = 1, \dots, k_i$ be points of colour i. Note that each $c_{i,j}$ lies in the boundary of exactly $2d_{i,j}$ components of $\widehat{\mathbb{C}} \setminus \Gamma$. Consider a small neighbourhood of a point $c_{i,j}$ with $d_{i,j} > 1$. For simplicity, assume that $c_{i,j} = 0$ and $\Gamma = \{z : \operatorname{Arg}(z) = 2\pi l/(2d_{i,j}) : l = 1, \dots, 2d_{i,j}\}$ near 0. Let U_l, $l = 1, \dots, 2d_{i,j}$ be the component of $\widehat{\mathbb{C}} \setminus \Gamma$ containing $\varepsilon \exp(2\pi \boldsymbol{i}(2l-1)/(4d_{i,j}))$ for a small $\varepsilon > 0$. We show that there exists $l_0 \in \{1, \dots, 2d_{i,j}\}$ such that

$$U_{l_0+1 \mod 2d_{i,j}} \neq U_{l_0-1 \mod 2d_{i,j}}. \tag{2}$$

To do it, suppose $U_{l_0+1 \mod 2d_{i,j}} = U_{l_0-1 \mod 2d_{i,j}} = U$. Then it is easy to see that $U \cup \{0\}$ separates U_{l_0} from other U_l, so $U_{l_0+2 \mod 2d_{i,j}} \neq U_{l_0}$ (recall that we assume $d_{i,j} > 1$). In this way we have proved (2).

For simplicity, assume $l_0 = 1$. Now we modify Γ, changing ∂U_1 near 0 so that

$$\begin{aligned}\partial U_1 &= \{r \exp(2\pi \boldsymbol{i} l/(2d_{i,j})) : r > \varepsilon, l \\ &= 0, 1\} \cup \{\varepsilon \exp(2\pi \boldsymbol{i} \theta) : \theta \in [0, 1/(2d_{i,j})]\}\end{aligned}$$

near 0. Then U_1 has changed into some smaller topological disc and U_2 and $U_{2d_{i,j}}$ have changed into one component V of $\widehat{\mathbb{C}} \setminus \Gamma$. By (2), before the modification, U_2 and $U_{2d_{i,j}}$ were disjoint topological discs. This easily implies that after the modification, the component V is also a topological disc. Since we have not modified any other component of $\widehat{\mathbb{C}} \setminus \Gamma$, it follows that Γ is still connected. Moreover, the branch point $c_{i,j} = 0$ has its branching degree decreased by one. Let $\varepsilon \in \Gamma$ be a new point of colour i and connect it to $c_{i,j} = 0$ by a straight line segment $\gamma_{i,1}$.

Now we repeat the procedure of modifying Γ near its branch points. After each step, there appears a new curve $\gamma_{i,m}$ and the branching degree of some branch point $c_{i,j}$ is decreased by one. Hence, after $2d-2$ steps, Γ is a compact connected 1-dimensional manifold without branching points, so it is a Jordan curve. Moreover, Γ contains d points of each colour $i = 1, \ldots, n$, suitable ordered and connected by $2d-2$ curves $\gamma_{i,m}$. Change the coordinates on $\widehat{\mathbb{C}}$ such that $\Gamma = \partial\mathbb{D}$ and coloured points have the appropriate positions. Then the curves $\gamma_{i,m}$ have required properties. □

3 The polynomial case

Using the method described in the previous section we now give a new geometric proof of the realizability of all polynomial branch data.

Theorem 3.1. *Every polynomial data is realizable.*

Proof. Let $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ be a polynomial branch data of degree d. We can assume $d_{n,1} = d$. According to Lemma 2.1, we will construct the suitable curves $\gamma_{i,m}$. It is obvious that we can connect all d points of colour n by $d-1$ curves $\gamma_{n,1}, \ldots, \gamma_{n,d-1}$ lying outside the unit circle, such that $\gamma_{n,m}$ connects $x_{n,m}$ to $x_{n,m+1}$ and $\gamma_{n,m}$ satisfy conditions of Lemma 2.1. Hence, it is sufficient to prove that we can connect points of colours $i = 1, \ldots, n-1$ by suitable $d-1$ curves $\gamma_{i,m}$ inside the unit circle.

The proof is by induction on d. The case $d = 2$ is trivial. Suppose we have a polynomial branch data $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ of degree $d > 2$ with $d_{n,1} = d$. Now we show that there exists $i_0 < n$, such that

$$k_i' < k_i \quad \text{for every } i \neq i_0, n. \tag{3}$$

Suppose the converse. Then there exist $i_1, i_2 < n$ such that $i_1 \neq i_2$ and $k_{i_1}' = k_{i_1}$, $k_{i_2}' = k_{i_2}$. This implies $d_{i,j} \geq 2$ for $i = i_1, i_2$, so $2k_i \leq \sum_j d_{i,j} = d$ for $i = i_1, i_2$. Hence,

$$2d-2 = \nu(\mathcal{D}) \geq \nu_{i_1}(\mathcal{D}) + \nu_{i_2}(\mathcal{D}) + \nu_n(\mathcal{D}) = 3d-1-k_{i_1}-k_{i_2} \geq 2d-1,$$

which is a contradiction. Thus, (3) holds. Permuting the index i assume $i_0 = n-1$. Then by (3), for every $i < n-1$ we have $d_{i,k_i} = 1$ (because $d_{i,1} \geq \cdots \geq d_{i,k_i}$).

Let

$$j_0 = \max\{j \in \{1, \ldots, k_{n-1}\} : d_{n-1,j} > 1\}.$$

Consider a table $\tilde{\mathcal{D}} = [\tilde{d}_{i,j}]_{(i,j)\in\tilde{I}}$ obtained from $\mathcal{D}$ in the following way: if $\nu_{n-1}(\mathcal{D}) > 1$, then remove numbers $d_{i,k_i} = 1$ for $i < n-1$, replace d_{n-1,j_0} by $d_{n-1,j_0} - 1$ and replace $d_{n,1} = d$ by $d-1$. If $\nu_{n-1}(\mathcal{D}) = 1$ (then $j_0 = 1$

and $d_{n-1,1} = 2$), remove numbers $d_{i,k_i} = 1$ for $i < n-1$, remove all $d_{i,j}$ with $i = n-1, n$ and set $\tilde{d}_{n-1,1} = d-1$. Now $\tilde{\mathcal{D}}$ is a polynomial branch data of degree $d-1$ for suitable $\tilde{I}$. Define $\tilde{n}, \tilde{k}_i, \tilde{k}'_i, \tilde{I}', \tilde{I}'_i$ in the suitable way.

Consider $\partial\mathbb{D}$ with points $x_{i,l}$, $i = 1, \dots, \tilde{n}$, $l = 1, \dots, d-1$. By induction, for $\tilde{\mathcal{D}}$ there exist suitable $d-2$ curves $\gamma_{i,m}$, $i = 1, \dots, \tilde{n}-1$, $m = 1, \dots, d-1-\tilde{k}_i$ inside the unit circle connecting points of colours $i = 1, \dots, \tilde{n}-1$.

Choose a point $x_{n-1,l_0} \in \partial\mathbb{D}$ in the following way. If $d_{n-1,j_0} > 2$, then $\tilde{n} = n$, $j_0 \in \tilde{I}'_{n-1}$ and we can take a point $x_{n-1,l_0} \in \Gamma_{n-1,j_0}$. Consider now the case $\nu_{n-1}(\mathcal{D}) > 1, d_{n-1,j_0} = 2$. Then $\tilde{n} = n$ and $j_0 \notin \tilde{I}'_{n-1}$, so $\tilde{k}'_{n-1} < \tilde{k}_{n-1}$. Note that $\bigcup_m \gamma_{n-1,m}$ contains exactly

$$\sum_{j \in \tilde{I}'_{n-1}} \tilde{d}_{n-1,j} = d-1-\sum_{j \notin \tilde{I}'_{n-1}} \tilde{d}_{n-1,j} = d-1-(\tilde{k}_{n-1}-\tilde{k}'_{n-1}) < d-1$$

points of colour $n-1$, so we can take a point x_{n-1,l_0} disjoint from $\bigcup_{i,m} \gamma_{i,m}$. Consider now the case $\nu_{n-1}(\mathcal{D}) = 1$ (then $\tilde{n} = n-1$). In this case let x_{n-1,l_0} be any point of colour $n-1$.

Rotating the unit circle, we can assume $l_0 = d-1$. Add $n-1$ new points $x_{1,d}, \dots, x_{n-1,d}$ situated in counter-clockwise order on $\partial\mathbb{D}$ between $x_{\tilde{n},d-1}$ and $x_{1,1}$. It is clear that we can connect $x_{n-1,d-1}$ to $x_{n-1,d}$ by a new curve $\gamma_{n-1,d-k_{n-1}}$ inside the unit circle, such that $\gamma_{n-1,d-k_{n-1}} \setminus \{x_{n-1,d-1}\}$ is disjoint from other $\gamma_{i,m}$. In this way we have constructed (up to a homeomorphism) the suitable curves for the branch data $\mathcal{D}$. By Lemma 2.1, $\mathcal{D}$ is realizable. □

We would like to indicate some consequence of the polynomial realizability in the theory of polynomial-like maps (see [DH] for definitions and properties). Namely, we can state the straightening lemma for "refined" polynomial-like maps.

Definition 3.2. Let $U_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, k_i$ and V_i, $i = 1, \dots, n$ be a family of bounded topological discs in $\mathbb{C}$ with smooth boundary, such that $\operatorname{cl} U_{i,j} \subset V_i$ for every i, j and $\operatorname{cl} U_{i,j}$ are pairwise disjoint. Let $f : \bigcup_{i,j} U_{i,j} \to \bigcup_i V_i$ be a proper holomorphic map extendable to the boundary, such that for every i, j there exist $l(i,j) \in \{1, \dots, n\}$ and a positive integer $d_{i,j}$, such that $f|_{\partial U_{i,j}}$ is a covering of $\partial V_{l(i,j)}$ of degree $d_{i,j}$. Suppose there exist positive integers d_i, $i = 1, \dots, n$, such that

- $\displaystyle\sum_{j:\, l(i,j)=l} d_{i,j} = d_i$ for every $i, l \in \{1, \dots, n\}$,
- $k_i = (n-1)d_i + 1$ for every $i \in \{1, \dots, n\}$.

Then we call f *a refined polynomial-like mapping* of degree $d = \sum_{i=1}^{n} d_i$.

Remark. The definition is formulated so that it is possible to conjugate f to a polynomial p of degree d. Then the first condition says that $p^{-1}(\bigcup_i V_i) = \bigcup_{i,j} U_{i,j}$ and the second one is necessary because of the Riemann-Hurwitz formula.

Corollary 3.3 (Refined straightening lemma). *Every refined polynomial-like mapping of degree d is hybrid equivalent to a polynomial of degree d.*

Proof. Let $V = \{z : |z| < r\}$, such that $V \supset \bigcup_i \operatorname{cl} V_i$ and define $\tilde{V}_i$ (resp. $\tilde{U}_{i,j}$) to be an open topological disc with smooth boundary, such that $\tilde{V}_i \supset \operatorname{cl} V_i$ (resp. $\tilde{U}_{i,j} \supset \operatorname{cl} U_{i,j}$) and $\partial\tilde{V}_i$ (resp. $\partial\tilde{U}_{i,j}$) is close to ∂V_i (resp. $\partial U_{i,j}$). Set $A = V \setminus \bigcup_i \operatorname{cl} V_i$, $A_i = V_i \setminus \bigcup_j \operatorname{cl} U_{i,j}$, $\tilde{A} = V \setminus \bigcup_i \operatorname{cl} \tilde{V}_i$, $\tilde{A}_i = V_i \setminus \bigcup_j \operatorname{cl} \tilde{U}_{i,j}$. Fix $i \in \{1, \ldots, n\}$. If $d_{i,j} = 1$ for every $j \in \{1, \ldots, k_i\}$, then by definition, $nd_i = k_i = (n-1)d_i + 1$, so $d_i = 1$, $k_i = n$. Hence, A_i is homeomorphic to A, so there exists a C^1 diffeomorphism $g_i : \operatorname{cl} A_i \xrightarrow{\text{onto}} \operatorname{cl} A$ consistent with f on $\partial U_{i,j}$, $j = 1, \ldots, k_i$. Suppose now there exists $j \in \{1, \ldots, k_i\}$, such that $d_{i,j} > 1$. Let

$$L_i = \{l \in \{1, \ldots, n\} : \text{ there exists } j \text{ such that } l(i,j) = l \text{ and } d_{i,j} > 1\}.$$

For simplicity, assume $L_i = \{1, \ldots, n_0\}$ for some $n_0 \leq n$. Define

$$I_i = \{(l,m) : l = 1, \ldots, n_0,\ m = 1, \ldots, \#\{j : l(i,j) = l\}\}.$$

Renumber $d_{i,j}$ for j such that $l(i,j) \leq n_0$ to $\tilde{d}_{l,m}$, $(l,m) \in I_i$ in such a way that $\tilde{d}_{l,m} = d_{i,j}$ for some j such that $l(i,j) = l$ and $\tilde{d}_{l,1} \geq \tilde{d}_{l,2} \geq \cdots$. It is easy to check that by the definition of f, $\mathcal{D}_i = [\tilde{d}_{l,m}]_{(l,m)\in I_i}$ is a polynomial branch data of degree d_i. Let p_i be a polynomial of degree d_i realizing $\mathcal{D}_i$ and let W_i be $\widehat{\mathbb{C}}$ with removed small disjoint disc-like neighbourhoods with smooth boundary of ∞, all critical values of p_i and $n - n_0$ non-critical values. Composing $p_i|_{p_i^{-1}(W_i)}$ with a suitable C^1 diffeomorphism, we can define a covering map $g_i : \tilde{A}_i \xrightarrow{\text{onto}} \tilde{A}$ of degree d_i, such that $g_i|_{\partial\tilde{U}_{i,j}}$ is a covering of $\partial\tilde{V}_{l(i,j)}$ of degree $d_{i,j}$ and $g_i|_{\partial V_i}$ is a covering of ∂V of degree d_i. Extend g_i to $\operatorname{cl} A_i$ in a C^1 way, such that $g_i|_{\operatorname{cl} A_i}$ is a covering of $\operatorname{cl} A$ of degree d_i, consistent with f on $\partial U_{i,j}$, $j = 1, \ldots, k_i$. In the similar way define a C^1 map $g : \operatorname{cl} A \xrightarrow{\text{onto}} \{z : |z| \leq r^d\} \setminus V$, such that g has one critical point of multiplicity $n-1$ in A, $g = g_i$ on ∂V_i and $g(z) = z^d$ on ∂V. Setting $g = f$ on $\bigcup_{i,j} U_{i,j}$, $g = g_i$ on A_i and $g(z) = z^d$ on $\widehat{\mathbb{C}} \setminus V$, we extend g to a C^1 branched covering of $\widehat{\mathbb{C}}$ of degree d, holomorphic outside $\operatorname{cl} A \cup \bigcup_i \operatorname{cl} A_i$, such that $\operatorname{cl} g(\widehat{\mathbb{C}} \setminus \operatorname{cl} V) \subset \widehat{\mathbb{C}} \setminus \operatorname{cl} V$ and $g^3(\operatorname{cl} A \cup \bigcup_i \operatorname{cl} A_i) \subset \widehat{\mathbb{C}} \setminus \operatorname{cl} V$. Now we can define a bounded g-invariant conformal structure and using the measurable Riemann theorem we find a polynomial p of degree d hybrid equivalent to g. □

4 The rational case

Now we show the realizability of some classes of rational branch data. Proposition 4.1 was also proved in [EKS].

Proposition 4.1. *Let $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ be a rational branch data of degree d. If there exists a set $K \subset \{1, \dots, n\}$, such that $\sum_{i\in K} \nu_i(\mathcal{D}) = d-1$, then $\mathcal{D}$ is realizable.*

Proof. We can assume $K = \{1, \dots, r\}$ for some $r < n$. Let

$$\mathcal{D}_1 = [[d_{1,1}, \dots, d_{1,k_1}], \dots, [d_{r,1}, \dots, d_{r,k_r}], [d]],$$
$$\mathcal{D}_2 = [[d_{1,r+1}, \dots, d_{1,k_{r+1}}], \dots, [d_{n,1}, \dots, d_{n,k_n}], [d]].$$

Then $\mathcal{D}_1, \mathcal{D}_2$ are polynomial branch data of degree d, so by the proof of Theorem 3.1, we can connect points of colours $i \in K$ by suitable $d-1$ curves inside the unit circle and points of colours $i \notin K$ by suitable $d-1$ curves outside the unit circle. There is no interference between the curves inside and outside, because they join points of different colours. By Lemma 2.1, $\mathcal{D}$ is realizable. □

Proposition 4.2. *If a rational branch data $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ of degree d is realizable, then $\mathcal{D}_1$ is realizable, where $\mathcal{D}_1$ is a rational branch data of degree $d+1$ obtained from $\mathcal{D}$ by increasing any fixed d_{i_0,j_0} by one, adding a new number $d_{i,k_i+1} = 1$ for every $i \le n, i \ne i_0$ and adding new numbers $d_{n+1,1} = 2$, $d_{n+1,2} = \cdots = d_{n+1,d} = 1$. Similarly, $\hat{\mathcal{D}}$ is realizable, where $\mathcal{D}_2$ is a data of degree $d+1$ obtained from $\mathcal{D}$ by adding $d_{i,k_i+1} = 1$ for every $i \le n$ and adding new numbers $d_{i,1} = 2$, $d_{i,2} = \cdots = d_{i,d} = 1$ for $i = n+1, n+2$.*

Proof. Let $\gamma_{i,m}$ be the curves from Lemma 2.1 corresponding to $\mathcal{D}$. If $d_{i_0,j_0} > 1$, choose $l_0 \in \{1, \dots, d\}$ such that $x_{i_0,l_0} \in \Gamma_{i_0,j_0}$. If $d_{i_0,j_0} = 1$, let $l_0 = d$. For simplicity, assume $i_0 = n$, $l_0 = d$. Add n new points $x_{1,d+1}, \dots, x_{n,d+1}$ situated in counter-clockwise order on $\partial\mathbb{D}$ between $x_{n,d}$ and $x_{1,1}$ and add another $d+1$ new points $x_{n+1,1}, \dots, x_{n+1,d+1} \in \partial\mathbb{D}$ such that $x_{n+1,l}$ lies between $x_{n,l}$ and $\tilde{x}_{1,(l \bmod (d+1))+1}$. Now we can connect $x_{n,d}$ to $x_{n,d+1}$ by a new curve inside the unit circle and $x_{n+1,d}$ to $x_{n+1,d+1}$

by another curve outside the unit circle to obtain the suitable set of curves for the data $\mathcal{D}_1$. By Lemma 2.1, $\mathcal{D}_1$ is realizable. The proof for $\mathcal{D}_2$ is analogous. $\square$

Theorem 4.3. *Let $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ be a rational branch data of degree d with $n \geq d$. Then $\mathcal{D}$ is realizable.*

Proof. We proceed similarly as in the proof of Theorem 3.1, using induction on d. The case $d = 2$ is obvious. Consider a data $\mathcal{D}$ of degree $d > 2$ with $n \geq d$. Suppose there exist $i_1 \neq i_2$ such that $k'_{i_1} = k_{i_1}$, $k'_{i_2} = k_{i_2}$. Then $d_{i,j} \geq 2$ for $i = i_1, i_2$, so $2k_i \leq \sum_j d_{i,j} = d$ for $i = i_1, i_2$. This implies $\nu_i(\mathcal{D}) = d - k_i \geq d/2$ for $i = i_1, i_2$, where the equality holds if and only if d is even and $d_{i,j} = 2$ for all j. Hence,

$$2d - 2 = \nu(\mathcal{D}) \geq \nu_{i_1}(\mathcal{D}) + \nu_{i_2}(\mathcal{D}) + n - 2 \geq 2d - 2,$$

so we must have d even, $n = d$, $d_{i,j} = 2$ for $i = i_1, i_2$, $j \geq 1$ and $i \neq i_1, i_2$, $j = 1$ and $d_{i,j} = 1$ for $i \neq i_1, i_2$, $j > 1$. Then $\mathcal{D}$ is realizable by Proposition 4.1.

Therefore, we can assume there exists i_0 such that $k'_i < k_i$ for every $i \neq i_0$, which implies $d_{i,k_i} = 1$ for $i \neq i_0$. Changing i_0 if necessary, we can also assume that either $\nu_{i_0}(\mathcal{D}) > 1$ or $\nu_i(\mathcal{D}) = 1$ for every $i \leq n$. Note also that we always have $\nu_{i_1}(\mathcal{D}) = 1$ for some i_1, because otherwise $\nu(\mathcal{D}) \geq 2n \geq 2d$. Consider a rational branch data $\tilde{\mathcal{D}} = [\tilde{d}_{i,j}]_{(i,j)\in\tilde{I}}$, $\tilde{I} = \{i \leq \tilde{n}, j \leq \tilde{k}_i\}$ of degree $d - 1$ obtained from $\mathcal{D}$ by removing $d_{i_1,j}$ for $j \leq k_{i_1}$, removing $d_{i,k_i} = 1$ for $i \neq i_0, i_1$ and either decreasing by one d_{i_0,j_0} for $j_0 = \max\{j : d_{i_0,j} > 1\}$ (in the case $\nu_{i_0}(\mathcal{D}) > 1$) or removing $d_{i_0,j}$ for $j \leq k_{i_0}$ (in the case $\nu_i(\mathcal{D}) = 1$ for every $i \leq n$). In the first case we have $\tilde{n} = n - 1 \geq d - 1$. In the second case, $\tilde{n} = n - 2$ and $n = 2d - 2$, which implies $\tilde{n} \geq d - 1$. Hence, $\tilde{D}D$ is realizable by induction, so $\mathcal{D}$ is realizable by Proposition 4.2. $\square$

To prove the next result, we need the following lemma:

Lemma 4.4. *Let $R_d = \{z \in \mathbb{C} : z^d = 1\}$ and let $A, B \subset R_d$. If $\#A \cdot \#B < d$, then there exists $z \in R_d$ such that $zA \cap B = \emptyset$ (where $zA = \{za : a \in A\}$).*

Proof. Suppose that for every $z \in R_d$ there exists $a(z) \in A$ such that $za(z) \in B$. Since R_d has d elements, there exists $b \in B$, such that $b = za(z)$ for k disjoint points $z \in R_d$ with $k \geq d/\#B$. Hence, there are k disjoint points $b/z = a(z) \in A$, so $\#A \geq k \geq d/\#B$. $\square$

Theorem 4.5. *Let $\mathcal{D} = [d_{i,j}]_{(i,j)\in I}$ be a rational branch data of degree d. Suppose there exist $K \subset \{1, \ldots, n\}$, $i_0 \in \{1, \ldots, n\} \setminus K$ and $J \subset I'_{i_0}$, such that*

- $\sum_{i \in K} \nu_i(\mathcal{D}) + \sum_{j \in J} (d_{i_0,j} - 1) = d - 1,$
- $\sum_{j \in J} d_{i_0,j} \sum_{j \in I'_{i_0} \setminus J} d_{i_0,j} < d.$

Then $\mathcal{D}$ is realizable.

Proof. We can assume $K = \{1, \ldots, r\}$ for some $r < n$. Let

$$\mathcal{D}_1 = [[d_{1,1}, \ldots, d_{1,k_1}], \ldots, [d_{r,1}, \ldots, d_{r,k_r}], D_1, [d]],$$
$$\mathcal{D}_2 = [[d_{1,r+1}, \ldots, d_{1,k_{r+1}}], \ldots, [d_{n,1}, \ldots, d_{n,k_n}], D_2, [d]],$$

where D_1 consists of numbers $d_{i_0,j}$, $j \in J$ and $d - \sum_{j \in J} d_{i_0,j}$ unities and D_2 consists of numbers $d_{i_0,j}$, $j \in I'_{i_0} \setminus J$ and $d - \sum_{j \in I'_{i_0} \setminus J} d_{i_0,j}$ unities, suitable ordered. Then by the first assumption, $\mathcal{D}_1, \mathcal{D}_2$ are polynomial branch data of degree d. By the proof of Theorem 3.1, we can draw suitable $d-1$ curves joining points of colours $i \in K \cup \{i_0\}$ according to the data $\mathcal{D}_1$ inside the unit circle and $d - 1$ curves joining points of colours $i \notin K$ according to the data $\mathcal{D}_2$ outside the unit circle. Note that for colours $i \neq i_0$ there is no interference between curves inside and outside. Moreover, there are exactly $\sum_{j \in J} d_{i_0,j}$ points of colour i_0 connected by curves inside the unit circle and exactly $\sum_{j \in I'_{i_0} \setminus J} d_{i_0,j}$ points of colour i_0 connected by curves outside the unit circle. By the second assumption and Lemma 4.4, we can rotate all curves inside the unit circle by an angle $2\pi s/d$ for some $s \in \{1, \ldots, d\}$, so that curves joining points of colour i_0 inside the unit circle will be disjoint from curves joining points of colour i_0 outside the unit circle. Then these curves correspond to the data $\mathcal{D}$, so $\mathcal{D}$ is realizable by Lemma 2.1. $\square$

Corollary 4.6. *Let $\mathcal{D} = [d_{i,j}]_{(i,j) \in I}$ be a rational branch data of degree d, such that $d_{i,j} \leq 2$ and $\nu_i(\mathcal{D}) \leq \sqrt{d/2}$ for every i, j. Then $\mathcal{D}$ is realizable.*

Proof. Let $K = \{1, \ldots, r\}$, where $r = \max\{p : \sum_{i=1}^{p} \nu_i(\mathcal{D}) \leq d - 1\}$. If $\sum_{i \in K} \nu_i(\mathcal{D}) = d-1$, then $\mathcal{D}$ is realizable by Proposition 4.1. If $\sum_{i \in K} \nu_i(\mathcal{D}) < d - 1$, let $i_0 = r + 1$ and let $J = \{1, \ldots, q\}$, where $q = d - 1 - \sum_{i=1}^{r} \nu_i(\mathcal{D})$. Then the first assumption of Theorem 4.5 is obviously satisfied. To check the other one, note that

$$\sum_{j \in J} d_{i_0,j} \sum_{j \in I'_{i_0} \setminus J} d_{i_0,j} = 4q(k'_{i_0} - q) < 2(k'_{i_0})^2 = 2(\nu_{i_0}(\mathcal{D}))^2 \leq d.$$

By Theorem 4.5, $\mathcal{D}$ is realizable. $\square$

References

[DH] A. Douady and J. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. 18 (1985), 287-343.

[EKS] A. L. Edmonds, R. S. Kulkarni and R. E. Stong, *Realizability of branched coverings of surfaces*, Trans. Amer. Math. Soc. (2) 282 (1984), 773-790.

[Ez] C. L. Ezell, *Branch point structure of covering maps onto nonorientable surfaces*, Trans. Amer. Math. Soc. 243 (1978), 123-133.

[Ge] S. M. Gersten, *On branched covers of the 2-sphere by the 2-sphere*, Proc. Amer. Math. Soc. (4) 101 (1987), 761-766.

[Hus] D. Husemoller, *Ramified coverings of Riemann surfaces*, Duke Math. J. 29 (1962), 167-174.

[Hur] A. Hurwitz, *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. 103 (1891), 1-60.

[Th] R. Thom, *L'équivalence d'une fonction différentiable et d'un polynome*, Topology (2) 3 (1965), 297-307.