ON UNIFORM APPROXIMATION BY POLYANALYTIC POLYNOMIALS AND THE DIRICHLET PROBLEM FOR BIANALYTIC FUNCTIONS

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Abstract. In this paper we give some necessary and sufficient conditions for uniform approximability of functions by polyanalytic polynomials on plane compact sets of special form. Also connections with the corresponding Dirichlet problem are considered.

1. Introduction

Let n be a natural number. We recall, that a function f is n-analytic (polyanalytic of order n) in an open set $G \subset \mathbb{C}$ if $\overline{\partial}^n f = 0$ in G in the classical sense, where $\overline{\partial}^n$ is the n-power of the Cauchy-Riemann operator $\overline{\partial} = \partial/\partial\overline{z}$. Then, clearly, f has a unique representation

$$f(z) = f_0(z) + \overline{z} f_1(z) + \cdots + \overline{z}^{n-1} f_{n-1}(z),\ z \in G,$$

where $f_0, \cdots, f_{n-1}$ are holomorphic in G. In particular, all n-analytic polynomials have a form $p_0(z) + \overline{z} p_1(z) + \cdots + \overline{z}^{n-1} p_{n-1}(z)$, where $p_0, \cdots, p_{n-1}$ are polynomials of a complex variable z. In [Ca] it has been proved the following result.

Theorem 1.1. *Let X be a compact set in $\mathbb{C}$ with connected complement $\mathbb{C} \setminus X$ and $n \geq 2$. Then each function f, continuous on X and n-analytic in the interior X° of X, can be uniformly approximated on X (with arbitrary accuracy) by n-analytic polynomials.*

Denote by $C(X)$ the space of all continuous complex valued functions on a compact set X with the uniform norm $\|f\|_X = \sup\{|f(z)| \ : \ z \in X\}$, and

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set

$$P_n = \{p : p \text{ is any n-analytic polynomial}\},$$
$$P_n(X) = \text{Closure}_{C(X)}\{p|_X \ : \ p \in P_n\},$$
$$A_n(X) = \{f \in C(X) : \overline{\partial}^n f = 0 \text{ on } X^\circ \text{ in the classical sense}\}.$$

Trivially, for $1 \leq s \leq n$, one has $P_s(X) \subset P_n(X) \subset A_n(X)$.

Problem 1.1. *To find a necessary and sufficient conditions on X in order that*

$$P_n(X) = A_n(X). \tag{1.1}$$

By Mergelyan's theorem [Me](see also [Ru, Chapter 20]) $P_1(X) = A_1(X)$ holds if and only if $\mathbb{C} \setminus X$ is connected. Theorem 1.1 says that (1.1) still is satisfied for all $n \geq 2$ whenever $\mathbb{C} \setminus X$ is connected, but the last condition, as it turns out [Fe], is already not necessary for (1.1) when $n \geq 2$. The situation is not simple even for the case, when X is a closed Jordan curve.

Throughout this paper Γ will denote a *contour* (closed Jordan curve, no necessary rectifiable) which surrounds a Jordan domain $D = D(\Gamma)$ in $\mathbb{C}$.

Definition 1.1. ([Fe]) *A rectifiable contour Γ is called a Nevanlinna contour (for short we write $\Gamma \in \mathcal{N}$) if there exist two bounded holomorphic functions u and v in D ($v \not\equiv 0$) such that $u(\zeta)/v(\zeta) = \overline{\zeta}$ on Γ (the last equality is understood in the sense of the angular boundary values almost everywhere (a.e.) with respect to the arc length measure on Γ).*

For example the unit circle $\Gamma_1 = \{t \in \mathbb{C} : \bar{t} = 1/t\} \in \mathcal{N}$, but any ellipse

$$\Gamma_{ab} = \{z = x + iy : x^2/a^2 + y^2/b^2 = 1\},\ 0 < b < a, \tag{1.2}$$

is not in $\mathcal{N}$.

Theorem 1.2. ([Fe]) *Let Γ be a rectifiable contour. Then for each $n \geq 2$,*

$$P_n(\Gamma) = C(\Gamma) \Longleftrightarrow \Gamma \notin \mathcal{N}.$$

In Section 2 we give a definition of Nevanlinna contours (no necessary rectifiable) and find a precise characterization of them in terms of the corresponding conformal mapping from the unit disk $D_1 = \{w : |w| < 1\}$ onto D. We also define some other classes of contours which will be needed later and give a short proof of a generalization of Theorem 1.2 for arbitrary contours.

In Section 3 we study the case $X = \Gamma \cup K$, where $\Gamma \notin \mathcal{N}$, $K \subset D$ is a compact set with $P_n(K) = A_n(K)$. In Theorem 3.1 we give two sufficient conditions on K and Γ in order that (1.1) is satisfied. Then we present several examples for which (1.1) fails, showing, in particular, that the mentioned above conditions of Theorem 3.1 are not necessary. This means that the solution of Problem 1.1 (which is still not found) should take into account some very complicated analytic properties of X, which even depend on n. For instance if $\Gamma = \Gamma_{ab}$ is an ellipse and K is the interval joining its focuses, then (for $X = \Gamma \cup K$) one has $P_2(X) \neq P_3(X) = C(X)$.

In Section 4 we discuss the relation between the Problem 1.1 (restricted to the case $X = \Gamma$ and $n = 2$) and the corresponding Dirichlet problem for 2-analytic functions in the domain D. A very simple example shows that these problems are not equivalent, which means that there exists (even analytic) contour Γ for which $P_2(\Gamma) = C(\Gamma)$, but D is not regular for the "classical" Dirichlet problem for the operator $\overline{\partial}^2$. Moreover, it is still not known if there exists (or not) at least one regular domain for the $\overline{\partial}^2$-Dirichlet problem. There is also no equivalence between the condition (1.1) for $X = \Gamma$ and the uniqueness property for the $\overline{\partial}^2$-Dirichlet problem in D. In Theorem 4.1 it is proved that $\Gamma \in \mathcal{N}$ implies nonuniqueness but the opposite is not true. In Example 4.1 we show that for some Nevanlinna contours Γ both functions u, v in Definition 1.1 cannot be taken from $A_1(\overline{D})$.

2. Background

Let $\Gamma_1 = \{t \in \mathbb{C} : |t| = 1\}$ and Γ be any contour. Fix a conformal mapping k from the unit disk D_1 onto $D = D(\Gamma)$, which always will be considered as already extended by Caratheodory theorem ([Ko], p. 59) to the corresponding homeomorphism of $\overline{D_1}$ onto $\overline{D}$. For any function g we will denote by g_* the function $g_*(w) = \overline{g(\overline{w})}$ where it is defined.

Lemma 2.1. *The following equality holds everywhere on* Γ*:*

$$\overline{\zeta} = k_*(1/k^{-1}(\zeta)),\ \zeta \in \Gamma. \tag{2.1}$$

Proof. Let $\zeta \in \Gamma$ and $t = k^{-1}(\zeta)$. Since $\overline{t} = 1/t$ on Γ_1 one gets $\overline{k^{-1}(\zeta)} = 1/k^{-1}(\zeta)$, but $\overline{k^{-1}(\zeta)} = k_*^{-1}(\overline{\zeta})$, which gives (2.1) □

Definition 2.1. *We say that* Γ *is a Nevanlinna (respectively locally Nevanlinna) contour if there exist two bounded holomorphic functions* u *and* v *in* D *(respectively if the functions* u *and* v *are bounded and holomorphic in* $D \setminus K_0$ *for some compact set* $K_0 \subset D$*),* $v \not\equiv 0$ *such that* $u(\zeta)/v(\zeta) = \overline{\zeta}$ *angularly a.e. on* Γ *in the sense of conformal mapping, which means that*

$$u(k(t))/v(k(t)) = \overline{k(t)} \tag{2.2}$$

angularly a. e. on Γ_1*, (clearly this is independent of the choice of* k*).*

We use the short notations $\Gamma \in \mathcal{N}$ and $\Gamma \in \mathcal{LN}$ for Nevanlinna and locally Nevanlinna contours respectively.

Proposition 2.1. *One has* $\Gamma \in \mathcal{N}$ *(respectively* $\Gamma \in \mathcal{LN}$*) if and only if a conformal mapping* k *from* D_1 *onto* D *can be extended to same function* $\tilde{k}$ *defined on* $\overline{\mathbb{C}}$ *(respectively on some neighborhood* U_1 *of* $\overline{D_1}$*) in the following sense: there exist bounded holomorphic functions* u_1 *and* v_1 *on* $\overline{\mathbb{C}} \setminus \overline{D_1}$ *(respectively on* $U_1 \setminus \overline{D_1}$*) such that* $\tilde{k}(w) = u_1(w)/v_1(w)$, $|w| > 1$ *and* $\tilde{k}(t) = k(t)$ *angularly from* $\mathbb{C} \setminus \overline{D_1}$ *a.e.* $t \in \Gamma_1$.

Proof. We shall consider only the necessity for $\Gamma \in \mathcal{LN}$, the other cases are similar. Let $\Gamma \in \mathcal{LN}$ and u, v, K_0 are from Definition 2.1. Taking into account (2.1) we set

$$\tilde{k}(w) = \frac{\overline{u(k(1/\overline{w}))}}{\overline{v(k(1/\overline{w}))}},\ |w| > 1,\ 1/\overline{w} \notin k^{-1}(K_0),$$

which is the desired extension. Then we just take $u_1(w) = (u(k(1/w)))_*$, $v_1(w) = (v(k(1/w)))_*$ and $U_1 = \overline{D_1} \cup \{w : |w| > 1, 1/\overline{w} \notin k^{-1}(K_0)\}$. Observe that (2.2) implies $\tilde{k}(w) \to k(t)$ if w goes angularly to t a.e $t \in \Gamma_1$. □

In particular, if k can be extended conformally to some neighborhood of $\overline{D_1}$ (which would exactly mean that Γ is an analytic contour), then the function

$$S(z) = k_*(1/k^{-1}(z))$$

is holomorphic in some connected neighborhood U of Γ, which is possible to choose such that

$$\Gamma = \{\zeta \in U : \overline{\zeta} = S(\zeta)\}. \tag{2.3}$$

The function S is called the Schwarz function of Γ and it is uniquely defined in some connected neighborhood of Γ (see [Da] for more insight). Observe that Proposition 2.1 implies that if Γ is analytic then: $\Gamma \in \mathcal{N}$ if and only if k is a rational function with poles off $\overline{D_1}$ (see also [Da]). We shall often take in use the uniqueness Luzin-Privalov theorem ([Ko, p. 84-85]. From this theorem it easily follows that the function u/v in Definition 1.1 and 2.1 is uniquely defined and also that if Γ is analytic with Schwarz function S, then $\Gamma \in \mathcal{N}$ if and only if $S = u/v$ has just a finite number of singularities (which are poles) in D. For these reasons we consider the function u/v in Definition 2.1 as a generalization of the notion of Schwarz function.

It is clear that if Γ is rectifiable then the Definitions 1.1 and 2.1 of Nevanlinna contour give the same. We illustrate Definition 2.1 and Proposition 2.1 with some examples. When Γ_{ab} is the ellipse (1.2) its Schwarz function can be easily found:

$$S(z) = \frac{(a^2+b^2)z - 2ab\sqrt{z^2-c^2}}{c^2}, \tag{2.4}$$

where $c = (a^2-b^2)^{1/2}, c > 0$ and we fixed holomorphic branch of $\sqrt{z^2-c^2}$ defined outside the interval $[-c, c]$ sending the point a to b. Then $\Gamma_{ab} \in \mathcal{LN} \setminus \mathcal{N}$. On the other hand, if Γ is any contour which contains two analytically independent subarcs (see [Fe], Proposition 2) then $\Gamma \notin \mathcal{LN}$.

REMARK 2.1 We do not know if the condition $\Gamma \in \mathcal{N}$ implies some additional geometric properties of Γ. We believe that if $\Gamma \in \mathcal{N}$ then the Hausdorff dimension of it is equal to 1 or even Γ is rectifiable.

Proposition 2.2. *If $\Gamma \in \mathcal{N}$ is rectifiable, then the function $u - \overline{z}v$ (see Definition 1.1) extends continuously from D to $\overline{D}$ and $u - \overline{z}v = 0$ on Γ.*

Proof. See the proof of Theorem 4.1 (1).

Now we give a short proof of a generalization of Theorem 1.2.

Theorem 2.1. *Let Γ be any contour, $n \geq 2$. Then*

$$P_n(\Gamma) \neq C(\Gamma) \Longleftrightarrow \Gamma \in \mathcal{N}.$$

Proof. Assume that $P_n(\Gamma) \neq C(\Gamma)$, so that also $P_2(\Gamma) \neq C(\Gamma)$. Then there exists a finite nonzero Borel measure μ on Γ, such that $\mu \perp P_1(\Gamma)$ and $\overline{\zeta}\mu \perp P_1(\Gamma)$. Now define the measures $k^{-1}(\mu)$ and $k^{-1}(\overline{\zeta}\mu)$ on Γ_1 transporting the corresponding measures by k^{-1}, i. e. $k^{-1}(\mu)(E) = \mu(k(E))$, $E \subset \Gamma_1$. Since

$$\int_{\Gamma_1} P(k(t))\, dk^{-1}(\mu)(t) = \int_{\Gamma} P(\zeta)\, d\mu(\zeta), \quad P \in C(\Gamma),$$

the Mergelyan theorem implies that $\mu, \overline{\zeta}\mu \perp P_1(\Gamma)$ if and only if $k^{-1}(\mu)$, $k^{-1}(\overline{\zeta}\mu) \perp P_1(\Gamma_1)$ and this is equivalent (by F. and M. Riesz theorem [Ko, p. 40]) to the existence of two functions $u_2, v_2 \in H^1(D_1)$ such that $k^{-1}(\mu) = v_2(t)\, dt$ and $k^{-1}(\overline{\zeta}\mu) \equiv \overline{k}[k^{-1}(\mu)] = u_2(t)\, dt$ and therefore $\overline{k(t)} = u_2(t)/v_2(t)$ a.e. t on Γ_1. By [Ru, p. 374] it it possible to find functions u_1 and v_1 bounded and holomorphic in D_1 such that $u_2/v_2 = u_1/v_1$ in D_1. Now we take $u = u_1 \circ k^{-1}$, $v = v_1 \circ k^{-1}$.

Oppositely, if $\Gamma \in \mathcal{N}$ and k, u, v are taken from Definition 2.1 then one can choose $a \in D$ such that $0 < |u(a) - \overline{a}v(a)|$. We claim that $\frac{(\overline{\zeta}-\overline{a})}{\zeta - a}^{n-1} |_\Gamma$

$\notin P_n(\Gamma)$. Assume that this is not true, then for each $\delta > 0$ there exist polynomials $p_0, \cdots, p_{n-1}$ such that

$$\left|\sum_{s=0}^{n-1} p_s(\zeta)\overline{\zeta}^s - \frac{(\overline{\zeta}-\overline{a})^{n-1}}{\zeta - a}\right| < \delta,\ \zeta \in \Gamma.$$

This implies

$$|\sum_{s=0}^{n-1} p_s(k(t))u(k(t))^s(k(t)-a)v(k(t))^{n-1-s} - (u(k(t)) - \overline{a}v(k(t)))^{n-1}| < \delta \sup_{\zeta\in\Gamma} |v(\zeta)|^{n-1}|\zeta - a|, \tag{2.5}$$

for almost all $t \in \Gamma$. But all functions in (2.5) are boundary values of bounded and holomorphic functions on D_1, then, by the maximum modules principle, we can put $w = k^{-1}(a)$ in (2.5) instead of t so that

$$|(u(a) - \overline{a}v(a))^{n-1}| < \delta \sup_{\zeta\in\Gamma} |v(\zeta)|^{n-1}|\zeta - a|,$$

this gives the desired contradiction for δ small enough. □

P. Ahern made us aware of the results of Douglas, Shapiro and Shields [DSS]. By terminology of this paper, the statement of Proposition 2.1 (the case $\Gamma \in \mathcal{N}$) means that the conformal mapping k has the Nevanlinna-type pseudocontinuation $\tilde{k}$ from D_1 to $\mathbb{C} \setminus \overline{D_1}$ across Γ_1. It happens if and only if k is nonciclic vector for the backward-shift operator $f \mapsto (f(z) - f(0))/z$ in $H^2(D_1)$ ([DSS], Theorem 2.2.1).

3. Approximation on the union of sets

Fix $n \geq 2$ natural. Let X_1 and X_2 be compact sets in $\mathbb{C}$ and let $X = X_1 \cup X_2$. When does $P_n(X) = A_n(X)$?

In this section we consider several examples connected with this question. For the first, some simple remarks are in order. In general the above question is not well-posed: everything can happen. For instance, both compacts $X_1 = \{z : |z| = 1,\ \mathrm{Re}\ z \leq 0\}$ and $X_2 = \{z : |z| = 1,\ \mathrm{Re}\ z \geq 0\}$ satisfy (1.1), but $X = X_1 \cup X_2$ does not. The opposite situation happens for $X_1 = \{z : |z-1| = 1,\ \text{or}\ |z+1| \leq 1\}$ and $X_2 = \{z : |z+1| = 1,\ \text{or}\ |z-1| \leq 1\}$, just take in use Theorems 1.1 and 1.2. On the other hand, if $X_1 \cap X_2 = \emptyset$ then, in order to have (1.1) for $X = X_1 \cup X_2$, it is necessary that (1.1) is satisfied by each X_1 and X_2 separately. Moreover, the last is also sufficient whenever $\hat{X}_1 \cap \hat{X}_2 = \emptyset$. Here and in the sequel $\hat{K}$ means the polynomial convex hull, i. e. the union of the compact set K and all its bounded complementary components (holes).

The rest of this section is devoted to the special case when $X_1 = \Gamma$ is a *rectifiable* non-Nevanlinna contour (so that $P_2(\Gamma) = C(\Gamma)$), and $X_2 = K$ is a compact set in D such that $P_n(K) = A_n(K)$. We will obtain two sufficient condition in order that $X = K \cup \Gamma$ satisfies (1.1), and several examples, when it is not the case. It will be clear that the main role here plays the position of K, related to the nonmeromorphic singularities of the Schwarz function of the contour Γ.

For a measure μ (all measures will be finite complex Borel) with compact support, sptμ, we denote by $\hat{\mu}$ its Cauchy transform:

$$\hat{\mu}(z) = \frac{1}{\pi z} * \mu = \frac{1}{\pi} \int \frac{d\mu(w)}{z - w},$$

which is holomorphic outside sptμ, $\hat{\mu} \in L^1_{\text{loc}}(\mathbb{C})$ and $\overline{\partial}\hat{\mu} = \mu$ in the distributional sense [Ga, p. 38].

Lemma 3.1. *Let Γ, K, X be as above and let μ be a measure with spt $\mu \subset K$. Then each measure σ with conditions spt $\sigma \subset X, \sigma_{|K} = \mu$ and $\sigma \perp P_1(X)$ (that is $\int p\, d\sigma = 0$ for each $p \in P_1$) has the following form:*

$$\sigma = \mu - \frac{1}{2i}\hat{\mu}(\zeta)\, d\zeta + h(\zeta)\, d\zeta, \tag{3.1}$$

where $d\zeta$ is the differential of complex variable ζ on Γ, $h \in H^1(D)$, and $h(\zeta)$ means the angular limit function of h on Γ (which always exists $d\zeta$ a.e. [Ko, p. 76]*). Conversely, for each μ on Γ and $h \in H^1(D)$ the measure σ (see (3.1)) is orthogonal to $P_1(X)$.*

Proof. By F. and M. Riesz theorem it is enough to prove that

$$\rho = \mu - \frac{1}{2i}\hat{\mu}(\zeta)\, d\zeta \perp P_1(X)$$

since then $\sigma - \rho$ (which is supported on Γ) would be orthogonal to $P_1(X)$ and also to $P_1(\Gamma)$.

Now, for each $p \in P_1$, one has

$$\int_X p(z)\, d\rho(z) = \int_K \left[p(z) - \frac{1}{2\pi i} \int_\Gamma \frac{p(\zeta)}{\zeta - z}\, d\zeta \right] d\mu(z) = 0$$

by Fubini and Cauchy theorems. □

The following result allows to find many concrete nontrivial examples of sets which satisfy (1.1) for $n \geq 2$.

Theorem 3.1. *Let Γ, K and X be as above.*

(1) *If $\Gamma \notin \mathcal{LN}$ then $P_n(X) = A_n(X)$ (the same n as for K).*
(2) *Let $\Gamma \in \mathcal{LN}$ and u, v, K_0 are taken from Definition 2.1. Then if u/v cannot be extended meromorphically from $D \setminus (\widehat{K_0 \cup K})$ to $D \setminus \hat{K}$, then $P_n(X) = A_n(X)$.*

Proof. Proceeding by contradiction suppose that $P_n(X) \neq A_n(X)$ so that there exists $\sigma \perp P_n(X)$ but $\sigma \not\perp A_n(X)$. Put $\mu = \sigma_{|K}$, so that, by lemma 3.1, one finds (3.1) for some $h \in H^1(D)$ and $\overline{z}^s\sigma \perp P_1(X)$ for each $s = 1, \cdots, n-1$, that is

$$\overline{z}^s\mu - \frac{1}{2i}\overline{\zeta}^s\hat{\mu}(\zeta)\,d\zeta + \overline{\zeta}^s h(\zeta)\,d\zeta \perp P_1(X). \tag{3.2}$$

On the other hand, by Lemma 3.1,

$$\overline{z}^s\mu - \frac{1}{2i}\hat{\mu}_s(\zeta)\,d\zeta \perp P_1(X),$$

where $\mu_s = \overline{z}^s\mu$. Therefore, for $s = 1, \cdots, n-1$:

$$\left(\frac{1}{2i}\hat{\mu_s}(\zeta) - \frac{1}{2i}\overline{\zeta}^s\hat{\mu}(\zeta) + \overline{\zeta}^s h(\zeta)\right)\,d\zeta \perp P_1(X),$$

but then these measures are also orthogonal to $P_1(\Gamma)$. Again by F. and M. Riesz theorem, one can find $h_s \in H^1(D)$ such that

$$\hat{\mu_s}(\zeta) - \overline{\zeta}^s\hat{\mu}(\zeta) + 2i\overline{\zeta}^s h(\zeta) = h_s(\zeta),\ d\zeta \text{ a. e. on } \Gamma,$$

or, which is the same,

$$\overline{\zeta}^s(2ih(\zeta) - \hat{\mu}(\zeta)) = h_s(\zeta) - \hat{\mu_s}(\zeta), \quad d\zeta \text{ a. e. on } \Gamma,\ s = 1, \cdots, n-1. \tag{3.3}$$

We claim that $2ih(z) - \hat{\mu}(z) \not\equiv 0$ in $D \setminus \hat{K}$ (and then $2ih(\zeta) - \hat{\mu}(\zeta) \neq 0$ a.e. on Γ). Indeed, otherwise, by (3.1) and (3.2) $\mu \perp P_n(X)$ and since $\operatorname{spt}\mu \subset K$ we have $\mu \perp P_n(K) = A_n(K)$, $\mu \neq 0$. But then, clearly, $\mu \perp A_n(X)$ which is the desired contradiction.

Thus, we have angularly a. e. on Γ for $s = 1, \cdots, n-1$:

$$\overline{\zeta}^s = (h_s(\zeta) - \hat{\mu_s}(\zeta))/(2ih(\zeta) - \hat{\mu}(\zeta)),\ s = 1, \cdots, n-1,$$

As before we can find bounded holomorphic in D functions u_0, u_1, v_0 and v_1 such that $h_1 = u_1/v_1$ and $h = u_0/v_0$, so that on Γ we get

$$\overline{\zeta} = \frac{(u_1(\zeta) - v_1(\zeta)\hat{\mu_1}(\zeta))v_0(\zeta)}{(2iu_0(\zeta) - v_0(\zeta)\hat{\mu}(\zeta))v_1(\zeta)}, \tag{3.4}$$

where the numerator and the denominator of (3.4) are holomorphic and bounded near Γ in D. This means $\Gamma \in \mathcal{LN}$ which contradicts to our assumptions in (1).

To prove (2) we use (3.4) and Luzin-Privalov theorem which give that u/v should coincide with the right hand side of (3.4) in $D \setminus (\widehat{K_0 \cup K})$ and then can be extended by (3.4) meromorphically on $D \setminus \hat{K}$. $\square$

Remark 3.1 Theorem 3.1 can be generalized to nonrectifiable contours Γ. In fact, we should "send the picture from $\overline{D}$ to $\overline{D_1}$ by k^{-1}, and instead of multiplication on $\overline{z}^s$ in (3.2) use the multiplication on $\overline{k(w)}^s$. Then, instead of (3.3) we shall have

$$\overline{k(t)}^s(2ih(t) - \hat{\mu_0}(t)) = h_s(t) - \widehat{(\overline{k}^s \mu_0)}(t),\ dt \text{ a. e. on } \Gamma_1,$$

where $\mu_0 = k^{-1}(\mu)$ and $h,\ h_1 \in H^1(D_1)$.

Example 3.1 Let S be the Schwarz function (2.4) of the ellipse (1.2), that is $\Gamma = \Gamma_{ab}$. Since the complete Weierstrass analytic extension of S from $\mathbb{C} \setminus [-c, c]$ has two branch points $\pm c$ of second order, then by theorem 3.1 (2), for each $n \geq 2$ one gets $P_n(K \cup \Gamma) = A_n(K \cup \Gamma)$ for any compact set K in D (satisfying (1.1)), such that $\hat{K}$ does not connect c and $-c$ (or simply $\hat{K}$ does not contain one of them).

It is interesting also to find an opposite example for the same Γ. Let γ be any simple analytic arc joining c and $-c$, with the property that the Schwarz function φ of γ (that is $\varphi(z) = \overline{z}$ on γ) is holomorphic in some neighborhood of $\overline{D}$, for example $\gamma = [-c, c], \varphi(z) = z$. We shall prove that if $K = \gamma$, then $P_2(X) \neq C(X)$ but $P_n(X) = C(X)$ for all $n \geq 3$. To see this we recall that, by Lemma 3.1, for each measure μ on γ and for each $h \in H^1(D)$ one has

$$\sigma = (\mu - \frac{1}{2i}\hat{\mu}(\zeta)\, d\zeta + h(\zeta)\, d\zeta) \perp P_1(X).$$

Since $\varphi \in P_1(\overline{D}) \subset P_1(X)$ then $\varphi^s \sigma \perp P_1(X), s = 1, \cdots, n-1$. Then $\sigma \perp P_n(X)$ if and only if $(\overline{z}^s \sigma - \varphi^s \sigma) \perp P_1(X)$ or, equivalently, (since $\overline{z} = \varphi(z)$ on γ):

$$((\overline{\zeta}^s - \varphi^s(\zeta))\hat{\mu}(\zeta)\, d\zeta - 2i(\overline{\zeta}^s - \varphi^s(\zeta))h(\zeta)\, d\zeta) \perp P_1(\Gamma), s = 1, \cdots, n-1.$$

Since $\overline{\zeta} = S(\zeta)$ on Γ we just need to find $\mu \neq 0$ on γ, $h \in H^1(D), h_s \in H^1(D)$ such that

$$(S^s(z) - \varphi^s(z))\hat{\mu}(z) - 2i(S^s(z) - \varphi^s(z))h(z) = h_s(z),\ s = 1, \cdots, n-1, \quad (3.5)$$

in $D \setminus \gamma$, (note that here S is defined and holomorphic on $\mathbb{C} \setminus \gamma$).

If $n \geq 3$ and $\mu \neq 0$ then (as above) we can assume that $2ih(z) - \hat{\mu}(z) \not\equiv 0$ in $D \setminus \gamma$ and so $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$ in D. But from (3.5) one sees that $h_2(z) = h_1(z)(S(z) + \varphi(z))$, which is impossible. Therefore, $P_n(X) = C(X)$, $n \geq 3$. Let now $n = 2$. One can easily check that $S(z) \neq \overline{z}$ in D, including the boundary values of S on γ, which means that (since $\varphi(z) = \overline{z}$ on γ) the function $1/(S(z) - \varphi(z))$ is bounded near γ, so that $S(z) - \varphi(z)$ has just a finite number of zeros in $W \setminus \gamma$ (where W is some neighborhood of $\overline{D}$, where also φ is holomorphic).

Now we choose h_1 to be some polynomial with the same zeros, and orders, as $S - \varphi$ in $W \setminus \gamma$. So we would have

$$\hat{\mu}(z) = 2ih(z) + \frac{h_1(z)}{S(z) - \varphi(z)}.$$

Let $\chi \in C_0^\infty(W)$ be such that $\chi = 1$ in some neighborhood W_1 of $\overline{D}$, $W_1 \subset W$, and define

$$\mu = \chi \overline{\partial} \left(\frac{h_1}{S - \varphi} \right) \not\equiv 0,$$

in the distributional sense. From Cauchy formula and the well known weak-*- limit construction of appropriate measures one can prove that μ is a measure on γ, absolutely continuous with respect to arc length on γ.

Finally, set

$$h = \frac{\hat{\mu}}{2i} - \frac{h_1}{2i(S - \varphi)}, \text{ in } W_1.$$

Then $h \in A_1(\overline{D}) \subset H^1(D)$. Since $P_2(\gamma) = C(\gamma)$ one can find a polynomial $p \in P_2$ with $\int p \, d\mu \neq 0$. Now take a function $f = p$ on K and $f(\zeta) = 0, \zeta \in \Gamma$. Then $f \in C(X)$, but $\int f \, d\sigma = \int f \, d\mu \neq 0$. By the construction $\sigma \perp P_2(X)$, therefore $C(X) \neq P_2(X)$.

Example 3.2 In this example we show that the sufficient conditions on X to satisfy (1.1), given in Theorem 3.1 are not necessary. Concretely, (1.1) can hold for $X = K \cup \Gamma$ when Γ is analytic and its Schwarz function S is holomorphic in $D \setminus \hat{K}, K = \hat{K}, K^\circ = \emptyset$.

Let Γ be the image of the unit circle under the mapping $z = \exp w$. For Γ one has $S(z) = \exp(1/\log z)$, where $\log z$ is the principal branch of the logarithm defined in the right half-plane. Since S has just one essential singular point 1 in D, then $P_2(\Gamma) = C(\Gamma)$. We claim that if $K = \hat{K}$ has planar Lebesgue measure zero, then $P_2(X) = C(X)$. By theorem 3.1 we need to check only the case $1 \in K$. Suppose that $P_2(X) \neq C(X)$. Then by (3.3), we can find a measure $\mu \neq 0$ on K, h and h_1 in $H^1(D)$ such that

$$\widehat{(\overline{z}\mu)} - S\hat{\mu} = h_1 - 2iSh \tag{3.6}$$

in $D \setminus K$. Since K has planar measure zero, then (3.6) hold in $D \setminus \{1\}$ in the distributional sense. Applying the $\overline{\partial}$ operator to (3.6), we obtain

$$\overline{z}\mu - S(z)\mu = 0,\, z \in D \setminus \{1\}.$$

But $\overline{z} \neq S(z)$ in D, so that spt$\mu = \{1\}$. Whence

$$\hat{\mu}(z) = \widehat{(\overline{z}\mu)} = \text{const}/(z-1),$$

which contradicts to (3.6), because S has at 1 an essential singularity. One sees that the same result holds if S has just finite number of singular points in D, ($\Gamma \notin \mathcal{N}$) and a finite number of points z which satisfy $\overline{z} = S(z)$.

4. Bianalytic polynomial approximation and $\overline{\partial}^2$-Dirichlet problem

Let Ω be any bounded domain in $\mathbb{C}$ and denote by T its boundary. Define

$$S_2(T) = \{g \in C(T) :\ \text{there exists}\ \ G \in A_2(\overline{\Omega}), G|_T = g\}.$$

In usual terminology $S_2(T)$ is the set of functions $g \in C(T)$ for which the $\overline{\partial}^2$-Dirichlet problem is solvable in the classical sense. One says that Ω is $\overline{\partial}^2$-regular if $S_2(T) = C(T)$, and $\overline{\partial}^2$-irregular otherwise. In fact we have:

Problem 4.1. *Does there exist at least one $\overline{\partial}^2$-regular domain in $\mathbb{C}$?*

The following proposition points in the direction that the last problem seems to have a negative answer.

Proposition 4.1. *Let Γ be a contour which contains some analytic arc γ, then D is $\overline{\partial}^2$-irregular.*

Proof. Let S be the Schwarz function for γ, that is S is holomorphic in some neighborhood U of γ and $\overline{z} = S(z)$ on γ. We additionally require that $U \cap D$ is connected. Take $a \in U \cap D$. We claim that $g = 1/(z-a)|_\Gamma \notin S_2(\Gamma)$. Suppose, by contradiction, that there exists $G \in A_2(\overline{D}), G|_T = g$. Then $G(z) = G_0(z) + \overline{z}G_1(z)$, where G_0 and G_1 are holomorphic in D. By [Ca, Lemma 3] one has

$$|G_1(z)| \leq A\frac{\omega(G, \text{dist }(z,\Gamma))}{\text{dist }(z,\Gamma)} \tag{4.1}$$

for all $z \in D$, where $\omega(G,\cdot)$ is the modulus of continuity of G on $\overline{D}$ and A is an absolute constant. Consider the coherent function $G_\Gamma(z) = G_0(z) + S(z)G_1(z)$ (see [Ba, p. 53]) holomorphic in $D \cap U$. By (4.1) we get

$$G(z) - G_\Gamma(z) = G_1(z)(\overline{z} - S(z)) \to 0,\ \text{ uniformly as } z \to \gamma_0,\, z \in D \cap U,$$

where γ_0 is some nondegenerate analytic subarc of γ. Hence G_Γ extends continuously on $D \cup \gamma_0$, $G_\Gamma = g$ on γ_0, so that $G_\Gamma(z) = 1/(z-a)$ in $D \cap U$, which is impossible. $\square$

Corollary 4.1. *(1) For each contour* Γ *one has*

$$S_2(\Gamma) \subset P_2(\Gamma) \subset C(\Gamma);$$

(2) for each $\Gamma \notin \mathcal{N}$ *which contains some analytic subarc*

$$S_2(\Gamma) \neq P_2(\Gamma) = C(\Gamma). \tag{4.2}$$

Proof. To prove (1) use Theorem 1.1; for (2) take Γ from Example 3.1 and use Theorem 2.1 and Proposition 4.1. $\square$

It is worth to compare (4.2) with the corresponding situation for harmonic functions in the scale of uniform and Lipschitz norms. Let Γ be any contour in $\mathbb{C}$. Then for each $q \in [0, 1/2)$ any function $f \in \mathrm{lip}_q(\Gamma)$ (where $\mathrm{lip}_0(\Gamma) = C(\Gamma)$) can be uniquely extended to a function $F \in \mathrm{lip}_q(\overline{D})$, which is harmonic in D (see [Jo]) and f is approximable in lip_q-norm by harmonic polynomials ([Pa], Corollary 1.2). When $q \in [1/2, 1)$ there exist a contour Γ and $f \in \mathrm{lip}_q(\Gamma)$ such that f is lip_q-approximable on Γ by harmonic polynomials, but the corresponding solution F for the Dirichlet problem is not in $\mathrm{Lip}_q(\Gamma)$ ([Pa], Proposition 3.5). This means that the situation, analogous to (4.2), for harmonic functions starts from the order of smoothness $q = 1/2$.

The following result tell us what are the spaces $S_2(\Gamma)$ and $P_2(\Gamma)$ in the particular case when Γ is an analytic Nevanlinna contour.

Proposition 4.2. *Let* Γ *be an analytic Nevanlinna contour and let* $S = u/v$ *be its Schwarz function (we can require that* u, v *are holomorphic in open set* U *containing* $\overline{D}$ *and* u, v *have no common zeros in* U*). Then*

$$S_2(\Gamma) = P_2(\Gamma) = \{f \in C(\Gamma) : P[fv] \in A_1(\overline{D})\}, \tag{4.3}$$

where P *is the Poisson operator of the domain* D.

Proof. Let us denote by $B(\Gamma)$ the right hand side set in (4.3). By Corollary 4.1 it is enough to show that $P_2(\Gamma) \subset B(\Gamma)$ and $B(\Gamma) \subset S_2(\Gamma)$. Let $f \in P_2(\Gamma)$, then there exist sequences of polynomials $\{P_n\}$ and $\{Q_n\}$ such that $f = \lim_{n\to\infty}(P_n + \overline{z}Q_n)$ uniformly on Γ. This implies that $fv = \lim_{n\to\infty}(P_n v + uQ_n)$ uniformly on Γ and so, by the maximum modulus principle, $\{P_n v + uQ_n\}$ converges uniformly on $\overline{D}$ to a function $F \in A_1(\overline{D})$. Then $P[fv] = F$.

Next we prove the other inclusion. By [Ru, p. 329] there exist a and b holomorphic in U such that $1 = av + bu$. This implies that for each $g \in A_1(\overline{D})$ one has $g = g_1 v + g_2 u$, with $g_1 = ga \in A_1(\overline{D})$ and $g_2 = gb \in A_1(\overline{D})$. Take now $f \in B(\Gamma)$ and consider $g = P[fv]$. The previous arguments tell us that

$$P[fv] = g_1 v + g_2 u \text{ on } \overline{D},$$

so $f = g_1 + g_2(u/v) = g_1 + g_2\overline{z}$ on Γ with $g_1, g_2 \in A_1(\overline{D})$. Consequently, $f \in S_2(\Gamma)$. This ends the proof of (4.3). $\square$

The following theorem shows that there is also no equivalence between approximability condition (1.1) for $X = \Gamma$ and the uniqueness property for the $\overline{\partial}^2$-Dirichlet problem in D.

Theorem 4.1. *(1) Let $\Gamma \in \mathcal{N}$, then there exists a function $f \in A_2(\overline{D})$, $f|_\Gamma = 0$, but $f \not\equiv 0$ in D;*

(2) there exists a rectifiable contour $\Gamma \notin \mathcal{LN}$ with the same property on f as in (1); this contour can be even analytic except one point.

REMARK 4.1 By [Ba, Theorem 3.8] each analytic contour $\Gamma \notin \mathcal{N}$ has the uniqueness property, that is $f \in A_2(\overline{D})$ and $f|_\Gamma = 0$ implies $f \equiv 0$ in D.

Proof of Theorem 4.1 (1). Let $\Gamma \in \mathcal{N}$. Like in the proof of Theorem 2.1 we can find a nonzero measure μ on Γ such that $\mu \perp P_2(\Gamma)$ and μ has no atoms. Take (see [TW])

$$\breve{\mu}(z) = \frac{1}{\pi}\int_\Gamma \frac{\overline{\zeta} - \overline{z}}{\zeta - z}\, d\mu(\zeta).$$

Since the kernel $(\overline{\zeta} - \overline{z})/(\zeta - z)$ is bounded and μ has no atoms, then $\breve{\mu}$ is everywhere defined and continuous. Fix $z \notin \overline{D}$, then Theorem 1.1 tell us that $\breve{\mu}(z) = 0$ so $\breve{\mu}$ vanishes outside D. But $\breve{\mu} \not\equiv 0$, because $\overline{\partial}^2 \breve{\mu} = \mu$ in the distributional sense (the function $\overline{z}/\pi z$ is the fundamental solution for the operator $\overline{\partial}^2$), and (1) is proved.

Proof of Proposition 2.2. Let $\Gamma \in \mathcal{N}$ be rectifiable and u, v are taken from Definition 1.1. By Cauchy formula, which holds for functions form $H^\infty(D)$, one has

$$u(z) - \overline{z}v(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\zeta} - \overline{z}}{\zeta - z} v(\zeta)\, d\zeta,\ z \in D,$$

so that, as just above, it is enough to note that the measure $v\, d\zeta$ is finite and contains no atoms.

Proof of Theorem 4.1 (2). Here we follow an idea of M. Mazalov [Ma], modifying his construction of a nowhere analytic Nevanlinna contour.

Take a sequence of different points $\{a_n\}$ such that $2/3 < |a_n| < 1$ and $|a_n| \nearrow 1$ as $n \to +\infty$ and a sequence of positive numbers $\{r_n\}$, which (both) will be specified later. Now we just need that all disks $B_n = B(a_n, r_n)$ (centered at a_n with radius r_n), $n \geq 1$, are pairwise disjoint and $r_n < d_n/2$, where $d_n = 1 - |a_n|$. In particular $\cup_{n=1}^{\infty} B_n \subset \{w : 1/2 < |w| < 1\}$.

We choose a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, such that

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{r_n} \leq 1, \tag{4.4}$$

which guarantees that the function

$$M(w) = \frac{1}{w} + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{w - a_n} \tag{4.5}$$

is meromorphic in D_1, holomorphic in $\overline{\mathbb{C}} \setminus \overline{D_1}$, and continuous on $\overline{\mathbb{C}} \setminus D_1$. Moreover,

$$|M(w)| \leq \frac{1}{|w|} + 1 \quad \text{if} \quad w \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_n \ \text{ and}$$
$$|M(w)| \leq 3 + \frac{\varepsilon_j}{|w - a_j|} \quad \text{if} \quad w \in B_j,\, j = 1, 2, \cdots$$

Define $k(w) = M_*(1/w)$ and let us estimate $|M(w) - \overline{k(w)}|$ if $1/2 \leq |w| < 1$ and $w \notin \cup_{n=1}^{\infty} B_n$:

$$\begin{aligned} |M(w) - \overline{k(w)}| &\leq |\frac{1}{w} - \overline{w}| + \sum_{n=1}^{\infty} \frac{\varepsilon_n(1 - |w|^2)}{|w - a_n||1 - a_n\overline{w}|} \\ &\leq 3(1 - |w|) + \sum_{n=1}^{\infty} \frac{2\varepsilon_n(1 - |w|)}{r_n d_n} \leq \\ &\leq 5(1 - |w|), \end{aligned} \tag{4.6}$$

as soon as we additionally require that

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{r_n d_n} \leq 1, \tag{4.7}$$

which implies (4.4). Since

$$k(w) = w - \sum_{n=1}^{\infty} \frac{\varepsilon_n w}{\overline{a}_n (w - a_n^*)}, \quad a_n^* = \frac{1}{\overline{a}_n},$$

then

$$k'(w) = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\overline{a}_n^2 (w - a_n^*)^2}.$$

From (4.7) it follows that the last derivative really exists in D_1, and k gives a conformal mapping from D_1 onto some Jordan domain with smooth boundary, since for $|w| \leq 1$ one gets

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{|a_n|^2 |w - a_n^*|^2} \leq \sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_n^2} \leq 1/2.$$

Now we need the following.

Lemma 4.1. *There exists a holomorphic function H on D_1, $H \not\equiv 0$, with the properties:*

$$|H(w)|(1 - |w|) = \eta(w) \to 0 \quad \text{as } |w| \to 1, \tag{4.8}$$

its zero set $\{a_n\}_{n=0}^{\infty}$ ($a_0 = 0$, $2/3 < |a_n| < 1$ for $n \geq 1$) does not satisfy the Blaschke condition, that is

$$\sum_{n=1}^{\infty} (1 - |a_n|) = +\infty, \tag{4.9}$$

and, moreover, $\lim_{n\to\infty} a_n = 1$.

Let us continue and finish the proof of Theorem 4.1, Lemma 4.1 will be proven thereafter.

Now we fix H and $\{a_n\}_{n=0}^{\infty}$ from Lemma 4.1. If $\lambda_n = \lim\limits_{z \to a_n} \frac{H(w)}{w - a_n}$, then we can find $\rho_n > 0$ such that for $|w - a_n| \leq \rho_n$ one has

$$|H(w)| \leq (|\lambda_n| + 1)|w - a_n| \leq \frac{1}{n},\ n \geq 1.$$

We already can fix $\{r_n\}$, which only must satisfy conditions given above (just before (4.4)) and, additionally, $r_n \leq \rho_n$, $n \geq 1$. Consider in D_1 the function

$$F(w) = H(w)(M(w) - \overline{k(w)}),$$

where ε_n in (4.5) are chosen with the condition (4.7).

For $w \in B_n$ one finds

$$\begin{aligned}|F(w)| &\le |H(w)||M(w)| + |H(w)||k(w)| \\ &\le |H(w)|(3 + \frac{\varepsilon_n}{|w - a_n|}) + 2|H(w)| \\ &\le \frac{5}{n} + \varepsilon_n(|\lambda_n| + 1).\end{aligned}$$

The final property on ε_n, additional to (4.7), is $(|\lambda_n| + 1)\varepsilon_n \le 1/n$, $n \ge 1$, from which we obtain

$$|F(w)| \le \frac{6}{n},\ w \in B_n.$$

But in $D_1 \setminus \cup_{n=1}^{\infty} B_n$, by (4.6) and (4.8) (also since $H(0) = 0$), we get

$$|F(w)| \le C(1 - |w|)\eta(w)/(1 - |w|) = C\eta(w) \to 0, \text{ as } |w| \to 1.$$

Hence F extends continuously on $\overline{D_1}$, $F|_{\Gamma_1} = 0$ but $F \not\equiv 0$ in D_1. Now define $\Gamma = k(\Gamma_1), D = D(\Gamma)$. The desired function $f \in A_2(\overline{D})$ is

$$f(z) = F(k^{-1}(z)) = H(k^{-1}(z))(M(k^{-1}(z)) - \overline{z}).$$

Let $\Gamma_1' = \Gamma_1 \setminus \{1\}$ and $\Gamma' = k(\Gamma_1')$. One can easily check that k is conformal really in some neighbouhood of $\overline{D_1} \setminus \{1\}$, so that Γ' is an analytic open arc. The function $M_1(z) = M(k^{-1}(z))$ is meromorphic in D, continuous on $D \cup \Gamma'$ and $M_1(\zeta) = \overline{\zeta}$ on Γ', because M is continuous on $\overline{D_1} \setminus \{1\}$ with $M = \overline{k}$ on Γ_1'.If, by contradiction, $\Gamma \in \mathcal{LN}$, u and v and K_0 are as in Definition 2.1, then by Luzin-Privalov theorem, one has $M_1 = u/v$ everywhere in $D \setminus K_0$, which means that $v(k(a_n)) = 0$ for big enough n. The last is impossible, because $v \circ k$ is bounded holomorphic in $D_1 \setminus k^{-1}(K_0)$ and then, by a localization argument, $\{a_n\}$ must satisfy Blaschke condition ([Ko, p. 92]), which contradicts (4.9).

Proof of Lemma 4.1 Consider a function

$$H_0(w) = \prod_{m=1}^{\infty}(1 + \alpha w^{\beta^m}),\ w \in D_1,$$

where $\alpha > 1$ and $\beta \ge 2$ is an integer. In ([Ho, p. 699], case (4.9)) one can find sufficient conditions on α and β (for example in the sequel we take $\alpha = 2, \beta = 6$) such that H_0 belongs to the Bergman space $A^2(D_1)$, i.e. the space of all holomorphic functions f on D_1, for which $|f|^2$ is integrable with respect to the area measure over D_1.

Therefore, the set W_0, which consists of all 6^m-roots of $-1/2$ for all $m = 1, 2, \cdots$, is a zero set for $A^2(D_1)$. Take

$$W = \{w \in W_0 : \text{there exists } m \geq 1 \text{ such that } w^{6^m} = -1/2, 0 < \arg w < \frac{2\pi}{m}\}.$$

Let now $\{a_n\}$ be the sequence enumerating the point of W in such a way that $|a_n| \nearrow 1$ as $n \to +\infty$. By ([Ho], Theorem 3, see also (7.9) Theorem there) any nonempty subset of any zero set of $A^p(D_1), p > 0$, is also a zero set for $A^p(D_1)$. Therefore, there exists a function $H \in A^2(D_1)$, $H \not\equiv 0$, such that $H(a_n) = 0, n \geq 1$, and also $H(0) = 0$.

To check that (4.8) holds, we just use mean value theorem and Schwarz inequality:

$$\pi r^2 |H(w)| = \left| \int_{B(w,r)} H(z)\, dxdy \right| \leq r\pi^{1/2} \left(\int_{B(w,r)} |H(z)|^2\, dxdy \right)^{\frac{1}{2}},$$

where $r = 1 - |w|$. Whence

$$|H(w)|(1 - |w|) \leq \left(\int_{\{1-2r<|z|<1\}} |H(z)|^2\, dxdy \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad r \to 0,$$

by dominated convergence theorem.

It remains to prove that the zeros of H satisfy (4.9). In fact, if $a_n \in W$ is a 6^m-root of $-1/2$ then $1 - |a_n| = 1 - \exp(-(\log 2)/6^m)$ and this number is equivalent to $(\log 2)/6^m$, but there are around $6^m/m$ such points a_n, so (4.9) holds. Also one has $\lim_{n\to\infty} a_n = 1$. This ends the proofs of Lemma 4.1 and Theorem 4.1. □

EXAMPLE 4.1 Now we can give an example of a rectifiable contour Γ, $\Gamma \in \mathcal{N}$, such that the corresponding functions u and v from Definition 1.1 cannot be (any of them) chosen from $A_1(\overline{D})$.

Consider the proof of Theorem 4.1 (2) but now, instead of H from Lemma 4.1, we will use a Blaschke product H_1 with the property that the set of accumulation points of its zero set $\{a_n\}$ ($a_0 = 0$, $2/3 < |a_n| < 1$, $n \geq 1$, each a_n is a simple zero for H_1) is an nondegenerated interval of Γ_1. As before, we can choose $\{r_n\}$ and $\{\varepsilon_n\}$ such that the function

$$F(w) = H_1(w)(M(w) - \overline{k(w)})$$

has a continuous extension up to the boundary of the unit disk with $F|_{\Gamma_1} = 0$ and k sends conformally the unit disk onto some Jordan domain D with smooth boundary Γ. By the maximum modulus principle $H_1 M \in H^\infty(D_1)$. Since angular limits of H_1 have module 1 a.e. on Γ_1 (see [Ru, p. 335]), then $M(w)$ has a. e. angular limits on Γ, which are equal to $\overline{k(w)}$. Take $u(z) = H_1(k^{-1}(z))M(k^{-1}(z))$, $v(z) = H_1(k^{-1}(z))$, so that $\overline{\zeta} = u(\zeta)/v(\zeta)$ angularly a. e. on Γ and therefore $\Gamma \in \mathcal{N}$. Suppose, by contradiction, that we can find same bounded holomorphic functions u_0 and v_0 in D such that u_0 or v_0 are in $A_1(\overline{D})$ and $\overline{\zeta} = u_0(\zeta)/v_0(\zeta)$ a. e. on Γ. Then $uv_0 = u_0 v$ in D. Since $v(k(a_n)) = 0$, but $u(k(a_n)) \neq 0$, we have $v_0(k(a_n)) = 0, n \geq 1$, so that v_0 cannot be in $A_1(\overline{D})$. But if $u_0 \in A_1(\overline{D})$, then $v_0(\zeta) = u_0(\zeta)/\overline{\zeta}$ is continuous on Γ_1 (we have assumed that $0 \notin \Gamma$), but then v_0 must be in $A_1(\overline{D})$, which is impossible. This ends the proof.

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References

[Ba] M. B. Balk, *Polyanalytic Functions*, Mathematical Research, vol 63, Akademie Verlag, Berlin, 1991.

[Ca] J.J. Carmona, *Mergelyan approximation theorem for rational modules*, J. Approx. Theory **44** (1985), 113–126.

[Da] P. Davis, *The Schwarz Function and Its Applications*, Carus Mathematical Monographs vol 17, 1974.

[DSS] R. G. Douglas, H. S. Shapiro, A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble) **20 (1)** (1970), 37–76.

[Fe] K. Yu. Fedorovski, *Uniform n-analytic polynomial approximations of functions on rectifiable contours in* $\mathbb{C}$, Math. Notes **59 (4)** (1996), 435–439.

[Ga] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics, vol. 297, Springer-Verlag, 1972.

[Ho] Ch. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693–710.

[Jo] E. H. Johnston, *The boundary modulus of continuity of harmonic functions*, Pacif. J. Math. **90** (1980), 87–98.

[Ko] P. Koosis, *Introduction to H_p spaces*, Lecture Notes Series Mathematics, vol. 40, Cambridge Univ. Press, 1980.

[Ma] M.Ya.Mazalov, *An example of a nonconstant bianalytic function vanishing everywhere on a nowhere analytic boundary*, Math. Notes. **62 (3-4)** (1997), 524–526.

[Me] S. N. Mergelyan, *Uniform approximation to functions of a complex variable*, Uspekhi Mat. Nauk.(Russian Math. Surveys) **7 (2)** (1952), 31–122.

[Pa] P. V. Paramonov, *C^m-approximations by harmonic polynomials on compact sets in $\mathbb{R}^n$*, Russian Acad. Sci. Sbornik Math. **78 (1)** (1994), 231–251.

[Ru] W. Rudin, *Real and Complex Analysis*, Second Edition, New Delhi, Tata McGraw-Hill, 1978.

[TW] T. Trent and J. L. Wang, *Uniform aproximation by rational modules on nowhere dense sets*, Proc. Amer. Math. Soc. **81** (1981), 62–64.

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