Bifurcation of Limit Cycles from Quadratic Isochronous Centers

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Abstract

The main objective of this paper is the study of the number of limit cycles that bifurcate from the periodic orbits of the quadratic isochronous centers when we perturb them inside the class of all polynomial systems of degree n.

1 Introduction and statement of the main results

In the qualitative theory of real planar differential systems the main open problem is the determination of limit cycles. Probably the more usual way to obtain limit cycles is perturbing the periodic orbits of a center, in such a way that the center is destroyed in the perturbed system which can exhibit limit cycles.

There are several methods for studying the bifurcated limit cycles from a center. The major part of the methods are based either on the Poincaré return map, or on the Poincaré-Melnikov integral or Abelian integral which are equivalent in the plane (see for instance [1]). Recently some other methods are presented, ones based on the inverse integrating factor (see [5]), and the others based in the reduction of the problem to a one dimensional differential equation (see [12] and [14]). In general these methods are difficult to apply for studying the limit cycles that bifurcate from the periodic orbits of a center when the system is integrable but not Hamiltonian. As far as we know few papers study the non-Hamiltonian centers, see for instance [2], [3], [6], [12] and [14].

By definition a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P and Q are polynomials with real coefficients. We say that $n = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system.

In what follows systems (1) of degree 2 are called *quadratic systems*. These last thirty years quadratic systems have been intensively studied, and more than one thousand of papers have been published about them (see for instance the bibliographical survey of Reyn [22]). Many authors have studied the limit cycles which bifurcate from periodic orbits of a center for a quadratic system, see for instance, [8], [16], [20], [21], [23] and [25].

In [12] we considered three subclasses of real planar polynomial differential systems of the form

$$\dot{x} = -y + P_m(x,y), \;\; \dot{y} = x + Q_m(x,y), \tag{2}$$

where P_m and Q_m are homogeneous polynomials of degree m having a center at the origin, and we study the limit cycles that bifurcate from their periodic orbits when we perturb such subclasses inside the class of all polynomial systems (2). In this paper we want to study the perturbation of these three subclasses of centers for $m = 2$ inside the full class of polynomial systems of degree n.

In order to be more precise we define the functions

$$\begin{aligned} f(\theta) &= \cos\theta P_m(\cos\theta, \sin\theta) + \sin\theta Q_m(\cos\theta, \sin\theta), \\ g(\theta) &= \cos\theta Q_m(\cos\theta, \sin\theta) - \sin\theta P_m(\cos\theta, \sin\theta). \end{aligned}$$

We say that a system (2) belongs to the class **F** if its function $f(\theta)$ is identically zero. We note that the orbits of those systems have the radial polar coordinate r equal to a constant. We say that a system (2) belongs to the subclass **G** if its function $g(\theta)$ is identically zero and $\int_0^{2\pi} f(\theta)\, d\theta = 0$. We note that if the degree m of system (2) is even, then the condition $\int_0^{2\pi} f(\theta)\, d\theta = 0$ always is satisfied. Also we remark that $g(\theta) \equiv 0$ means that the infinity of system (2) in the Poincaré compactification is fullfiled of singular points. For more details on the Poincaré compactification, see for instance [7]. Additionally $g(\theta) \equiv 0$ implies that the angular polar coordiante θ satisfies $\dot{\theta} = 1$, and consequently the center is isochronous. We say that a system (2) belongs to the subclass **C** if the function $(n-1)f(\theta) - g'(\theta)$ is identically zero. For more details about the centers of these three classes see Proposition 2 of [12]. For $m = 2$ the classes **F**, **G** and **C** are described in the next proposition.

Proposition 1. *After a linear change of variables and rescaling of the time (if necessary) any quadratic system of the class* **F**, **G** *or* **C** *goes over to*

$$
\begin{array}{lllllll}
\dot{x} & = & -y(1+x), & \qquad & \dot{y} & = & x(1+x); \\
\dot{x} & = & -y + x^2, & \qquad & \dot{y} & = & x(1+y); \\
\dot{x} & = & -y + \frac{1}{2}x^2 - \frac{1}{2}y^2, & \qquad & \dot{y} & = & x(1+y);
\end{array}
$$

respectively. Moreover, any quadratic system having a center such that all their orbits lie on conics after a linear change of variables and rescaling of the time (if necessary) can be written as one of the above three systems.

Proposition 1 is proved in Section 2.

The three quadratic systems of Proposition 1 will be denote by F, G and C respectively. We note that the centers of systems G and C are isochronous. Using the notation of paper [18] these two centers correspond to the classes S_2 and S_1 of quadratic isochronous centers respectively.

From the classification of quadratic isochronous centers due to Loud [15], in addition to systems F and G there are two other quadratic isochronous centers, the centers S_3 and S_4 in the notation of [18]. The center of system F is isochronous if we remove the straight line of singular points $1+x=0$. So, in what follows we consider such a center also as a quadratic isochronous center. In short, for us there will be five quadratic isochronous centers, namely F, G, C, S_3 and S_4 where these last two systems are given by

$$
\begin{array}{lllllll}
\dot{x} & = & -y + \frac{1}{4}x^2, & \qquad & \dot{y} & = & x(1+y); \\
\dot{x} & = & -y + 2x^2 - \frac{1}{2}y^2, & \qquad & \dot{y} & = & x(1+y);
\end{array}
$$

respectively.

In this paper we study how many limit cycles bifurcate from the periodic orbits of the quadratic isochronous centers of type F, G, C, S_3 and S_4 when we perturb one of these systems inside the class of all polynomial systems of degree n. The perturbation of the centers G, C, S_3 and S_4 inside the class of all quadratic systems has been studied by Chicone and Jacobs [2]. We note that the results for quadratic isochronous centers may be also valid for some other kinds of isochronous centers, if these can be transformed to quadratic ones. As an example, we will study the polynomial perturbations of the following Kukles system:

$$\dot{x} = -y\,, \qquad \dot{y} = x + 3xy + x^3\,.$$

The technique that we use is a classical one. It consists in writing the non-Hamiltonian quadratic isochronous center in a Hamiltonian one multiplying the non-Hamiltonian system by an integrating factor, and then to use the method based on computing the zeros of an Abelian integral for determining the limit cycles bifurcating from periodic orbits of the center. The key point in our approach is that, by using Green's Theorem, we will compute the Abelian integral through a double integral. These double integrals for the quadratic isochronous centers of type F, G, C and S_3 are very easy to compute in comparison with the usual simple Abelian integral. The method for case S_4 is different from the previous one, here the Abelian integral is divided into two parts: the first part follows by direct calculation as the previous one, and the second part follows by using Picard-Fuchs equation. Then we combine together both parts and deduce a Ricatti equation. Thus, our main result is the following one.

Theorem 2. *The least upper bound for the number of zeros (taking into account their multiplicity) of the Abelian integral associated to system:*

(a) F *is* n *for all* $n \geq 0$.

(b) G *is* 0 *if* $n = 0, 1$; *and* n *for* $n \geq 2$.

(c) C *is* 0 *if* $n = 0$; 1 *if* $n = 1, 2, 3$; $n-2$ *for* $n \geq 4$; $[(n-1)/2]$ *if the perturbation is symmetrical with respect to the invariant line* $y = -1$.

(d) S_3 *is* n *for all* $n \geq 0$.

(e) S_4 *is* $\leq 14n + 11$.

(f) Kukles system is $\leq 2n$, *for* $n \geq 0$.

Moreover, for any of the first four isochronous quadratic centers there are perturbations such that the indicated maximum number of continuous families of limit cycles can be made to emerge from a corresponding number of arbitrarily prescribed periodic orbits within the period annulus of the isochronous center.

As usual $[x]$ denotes the integer part function of the real number x.

Statements (a), (b) together with (f) , (c), (d) and (e) of Theorem 2 are proved in Sections 3, 4, 5, 6 and 7 respectively.

In spite of our systems are non-Hamiltonian the upper bounds for the number of zeros of the Abelian integrals given in Theorem 2 depend all linearly with respect to the degree n of the polynomial perturbation, that agrees with traditional prediction and some known results for the perturbation of Hamiltonian centers, see for instants [4], [9], [11], [13], [19], [20], [21], [24].

2 Proof of Proposition 1

By Proposition 6, 7 and 8 in [12], every quadratic system of the class F, G and C can be written in the following forms respectively:

$$F: \quad \begin{aligned} \dot{x} &= -y - axy - by^2\,, \\ \dot{y} &= x + ax^2 + bxy\,; \end{aligned} \tag{3}$$

$$G: \quad \begin{aligned} \dot{x} &= -y + bx^2 + cxy\,, \\ \dot{y} &= x + bxy + cy^2\,; \end{aligned} \tag{4}$$

$$C: \quad \begin{aligned} \dot{x} &= -y + lx^2 - 2axy - ly^2\,, \\ \dot{y} &= x + ax^2 + 2lxy - ay^2\,. \end{aligned} \tag{5}$$

By the definition of class F, G, and C, under rotation of coordinate:

$$(x,y) \to (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta),$$

each class is transformed into itself, then suitable angle θ can be chosen to eliminate all the terms with parameter a in F, c in G and a in C respectively. After this by rescaling x and y, the parameter b in F, c in G and l in C can be changed to 1 and $1/2$ respectively, thus the first conclusion of Proposition 1 is proved.

The second conclusion is directly deduced from the complete classification of quadratic integrable systems with centers by I.D. Iliev [10].

3 Bifurcation of limit cycles from system F

Consider the polynomial perturbations of degree n of system F:

$$\begin{aligned} \dot{x} &= -y(1+x) + \varepsilon P(x,y)\,, \\ \dot{y} &= x(1+x) + \varepsilon Q(x,y)\,, \end{aligned} \tag{6}$$

where $P(x,y) = \sum_{i+j\le n} a_{ij}x^iy^j$, $Q(x,y) = \sum_{i+j\le n} b_{ij}x^iy^j$. For $\varepsilon = 0$, the first integral of (6) is $H = x^2 + y^2$, and the integrating factor is $\frac{1}{1+x}$. The Abelian integral of (6) is

$$M(h) = \oint_{H=1-h} \frac{P}{1+x}dy - \frac{Q}{1+x}dx\,, \qquad 0 < h \le 1\,,$$

where $H = 1-h$ $(0 < h < 1)$ corresponds to the circle $x^2 + y^2 = 1 - h$, and $H = 0$ corresponds to the center $(0,0)$. The straight line $x = -1$ is full of singular points, the integral is taken over an oval lying on a level curve $H = 1 - h$.

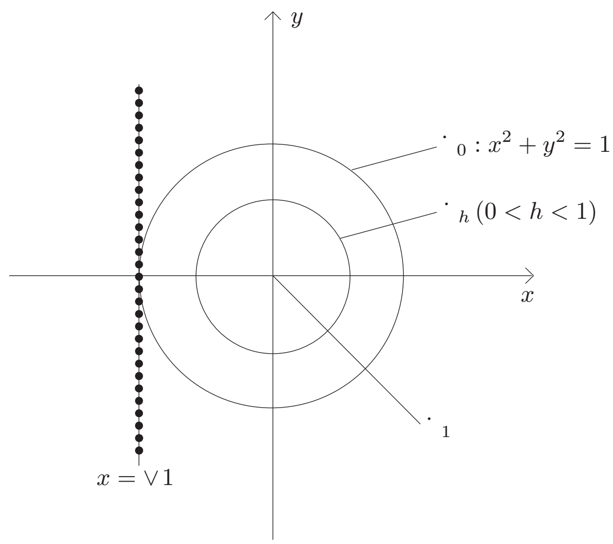

Figure 1: The phase portrait of system (6) for $\varepsilon = 0$.

Proposition 3. *For the perturbed system* (6) *the following statements hold:*

(a) The Abelian integral verifies

$$M(h) = \sum_{i=-1}^{n} \alpha_i h^{\frac{i}{2}}\,,$$

where $\sum_{i=-1}^{n} \alpha_i = 0$.

(b) The linear map

$$K : (a_{ij}, b_{ij}) \to (\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n)$$

from the coefficients of polynomials P and Q to the coefficients of Abelian integral $M(h)$ under restriction $\sum_{i=-1}^{n} \alpha_i = 0$ *is surjective.*

Proof. By Green's formula,

$$
\begin{aligned}
M(h) &= \iint_{H\le 1-h} \left[\frac{\partial}{\partial x}\left(\frac{P}{(1+x)}\right) + \frac{\partial}{\partial y}\left(\frac{Q}{(1+x)}\right)\right] dx\, dy \\
&= \iint_{H\le 1-h} \left[\frac{1}{1+x}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) - \frac{P}{(1+x)^2}\right] dy\, dx \\
&= \iint_{H\le 1-h} \frac{1}{1+x}\left(\sum_{i+j\le n} i a_{ij} x^{i-1} y^j + \sum_{i+j\le n} j b_{ij} x^i y^{j-1}\right. \\
&\qquad \left. -\frac{1}{(1+x)^2}\sum_{i+j\le n} a_{ij} x^i y^j\right) dy\, dx \\
&= \iint_{H\le 1-h} \left(\frac{1}{(1+x)^2}\sum_{i+j\le n} a_{ij}(i(1+x)x^{i-1} - x^i) y^j\right. \\
&\qquad \left. + \frac{1}{1+x}\sum_{i+j\le n} j b_{ij} x^i y^{j-1}\right) dy dx \\
&= \iint_{H\le 1-h} \left[\frac{1}{(1+x)^2}\sum_{i+2j\le n} a_{i\ 2j}(i x^{i-1} + (i-1)x^i) y^{2j}\right. \\
&\qquad \left. + \frac{1}{1+x}\sum_{i+2j+1\le n} (2j+1) b_{i\ 2j+1} x^i y^{2j}\right] dy dx \quad \text{(by symmetricity)} \\
&= \int_{-\sqrt{1-h}}^{\sqrt{1-h}} \frac{1}{(1+x)^2}\sum_{i+2j\le n} \frac{2}{2j+1} a_{i\ 2j}(i x^{i-1} + (i-1)x^i) \\
&\qquad\qquad (1-h-x^2)^j \sqrt{1-h-x^2}\, dx \\
&\qquad + \int_{-\sqrt{1-h}}^{\sqrt{1-h}} \frac{2}{1+x}\sum_{i+2j\le n-1} b_{i\ 2j+1} x^i (1-h-x^2)^j \sqrt{1-h-x^2}\, dx \\
&= \sum_{k=0}^{n} \overline{m}_k(h) \int_{-\sqrt{1-h}}^{\sqrt{1-h}} (1+x)^{k-2}\sqrt{1-h-x^2}\, dx \\
&\qquad + \sum_{k=0}^{n-1} \widetilde{m}_k(h) \int_{-\sqrt{1-h}}^{\sqrt{1-h}} (1+x)^{k-1}\sqrt{1-h-x^2}\, dx\,,
\end{aligned}
$$

where $\overline{m}_k$, $\widetilde{m}_k$ are polynomials of h with degree:

$$\deg \overline{m}_k \le \left[\frac{n-k}{2}\right], \qquad \deg \widetilde{m}_k \le \left[\frac{n-1-k}{2}\right].$$

Indeed, let

$$x^i(1-h-x^2)^j = \sum_{k=0}^{n} f_k^{ij}(h)(x+1)^k,$$

obviously $f_k^{ij} = 0$, for $k > 2j+i$. By Newton formula,

$$x^i(1-h-x^2)^j = \sum_{s=0}^{j} C_j^s (1-h)^{j-s}(-1)^s x^{2s+i}.$$

Taking derivative k times with respect to x, we get

$$\frac{d^k}{dx^k}[x^i(1-h-x^2)^j] =$$
$$= \sum_{s=\left[\frac{n-i+1}{2}\right]}^{j} C_j^s (1-h)^{j-s}(-1)^s (2s+i)\cdots(2s+i-k+1)x^{2s+i-k}.$$

Substituting $x=-1$ in both sides, we get

$$f_k^{ij}(h) = \frac{1}{k!} \sum_{s=\left[\frac{k-i+1}{2}\right]}^{j} C_j^s (1-h)^{j-s}(2s+i)\cdots(2s+i-k+1)(-1)^{s+i-k},$$

which implies

$$\begin{aligned}\deg f_k^{ij} &\le j - \left[\frac{k-i+1}{2}\right] \\ &\le j - \left(\frac{k-i+1}{2} - \frac{1}{2}\right) \\ &= \frac{2j+i-k}{2} \le \frac{n-1-k}{2},\end{aligned}$$

so

$$\deg f_k^{ij} \le \left[\frac{n-1-k}{2}\right],$$

and

$$\deg \widetilde{m}_k = \deg \left(\sum_{i+2j \le n-1} b_{i\ 2j+1} f_k^{ij}(h) \right) \le \left[\frac{n-1-k}{2} \right] .$$

Similarly, we can prove

$$\deg \overline{m}_k \le \left[\frac{n-k}{2} \right] .$$

Let

$$I_k = \int_{-\sqrt{1-h}}^{\sqrt{1-h}} (1+x)^k \sqrt{1-h-x^2}\, dx ,$$

then

$$\begin{aligned} M(h) &= \sum_{k=0}^{n} \overline{m}_k(h) I_{k-2} + \sum_{k=0}^{n-1} \widetilde{m}_k(h) I_{k-1} \\ &= \sum_{k=0}^{n} m_k(h) I_{k-2} , \end{aligned}$$

where

$$\deg m_k \le \left[\frac{n-k}{2} \right] . \tag{7}$$

For $k > 0$,

$$\begin{aligned} I_k &= \int_{-\sqrt{1-h}}^{\sqrt{1-h}} (1+x)^k \sqrt{1-h-x^2}\, dx \\ &= \int_{y_1}^{y_2} y^k \sqrt{(y_2-y)(y-y_1)} dy , \end{aligned}$$

where $1+x=y$, $y_1 = 1-\sqrt{1-h}$, $y_2 = 1+\sqrt{1-h}$.

Let $\sqrt{(y_2-y)(y-y_1)} = t(y-y_1)$, then

$$I_k(h) = 2(y_2-y_1)^2 J_k ,$$

where

$$\begin{aligned} J_k(y_1, y_2) &= \int_0^\infty \frac{t^2(y_2+y_1t^2)^k}{(1+t^2)^{3+k}} dt \quad (\text{let } t = \frac{1}{u}) \\ &= \int_0^\infty \frac{u^2(y_2u^2+y_1)^k}{(1+u^2)^{3+k}} du = J_k(y_2, y_1) . \end{aligned}$$

This means that J_k's are homogeneous symmetrical polynomials in the variables y_1, y_2 of degree k. Therefore, there exist constants $C_{0,k}$, $C_{1,k}$, $\cdots, C_{[\frac{k}{2}],k}$, such that

$$J_k = \sum_{j=0}^{[\frac{k}{2}]} C_{j,k}(y_1+y_2)^{k-2j}(y_1y_2)^j \,, \tag{8}$$

where

$$C_{[\frac{k}{2}],k} = C_k^{[\frac{k}{2}]} \int_0^\infty \frac{t^{2[\frac{k}{2}]+2}}{(1+t^2)^{3+k}}\, dt > 0 \,. \tag{9}$$

Notice that $y_1 + y_2 = 2$, $y_1 y_2 = h$, we immediately get

$$I_k = 8(1-h)\sum_{j=0}^{[\frac{k}{2}]} 2^{k-2j} C_{j,k} h^j \,, \qquad k > 0 \,.$$

It is not difficult to compute:

$$\begin{aligned} I_{-2}(h) &= (h^{-1/2} - 1)\pi \,, \\ I_{-1}(h) &= (1 - \sqrt{h})\pi \,, \\ I_0(h) &= \frac{1}{2}(1-h)\pi \,. \end{aligned} \tag{10}$$

From formulas (7) and (10), we have

$$\begin{aligned} M(h) &= m_0(h)I_{-2}(h) = \alpha_0(h^{-1/2} - 1) \,, \quad \text{for} \quad n = 0 \,, \\ M(h) &= m_0(h)I_{-2}(h) + m_1(h)I_{-1}(h) \\ &= \alpha_{-1}h^{-1/2} + \alpha_0 + \alpha_1 h^{1/2} \,, \quad \text{for} \quad n = 1 \,, \end{aligned}$$

and

$$\begin{aligned} \deg(m_k(h)I_{k-2}(h)) &= \deg m_k(h) + \deg I_{k-2}(h) \\ &\le \left[\frac{n-k}{2}\right] + \left[\frac{k-2}{2}\right] + 1 \\ &\le \left[\frac{n}{2}\right] \,, \quad \text{for} \quad k \ge 2 \,, \end{aligned}$$

therefore, for $n \geq 2$, by formula (7), we get

$$\begin{aligned} M(h) &= \sum_{k=0}^{n} m_k I_{k-2} = m_0 I_{-2} + m_1 I_{-1} + \sum_{k=2}^{n} m_k I_{k-2} \\ &= \sum_{i=-1}^{n} \alpha_i h^{i/2} \,. \end{aligned}$$

Since $I_k(1) = 0$, for any $k \geq -2$, it implies that

$$\sum_{i=-1}^{n} \alpha_i = M(1) = 0 \,.$$

Hence statement (a) of Proposition 3 is proved.

Now, we are going to prove statement (b); that is, the linear map K is surjective. We need the following result.

Lemma 4. *Let*

$$\iint_{H \leq 1-h} \frac{\partial}{\partial x} \left(\frac{y^{2m}}{1+x} \right) dx\, dy = d_{2m} h^m + \cdots$$
$$\iint_{H \leq 1-h} \frac{\partial}{\partial y} \left(\frac{y^{2m+1}}{1+x} \right) dx\, dy = d_{2m+1} h^{m+\frac{1}{2}} + \cdots$$

where the dots denote the lower order terms of h. Then $d_{2m}, d_{2m+1} \neq 0$.

Proof. First, we prove that $d_{2m} \neq 0$. Let

$$(1-h-x^2)^m = \sum_{k=0}^{2m} f_k(h)(1+x)^k \,, \tag{11}$$

we claim that $\deg f_k = m - \left[\frac{k+1}{2}\right] = l$ and $f_k(h) = a_k h^l + \cdots$, where

$$a_k = (-1)^{m-k} C_m^{\left[\frac{k+1}{2}\right]} C_{2\left[\frac{k+1}{2}\right]}^{k} \,.$$

Indeed,

$$(1-h-x^2)^m = \sum_{i=0}^{m} C_m^i (1-h)^{m-i} (-1)^i x^{2i} \,.$$

Taking derivative k times with respect to x, we get

$$\frac{d^k}{dx^k}(1-h-x^2)^m = \sum_{i=\left[\frac{k+1}{2}\right]}^{m} C_m^i (1-h)^{m-2} (-1)^{i-k} \frac{(2i)!}{(2i-k)!} x^{2i-k} \,.$$

Substituting $x = -1$, we have

$$k! f_k(h) = \sum_{i=\left[\frac{k+1}{2}\right]}^{m} C_m^i (1-h)^{m-i} (-1)^{i-k} \frac{(2i)!}{(2i-k)!} \,,$$

so deg $f_k(h) = m - \left[\frac{k+1}{2}\right]$, and the coefficient of $h^{m-\left[\frac{k+1}{2}\right]}$ is

$$a_k = C_m^{\left[\frac{k+1}{2}\right]} \cdot C^k_{2\left[\frac{k+1}{2}\right]} (-1)^{m-k} \,.$$

Also

$$\begin{aligned}
&\iint_{H \le 1-h} \frac{\partial}{\partial x} \left(\frac{y^{2m}}{1+x} \right) dx\, dy \\
&= -\iint_{H \le 1-h} \left(\frac{y^{2m}}{(1+x)^2} \right) dx\, dy \\
&= -2 \int_{-\sqrt{1-h}}^{\sqrt{1-h}} \frac{(1-h-x^2)^m}{(1+x)^2} \sqrt{1-h-x^2}\, dx \\
&= -2 \sum_{k=0}^{2m} f_k(h) I_{k-2} \,.
\end{aligned}$$

Note that for k odd, the maximum power of h in $f_k I_{k-2}$ is less than m, therefore

$$\begin{aligned}
&\iint_{H \le 1-h} \frac{\partial}{\partial x} \left(\frac{y^{2m}}{1+x} \right) dx\, dy \\
&= \left(16 \sum_{k=0}^{m} a_{2k} C_{k-1,2k-2} \right) h^m + \cdots \,.
\end{aligned}$$

Since by (9) $C_{k-1,2k-2} > 0$, and in the above expression of $f_k(h)$, a_{2k} have the same sign for all k, we obtain

$$d_{2m} = 16 \sum_{k=0}^{m} a_{2k} C_{k-1,2k-2} \neq 0 \,.$$

Next we prove $d_{2m+1} \neq 0$.

$$\iint_{H \le 1-h} \frac{\partial}{\partial y} \left(\frac{y^{2m+1}}{1+x} \right) dx\, dy$$

$$= 2\int_{-\sqrt{1-h}}^{\sqrt{1-h}} \frac{(1-h-x^2)^m}{1+x}\sqrt{1-h-x^2}\,dx$$
$$= 2\sum_{k=0}^{2m} f_k(h) I_{k-1}\,.$$

Note that the maximum power of h in $f_k I_{k-1}$ is $m+\frac{1}{2}$ and which appears only in the term $f_0 I_{-1}$, so

$$\iint_{H\le 1-h} \frac{\partial}{\partial y}\left(\frac{y^{2m+1}}{1+x}\right) dx\,dy = -2\pi(-1)^m h^{m+\frac{1}{2}} + \cdots\,,$$

which means $d_{2m+1} = -2\pi(-1)^m \neq 0$. So the lemma 4 is proved. ■

Let $p = \sum_{i=0}^{[\frac{n}{2}]} a_{0\ 2i} y^{2i}$, $Q = \sum_{i=0}^{[\frac{n-1}{2}]} b_{0\ 2i+1} y^{2i+1}$, then

$$\begin{pmatrix} \alpha_{-1} \\ \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} -d_0 & & & \\ & d_1 & * & \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & d_n \end{pmatrix} \begin{pmatrix} a_{00} \\ b_{01} \\ a_{02} \\ b_{03} \\ \vdots \end{pmatrix}.$$

By Lemma 4, $d_i \neq 0$, $i = 0, 1, \ldots, n$, therefore K is surjective. The proof of Proposition 3 is complete. ■

Lemma 5. *The least upper bound of the number of isolated zeros (taking into account their multiplicity) of the Abelian integral*

$$M(h) = \sum_{i=-1}^{n} \alpha_i h^{i/2}\,, \qquad \sum_{i=-1}^{n} \alpha_i = 0\,,$$

in the interval $(0,1)$ *is* n*. Moreover for arbitrary* n *real numbers* $h_1, h_2, \ldots,$ $h_n \in (0,1)$*, there exist constants* $\alpha_{-1}, \alpha_0, \ldots, \alpha_n$ *with* $\sum_{i=-1}^{n} \alpha_i = 0$ *and* $\sum_{i=-1}^{n} \alpha_i^2 > 0$*, such that*

$$M(h_i) = 0\,, \qquad i = 1, 2, \ldots, n\,.$$

Proof. As usual $\#$ denotes the cardinal of the set A. Since

$$\#\{h > 0 | M(h) = 0\} = \#\{h > 0 | h^{1/2} M(h) = 0\}\,.$$

and $h^{1/2}M(h)$ is a polynomial in the variable $\sqrt{h}$ of degree $n+1$, we have

$$\#\{h > 0 | M(h) = 0\} \le n+1\,.$$

On the other hand, $M(1) = 0$, so

$$\#\{0 < h < 1 | M(h) = 0\} \leq n \,.$$

Finally, for arbitrary real numbers $h_i \in (0,1)$, $(i = 1, 2, \ldots, n)$ take

$$\begin{aligned} f(h) &= (\sqrt{h} - 1) \prod_{i=1}^{n} (\sqrt{h} - \sqrt{h_i}) \\ &= \sum_{i=-1}^{n} \alpha_i h^{\frac{i+1}{2}} \,. \end{aligned}$$

Then $M(h) = \sum_{i=-1}^{n} \alpha_i h^{i/2} = h^{-1/2} f(h)$ has the zero points h_i for $i = 1, 2, \ldots, n$.

The proof of Lemma 5 is complete. ■

Statement (a) of Theorem 2 for system F is a corollary of Proposition 3 and Lemma 5.

4 Bifurcation of limit cycles from system G

Consider the polynomial perturbation of degree n of system G:

$$\begin{aligned} \dot{x} &= -y + x^2 + \varepsilon P(x, y) \,, \\ \dot{y} &= x + xy + \varepsilon Q(x, y) \,, \end{aligned} \tag{12}$$

where $P(x, y) = \frac{1}{2} \sum_{i+j \leq n} a_{ij} x^i y^j$, $Q(x, y) = \frac{1}{2} \sum_{i+j \leq n} b_{ij} x^i y^j$. For $\varepsilon = 0$, system (12) is an integrable system with first integral $H(x, y) = \frac{2y+1-x^2}{(1+y)^2}$. Multiplying by the integrating factor $\frac{2}{(1+y)^3}$, (12) can be changed to the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} + \varepsilon \overline{P}(x, y) \,, \\ \dot{y} &= -\frac{\partial H}{\partial x} + \varepsilon \overline{Q}(x, y) \,, \end{aligned} \tag{13}$$

where $\overline{P} = \frac{1}{(1+y)^3} \sum_{i+j \leq n} a_{ij} x^i y^j$ $\overline{Q} = \frac{1}{(1+y)^3} \sum_{i+j \leq n} b_{ij} x^i y^j$.

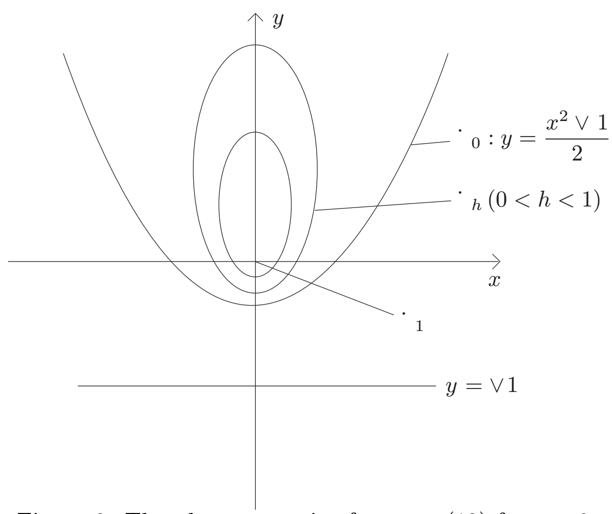

Figure 2: The phase portrait of system (13) for $\varepsilon = 0$.

The Abelian integral for system (13) in the form of double integral is the following:

$$M(h) = \iint_{H \geq h} \left(\frac{\partial \overline{P}}{\partial x} + \frac{\partial \overline{Q}}{\partial y} \right) dx\, dy\,, \qquad 0 < h \leq 1\,.$$

Proposition 6. *For $n = 0, 1$ $M(h) = \alpha_0 + \alpha_1 h$, $\alpha_0 + \alpha_1 = 0$; for $n = 2, 3$, $M(h) = \sum_{i=2-n}^{3} \alpha_i h^{i/2}$, $\sum_{i=2-n}^{3} \alpha_i = 0$; for $n \geq 4$,*

$$M(h) = \sum_{i=2-n}^{-1} \alpha_i h^{i+1/2} + \sum_{i=0}^{3} \alpha_i h^{i/2}\,, \qquad \sum_{i=2-n}^{3} \alpha_i = 0\,.$$

Moreover, the linear maps K:

$$(a_{ij}, b_{ij})_{i+j \leq n} \mapsto \alpha_1\,, \quad \textit{for} \quad n = 0, 1\,;$$

and

$$(a_{ij}, b_{ij})_{i+j \leq n} \mapsto (\alpha_i)_{\substack{2-n \leq i \leq 3 \\ i \neq 0}}\,, \quad \textit{for} \quad n \geq 2\,,$$

are surjective.

Proof. Since

$$\frac{\partial \overline{P}}{\partial x} = \sum_{i+j \leq n} \frac{i a_{ij} x^{i-1} y^j}{(1+y)^3}\,, \qquad \frac{\partial \overline{Q}}{\partial y} = \sum_{i+j \leq n} \frac{j b_{ij} x^i y^{j-1} + (j-3) b_{ij} x^i y^j}{(1+y)^4}\,,$$

we obtain

$$M(h) = \sum_{2m+1+j\le n} \iint_{H\ge h} \frac{(2m+1)a_{2m+1,j}x^{2m}y^j}{(1+y)^3}\,dx\,dy$$
$$+ \sum_{2m+j\le n} \iint_{H\ge h} \left(\frac{jb_{2m,j}y^{j-1}+(j-3)b_{2m,j}y^j}{(1+y)^4}\right) x^{2m}\,dx\,dy$$
$$\text{(by symmetricity)}\,.$$

Denote by $y_1 = -1 + \frac{1-\sqrt{1-h}}{h}$, $y_2 = -1 + \frac{1+\sqrt{1-h}}{h}$ the two roots of equation $2y+1-h(1+y)^2 = 0$, then

$$M(h) = \sum_{2m+1+j\le n} a_{2m+1j} \int_{y_1}^{y_2} \frac{2y^j}{(1+y)^3}(2y+1-h(1+y)^2)^m \sqrt{2y+1-h(1+y)^2}\,dy$$
$$+ \sum_{2m+j\le n} b_{2mj} \int_{y_1}^{y_2} \frac{2(jy^{j-1}+(j-3)y^j)}{(2m+1)(1+y)^4}(2y+1-h(1+y)^2)^m \sqrt{2y+1-h(1+y)^2}\,dy$$
$$= \sum_{k=0}^{n} m_k(h) \int_{y_1}^{y_2} (1+y)^{k-4}\sqrt{2y+1-h(1+y)^2}\,dy\,,$$

where $m_k(h)$ is a polynomial of h with degree $\le \left[\frac{k}{2}\right]$.

Let $I_k(h) = \int_{y_1}^{y_2}(1+y)^k\sqrt{1+2y-h(1+y)^2}\,dy$, then

$$M(h) = \sum_{k=0}^{n} m_k(h) I_{k-4}(h)\,, \quad \text{where} \quad \deg m_k \le \left[\frac{k}{2}\right]\,. \tag{14}$$

Let $z = 1+y$, $z_1 = \frac{1-\sqrt{1-h}}{h}$, $z_2 = \frac{1+\sqrt{1-h}}{h}$, then

$$I_k(h) = \int_{z_1}^{z_2} z^k \sqrt{-h(z-z_1)(z-z_2)}\,dz\,.$$

Let $\sqrt{-h(z-z_1)(z-z_2)} = -\sqrt{h}t(z-z_2)$, then

$$I_k(h) = 2h^{1/2}(z_2-z_1)^2 J_k\,, \qquad J_k = \int_0^\infty \frac{t^2(z_2t^2+z_1)^k}{(t^2+1)^{3+k}}\,dt\,.$$

It is not difficult to do the following five integrations:

$$\begin{aligned}
I_{-4} = I_{-3} &= \frac{1}{2}(1-h)\pi\,, \\
I_{-2} &= (1-\sqrt{h})\pi\,, \\
I_{-1} &= (h^{-1/2}-1)\pi\,, \\
I_0(h) &= \frac{1}{2}h^{-3/2}(1-h)\pi\,.
\end{aligned}$$

J_k are homogeneous symmetrical polynomials in the variables z_1 and z_2 of degree k, and $z_1+z_2=\frac{2}{h}$, $z_1z_2=\frac{1}{h}$, from formula (9), we have

$$I_k = h^{-k-3/2}(1-h)\sum_{i=0}^{\left[\frac{k}{2}\right]} d_{ik}h^i\,,$$

where $d_{0k}=2^{3+k}\int_0^\infty \frac{t^{2+2k}}{(t^2+1)^{3+k}}\,dt>0$. From (14), we get

$$\begin{aligned}
&M(h) = m_0(h)I_{-4}(h) = \alpha_0+\alpha_1 h\,, \quad \alpha_0+\alpha_1=0\,, \quad \text{if} \quad n=0\,; \\
&M(h) = m_0(h)I_{-4}(h)+m_1(h)I_{-3}(h) = \alpha_0+\alpha_1 h\,, \quad \alpha_0+\alpha_1=0\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad n=1\,.
\end{aligned}$$

For $n=2$,

$$M(h) = \sum_{k=0}^{2} m_k(h)I_{k-4}(h) = \sum_{i=0}^{3}\alpha_i h^{i/2}\,, \qquad \sum_{i=0}^{3}\alpha_i = 0\,.$$

For $n=3$,

$$M(h) = \sum_{k=0}^{3} m_k(h)I_{k-4}(h) = \sum_{i=-1}^{3}\alpha_i h^{i/2}\,, \qquad \sum_{i=-1}^{3}\alpha_i = 0\,.$$

For $k\ge 4$,

$$\begin{aligned}
m_k(h)I_{k-4}(h) &= m_k(h)h^{\frac{5}{2}-k}(1-h)\sum_{i=0}^{\left[\frac{k}{2}-2\right]} d_{ik}h^i \\
&= \begin{cases} \sum_{i=-k+2}^{1}\beta_i^k h^{i+1/2}\,, & \text{for } k \text{ even}\,; \\ \sum_{i=-k+2}^{0}\beta_i^k h^{i+1/2}\,, & \text{for } k \text{ odd}\,. \end{cases}
\end{aligned}$$

Thus, for $n \geq 4$,

$$\begin{aligned} M(h) &= \sum_{k=0}^{n} m_k(h) I_{k-4}(h) \\ &= \sum_{k=0}^{3} m_k(h) I_{k-4}(h) + \sum_{k=4}^{n} m_k(h) I_{k-4}(h) \\ &= \sum_{i=-1}^{3} \alpha_i' h^{i/2} + \sum_{k=4}^{n} \sum_{i=-k+2}^{1} \beta_i^k h^{i+1/2} \\ &= \sum_{i=-1}^{3} \alpha_i' h^{i/2} + \sum_{i=-n+2} \beta_i h^{i+\frac{1}{2}} \\ &= \sum_{i=2-n}^{-1} \alpha_i h^{i+\frac{1}{2}} + \sum_{i=0}^{3} \alpha_i' h^{i/2} \,. \end{aligned}$$

Next, we prove that the linear map K is surjective. Let $\overline{P} = 0$, $\overline{Q} = 1/(1+y)^3$, then

$$\begin{aligned} \iint_{H \geq h} \left(\frac{\partial \overline{P}}{\partial x} + \frac{\partial \overline{Q}}{\partial y} \right) dx\, dy &= -3 \iint_{H \geq h} \frac{1}{(1+y)^4}\, dx\, dy \\ &= -6 \int_{y_1}^{y_2} (1+y)^{-4} \sqrt{2y + 1 - h(1+y)^2}\, dy \\ &= -6 I_{-4} = -3(1-h)\pi \,. \end{aligned}$$

This implies that for $n = 0, 1$, the linear map K is surjective. Now we consider

$$\begin{aligned} &\iint_{H \geq h} \frac{\partial (x^2/(1+y)^3)}{\partial y}\, dx\, dy \\ &= -3 \iint_{H \geq h} \frac{x^2}{(1+y)^4}\, dx\, dy \\ &= -2 \int_{y_1}^{y_2} \frac{2y + 1 - h(1+y)^2}{(1+y)^4} \sqrt{2y + 1 - h(1+y)^2}\, dy \\ &= -2 \left(-I_{-4} + 2I_{-3} - hI_{-2} \right) \\ &= \pi(-1 + 3h - 2h^{\frac{3}{2}}) \,, \end{aligned}$$

and

$$\iint_{H \geq h} \frac{\partial (x y^{m-1}/(1+y)^3)}{\partial x}\, dx\, dy$$

$$
\begin{aligned}
&= \iint_{H\geq h} \frac{y^{m-1}}{(1+y)^3}\,dx\,dy \\
&= 2\int_{y_1}^{y_2} \frac{y^{m-1}}{(1+y)^3}\sqrt{2y+1-h(1+y)^2}\,dy \\
&= 2\int_{y_1}^{y_2} \frac{\sum_{i=0}^{m-1} C_{m-1}^i(-1)^i(y+1)^{m-1-i}}{(1+y)^3}\sqrt{2y+1-h(1+y)^2}\,dy \\
&= 2\sum_{i=0}^{m-1}(-1)^i C_{m-1}^i I_{m-i-4} \\
&= 2d_{0\ m-4}h^{5/2-m} + \cdots ,
\end{aligned}
$$

where the dots denote the higher order terms of h.

The above calculations show that, if we choose

$$
P = \sum_{i=1}^{n-1} a_{1i}xy^i , \qquad Q = b_{00} + b_{20}x^2 ,
$$

then we have

$$
\begin{pmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_3 \\ \alpha_{-1} \\ \alpha_{-2} \\ \vdots \\ \alpha_{-n+2} \end{pmatrix} = K(P,Q) =
$$

$$
= \begin{pmatrix} 3\pi & \pi & 3\pi & & & & \\ 0 & -2\pi & 0 & & & & \\ 0 & 0 & -2\pi & & & * & \\ & & & 2d_{0-1} & & & \\ & & & & 2d_{00} & & \\ & 0 & & & & \ddots & \\ & & & & & & 2d_{0n-4} \end{pmatrix} \begin{pmatrix} b_{00} \\ a_{11} \\ b_{20} \\ a_{12} \\ a_{13} \\ \vdots \\ a_{1\ n-1} \end{pmatrix} .
$$

The above matrix is an upper triangular one with nonzero elements on the diagonal, which implies that K is surjective.

The proof of Proposition 6 is complete. ■

Lemma 7. *The least per bound of the number of isolated zeros (taking into account their multiplicity) of $M(h)$ in $(0,1)$ is*

$$\begin{array}{lll} 0\,, & \textit{if} & n=0,1\,; \\ n\,, & \textit{for} & n\geq 2\,. \end{array}$$

Moreover, for any given $h_1, h_2, \dots, h_n \in (0,1)$, $n \geq 2$, there exist constants $\alpha_{2-n}, \alpha_{3-n}, \dots, \alpha_3$, with $\sum_{i=2-n}^{3} \alpha_i = 0$ and $\sum_{i=2-n}^{3} \alpha_i^2 > 0$, such that

$$M(h_i) = 0\,, \qquad i = 2-n, 3-n, \dots, 3\,.$$

Proof. Denote by N the least upper bound of the number of isolated zeros of $M(h)$ in the interval $(0,1)$, then obviously $N = 0$, for $n = 0, 1$.

Next we prove $N = n$ for $n \geq 2$.

$$\#\{0 < h \leq 1 \,|\, M(h) = 0\} = \#\{0 < h \leq 1 \,|\, f(h) = h^{n-5/2} M(h) = 0\}\,.$$

Note that $f^{(n-3)}(h)$ is a polynomial in the variable $\sqrt{h}$ of degree 4. We get

$$\#\{h > 0 \,|\, f^{(n-3)}(h) = 0\} \leq 4\,.$$

By Rolle theorem,

$$\#\{h > 0 \,|\, f(h) = 0\} \leq n+1\,.$$

On the other hand, $f(1) = M(1) = 0$, therefore

$$N = \sup \#\{0 < h < 1 \,|\, f(h) = 0\} \leq n\,.$$

Now we are going to prove the last conclusion of Lemma 7.

Given n points $h_j \in (0,1)$ $j = 1, 2, \dots, n$, we consider the system of linear equations:

$$M(h_j) = \sum_{i=2-n}^{-1} \alpha_i (h_j^{i+\frac{1}{2}} - 1) + \sum_{i=1}^{3} \alpha_i (h_j^{\frac{i}{2}} - 1) = 0\,, \qquad j = 1, 2, \dots, n\,.$$

Since the number of unknown variables is greater than the number of equations in the above system, so there exists nonzero solutions $(\alpha_i)_{2-n \leq i \leq 3}$, $i \neq 0$. Now let $\alpha_0 = -\sum_{i\neq 0} \alpha_i$, then

$$M(h) = \sum_{i=2-n}^{-1} \alpha_i h^{i+\frac{1}{2}} + \sum_{i=0}^{3} \alpha_i h^{\frac{i}{2}}$$

has zeros h_i, for $i = 1, 2, \dots, n$. The proof of the lemma is complete. ∎

Statement (b) of Theorem 2 for system G is a corollary of Proposition 6 and Lemma 7.

Consider now the polynomial perturbation of the Kukles system

$$\begin{aligned}\dot{x} &= -y + \varepsilon P(x,y)\,,\\ \dot{y} &= x + 3xy + x^3 + \varepsilon Q(x,y)\,,\end{aligned}$$

where

$$P(x,y) = \sum_{i+j\le n} a_{ij}x^i y^j\,, \qquad Q(x,y) = \sum_{i+j\le n} b_{ij}x^i y^j\,.$$

Let $u = x$, $v = x^2 + y$, then the above system is transformed to the following one:

$$\begin{aligned}\dot{u} &= -v + u^2 + \varepsilon \overline{P}(u,v)\,,\\ \dot{v} &= u + uv + \varepsilon \overline{Q}(u,v)\,,\end{aligned}$$

where

$$\overline{P}(u,v) = P(u, v-u^2)\,, \qquad \overline{Q}(u,v) = 2uP(u, v-u^2) + Q(u, v-u^2)\,.$$

Note that

$$\deg \overline{P} = 2n\,, \quad \overline{Q} = 2(-1)^n a_{0n} u^{2n+1} + \widetilde{Q}\,, \quad \deg \widetilde{Q} = 2n\,,$$

and

$$\begin{aligned}M(h) &= \iint_{H\ge h} \left(\frac{\partial(\overline{P}/(1+v)^3)}{\partial u}\right) + \left(\frac{\partial(\overline{Q}/(1+v)^3)}{\partial v}\right) du\, dv\\ &= \iint_{H\ge h} \left(\frac{\partial(\overline{P}/(1+v)^3)}{\partial u}\right) + \left(\frac{\partial(\widetilde{Q}/(1+v)^3)}{\partial v}\right) du\, dv\,.\end{aligned}$$

By using Lemma 7, we can immediately prove statement (f) of Theorem 2.

5 Bifurcation of limit cycles from system C

By shifting the invariant line $y = -1$ to $y = 0$, i.e. to the x-axis, the system of class C can be written in the form

$$\begin{aligned}\dot{x} &= y^2 - x^2 - 1\,,\\ \dot{y} &= -2xy\,.\end{aligned} \tag{15}$$

The first integral of (15) is $H(x,y) = \frac{x^2+y^2+4y+1}{y}$. Denote by $\Gamma_h \colon H = 4+h$ the integral curves of (15). For $\pm h > 2$, Γ_h are circles in the upper (lower) half-plane with radius $\sqrt{\frac{h^2}{4}-1}$ and center at point $(0, \frac{h}{2})$ for $h > 2$ $(h < -2)$. $\Gamma_{\pm 2}$ has center at the point $(0, \pm 1)$.

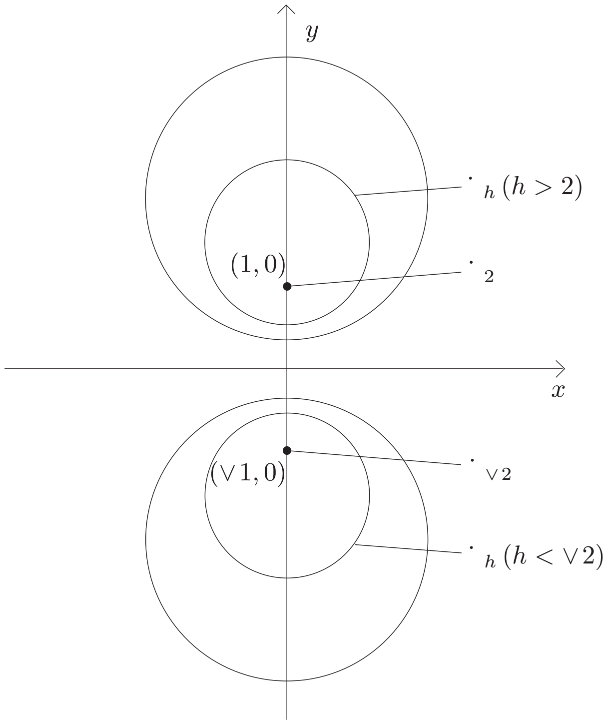

Figure 3: The phase portrait of system (15) for $\varepsilon = 0$.

Consider now the polynomial perturbation of degree n of system (15):

$$\begin{aligned} \dot{x} &= y^2 - x^2 - 1 + \varepsilon P(x,y)\,, \\ \dot{y} &= -2xy + \varepsilon Q(x,y)\,, \end{aligned} \tag{16}$$

where $P(x,y) = \sum_{i+j\le n} a_{ij}x^iy^j$, $Q(x,y) = \sum_{i+j\le n} b_{ij}x^iy^j$. Multiplying by the integrating factor $\frac{1}{y^2}$, (16) is changed to the form

$$\dot{x} = \frac{\partial H}{\partial y} + \varepsilon \frac{P}{y^2}\,, \qquad \dot{y} = -\frac{\partial H}{\partial x} + \varepsilon \frac{Q}{y^2}\,. \tag{17}$$

We only calculate the Abelian integral for the case $h > 2$. The case $h < -2$ can be studied by the same way. The Abelian inegral of (17) is

$$M(h) = \oint_{\Gamma_h} \left(\frac{P}{y^2} dy - \frac{Q}{y^2} dx \right) . \tag{18}$$

Proposition 8. *For the perturbed system* (17) *the following statements hold.*

(a) The Abelian integral

$$M(h) = \begin{cases} -b_{00}\pi(h-2)(h+2)/2 , & \textit{for} \quad n = 0 ; \\ (h-2)(\alpha_0 + \alpha_1 h) , & \textit{for} \quad n = 1, 2, 3 ; \\ (h-2) \left(\sum_{i=0}^{n-2} \alpha_i h^i \right) , & \textit{for} \quad n \geq 4 . \end{cases}$$

(b) The linear maps K:

$$(a_{ij}, b_{ij})_{i+j \leq n} \mapsto (\alpha_0, \alpha_1) , \qquad \textit{for} \quad n = 1, 2, 3$$

and

$$(a_{ij}, b_{ij})_{i+j \leq n} \mapsto (\alpha_0, \alpha_1, \ldots , \alpha_{n-2}) , \qquad \textit{for} \quad n \geq 4$$

are surjective.

Proof. Denote by $y_1 = \frac{h - \sqrt{h^2-4}}{2}$, $y_2 = \frac{h+\sqrt{h^2-4}}{2}$ the y-coordinate of the intersection $\Gamma_h \cap \{x = 0\}$. By Green's formula we have

$$\begin{aligned} M(h) &= \iint_{H \geq h+4} \frac{\partial}{\partial x} \left(\frac{P}{y^2} \right) + \frac{\partial}{\partial y} \left(\frac{Q}{y^2} \right) dx\, dy \\ &= \iint_{H \geq h+4} \left(\frac{1}{y^2} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) - \frac{2Q}{y^3} \right) dxdy \\ &= \iint_{H \geq h+4} \frac{1}{y^2} \left(\sum_{i+j \leq n} i a_{ij} x^{i-1} y^j \right) dxdy \\ &\quad + \iint_{H \geq h+4} \frac{1}{y^3} \left(\sum_{i+j \leq n} j b_{ij} x^i y^j - \sum_{i+j \leq n} 2 b_{ij} x^i y^j \right) dx\, dy \\ &= \sum_{2i+1+j \leq n} 2a_{2i+1\, j} \int_{y_1}^{y_2} \frac{y^j (hy - 1 - y^2)^i}{y^2} \sqrt{hy - 1 - y^2}\, dy \end{aligned}$$

$$+\sum_{2i+j\le n}\frac{2(j-2)b_{2ij}}{(2i+1)}\int_{y_1}^{y_2}\frac{y^j(hy-1-y^2)^i}{y^3}\sqrt{hy-1-y^2}\,dy$$

$$=\sum_{k=0}^{n}m_k(h)\int_{y_1}^{y_2}y^{k-3}\sqrt{hy-1-y^2}\,dy\,,$$

where

$$\deg m_k(h)\le\min\{k,n-k\}\,. \tag{19}$$

Let $y_1=(h-\sqrt{h^2-4})/2$, $y_2=(h+\sqrt{h^2-4})/2$ and $\sqrt{-(y-y_1)(y-y_2)}=-t(y-y_2)$. Then

$$I_k(h)=\int_{y_1}^{y_2}y^{k-3}\sqrt{-(y-y_1)(y-y_2)}\,dy$$
$$=2(y_2-y_1)^2J_k\,,$$

where

$$J_k=\int_0^{\infty}\frac{t^2(y_2t^2+y_1)^k}{(t^2+1)^{k+3}}\,dt\,.$$

It is easy to compute

$$\begin{aligned} I_{-1}&=I_{-2}=\frac{\pi}{2}(h-2)\,,\\ I_{-3}&=I_0=\frac{\pi}{8}(h^2-4)\,,\\ I_{-4}&=\frac{\pi}{16}h(h^2-4)\,. \end{aligned} \tag{20}$$

J_k are homogeneous symmetrical polynomials in the variables y_1 and y_2 of degree k, and $y_1+y_2=h$, $y_1y_2=1$. For $k\ge 0$, by formula (9), we obtain

$$I_k(h)=2(h^2-4)\sum_{j=0}^{[\frac{k}{2}]}C_{j,k}h^{k-2j}\,, \tag{21}$$

where

$$C_{0k}=\int_0^{\infty}\frac{t^2}{(t^2+1)^k}\,dt>0\,.$$

Therefore $M(h)=\sum_{k=0}^{n}m_k(h)I_{k-3}$ is a polynomial in h.

For $n=0$,

$$M(h) = -2b_{00} \iint_{H\geq 4+h} \frac{dx\,dy}{y^3} = -4b_{00} I_3 = -\frac{\pi}{2} b_{00}(h^2-4)\,.$$

For $n=1,2,3$, by (19) and (20) we get

$$\deg M(h) \leq \max_{0\leq k\leq 3} (\deg m_k + \deg I_{k-3}) \leq 2\,.$$

For $n\geq 4$,

$$\begin{aligned}
\deg M(h) &= \deg\left(\sum_{k=0}^{n} m_k(h) I_{k-3}\right) \\
&\leq \max_{0\leq k\leq n} \{\deg m_k(h) I_{k-3}\} \\
&= \max_{3\leq k\leq n} \{\min(k, n-k) + k - 1\} \\
&= \max_{3\leq k\leq n} \min\{2k-1, n-1\} = n-1\,.
\end{aligned}$$

On the other hand $M(2)=0$, thus statement (a) is proved.

Next we prove statement (b), that is the linear maps K are surjective.

For $m\geq 1$,

$$\begin{aligned}
&\iint_{H\geq 4+h} \frac{\partial(xy^{m-1}/y^2)}{\partial x}\,dx\,dy \\
&= 2\int_{y_1}^{y_2} y^{m-3}\sqrt{hy-1-y^2} \\
&= 2I_{m-3} = \begin{cases} \pi(h-2)\,, & \text{for} \quad m=1,2\,; \\ 4(h^2-4)\sum_{j=0}^{\left[\frac{m-3}{2}\right]} C_{j,m-3} h^{m-3-2j}\,, & \text{for} \quad m\geq 3\,. \end{cases}
\end{aligned}$$

Let

$$P = a_{11}x + \sum_{i=3}^{n-1} a_{1i} x y^i\,, \qquad Q = b_{00}\,.$$

Then the linear map K has the following form:

$$(\alpha_0, \alpha_1, \ldots, \alpha_{n-2})^T = \begin{pmatrix} -\pi & \pi & & & \\ \frac{\pi}{2} & 0 & & * & \\ & & 4C_{0,0} & & \\ & 0 & & \ddots & \\ & & & & 4C_{0,n-3} \end{pmatrix} \begin{pmatrix} b_{00} \\ a_{11} \\ a_{13} \\ \vdots \\ a_{1,n-1} \end{pmatrix}\,.$$

This matrix is nonsingular, so K is surjective. The proof of Proposition 8 is complete. ∎

Next we consider the symmetrical polynomial perturbation of (15):

$$\begin{aligned} \dot{x} &= y^2 - x^2 - 1 + \varepsilon P(x,y)\,, \\ \dot{y} &= -2xy + \varepsilon Q(x,y)\,, \end{aligned} \tag{22}$$

where

$$P(x,y) = \sum_{i+2j\le n} a_{i\,2j} x^i y^{2j}\,, \qquad Q(x,y) = \sum_{i+2j+1\le n} b_{i\,2j} x^i y^{2j+1}\,.$$

The Abelian integral $M(h)$ is defined as (18).

Proposition 9. *For system* (22) *the following statements hold.*

(a)

$$M(h) = \begin{cases} 0\,, & \text{if} \quad n=0\,; \\ \alpha_0(h-2)\,, & \text{if} \quad n=1,2\,; \\ \alpha_0(h-2) + (h^2-4)\sum_{i=1}^{\left[\frac{n-1}{2}\right]} \alpha_i h^{2i-2}\,, & \text{if} \quad n\ge 3\,. \end{cases}$$

(b) The linear map K:

$$(a_{i\,2j}, b_{i\,2j+1}) \mapsto (\alpha_0, \alpha_1, \dots, \alpha_{\left[\frac{n-1}{2}\right]})$$

is surjective.

Proof. By Green's formula we have

$$\begin{aligned} M(h) &= \iint_{H\ge h+4} \frac{\partial}{\partial x}\left(\frac{P}{y^2}\right) + \frac{\partial}{\partial y}\left(\frac{Q}{y^2}\right)\,dx\,dy \\ &= \iint_{H\ge h+4} \frac{1}{y^2}\left(\sum_{2i+2j\le n-1} C_{ij} x^{2i} y^{2j}\right)\,dx\,dy \\ &= \int_{y_1}^{y_2} \frac{2}{y^2}\left(\sum_{2i+2j\le n-1} \frac{C_{ij}}{2i+1}(hy-1-y^2)^i y^{2j}\sqrt{hy-1-y^2}\right)\,dy \\ &= \int_{y_1}^{y_2} \sum_{k=0}^{2\left[\frac{n-1}{2}\right]} m_k(h) y^{k-2}\sqrt{hy-1-y^2}\,dy \\ &= \sum_{k=0}^{2\left[\frac{n-1}{2}\right]} m_k(h) I_{k-2}\,, \end{aligned}$$

where $C_{ij} = (2i+1)a_{2i+1,2j} + (2j-1)b_{2i,2j+1}$, and $m_k(h)$ is an even (respectively odd) polynomial of degree $\leq \min\{k, 2[(n-1)/2]-k\}$, if k is even (respectively odd).

Obviously, if $n = 0$, $M(h) \equiv 0$; if $n = 1, 2$,

$$M(h) = m_0(h)I_{-2} = \alpha_0(h-2)\,.$$

For $k \geq 2$, from (21) $m_k(h)I_{k-2}$ is an even polynomial of degree $\leq \min\{2k, 2[(n-1)/2]\}$. Therefore, for $n \geq 3$,

$$\begin{aligned}
M(h) &= \sum_{k=0}^{1} m_k(h)I_{k-2} + \sum_{k=2}^{2[\frac{n-1}{2}]} m_k(h)I_{k-2} \\
&= (m_0(h) + m_1(h))I_{-2} + (h^2-4)g(h^2) \\
&= (a+bh)(h-2) + (h^2-4)g(h^2) \\
&= (a-2b)(h-2) + (h^2-4)(g(h^2)+b) \\
&= \alpha_0(h-2) + (h^2-4)\sum_{i=1}^{[\frac{n-1}{2}]} \alpha_i h^{2i-2}\,,
\end{aligned}$$

where $\deg g \leq [(n-1)/2] - 1$.

Statement (a) of Proposition 9 is proved.

Next we prove statement (b), that is the linear map K is surjective. We consider

$$\begin{aligned}
&\iint_{H \geq h+4} \frac{\partial(xy^{2m-2})}{\partial x}\, dx\, dy \\
&= \iint_{H \geq h+4} y^{2m-2} dx\, dy = 2\int_{y_1}^{y_2} y^{2m-2}\sqrt{hy - 1 - y^2}\, dy \\
&= 2I_{2m-2} = \begin{cases} \pi(h-2)\,, & \text{if} \quad m = 0\,, \\ 4(h^2-1)\sum_{i=0}^{m-1} C_{i,2m-2}h^{2m-2-2i}\,, & \text{if} \quad m \geq 1\,. \end{cases}
\end{aligned}$$

Let

$$P = \sum_{i=0}^{[\frac{n-1}{2}]} a_{1,2i}xy^{2i}\,, \qquad Q = 0\,,$$

then the linear map K has the form

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \vdots \\ \alpha_{[\frac{n-1}{2}]} \end{pmatrix} = \begin{pmatrix} \pi & & & & \\ & 4C_{0,0} & & & * \\ & & 4C_{0,2} & & \\ & & & \ddots & \\ & 0 & & & 4C_{0,2[\frac{n-1}{2}]-2} \end{pmatrix} \begin{pmatrix} a_{10} \\ a_{12} \\ \vdots \\ \vdots \\ a_{1,2[\frac{n-1}{2}]} \end{pmatrix},$$

where $C_{0,2j} > 0$, which implies that K is surjective. The proof of Proposition 9 is complete. ■

Lemma 10. *Let $M(h)$ be a function as in statement (a) of Proposition 9, then the followings hold. The least upper bound of the number of the isolated zeros of $M(h)$ in $(2, +\infty)$ is $[(n-1)/2]$. Moreover, given $[(n-1)/2]$ arbitrary points $h_1, h_2, \dots, h_{[\frac{n-1}{2}]} \in (2, +\infty)$, there exist constants $\alpha_0, \alpha_1, \dots,$ $\alpha_{[\frac{n-1}{2}]}$ with $\sum_{i=0}^{[\frac{n-1}{2}]} \alpha_i^2 > 0$, such that*

$$M(h_i) = 0\,, \qquad i = 1, 2, \dots, \left[\frac{n-1}{2}\right].$$

Proof. Denote by N the least upper bound. Since $M''(h)$ is an even polynomial of degree $2[(n-1)/2] - 2$, we have

$$\#\{h > 0 \,|\, M''(h) = 0\} \le \left[\frac{n-1}{2}\right] - 1\,.$$

By Rolle theorem,

$$\#\{h > 0 \,|\, M(h) = 0\} \le \left[\frac{n-1}{2}\right] + 1\,.$$

Note that $M(2) = 0$, we get

$$\#\{h > 2 \,|\, M(h) = 0\} \le \left[\frac{n-1}{2}\right],$$

which implies $N \le [(n-1)/2]$.

Given $[(n-1)/2]$ arbitrary points $h_j \in (2, +\infty)$, $j = 1, 2, \dots, [(n-1)/2]$, consider the system of linear equations for α_i:

$$M(h_j) = (h_j - 2)\alpha_0 + (h_j^2 - 4) \sum_{i=1}^{[\frac{n-1}{2}]} \alpha_i h_j^{2i-2} = 0\,, \quad j = 1, 2, \dots, \left[\frac{n-1}{2}\right].$$

Since the number of unknown variables is greater than the number of equations in the above system, there exist solution $\{\alpha_i\}_{0\le i\le[\frac{n-1}{2}]}$ with $\sum_{i=0}^{[\frac{n-1}{2}]}\alpha_i^2>0$. The proof of Lemma 10 is complete. ■

Statement (c) of Theorem 2 for system C is a corollary of Propositions 8, 9 and Lemma 10.

6 Bifurcation of limit cycles from system S_3

Consider the polynomial perturbation of degree n of system S_3:

$$\begin{aligned} \dot{u} &= -v+\frac{1}{4}u^2-\varepsilon P(u,v)\,, \\ \dot{v} &= u(1+v)+\varepsilon Q(u,v)\,, \end{aligned} \tag{23}$$

where P, Q are polynomials of degree n. Let $u=-2x$, $v=y^2-1$, $t=2\tau$, then system (23) is transformed into the following perturbed system C:

$$\begin{aligned} \dot{x} &= y^2-x^2-1+\varepsilon P(-2x,y^2-1)\,, \\ \dot{y} &= -2xy+\varepsilon\frac{1}{y}Q(-2x,y^2-1)\,. \end{aligned} \tag{24}$$

Multiplying by the integrating factor $1/y^2$, (24) can be written in the form:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}+\varepsilon\frac{P}{y^2}\,, \\ \dot{x} &= -\frac{\partial H}{\partial x}+\varepsilon\frac{Q}{y^3}\,, \end{aligned} \tag{25}$$

where $H=\frac{1}{y}(x^2+y^2+4y+1)$. Let

$$\begin{aligned} P(-2x,y^2-1) &= \sum_{i+j\le n} a_{ij}x^iy^{2j}\,, \\ Q(-2x,y^2-1) &= \sum_{i+j\le n} b_{ij}x^iy^{2j}\,. \end{aligned}$$

The Abelian integral for system (25) is defined as

$$M(h)=\oint_{H\ge h+4}\frac{P}{y^2}\,dy-\frac{Q}{y^3}\,dx\,.$$

Proposition 11. *For the perturbed system* (25) *the following statements hold.*

(a)

$$M(h)=\begin{cases} -\frac{3\pi}{8}(h-2)[b_{00}(h^2+2h)-\frac{8}{3}(a_{10}-b_{01})]\,, & \textit{for } n=0,1\,;\\ (h-2)(\alpha_{-2}+\alpha_{-1}h+\alpha_0 h^2)\,, & \textit{for } n=2\,;\\ (h-2)(\alpha_{-2}+\alpha_{-1}h+\alpha_0 h^2)+(h^2-4)\sum_{i=1}^{n-2}\alpha_i h^{2i}\,, & \textit{for } n\geq 3\,. \end{cases}$$

(b) The linear map K:

$$(a_{ij},b_{ij})_{i+j\leq n}\mapsto(\alpha_{-2},\alpha_{-1},\dots,\alpha_{n-2})$$

is surjective.

Proof. By Green's formula we have

$$\begin{aligned} M(h) &= \iint_{H\geq h+4}\left[\frac{P_x y^2+yQ_y-3Q}{y^4}\right]dx\,dy\\ &= \iint_{H\geq h+4}\frac{1}{y^4}\left[\sum_{2i+1+j\leq n}(2i+1)a_{2i+1\;j}x^{2i}y^{2j}+\right.\\ &\qquad\left.+\sum_{i+j\leq n}(2j-3)b_{ij}x^iy^{2j}\right]dx\,dy\\ &= \iint_{H\geq h+4}\left[\frac{1}{y^2}\sum_{2i+1+j\leq n}(2i+1)a_{2i+1\;j}x^{2i}y^{2j}+\right.\\ &\qquad\left.+\frac{1}{y^4}\sum_{2i+j\leq n}(2j-3)b_{2i\;j}x^{2i}y^{2j}\right]dx\,dy\,. \end{aligned}$$

For $n=1$,

$$\begin{aligned} M(h) &= \iint_{H\geq h+4}\left[\frac{1}{y^2}a_{10}+\frac{1}{y^4}(-3b_{00}-b_{01}y^2)\right]dx\,dy\\ &= 2(a_{10}-b_{01})I_{-2}-6b_{00}I_{-4}\\ &= -\frac{3\pi}{8}(h-2)\left[b_{00}(h^2+2h)-\frac{8}{3}(a_{10}-b_{01})\right]\,, \end{aligned}$$

where in the last equality we have used (20). For $n \geq 2$,

$$M(h) = \sum_{2i+1+j \leq n} 2a_{2i+1\,j} \int_{y_1}^{y_2} (hy - 1 - y^2)^i y^{2j-2} \sqrt{hy - 1 - y^2}\, dy$$
$$+ \sum_{2i+j \leq n} (2j-3) b_{2i\,j} \int_{y_1}^{y_2} (hy - 1 - y^2)^i y^{2j-4} \sqrt{hy - 1 - y^2}\, dy$$
$$= \sum_{k=0}^{2(n-1)} m_k(h) \int_{y_1}^{y_2} y^{k-4} \sqrt{hy - 1 - y^2}\, dy = \sum_{k=0}^{2(n-1)} m_k(h) I_{k-4}\,,$$

where

$$I_k = \int_{y_1}^{y_2} y^k \sqrt{hy - 1 - y^2}\, dy\,,$$

and m_k is an even (respectively odd) polynomial, if k is even (respectively odd).

Moreover

$$\deg m_k \leq \min\left\{k, \left[\frac{2n-k}{3}\right]\right\}\,. \tag{26}$$

If $n = 2$,

$$\deg M(h) \leq \max_{0 \leq k \leq 4} \deg(m_k I_{k-4}) \leq 3\,.$$

Note that $M(2) = 0$, hence

$$M(h) = (h-2)(\alpha_{-2} + \alpha_{-1} h + \alpha_0 h^2)\,.$$

For $k \geq 4$, $m_k I_{k-4}$ is an even polynomial of degree $\leq 2n-2$. Therefore, for $n \geq 3$, by using the estimation of $\deg M(h)$ for system C, we get

$$\deg M(h) \leq \max_{0 \leq k \leq 2n} \{m_k I_{k-4}\} \leq 2n - 2\,.$$

$$M(h) = \sum_{k=0}^{2n} m_k(h) I_{k-4} = \sum_{k=0}^{3} m_k(h) I_{k-4} + \sum_{k=4}^{2n} m_k(h) I_{k-4}$$
$$= (h-2)(\alpha_{-2} + \alpha_{-1} h + \alpha_0 h^2) + (h^2 - 4) \sum_{i=1}^{n-2} \alpha_i h^{2i}\,.$$

Statement (a) is proved.

Next, we prove statement (b).

For $m \geq 4$,

$$\begin{aligned}
&\iint_{H\geq h+4} \frac{\partial(y^m/y^3)}{\partial y}\,dx\,dy \\
&= (m-3)\iint_{H\geq h+4} y^{m-4}\,dx\,dy \\
&= 2(m-3)I_{m-4} \\
&= (h^2-4)\sum_{j=0}^{[\frac{m}{2}]} d_{j,m-1}h^{m-4-2j}\,,
\end{aligned}$$

where

$$d_{0,m-1} = 4(m-3)\int_0^\infty \frac{t^2}{(t^2+1)^{m-1}}\,dt > 0\,.$$

If we choose $P=0$, $Q=\sum_{i=0}^n b_{0i}y^{2i}$, then the linear map K has the following form:

$$\begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \vdots \\ \vdots \\ \alpha_{n-2} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}\pi & -\pi & -\pi & & & \\ -\frac{3}{8}\pi & 0 & 0 & & * & \\ 0 & 0 & \frac{\pi}{4} & & & \\ & & & d_{0,5} & & \\ & 0 & & & \ddots & \\ & & & & & d_{0,2n-1} \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{01} \\ \vdots \\ \vdots \\ b_{0n} \end{pmatrix}.$$

This matrix is nonsingular, so k is surjective. The proof of Proposition 11 is complete. ■

Lemma 12. *The least upper bound of the number of isolated zeros in the interval* $(2,\infty)$ *of the function* $M(h)$ *of Proposition 11 is* n. *Moreover, given* n *arbitrary numbers* $h_1, h_2\cdots, h_n \in (2,+\infty)$, *there exist constants* $\alpha_{-2}, \alpha_{-1}, \dots, \alpha_{n-2}$ *with* $\sum_{i=-2}^{n-2}\alpha_i^2 > 0$, *such that*

$$M(h_i) = 0\,, \qquad \textit{for} \quad i = 1, 2, \dots, n\,.$$

Proof. Let N be the least upper bound. For $n=0$, $M(h) = -\frac{3\pi}{8}b_{00}(h-2)(h^2+2h)$, obviously, $N=0$.

For $n=1$,

$$M(h) = -\frac{3\pi}{8}(h-2)\left[b_{00}(h^2+2h) - \frac{8}{3}(a_{10}-b_{01})\right].$$

Since h^2+2h is monotonic in $(2,\infty)$, we obtain $N \le 1$.

For $n=2$, $M(h)$ is a polynomial of degree 3 and $M(2)=0$, so $N \le 2$.

For $n \ge 3$,

$$M^{(4)}(h) \quad \text{is an even polynomial of degree } 2n-6\,,$$

hence,

$$\#\{h>0 \,|\, M^{(4)}(h)=0\} \le n-3\,.$$

By Rolle theorem,

$$\#\{h>0 \,|\, M(h)=0\} \le n+1\,,$$

which, together with the fact $M(2)=0$, implies

$$\#\{h>2 \,|\, M(h)=0\} \le n\,.$$

The last conclusion of the lemma can be proved by the same method that in Lemma 10.

Statement (d) in Theorem 2 for system S_3 is a corollary of Proposition 11 and Lemma 12. ∎

7 The bifurcation of Limit cycles from system S_4

Consider the polynomial perturbation of degree n of system S_4:

$$\begin{aligned} \dot{x} &= -y+2x^2-\frac{y^2}{2}+\varepsilon P(x,y)\,, \\ \dot{y} &= x(1+y)+\varepsilon Q(x,y)\,, \end{aligned} \tag{27}$$

where, P, Q are polynomials of degree n. For $\varepsilon=0$, system (27) has a first integral $H=\frac{4x^2-2(y+1)^2+1}{(y+1)^4}$ with integrating factor $(1+y)^{-5}$. For $-1<h<0$, $\Gamma_h=\{(x,y) \,|\, H(x,y)=h \quad \text{are periodic orbits around the origin} \quad (0,0)\}$.

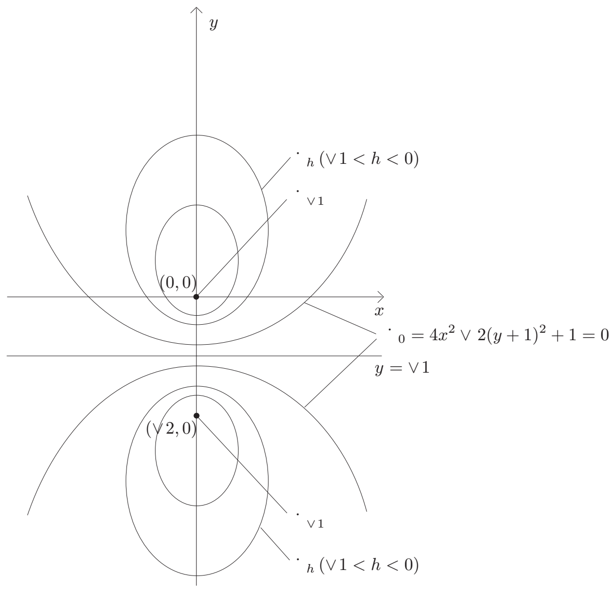

Figure 4: The phase portrait of system (27) for $\varepsilon = 0$

The Abelian integral of (27) is defined as

$$M(h) = \int_{H=h} \left(\frac{P}{(1+y)^5}\, dy - \frac{Q}{(1+y)^5}\, dx \right) , \qquad -1 < h < 0 .$$

Proposition 13. *The number of isolated zeros of the Abelian integral is* $\leq 14n + 11$, *for* $n \geq 4$.

Proof. By Green's formula, we have

$$\begin{aligned} M(h) &= \iint_{H \leq h} \frac{\partial}{\partial x} \left(\frac{P}{(1+y)^5)} \right) + \frac{\partial}{\partial y} \left(\frac{Q}{(1+y)^5} \right) dx\, dy \\ &= \iint_{H \leq h} \left[\frac{P_x + Q_y}{(1+y)^5} - \frac{5Q}{(1+y)^6} \right] dx\, dy \\ &= \iint_{H \leq h} \frac{(1+y)(P_x + Q_y) - 5Q}{(1+y)^6}\, dx\, dy . \end{aligned}$$

Let $(1+y)(P_x+Q_y)-5Q=\sum_{i+j\le n} a_{ij}x^i(1+y)^j$, then

$$\begin{aligned}M(h) &= \sum_{i+j\le n} a_{ij}\iint_{H\le h} x^i(1+y)^{j-6}dx\,dy\\ &= \sum_{2i+j\le n} a_{2i\,j}\iint_{H\le h} x^{2i}(1+y)^{j-6}dx\,dy\\ &= \sum_{2i+j\le n}\frac{2a_{2i\,j}}{(2i+1)}\int_{y_1}^{y_2}\left(\frac{h(1+y)^4+2(1+y)^2-1}{4}\right)^i\\ &\quad \frac{\sqrt{h(1+y)^4+2(1+y)^2-1}}{2}(1+y)^{j-6}\,dy\,,\end{aligned}$$

where y_1, y_2 are the zeros of the equation $h(1+y)^4+2(1+y)^2-1=0$. Let

$$I_k(h)=\int_{y_1}^{y_2}(1+y)^k\sqrt{h(1+y)^4+2(1+y)^2-1}\,dy\,,$$

then

$$M(h)=\sum_{k=0}^{n+2\left[\frac{n}{2}\right]} m_k(h)I_{k-6}\,,$$

where m_k are polynomials of h with

$$\deg m_k\le\left[\frac{k}{4}\right]\,. \tag{28}$$

Let $y+1=u$, $u_i=y_i+1$, then

$$I_k(h)=\int_{u_1}^{u_2}u^k\sqrt{hu^4+2u^2-1}\,du\,.$$

Next we calculate I_k.

For k odd, i.e. $k=2m+1>0$, we have

$$I_{2m+1}=\frac{1}{2}\int_{\overline{u}_1}^{\overline{u}_2}u^m\sqrt{hu^2+2u-1}\,du\,,$$

where

$$\begin{aligned}\overline{u}_1 &= \frac{\sqrt{h+1}-1}{h}\,,\\ \overline{u}_2 &= \frac{-(\sqrt{h+1}+1)}{h}\,.\end{aligned}$$

Again let $\sqrt{hu^2+2u-1} = t\sqrt{-h}(u-\overline{u}_1)$, then

$$I_{2m+1} = (\overline{u}_2-\overline{u}_1)^2\sqrt{-h}\int_0^{\infty}\frac{(t^2\overline{u}_1+\overline{u}_2)^m t^2}{(1+t^2)^{m+3}}\,dt\,.$$

By formula (8), and $\overline{u}_1+\overline{u}_2 = \frac{-2}{h}$, $\overline{u}_1\overline{u}_2 = \frac{-1}{h}$, we get

$$\begin{aligned}
I_{2m+1} &= 4\sqrt{-h}\left(\frac{1+h}{h^2}\right)\sum_{i=0}^{\left[\frac{m}{2}\right]} C_{i,m}(\overline{u}_1+\overline{u}_2)^{m-2i}(\overline{u}_1\overline{u}_2)^i \\
&= 4(-h)^{-\frac{3}{2}}(h+1)\sum_{i=0}^{\left[\frac{m}{2}\right]} C_{i,m}2^{m-2i}(-h)^{2i-m}(-h)^{-i} \\
&= 4(-h)^{-\frac{3}{2}}(h+1)\sum_{i=0}^{\left[\frac{m}{2}\right]} d_{i,m}(-h)^{i-m}\,.
\end{aligned}$$

Easy computations show that,

$$\begin{aligned}
I_{-1} &= \frac{1}{2}\left[(-h)^{-\frac{1}{2}}-1\right]\pi\,, \\
I_{-3} &= \frac{1}{2}(1-\sqrt{-h})\pi\,, \\
I_{-5} &= \frac{1}{4}(1+h)\pi\,.
\end{aligned}$$

For k even, doing the change $u = x/\sqrt{-h}$ we obtain

$$\begin{aligned}
I_k &= \int_{u_1}^{u_2} u^k\sqrt{hu^4+2u^2-1}\,du \\
&= (-h)^{-\frac{k}{2}-1}\int_{x_1}^{x_2} x^k\sqrt{-x^4+2x^2+h}\,dx \\
&= (-h)^{-\frac{k}{2}-1}J_k\,,
\end{aligned}$$

where $x_1 < x_2$ are the zeros of $-x^4+2x^2+h=0$, and

$$J_k = \int_{x_1}^{x_2} x^k y\,dx\,, \qquad y = \sqrt{-x^4+2x^2+h}\,.$$

For any integer m,

$$\begin{aligned}
-4J_{m+3}+4J_{m+1} &= \int_{x_1}^{x_2} x^m y(-4x^3+4x)\,dx \\
&= \int_{x_1}^{x_2} x^m y dy^2 \\
&= \frac{2}{3}\int_{x_1}^{x_2} x^m dy^3 \\
&= -\frac{2m}{3}\int_{x_1}^{x_2} y^3 x^{m-1} dx \\
&= -\frac{2m}{3}\int_{x_1}^{x_2} y x^{m-1}(-x^4+2x^2+h)\,dx \\
&= \frac{2m}{3}J_{m+3}-\frac{4m}{3}J_{m+1}-\frac{2m}{3}hJ_{m-1}\,,
\end{aligned}$$

so we get

$$\left(4+\frac{2m}{3}\right)J_{m+3}=\left(\frac{4m}{3}+4\right)J_{m+1}+\frac{2mh}{3}J_{m-1}\,, \tag{29}$$

which implies

$$J_{2m}=f_m J_2+g_m J_0\,, \qquad m\geq 1\,,$$

where f_m, g_m are polynomials in h with $\deg f_m=\left[\frac{m-1}{2}\right]$, $\deg g_m=\left[\frac{m}{2}\right]$. From (29), we have

$$\begin{aligned}
J_{-2} &= \frac{1}{h}(4J_0-5J_2)\,, \\
J_{-4} &= -\frac{1}{h}J_0\,, \\
J_{-6} &= \frac{1}{h^2}J_2\,,
\end{aligned}$$

which implies

$$\begin{aligned}
I_{-2} &= \frac{1}{h}(4J_0-5J_2)\,, \\
I_{-4} &= J_0\,, \\
I_{-6} &= J_2\,.
\end{aligned} \tag{30}$$

In the following, we denote by P_k, $\widetilde{P}_k$ or $\bar{P}_k$ the polynomials of degree k.

For $n \geq 4$ even,

$$M(h) = \sum_{k=0}^{2n} m_k(h) I_{k-6} = \sum_{i=0}^{n-1} m_{2i+1}(h) I_{2i-5} + \sum_{i=0}^{n} m_{2i}(h) I_{2i-6} .$$

Note that for $i \geq 3$,

$$\begin{aligned} m_{2i+1}(h) I_{2i-5} &= P_{\left[\frac{2i+1}{4}\right]}(-h)^{-\frac{3}{2}}(h+1) \sum_{j=0}^{\left[\frac{i-3}{2}\right]} d_{j,i-3}(-h)^{j-i+3} \\ &= (-h)^{\frac{3}{2}-i}(h+1) P_{\left[\frac{i}{2}\right]+\left[\frac{i-3}{2}\right]}(h) \\ &= (-h)^{\frac{3}{2}-i}(h+1) P_{i-2}(h) . \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{n-1} m_{2i+1} I_{2i-5} &= \sum_{i=0}^{2} m_{2i+1} I_{2i-5} + \sum_{i=3}^{n-1} m_{2i+1} I_{2i-5} \\ &= (-h)^{-\frac{1}{2}} P_3(\sqrt{-h}) + (-h)^{\frac{5}{2}-n}(h+1) P_{n-3}(h) \\ &= \left((-h)^{-\frac{1}{2}} - 1\right) P_2(\sqrt{-h}) + (-h)^{\frac{5}{2}-n}(h+1) P_{n-3}(h) , \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{n} m_{2i}(h) I_{2i-6} &= \sum_{i=0}^{2} m_{2i} I_{2i-6} + \sum_{i=3}^{n} m_{2i} I_{2i-6} \\ &= \frac{c}{h}(4J_0 - 5J_2) + C_0 J_0 + C_1 J_2 \\ &\quad + \sum_{i=3}^{n} P_{\left[\frac{i}{2}\right]}(h)(-h)^{-i+2} \left(P_{\left[\frac{i-4}{2}\right]} J_2 + P_{\left[\frac{i-3}{2}\right]} J_0\right) \\ &= (-h)^{2-n}(P_{n-2}(h) J_2 + \overline{P}_{n-2}(h) J_0) . \end{aligned}$$

Hence, for $n \geq 4$ even, we have:

$$\begin{aligned} M(h) = &\left((-h)^{-\frac{1}{2}} - 1\right) P_2(\sqrt{-h}) + (-h)^{\frac{5}{2}-n}(h+1) \widetilde{P}_{n-3}(h) \\ &+ (-h)^{2-n}(P_{n-2}(h) J_2 + \overline{P}_{n-2}(h) J_0) . \end{aligned} \tag{31}$$

Similarly, for $n \geq 5$ odd, we get

$$M(h) = \left((-h)^{-\frac{1}{2}} - 1\right) P_2(\sqrt{-h}) + (-h)^{\frac{5}{2}-n}(h+1)\widetilde{P}_{n-3}(h) + (-h)^{3-n}(P_{n-3}(h)J_2 + \overline{P}_{n-3}(h)J_0)\,. \tag{32}$$

We claim that J_0, J_2 satisfy the following Picard-Fuchs equation:

$$4h(h+1)\begin{pmatrix} J_0' \\ J_2' \end{pmatrix} = \begin{pmatrix} 4+3h & -5 \\ -h & 5h \end{pmatrix}\begin{pmatrix} J_0 \\ J_2 \end{pmatrix}\,. \tag{33}$$

Indeed, from (29),

$$J_4 = \frac{8}{7}J_2 + \frac{h}{7}J_0\,, \quad J_6 = \frac{4h}{21}J_0 + \frac{32+7h}{21}J_2\,, \tag{34}$$

$$\begin{aligned} J_0 &= \int_{x_1}^{x_2} y dx = \int_{x_1}^{x_2} \frac{y^2}{y}\,dx = \int_{x_1}^{x_2} \frac{-x^4+2x^2+h}{y}\,dx \\ &= -2J_4' + 4J_2' + 2hJ_0' \\ &= -\frac{2}{7}J_0 + \frac{12h}{7}J_0' + \frac{12}{7}J_2'\,, \end{aligned}$$

where we have used (34) in the last equality. Then

$$3J_0 = 4hJ_0' + 4J_2'\,. \tag{35}$$

Now we consider

$$\begin{aligned} J_2 &= \int_{x_1}^{x_2} x^2 y dx = \int_{x_1}^{x_2} x^2\frac{y^2}{y}\,dx \\ &= \int_{x_1}^{x_2} \frac{x^2}{y}(-x^4+2x^2+h)\,dx \\ &= -2J_6' + 4J_4' + 2hJ_2' \\ &= \frac{4}{21}J_0 - \frac{2}{3}J_2 + \frac{4h}{21}J_0' + \left(\frac{32}{21} + \frac{4}{3}h\right)J_2'\,, \end{aligned}$$

or

$$5J_2 - \frac{4}{7}J_0 = \frac{4h}{7}J_0' + (\frac{32}{7} + 4h)J_2'\,. \tag{36}$$

From (35), (36) we obtain (33).

Finally, we estimate the number of zeros of

$$M(h) = R_1(h) + R_2(h)J_2 + R_0(h)J_0\,,$$

where from (31) and (32),

$$\begin{aligned} R_1 &= ((-h)^{-\frac{1}{2}} - 1)P_2(\sqrt{-h}) + (-h)^{\frac{5}{2}-n}(1+h)\widetilde{P}_{n-3}(h)\,, \\ R_2 &= (-h)^{2-n}P_{n-2}(h)\,, \\ R_0 &= (-h)^{2-n}\overline{P}_{n-2}(h)\,. \end{aligned}$$

The method, which we are using here, is different from the previous one. Here we divide $M(h)$ into two parts, $R_1(h)$ and $R_2(h)J_2 + R_0(h)J_0$, the first part by Rolle theorem, the second part, using Picard-Fuchs equation (33), deducing a Ricatti equation. Again by Rolle theorem, we prove the proposition. Thus

$$\begin{aligned} &\#\{-1<h<0 \,|\, M=0\} \le \\ &\qquad \le \#\left\{-1<h<0 \,|\, \frac{M}{R_1}=0\right\} + \#\{-1<h<0 \,|\, R_1=0\}\,. \end{aligned} \tag{37}$$

We claim that

$$\#\{-1<h<0 \,|\, R_1=0\} \le n-1\,. \tag{38}$$

Indeed, let $\sqrt{-h}=t$, then

$$R_1 = (t^{-1}-1)P_2(t) + t^{5-2n}(1-t^2)\widetilde{P}_{n-3}(t^2)\,.$$

Computing the fourth derivative, we get

$$(tR_1)^{(4)} = t^{-6}P_{n-4}\left(\frac{1}{t^2}\right)\,,$$

which implies

$$\#\{t>0 \,|\, (tR_1)^4 = 0\} \le n-4\,.$$

By Rolle theorem,

$$\#\,\{R_1=0\} \le \#\{tR_1=0\} \le n\,.$$

Since $R_1\,|_{t=1}=0$, (38) holds.

By (33),

$$\begin{aligned} \left(\frac{M}{R_1}\right)' &= \frac{R_2'R_1 - R_1'R_2}{R_1^2}J_2 + \frac{R_2}{R_1}\left(\frac{-hJ_0+5hJ_2}{4h(h+1)}\right) \\ &\quad + \frac{R_0'R_1 - R_1'R_0}{R_1^2}J_0 + \frac{R_0}{R_1}\frac{(4+3h)J_0 - 5J_2}{4h(h+1)} \\ &= U_0J_0 + U_2J_2\,, \end{aligned}$$

where

$$U_0 = \frac{1}{4R_1^2 h(h+1)}(4h(h+1)(R_0' R_1 - R_1' R_0) - 4R_1 R_2 h + R_0 R_1(4+3h))\,,$$
$$U_2 = \frac{1}{4R_1^2 h(h+1)}(4h(h+1)(R_2' R_1 - R_1' R_2) + 5R_1 R_2 h - 5R_0 R_1)\,.$$

By straightforward computations we have

$$U_0 = \frac{(-h)^{1-2n} P_{4n+3}(\sqrt{-h})}{4R_1^2 h(h+1)}\,,$$
$$U_2 = \frac{(-h)^{1-2n} \overline{P}_{4n+3}(\sqrt{-h})}{4R_1^2 h(h+1)}\,.$$

Let

$$g = P_{4n+3}(\sqrt{-h})J_0 + \overline{P}_{4n+3}(\sqrt{-h})J_2\,,$$

then

$$\#\left\{\left(\frac{M}{R_1}\right)' = 0\right\} \leq \#\{g=0\}\,. \tag{39}$$

Let P_m be the greatest common divisor of P_{4n+3} and $\overline{P}_{4n+3}$:

$$\overline{P}_{4n+3} = P_m \overline{P}_{4n+3-m}\,, \qquad P_{4n+3} = P_m \overline{P}_{4n+3-m}\,,$$

and if

$$V = P_{4n+3-m}J_0 + \overline{P}_{4n+3-m}J_2\,,$$

then

$$\#\{g=0\} \leq \#\{V=0\} + m\,. \tag{40}$$

Put

$$U = \frac{P_{4n+3-m}}{\overline{P}_{4n+3-m}} + \frac{J_2}{J_0}\,,$$

then

$$\#\{V=0\} = \#\{U=0\}\,. \tag{41}$$

By (33), U satisfies the following Ricatti equation:

$$\frac{dU}{dh} = \frac{1}{4h(h+1)}\left[5U^2 + (2h-4-10)\frac{P_{4n+3-m}}{\overline{P}_{4n+3-m}}U\right] + \frac{P_{8n+8-2m}(\sqrt{-h})}{4h(h+1)\overline{P}^2_{4n+3-m}(\sqrt{-h})},$$

Therefore, by general Rolle theorem,

$$\begin{aligned}\#\{u=0\} &\le \deg P_{8n+8-2m}(\sqrt{-h}) + \deg \overline{P}_{4n+3-m}(\sqrt{-h}) + 1\\ &= 8n+8-2m+4n+3-m+1\\ &= 12n-3m+12,\end{aligned}$$

which, together with (40) and (41), implies

$$\#\{g=0\} \le 12n+12.$$

From (39),

$$\#\left\{\left(\frac{M}{R_1}\right)' = 0\right\} \le 12n+12.$$

By Rolle theorem,

$$\begin{aligned}\#\left\{\frac{M}{R_1} = 0\right\} &\le \#\{R_1 = 0\} + 12n + 12 + 1\\ &\le 13n + 12,\end{aligned}$$

which, together with (37), implies

$$\#\{-1 < h < 0 \,|\, M(h) = 0\} \le 14n + 11 \quad \text{for} \quad n \ge 4.$$

■

Statement (e) of Theorem 2 for system S_4 is directly implied from Proposition 13.

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