Radial growth of solutions to Poisson equation

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1 Introduction

Let u be a C^2 function of the unit ball B^n that satisfies the Bloch condition

$$|\nabla u(x)| \leq \frac{C}{1-|x|} \tag{1}$$

in B^n. Integration of (1) along radii gives a logarithmic upper bound on the growth of $|u(x)|$. This is the best possible conclusion without additional assumptions on u as is seen by considering the Bloch function $u(x) = \log(1/(1-|x|))$. If we assume in addition that u is harmonic, then we obtain a miracuously improved upper bound on the growth along almost all radii: by the celebrated iterated law of the logarithm

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\log(1/(1-r))\log\log\log(1/(1-r))}} \leq C \tag{2}$$

for almost all $\zeta \in S^{n-1}$. This was first proven by Makarov [3] for analytic Bloch functions of the disk and later on extended to higher dimensions.
The harmonicity assumption on u is very restrictive but as the function $u(x) = \log(1/(1-|x|))$ shows, some additional condition on u is really needed. Computing the Laplacian of this function we notice that a function u satisfying the Bloch condition (1) and the estimate

$$|\Delta u(x)| \leq \frac{C}{(1-|x|)^2}$$

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can grow as a logarithm on each radius. Somewhat surprisingly we have better growth estimates on almost all radii as soon as we assume a bound on the Laplacian better than that for the logarithm. For simplicity we formulate our results only for growth conditions of a special form.

Theorem 1.1. *Let u be a C^2 function of the unit ball B^n that satisfies the Bloch condition (1) and assume in addition that*

$$|\Delta u(x)| \leq \frac{C}{(1-|x|)^2(\log(2/(1-|x|)))^\gamma}$$

for all $x \in B^n$, where $0 < \gamma \leq 1$. Then

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{(\log(1/(1-r)))^{2-\gamma}\log\log(1/(1-r))}} \leq c$$

for almost all $\zeta \in S^{n-1}$. Here c depends only on C, n.

Thus we obtain a more restrictive growth estimate on almost all radii as soon as the Laplacian is "better" than the Laplacian of $u(x) = \log(1/(1-|x|))$. When the statement of Theorem 1.1 is compared with (2) we notice that the only essential difference is the length of the iterations in the iterated logarithm: in the harmonic case we have a string of three logarithms and for solutions to the Poisson equation with the given bound on the Laplacian we have a string of two logarithms. We do not know of any examples of solutions to the Poisson equation satisfying the assumptions of Theorem 1.1 that do not satisfy (2), but we believe that such functions exist. It would be interesting to know if the conclusion of Theorem 1.1 is the best possible. We will deduce Theorem 1.1 from the following more general result that allows for growth rates different from (1) on $|\nabla u|$. Notice that the functions considered need not satisfy (1).

Theorem 1.2. *Let $\gamma \leq 1$. Suppose that u is a C^2 function of the unit ball B^n with*

$$|\nabla u(x)|^2 + |u(x)\Delta u(x)| \leq \frac{C}{(1-|x|)^2(\log(2/(1-|x|)))^\gamma}$$

for all $x \in B^n$. Then, for almost all $\zeta \in S^{n-1}$,

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{(\log(1/(1-r)))^{1-\gamma}\log\log(1/(1-r))}} \leq c$$

if $\gamma < 1$ and

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\log\log(1/(1-r))} \leq c$$

when $\gamma = 1$. Here c depends only on n, C, γ.

The reader familiar with the work of Makarov on analytic Bloch functions may wonder if Theorem 1.2 allows for an improvement on the size of the set where the control on the growth of $|u(x)|$ is obtained. This is indeed the case. However, our estimate on the size of the exceptional set of radii appears not to be sharp. Thus we state it only in the Bloch setting and for a single growth order on $|u(x)\Delta u(x)|$.

Theorem 1.3. *Let u be a C^2 function of the unit ball B^n that satisfies the Bloch condition (1) and assume in addition that*

$$|\Delta u(x)| \leq \frac{C}{(1-|x|)^2 \log(2/(1-|x|))}$$

for all $x \in B^n$. Then, given $1/2 < s < 1$,

$$\limsup_{r\to 1} |u(r\zeta)| \log(1/(1-r))^{-s} = 0$$

for H_h-a.e. $\zeta \in S^{n-1}$ for the gauge function

$$h(t) = t^{n-1} \exp(c(\log \frac{1}{t})^{2s-1})$$

where c is a constant.

Yet another possible improvement on the growth estimates would be to allow for larger approach regions than simply the radii. It is immediate from the assumptions of Theorem 1.2 that one, in fact, has the indicated growth (with a larger value of c) in Stolz cones of fixed opening angle. One can then inquire if the nontangential approach can be replaced with some kind of tangential approach, that is, do we for example have

$$\frac{|u(x)|}{\sqrt{(\log(1/(1-|x|)))^{2-\gamma} \log\log(1/(1-|x|))}} \leq c'$$

for almost all $\zeta \in S^{n-1}$ and all $x \in \Omega_\zeta$, where Ω_ζ is tangential of fixed order at ζ in the setting of Theorem 1.1. Integrating the Bloch condition along hyperbolic geodesics and using Theorem 1.1 we obtain this estimate with $c' = 2c$ for

$$\Omega_\zeta = \{x \in B^n : |x-\zeta| \leq (1-|x|)\exp(\frac{c}{C}\sqrt{\phi(1-|x|)})\} \cap B(\zeta, r)$$

where $r > 0$ is sufficiently small (depending on ζ) and

$$\phi(1-|x|) = (\log(1/(1-|x|)))^{2-\gamma} \log\log(1/(1-|x).$$

In the general situation of Theorem 1.2 one cannot use this argument. However, the proof of Theorem 1.2 can be modified so as to give certain tangential approach regions. It would be interesting to know the allowable shapes for tangential approach.

The idea behind the proof of Theorem 1.2 is the following. One first differentiates the integral means of u^2 and uses the divergence theorem to tranfer the resulting surface integral to a volume integral. Next one integrates the derivative of the integral means and uses the growth condition on $|\nabla u(x)|^2 + |u(x)\Delta u(x)|$. This results in a growth rate on the integral means of u^2 that is better than the one obtained simply by estimating the growth of u^2 by integrating the upper bound on $|\nabla u|$. The estimates so far are essentially sharp. By induction one then obtains good estimates on the growth of the integral means of large powers of u that lead to upper bounds on the growth of u.
Theorem 1.3 is obtained as a consequence of the exponential integrability of a scaled version of u^2 that follows from the growth estimates on the integral means of powers of u.
These ideas are distilled from the earlier works by Makarov [3], Clunie and MacGregor [1], Korenblum [2], and Pommerenke [4].
Notice that in the case of harmonic functions, (2) admits an elegant proof based on martingales. We do not know if such an approach is applicable in the setting of Theorem 1.2.

2 Growth of the integral means

We begin with a basic result on the growth of the integral means.

Theorem 2.1. *Let u be a C^2 function in B^n with $u(0) = 0$. If*

$$|\nabla u(x)| \leq \frac{c}{1-|x|}$$

in B^n, then

$$\int_{S^{n-1}} |u(r\zeta)|^2 \, d\sigma \leq C \left(\log \frac{1}{1-r} + \int_{B(0,r)} |u(x)||\Delta u(x)|(1-|x|)\, dx \right)$$

for all $\frac{1}{2} < r < 1$, where $C = C(n, c)$.

Before giving the proof of Theorem 2.1 let us briefly consider the function $u_\alpha(x) = (\log \frac{1}{1-|x|})^\alpha$, where $0 < \alpha \leq 1$. Then

$$|\Delta u_\alpha(x)| \leq c(1-|x|)^{-2}(\log \frac{1}{1-|x|})^{\alpha-1}$$

and $|\nabla u_\alpha(x)| \leq \frac{c}{1-|x|}$. A computation gives

$$\int_{B(0,r)} |u_\alpha(x)||\Delta u_\alpha(x)|(1-|x|)\,dx \leq c(\log\frac{1}{1-|x|})^{2\alpha}.$$

On the other hand,

$$\int_{S^{n-1}} |u_\alpha(r\zeta)|^2\ d\sigma \approx (\log\frac{1}{1-r})^{2\alpha}.$$

Thus the conclusion in Theorem 2.1 is sharp for this function when $\frac{1}{2} \leq \alpha \leq 1$.

Proof. Differentiating the integral means of u^2 and applying the divergence theorem we obtain

$$\begin{aligned}\frac{d}{dt}\int_{S^{n-1}} u^2(t\zeta)\,d\sigma &= 2\int_{S^{n-1}} u(t\zeta)\frac{\partial u}{\partial n}(t\zeta)\,d\sigma \\ &= 2t^{1-n}\int_{S^{n-1}(0,t)} u(w) < \nabla u(w), n > \ d\sigma \\ &= 2t^{1-n}\int_{B(0,t)} |\nabla u(x)|^2 + u(x)\Delta u(x)\,dx.\end{aligned}$$

Fix a radius r with $\frac{1}{2} < r < 1$. Integrating the above estimate with respect to t and using the Fubini theorem we compute

$$\begin{aligned}&\int_{S^{n-1}} u^2(r\zeta)\,d\sigma - \int_{S^{n-1}} u^2(\frac{1}{2}\zeta)\,d\sigma = \\ &= \int_{\frac{1}{2}}^{r} \frac{d}{dt}\int_{S^{n-1}} u^2(t\zeta)\,d\sigma\,dt \\ &\leq \int_{\frac{1}{2}}^{r} 2t^{1-n}\int_{B(0,t)} |\nabla u(x)|^2 + u(x)\Delta u(x)\,dx\,dt \\ &\leq 2^n\int_0^r\int_{B(0,t)} |\nabla u(x)|^2 + |u(x)||\Delta u(x)|\,dx\,dt \\ &\leq 2^n\int_{B(0,r)} (|\nabla u(x)|^2 + |u(x)||\Delta u(x)|)(1-|x|)\,dx \\ &\leq C(\log\frac{1}{1-|x|} + \int_{B(0,r)} |u(x)||\Delta u(x)|)(1-|x|)\,dx),\end{aligned}$$

and the claim easily follows.

Theorem 2.2. *Let u be a C^2 function on B^n so that $u(0) = 0$ and*

$$A(x) := |\nabla u(x)|^2 + |u(x)||\Delta u(x)| \leq c(1-|x|)^{-2}(1+\log\frac{1}{1-|x|})^{-\gamma} \qquad (3)$$

in B^n for some $\gamma < 1$. Then, for all $0 < r < 1$ and each positive integer k,

$$\int_{S^{n-1}} |u(r\zeta)|^{2k}\ d\sigma \leq w_{n-1} C^k k! (1+\log\frac{1}{1-r})^{k(1-\gamma)},$$

where $C = \frac{4c}{1-\gamma}$.

Proof. We will use an induction argument. Let first $k = 1$. As in the proof of Theorem 2.1

$$\int_0^r \frac{d}{dt}\int_{S^{n-1}} u^2(t\zeta)\, d\sigma\, dt \leq \int_0^r 2t^{1-n}\int_{B(0,t)} A(x)\, dx\, dt,$$

and thus

$$\int_{S^{n-1}} u^2(r\zeta) d\sigma \leq \int_0^r 2t^{1-n}\int_0^t \rho^{n-1}\int_{S^{n-1}} A(\rho\zeta)\, d\sigma(\zeta)\, d\rho dt.$$

Since $\rho < t$, the Fubini theorem gives

$$\begin{aligned}
\int_{S^{n-1}} u^2(r\zeta) d\sigma &\leq 2cw_{n-1}\int_0^r\int_0^t (1-\rho)^{-2}(1+\log\frac{1}{1-\rho})^{-\gamma}\, d\rho\, dt \\
&\leq 2cw_{n-1}\int_0^r (1-\rho)^{-1}(1+\log\frac{1}{1-\rho})^{-\gamma}\, d\rho \\
&\leq 2cw_{n-1}(1-\gamma)^{-1}(1+\log\frac{1}{1-r})^{1-\gamma}.
\end{aligned}$$

Thus the desired estimate holds for $k = 1$.
Assume now that it holds for a fixed k. Using the divergence theorem as in the proof of Theorem 2.1 we compute

$$\begin{aligned}
&\frac{d}{dt}\int_{S^{n-1}} u^{2k+2}(t\zeta)\ d\sigma = \\
&\quad = (2k+2)t^{1-n}\int_{B(0,t)} (2k+1)u^{2k}|\nabla u(x)|^2 + u^{2k+1}\Delta u(x)\, dx \\
&\quad \leq (2k+2)(2k+1)t^{1-n}\int_{B(0,t)} |u(x)|^{2k} A(x)\, dx.
\end{aligned}$$

We integrate both sides of this inequality from 0 to r, apply the induction hypothesis and the Fubini theorem:

$$\begin{aligned}
&\int_{S^{n-1}} u^{2k+2}(t\zeta)\, d\sigma \leq \\
&\quad \leq 4(k+1)^2 \int_0^r t^{1-n} \int_0^t \rho^{n-1} \int_{S^{n-1}} |u(\rho\zeta)|^{2k} A(\rho\zeta)\, d\sigma(\zeta) d\rho dt \\
&\quad \leq 4(k+1)^2 c w_{n-1} (4c/(1-\gamma))^k k! \int_0^r (1-\rho)^{-1} (1+\log\frac{1}{1-\rho})^{k(1-\gamma)-\gamma} d\rho \\
&\quad \leq w_{n-1} C^{k+1} (k+1)! (1+\log\frac{1}{1-\rho})^{(k+1)(1-\gamma)}.
\end{aligned}$$

We continue with a version of Theorem 2.2 for $\gamma = 1$.

Theorem 2.3. *Let u be a C^2 function on B^n so that $u(0) = 0$ and*

$$A(x) := |\nabla u(x)|^2 + |u(x)||\Delta u(x)| \leq c(1-|x|)^{-2}(1+\log\frac{1}{1-|x|})^{-1} \qquad (4)$$

in B^n. Then, for all $0 < r < 1$ and each positive integer k,

$$\int_{S^{n-1}} |u(r\zeta)|^{2k}\, d\sigma \leq w_{n-1}(4c)^k k! (\log(1+\log\frac{1}{1-r}))^k.$$

Proof. We proceed as in the proof of Theorem 2.2, using an induction argument. When $k = 1$, we have

$$\begin{aligned}
\int_{S^{n-1}} u^2(r\zeta) d\sigma &\leq 2cw_{n-1} \int_0^r (1-\rho)^{-1}(1+\log\frac{1}{1-\rho})^{-1}\, d\rho \\
&= 2cw_{n-1} \log(1+\log\frac{1}{1-r}).
\end{aligned}$$

Assuming the induction hypothesis for k, we obtain

$$\begin{aligned}
&\int_{S^{n-1}} u^{2k+2}(r\zeta) d\sigma \leq \\
&\leq 4c(k+1)^2 \int_0^r \int_0^t (1-\rho)^{-2}(1+\log\frac{1}{1-\rho})^{-1} \int_{S^{n-1}} u^{2k}(\rho\zeta)\, d\sigma(\zeta) d\rho dt \\
&\leq 4c(k+1)^2 w_{n-1}(4c)^k k! \int_0^r (1-\rho)^{-1}(1+\log\frac{1}{1-\rho})^{-1}(\log(1+\log\frac{1}{1-\rho}))^k \\
&= w_{n-1}(4c)^{k+1}(k+1)!(\log(1+\log\frac{1}{1-r}))^{k+1}.
\end{aligned}$$

For the proof of Theorem 1.2 we need versions of Theorems 2.2 and 2.3 for general exponents. Such estimates immediately follow from the estimates on the integral means in Theorems 2.2 and 2.3 by way of Hölder's inequality.

Corollary 2.4. *Let $\lambda > 0$, and let k be the least integer with $\lambda < 2k$. If u satisfies (3), then*

$$\int_{S^{n-1}} |u(r\zeta)|^\lambda \ d\sigma \leq c_n C^k k!(1+\log\frac{1}{1-r})^{\frac{\lambda}{2}(1-\gamma)},$$

and if u satisfies (4), then

$$\int_{S^{n-1}} |u(r\zeta)|^\lambda \ d\sigma \leq c_n C^k k!(\log(1+\log\frac{1}{1-r}))^{\frac{\lambda}{2}}.$$

We close this section with an estimate on the integral of $\exp(u^2)$.

Corollary 2.5. *Assume that u is as in (3). Then there exist constants λ and C_0 so that, for all $0 < r < 1$,*

$$\int_{S^{n-1}} \exp(\lambda u^2(r\zeta)(\log\frac{1}{1-r})^{\gamma-1})\, d\sigma \leq C_0,$$

when $\gamma < 1$, and

$$\int_{S^{n-1}} \exp(\lambda u^2(r\zeta)(\log\log\frac{1}{1-r})^{-1})\, d\sigma \leq C_0,$$

when $\gamma = 1$.

Proof. We only consider the case $\gamma < 1$ and leave it to the reader to modify the argument below so as to apply to the case $\gamma = 1$. It suffices to prove the indicated estimate for $r > \frac{1}{2}$. For those values of r, $\log\frac{1}{1-r} \approx 1+\log\frac{1}{1-r}$, and thus

$$\int_{S^{n-1}} |u(r\zeta)|^{2k} \ d\sigma \leq w_{n-1} C^k k!(\log\frac{1}{1-r})^{k(1-\gamma)}$$

when $\frac{1}{2} < r < 1$. Therefore, for all $k \geq 1$ and each $\frac{1}{2} < r < 1$,

$$\int_{S^{n-1}} \left(|u(r\zeta)|^2(\log\frac{1}{1-r})^{(\gamma-1)}(2C)^{-1}\right)^k \frac{1}{k!} d\sigma \leq w_{n-1}2^{-k}.$$

The claim follows by summing over k.

3 Radial growth

We establish growth estimates on almost all radii by modifying the corresponding arguments for harmonic Bloch functions of the unit disk given in [4]. In our setting, there is no maximum principle available and this forces us to give the details for the arguments; the lack of the maximum principle seems to be the reason for the appearance of $\log\log$ instead of $\log\log\log$ in the growth estimate. Theorem 1.2 could alternatively be proven by combining Corollary 2.5 and ideas from the proof of Theorem 3 in [2].

Theorem 3.1. *Let $\gamma < 1$. Suppose that $u \in C^1(B^n)$ satisfies*

$$|\nabla u(x)| \leq c(1-|x|)^{-1}(\log \frac{1}{1-|x|})^{-\gamma/2} \tag{5}$$

in B^n and that

$$\int_{S^{n-1}} |u(r\zeta)|^{2k} \; d\sigma \leq C^k k!(1+\log \frac{1}{1-r})^{k(1-\gamma)} \tag{6}$$

for all $0 < r < 1$ and all positive integers k. Then

$$\limsup_{r \to 1} \frac{|u(r\zeta)|}{\sqrt{(\log(1/(1-|x|)))^{1-\gamma} \log\log(1/(1-|x|))}} \leq c$$

for almost all $\zeta \in S^{n-1}$, where c depends only on n, C, γ.

Proof. Define a function f by setting

$$f(x) = u(x)^{2k}(\log \frac{1}{1-|x|})^{-k(1-\gamma)-\frac{1}{2}}(\log\log \frac{1}{1-|x|})^{-2}$$

when $x \in B^n$. Integrating inequality (5) we see that

$$|u(x)| \leq C_1(\log \frac{1}{1-|x|})^{1-\gamma/2}$$

in B^n. Using this inequality and inequality (5) a simple computation shows that

$$|\frac{\partial f(x)}{\partial r}| \leq$$
$$\leq C_2 k|u(x)|^{2k-1}(1-|x|)^{-1}(\log \frac{1}{1-|x|})^{-k(1-\gamma)-\frac{1}{2}-\gamma/2}(\log\log \frac{1}{1-|x|})^{-2},$$

where C_2 depends only on C_1, c. Combining this inequality with the first estimate in Corollary 2.4 that follows from (6) and using the Fubini theorem we arrive at

$$\int_{S^{n-1}} \int_{\frac{1}{2}}^{1} |\frac{\partial f(r\zeta)}{\partial r}| \, dr \, d\sigma(\zeta) \leq C_3^k k! k,$$

where C_3 depends only on n, γ, C. Hence, there exists a set $A_k \subset S^{n-1}$ so that

$$\sigma(A_k) \geq w_{n-1} - k^{-2}$$

and for all $\zeta \in A_k$,

$$|f(r\zeta) - f(\zeta/2)| \leq \int_{\frac{1}{2}}^{1} |\frac{\partial f(\rho\zeta)}{\partial \rho}| \, d\rho \leq C_3^k k! k^3.$$

Since $|f(\zeta/2)| \leq C_4^k$, we have that, for all $\zeta \in A_k$ and $\frac{1}{2} < r < 1$,

$$|f(r\zeta)| \leq C_5^k k! k^3.$$

Let

$$A = \cup_{j=1}^{\infty} \cap_{k=j}^{\infty} A_k.$$

Then $\sigma(A) = w_{n-1}$ and it suffices to show that the desired growth estimate holds for points $\zeta \in A$.
Let $\zeta \in A$. Then there exists j_0 so that $\zeta \in A_k$ for all $k \geq j_0$. Therefore, for such a ζ and all such k we have

$$u(r\zeta)^{2k} (\log \frac{1}{1-r})^{-k(1-\gamma)-\frac{1}{2}} (\log\log \frac{1}{1-r})^{-2} \leq C_5^k k! k^3 \tag{7}$$

if $r > 1/2$.
Fix $\frac{1}{2} < r_0 < 1$ such that $\log\log \frac{1}{1-r_0} > j_0$. Then, for $r_0 < r < 1$, setting k to be the integer part of $\log\log \frac{1}{1-r}$ we have from inequality (7)

$$u(r\zeta)^{2k} (\log \frac{1}{1-r})^{-k(1-\gamma)} (\log\log \frac{1}{1-r})^{-k} \leq C_5^k k^{-k} e^{(k+1)/2} (k+1)^2 k! k^3.$$

Using Stirlings formula, we obtain for all $r > r_0$,

$$|u(r\zeta)|^2 (\log \frac{1}{1-r})^{\gamma-1} (\log\log \frac{1}{1-r})^{-1} \leq c(C, n, \gamma).$$

The following result is the counterpart of Theorem 3.1 for $\gamma = 1$.

Theorem 3.2. *Suppose that $u \in C^1(B^n)$ satisfies*

$$|\nabla u(x)| \leq C(1-|x|)^{-1}(\log \frac{1}{1-|x|})^{-1/2} \tag{8}$$

in B^n and that

$$\int_{S^{n-1}} |u(r\zeta)|^{2k} \, d\sigma \leq C^k k!(\log(1+\log\frac{1}{1-r}))^k$$

for all $0<r<1$ and all positive integers k. Then

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\log\log(1/(1-r|))} \leq c$$

for almost all $\zeta \in S^{n-1}$, where c depends only on n, C.

The proof is very similar to the proof of Theorem 3.1 and we leave it to the reader to provide the details with the following hint: in this case, define f by the formula $f(x) = u(x)^{2k}(\log\log\frac{1}{1-|x|})^{-k-1}(\log\frac{1}{1-|x|})^{-1/2}$.

Proofs of Theorems 1.1 and 1.2. Theorem 1.2 follows directly from Theorems 2.2, 2.3, 3.1, and 3.2.
Finally, Theorem 1.1 is deduced from Theorem 1.2 by the following observation. Integration of the Bloch condition (1) along radii results in a logarithmic upper bound on the growth of $|u(x)|$. This together with the given upper bound on $|\Delta u(x)|$ and the Bloch condition show that the assumptions of Theorem 1.2 are satisfied for $\tilde{\gamma} = \gamma - 1$. The conclusion of Theorem 1.2 for this value of $\tilde{\gamma}$ gives the desired growth rate on $|u(r\zeta)|$.
Proof of Theorem 1.3 Let $\epsilon > 0$ and define

$$A = \{\zeta \in S^{n-1} : \limsup_{r\to 1} \frac{|u(r\zeta)|}{(\log(1/(1-r)))^s} > \epsilon\}.$$

It suffices to show that the h-Hausdorff measure of A is zero. Define A_k by setting

$$A_k = \{\zeta \in S^{n-1} : |u(r\zeta)| > \epsilon(\log\frac{1}{1-r})^s \text{ for some } 1-2^{-k} < r < 1-2^{-k-1}\}.$$

Note that $A = \cap_{j=1}^{\infty} \cup_{k\geq j} A_k$.
Let W be the family of all "dyadic cubes" in S^{n-1} of side length 2^{-k} that intersect A_k. Denote by n_k the number of such cubes. The Bloch condition (1) implies that the oscillation of u is uniformly bounded on each ball $B_x = B(x,(1-|x|)/2) \subset B^n$. Thus we deduce from Corollary 2.5 with $\gamma = 0$ that

$$n_k \exp(Kk^{2s-1})2^{-k(n-1)} \leq c_0,$$

where K and c_0 are independent of k. Define

$$h(t) = t^{n-1} \exp(c(\log \frac{1}{t})^{2s-1})(\log \frac{1}{t})^{-2},$$

where $c = K/(\log 2)^{2s-1}$. Then we have the estimate

$$H_h^\infty(A_k) \leq n_k h(2^{-k}) \leq c_0 (\log 2)^{-\delta} k^{-2}$$

on the Hausdorff h-content of A_k which implies that $H_h(A) = 0$ since

$$\sum_{k=1}^{\infty} k^{-\delta} < \infty.$$

Finally notice that

$$\lim_{t \to 0} \exp(c(\log \frac{1}{t})^{2s-1})(\log \frac{1}{t})^{-2} = \infty.$$

Thus the claim follows when we replace c with $2c$ in the definition of the gauge function h above.

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