Linear Estimation of the Number of Zeros of Abelian Integrals for some Cubic Isochronous Centers

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Abstract

We study the number of limit cycles that bifurcate from the periodic orbits of cubic reversible isochronous centers having all their orbits formed by conics, when we perturb such systems inside the class of all polynomial systems of degree n.

1 Introduction and statement of the main results

The main open problem in the qualitative theory of real planar differential systems is the determination of limit cycles. A classical way to obtain limit cycles is perturbing the periodic orbits of a center. There are several methods for studying the bifurcated limit cycles from a center. The major part of the methods are based either on the Poincaré return map, or on the Poincaré-Melnikov integral or Abelian integral which are equivalent in the plane (see for instance [1]). Recently some other methods are presented, ones based on the inverse integrating factor (see [6]), others are based in the reduction of the problem to a one dimensional differential equation (see [12] and [15]). In general these methods are difficult to apply for studying the limit cycles that bifurcate from the periodic orbits of a center when the system is integrable but not Hamiltonian. As far as we know few papers study the non-Hamiltonian centers, see for instance [2], [3], [7], [10], [12], [13] and [15].

By definition a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P and Q are polynomials with real coefficients. We say that $n = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system.

In what follows for $n = 2$ or $n = 3$ systems (1) are called *quadratic* or *cubic* systems respectively. Many authors have studied the limit cycles which bifurcate from periodic orbits of a center for a quadratic system, see for instance, [8], [14], [13], [18], [21], [22], [23] and [25].

In this paper we will study the number of zeros of Abelian integrals for some cubic isochronous centers, which are characterized in the following proposition.

Proposition 1. *After a linear change of variables and a rescaling of the time variable (if necessary) a polynomial system* (1) *satisfying*

(i) the polynomials P and Q are coprime inside the ring of real polynomials in two variables,

(ii) all its orbits are conics, and

(iii) it has a center;

goes over to

$$\dot{x} = -y, \qquad \dot{y} = x; \tag{2}$$

$$\dot{x} = -y + x^2, \qquad \dot{y} = x(1+y); \tag{3}$$

$$\dot{x} = -y + \frac{1}{2}x^2 - \frac{1}{2}y^2, \qquad \dot{y} = x(1+y); \tag{4}$$

$$\dot{x} = -y + bx^2 - 2axy - by^2 + x^2y, \quad \dot{y} = x + ax^2 + 2bxy - ay^2 + xy^2; \tag{5}$$

where $(a,b) \in \mathbf{R}^2 \setminus \{(0,\pm 1)\}$. *Moreover all these polynomial systems are isochronous.*

Proposition 1 will be proved in Section 2.

We remark that multiplying $\dot{x}$ and $\dot{y}$ in the systems of Proposition 1 by an arbitrary polynomial we get all polynomial systems with a center such that all their orbits are conics. We also note that Zoladek in [25] and Mardesic, Rousseau and Toni in [19] mention implicitely (at least for quadratic systems) some relationship between the fact that all orbits of a center are conics and the fact that the center is isochronous.

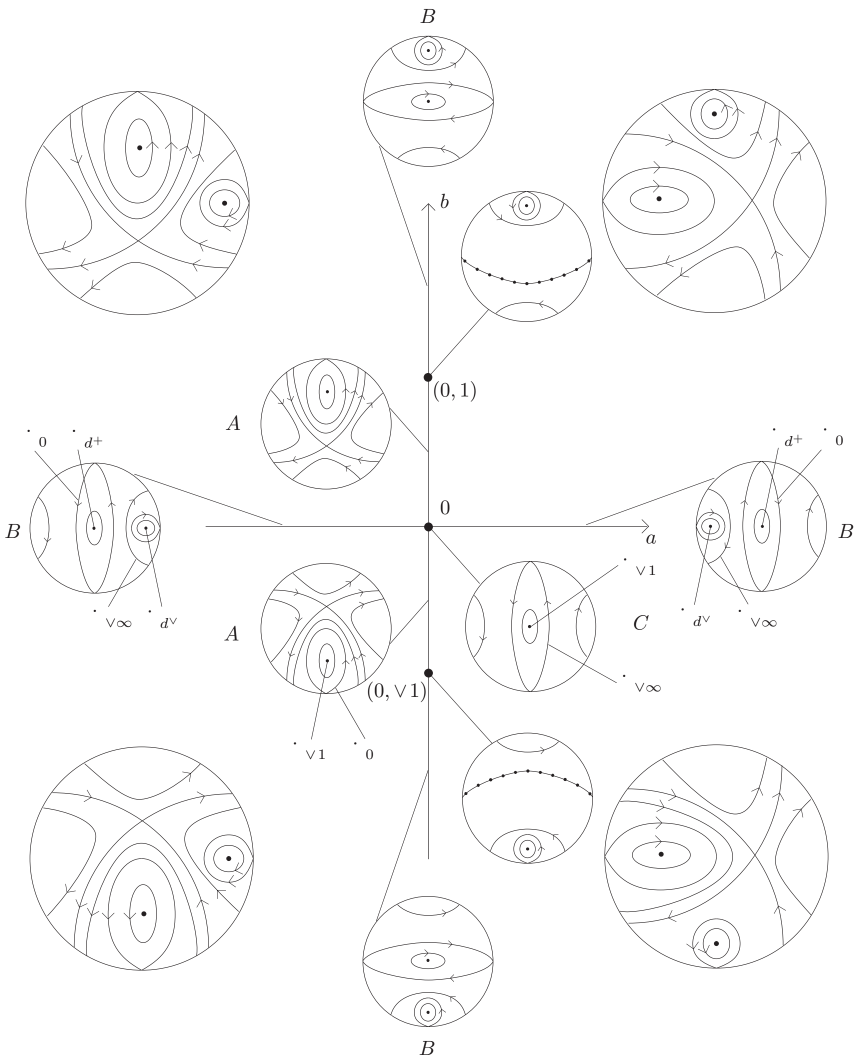

Figure 1: The phase portraits of the isochronous centers of type (5).

We note that the perturbation of the linear center of Proposition 1 inside the class of all polynomial systems of degree n has been studied by several authors, see for instance [1] and [5]. There are four quadratic isochronous centers, systems (3) and (4) in Proposition 1 are two of them, the other two can be transformed to (3) and (4) respectively. The perturbation of the quadratic isochronous centers inside the class of all polynomials systems of degree n has been studied in [13], the case $n = 2$ was studied by Chicone and Jacobs [2]. In [13] we obtain the exact upper bounds of the number of zeros of Abelian integrals for three of these centers and an upper bound for the remaining one, all the bounds depend linearly with the degree of perturbations.

The main result of this paper is to study the number of the limit cycles that bifurcate from the periodic orbits of cubic reversible isochronous centers of Proposition 1 when we perturb such systems inside the class of all polynomial systems of degree n. The phase portraits of systems (5) are given in Figure 1. Here we say that a center of a polynomial system (1) is *reversible* if there exists an straight line through the center such that the solutions of the system are symmetric with respect to this straight line and a reversibility of the time variable.

Obviously, system (5) is reversible if and only if $ab = 0$. There are the following three different phase portraits for reversible systems inside the class (5):

(A) $a = 0$, $0 < |b| < 1$;

(B) $a = 0$, $|b| > 1$ or $b = 0$, $a \neq 0$,

(C) $a = b = 0$.

The technique used in this paper for studying the perturbation of integrable but non-Hamiltonian systems is classical. It consists of writing a non-Hamiltonian center in a Hamiltonian one multiplying by a suitable integrating factor. The key point in our approach is that, by Green's Theorem, we will compute the Abelian integral through a double integral. These double integrals for these cubic isochronous centers are very easy to compute in comparison with the usual single Abelian integral. This technique was used by first time in [13]. Thus, our main result is the following one.

Theorem 2. *When we perturb the three cubic reversible isochronous systems (A), (B) and (C) inside the class of all polynomial systems of degree n an upper bound for the number of zeros (taking into account their multiplicity) of the Abelian integral associated to system:*

(A) is $(3n+1)/2$ *if* $n \geq 5$ *odd, and* $(3n/2) - 1$ *if* $n \geq 6$ *even;*

(B) is 1 *for* $n = 0, 1, 2$*;* 4 *for* $n = 3$*;* 5 *for* $n = 4$*; and for* $n \geq 5$ *is* $3(n-1)/2$ *if* n *is odd, and* $(3n/2) - 1$ *if* n *is even;*

(C) is 0 *for* $n = 0$*;* 1 *for* $n = 1, 2$*; and* $2\,[(n+1)/2]$ *for* $n \geq 3$*.*

Statements (A), (B), and (C) of Theorem 2 are proved in Sections 3, 4, and 5 respectively.

We note that in spite of our systems are non–Hamiltonian the upper bounds for the number of zeros of the Abelian integrals given in Theorem 2 are all linear in the degree n of the polynomial perturbation, which accord with the traditional prediction and some known results for the perturbation of Hamiltonian centers, see for instants [4], [9], [14], [15], [16], [20], [21], [22] and [24].

2 Proof of Proposition 1

Assume that all the orbits of polynomial system (1) are conics, and that P and Q are coprime. By Jouanolou results [11], if a polynomial system of degree m admits $q > 2 + [m(m+1)/2]$ algebraic solutions, then it has a rational first integral $H = f/g$, where f, g are polynomials. By the assumptions of Proposition 1, for any constant h, $f/g = h$ determines a conic, therefore, $\max\{\deg f, \deg g\} = 2$. Without loss of generality, we may assume that $h = 0$ corresponds to the center $(0, 0)$, which implies that $g(0, 0) \neq 0$, $f(0, 0) = 0$. Therefore, after a rotation of coordinates and a rescaling of x and y, f can be transformed into the form $f(x, y) = x^2 + y^2$, and the first integral can be written into the form

$$H(x, y) = \frac{x^2 + y^2}{1 + ax + by + cxy + dx^2 + ey^2}\,.$$

Taking another rotation of coordinates, we can eliminate the term xy in the denominator. Obviously, the function

$$\frac{H}{1 - dH} = \frac{x^2 + y^2}{1 + ax + by + (e - d)y^2}$$

is also a first integral also. In short, we get a first integral of system (1) as follows:

$$H(x, y) = \frac{f}{g}\,, \qquad f = x^2 + y^2\,, \qquad g = 1 + ax + by + cy^2\,. \tag{6}$$

Obviously, the function H in (6) is a first integral of the following cubic system

$$\begin{aligned} \dot{x} &= g'_y f - f'_y g \,, \\ \dot{y} &= f'_x g - g'_x f \,. \end{aligned} \tag{7}$$

By removing the common factor in $\dot{x}$ and $\dot{y}$ (if they have), we get a polynomial system of degree ≤ 3 as follows:

$$\begin{aligned} \dot{x} &= \widetilde{P}(x, y) \,, \\ \dot{y} &= \widetilde{Q}(x, y) \,, \end{aligned} \tag{8}$$

where $\widetilde{P}$ and $\widetilde{Q}$ are coprime. Since (1) and (8) have a common first integral H, we obtain

$$\frac{P}{Q} \equiv \frac{\widetilde{P}}{\widetilde{Q}} \,,$$

which, by the fact that $(P, Q) = 1$ and $(\widetilde{P}, \widetilde{Q}) = 1$, implies

$$\frac{P}{\widetilde{P}} = \frac{Q}{\widetilde{Q}} = \text{constant}.$$

Next we prove that system (8) can be written into the forms given in Proposition 1.

(i) if $g \equiv 1$, then system (8) is the linear center (2).

(ii) if $c = 0$, and $a^2 + b^2 > 0$, then after a rotation of coordinates, $g = 1 + y$. Therefore system (8) is the system (4).

(iii) if $a = 0$ and $b^2 = 4c \neq 0$, then after a rescaling of y and x, $g = (1+y)^2$. Hence system (8) is the system (3).

(iv) if $c \neq 0$ and $a^2 + (b^2 - 4c)^2 > 0$, then rescaling x and y, we can get $c = \pm 1$ and system (8) becomes

$$\begin{aligned} \dot{x} &= -y + bx^2 - 2axy - by^2 + cx^2 y \,, \\ \dot{y} &= x + ax^2 + 2bxy - ay^2 + cxy^2 \,, \qquad c = \pm 1 \,. \end{aligned}$$

If $c = -1$, we can change the sign of c using the transformation $(x, y, t) \to (y, x, -t)$. Thus we get the system (5).

Now we prove that all centers given in Proposition 1 are isochronous. The isochronousity of systems (2), (3) and (4) is well known (see [17]), here we only prove it for system (5).

System (5) has the first integral

$$H(x,y) = \frac{x^2+y^2}{1+2ax+2by+y^2}\,.$$

Let

$$u = \frac{x}{\sqrt{1+2ax+2by+y^2}}\,, \qquad v = \frac{y}{\sqrt{1+2ax+2by+y^2}}\,,$$

Then

$$x = -\frac{(au+bv+\sqrt{\Delta})u}{(v^2-1)}\,, \qquad y = -\frac{(au+bv+\sqrt{\Delta})v}{(v^2-1)}\,,$$

where $\Delta = (au+bv)^2 - v^2 + 1$, and system (5) goes over to

$$\begin{aligned} \dot{u} &= -v(1+ax+by)\,, \\ \dot{v} &= u(1+ax+by)\,. \end{aligned}$$

All periodic orbits of this system are concentric circles. Next we prove that they have the same period.

Let $u = r\cos\theta$, $v = r\sin\theta$, then

$$\begin{aligned} \frac{d\theta}{dt} &= 1+ax+by \\ &= 1 - \left(\frac{au+bv+\sqrt{\Delta}}{(v^2-1)}\right)(au+bv) \\ &= -\frac{\Delta+\sqrt{\Delta}(au+bv)}{(v^2-1)} \\ &= \frac{\sqrt{\Delta}}{\sqrt{\Delta}-(au+bv)}\,. \end{aligned}$$

Then

$$\begin{aligned} T &= \int_0^T dt = \int_0^{2\pi} \frac{\sqrt{\Delta}-(au+bv)}{\sqrt{\Delta}}\,d\theta \\ &= 2\pi - \int_0^{2\pi} \frac{\operatorname{arccos}\theta + br\sin\theta}{\sqrt{(\operatorname{arccos}\theta+br\sin\theta)^2 - r^2\sin^2\theta + 1}}\,d\theta \\ &= 2\pi\,. \end{aligned}$$

The proof of Proposition 1 is complete. ∎

3 Linear estimation of the number zeros of Abelian integral for system A

Consider the polynomial perturbation of system A:

$$\begin{aligned} \dot{x} &= -y + bx^2 - by^2 + x^2 y + \varepsilon P(x,y)\,, \\ \dot{y} &= x + 2bxy + xy^2 + \varepsilon Q(x,y)\,, \end{aligned} \tag{9}$$

where $0 < |b| < 1$, P and Q are polynomials of degree n. For $\varepsilon = 0$, system (9) has a first integral $H = (x^2 - 1 - 2by)/(1 + 2by + y^2)$. For $-1 < h < 0$, the level curves $\Gamma_h\colon H = h$ are periodic orbits which surround the center $\Gamma_{-1}\colon (0,0)$ (see Figure 1). Multiplying by the integrating factor $2/(1+2by+y^2)$, system (9) is changed to the form

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial y} + \varepsilon \frac{2P}{(1+2by+y^2)^2}\,, \\ \dot{y} &= \frac{\partial H}{\partial x} + \varepsilon \frac{2Q}{(1+2by+y^2)^2}\,. \end{aligned}$$

The Abelian integral for the above system is defined as

$$M(h) = \oint_{H=h} \left(\frac{2P}{(1+2by+y^2)^2}\, dy - \frac{2Q}{(1+2by+y^2)^2}\, dx \right)\,, \quad -1 \le h < 0\,.$$

By Green's formula

$$\begin{aligned} M(h) &= 2 \iint_{H \le h} \left[\frac{\partial}{\partial x} \left(\frac{P}{(1+2by+y^2)^2} \right) + \frac{\partial}{\partial y} \left(\frac{Q}{(1+2by+y^2)^2} \right) \right] dx\, dy \\ &= 2 \iint_{H \le h} \left[\frac{P'_x + Q'_y}{(1+2by+y^2)^2} - \frac{4(b+y)Q}{(1+2by+y^2)^3} \right] dx\, dy\,. \end{aligned}$$

Set $x = u$, and $y = \sqrt{1-b^2}\, v - b$. Then

$$H(u,v) = H(u, \sqrt{1-b^2}\, v - b) = \frac{u^2 - 1 - 2b\sqrt{1-b^2} v + 2b^2}{(1-b^2)(1+v^2)}\,,$$

and

$$M(h) = 2 \iint_{H \le h} \left[\frac{P'_x + Q'_y}{(1-b^2)^2(1+v^2)^2} - \frac{4(b+y)Q}{(1-b^2)^3(1+v^2)^3} \right] \sqrt{1-b^2}\, du\, dv\,.$$

Let

$$\frac{2P'_x + 2Q'_y}{(1-b^2)} = \sum_{i+j\le n-1} (1-b^2)^{-\frac{i}{2}} a_{ij} u^i v^j \, ,$$
$$\frac{8(b+y)Q}{(1-b^2)^2} = \sum_{i+j\le n} (1-b^2)^{-\frac{i}{2}} b_{ij} u^i v^{j+1} \, .$$

Then

$$\begin{aligned} M(h) &= \sum_{i+j\le n-1} a_{ij}(1-b^2)^{-\frac{i}{2}-\frac{1}{2}} \iint_{H\le h} \frac{u^i v^j}{(1+v^2)^2}\, du\, dv \\ &\quad - \sum_{i+j\le n} b_{ij}(1-b^2)^{-\frac{i}{2}-\frac{1}{2}} \iint_{H\le h} \frac{u^i v^{j+1}}{(1+v^2)^3}\, du\, dv \\ &= \sum_{2i+j\le n-1} a_{2i\,j}(1-b^2)^{-i-\frac{1}{2}} \iint_{H\le h} \frac{u^{2i} v^j}{(1+v^2)^3}\, du\, dv \\ &\quad - \sum_{2i+j\le n} b_{2i\,j}(1-b^2)^{-i-\frac{1}{2}} \iint_{H\le h} \frac{u^{2i} v^{j+1}}{(1+v^2)^3}\, du\, dv \\ &= \sum_{2i+j\le n-1} \frac{2a_{2i\,j}}{(2i+1)} \int_{v_1}^{v_2} \frac{v^j \Delta^i \sqrt{\Delta}}{(1+v^2)^2}\, dv \\ &\quad - \sum_{2i+j\le n} \frac{2b_{2i\,j}}{(2i+1)} \int_{v_1}^{v_2} \frac{v^{j+1} \Delta^i \sqrt{\Delta}}{(1+v^2)^3}\, dv \, , \end{aligned}$$

where $\Delta = h(1+v^2) + 2bv/\sqrt{1-b^2} + (1-2b^2)/(1-b^2)$, v_1, v_2 are two roots of the equation $\Delta = 0$, such that

$$\begin{aligned} v_1 + v_2 &= \frac{\overline{b}}{h}\, , \qquad v_1 v_2 = 1 + \frac{\overline{a}}{h}\, , \\ \overline{b} &= -\frac{2b}{\sqrt{1-b^2}}\, , \qquad \overline{a} = \frac{1-2b^2}{1-b^2}\, . \end{aligned} \tag{10}$$

If

$$\sum_{2i+j\le n-1} \frac{2a_{2i\,j}}{2i+1}(1+v^2) v^j \Delta^i - \sum_{2i+j\le n} \frac{2b_{2i\,j}}{(2i+1)} v^{j+1}\Delta^i =$$
$$\sum_{k=0}^{\left[\frac{n}{2}\right]} m_k(h) v (1+v^2)^k + \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \overline{m}_k(h)(1+v^2)^k \, ,$$

where m_k, $\overline{m}_k$ are polynomials of h with

$$\deg m_k \leq k\,, \qquad \deg \overline{m}_k \leq k\,, \tag{11}$$

then

$$M(h) = \sum_{k=0}^{\left[\frac{n}{2}\right]} m_k(h) J_{k-3} + \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \overline{m}_k(h) I_{k-3}\,, \tag{12}$$

where

$$I_k = \sqrt{-h}\int_{v_1}^{v_2} (1+v^2)^k \sqrt{(v-v_1)(v_2-v)}\,dv\,,$$
$$J_k = \sqrt{-h}\int_{v_1}^{v_2} v(1+v^2)^k \sqrt{(v-v_1)(v_2-v)}\,dv\,.$$

If $\sqrt{(v_2-v)(v-v_1)} = t(v-v_1)$, then

$$I_k = 2\sqrt{-h}(v_2-v_1)^2 \int_0^\infty \frac{t^2[(1+t^2)^2+(v_2+v_1t^2)^2]^k}{(1+t^2)^{2k+3}}\,dt\,,$$
$$J_k = 2\sqrt{-h}(v_2-v_1)^2 \int_0^\infty \frac{t^2(v_2+v_1t^2)[(1+t^2)^2+(v_1+v_1t^2)^2]^k}{(1+t^2)^{2k+4}}\,dt\,.$$

Consequently we have that

$$\begin{aligned} I_k(v_1,v_2) &= 2\sqrt{-h}(v_2-v_1)^2 \int_0^\infty \frac{t^2[(1+t^2)^2+(v_1+v_1t^2)^2]^k}{(1+t^2)^{2k+3}}\,dt \quad (t=u^{-1}) \\ &= 2\sqrt{-h}(v_2-v_1)^2 \int_0^\infty \frac{u^2[(1+u^2)^2+(v_1+v_2t^2)^2]^k}{(1+t^2)^{2k+3}}\,dt \\ &= I_k(v_2,v_1)\,. \end{aligned}$$

Similarly, we have $J_k(v_1,v_2) = J_k(v_2,v_1)$. These mean that for $k \geq 0$, I_k and J_k are symmetric polynomials of (v_1,v_2) of degree $2k+2$ and $2k+3$, respectively. Therefore, theres exist constants c_{ij}, d_{ij} such that

$$\begin{aligned} I_k &= 2\sqrt{-h}(v_2-v_1)^2 \sum_{i+2j\leq 2k} c_{ij}(v_1+v_2)^i(v_1v_2)^j \quad \text{(by (10))} \\ &= 2\sqrt{-h}(v_2-v_1)^2 \sum_{i+2j\leq 2k} c_{ij}\frac{\overline{b}^i}{h^i}(1+\frac{\overline{a}}{h})^j \\ &= 2\sqrt{-h}(v_2-v_1)^2 h^{-2k} P_{2k}(h)\,, \end{aligned} \tag{13}$$

and

$$\begin{aligned} J_k &= 2\sqrt{-h}(v_2-v_1)^2 \sum_{i+2j\le 2k+1} d_{ij}(v_1+v_2)^i(v_1v_2)^j \\ &= 2\sqrt{-h}(v_2-v_1)^2 \sum_{i+2j\le 2k+1} d_{ij}\frac{\overline{b}^i}{h^i}\left(1+\frac{\overline{a}}{h}\right)^j \\ &= 2\sqrt{-h}(v_2-v_1)^2 h^{-2k-1}P_{2k+1}(h)\,, \end{aligned} \tag{14}$$

where P_k denote a polynomial of degree k.

Next we calculate $I_k,\ J_k$ for $k<0$. Note that

$$(1+t^2)^2+(v_2+v_1t^2)^2=(1+v_1^2)(t^2+\alpha t+\beta)(t^2-\alpha t+\beta)\,,$$

$$\alpha=\sqrt{\frac{2(\sqrt{(1+v_1^2)(1+v_2^2)}-1-v_1v_2)}{(1+v_1^2)}}\,,\qquad \beta=\sqrt{\frac{1+v_2^2}{1+v_1^2}}\,, \tag{15}$$

we obtain

$$I_k=2\sqrt{-h}(v_2-v_1)^2(1+v_1^2)^kU_k\,, \tag{16}$$

$$J_k=2\sqrt{-h}(v_2-v_1)^2(1+v_1^2)^kV_k\,, \tag{17}$$

where

$$\begin{aligned} U_k &= \int_0^\infty \frac{t^2(t^2+\alpha t+\beta)^k(t^2-\alpha t+\beta)^k}{(1+t^2)^{2k+3}}\,dt\,, \\ V_k &= \int_0^\infty \frac{t^2(v_2+v_1t^2)(t^2+\alpha t+\beta)^k(t^2-\alpha t+\beta)^k}{(1+t^2)^{2k+4}}\,dt\,. \end{aligned}$$

Computing we get

$$\begin{aligned} U_{-1} &= -\frac{\pi}{2}\left[\frac{\sqrt{\Delta}-\beta-1}{(1-2\beta+\beta^2+\alpha^2)\sqrt{\Delta}}\right]\,, \\ U_{-2} &= \frac{\pi}{4}\frac{\beta+1}{\beta\Delta^{\frac{3}{2}}}\,, \\ U_{-3} &= -\frac{\pi}{16}\frac{-9+\alpha^2-7\beta^{-2}-7\beta+\beta^{-3}\alpha^2-9\beta^{-1}}{\Delta^{\frac{5}{2}}}\,, \end{aligned}$$

and

$$
\begin{aligned}
V_{-1} = & \frac{\pi}{(\beta^2 - 2\beta + 1 + \alpha^2)^2\sqrt{\Delta}} \\
& \left[\left(-\frac{3}{4}\beta^2\sqrt{\Delta} - \frac{3}{2}\beta + \frac{1}{2}\beta\sqrt{\Delta} + \frac{1}{2}\alpha^2 + \right. \right. \\
& \qquad \left. \beta^2 + \frac{1}{2}\beta^3 + \frac{1}{4}\sqrt{\Delta} - \frac{1}{4}\alpha^2\sqrt{\Delta} \right) v_1 + \\
& \left(\frac{1}{4}\beta^2\sqrt{\Delta} + \frac{1}{2}\alpha^2\beta + \beta + \frac{1}{2}\beta\sqrt{\Delta} + \right. \\
& \qquad \left. \left. \frac{1}{2} - \frac{3}{4}\sqrt{\Delta} - \frac{3}{2}\beta^2 - \frac{1}{4}\alpha^2\sqrt{\Delta} \right) v_2 \right] , \\
V_{-2} = & \frac{\pi}{4}\frac{(v_1 + \beta^{-1}v_2)}{\Delta^{\frac{3}{2}}} , \\
V_{-3} = & \frac{\pi}{16\Delta^{\frac{5}{2}}} \left[(7\beta + 6 - \alpha^2 + 3\beta^{-1})v_1 + (3 + 6\beta^{-1} - \beta^{-3}\alpha^2 + 7\beta^{-2})v_2 \right] .
\end{aligned}
$$

Substituting formulas (10) and (15) into U_{-i} and V_{-i}, and using (16) and (17), we get

$$
\begin{aligned}
I_{-1} &= \frac{\pi(h+1))}{1+\sqrt{-h}} , \\
I_{-2} &= (h+1)P_0(h) , \\
I_{-3} &= (h+1)P_1(h) ,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{-1} &= \frac{(h+1)}{\sqrt{-h}(\sqrt{-h}+1)}\overline{P}_0(h) , \\
J_{-2} &= (h+1)\widetilde{P}_o(h) , \\
J_{-3} &= (h+1)P_2(h) ,
\end{aligned}
\tag{18}
$$

where from now on P_k, $\overline{P}_k$, $\widetilde{P}_k$ denote polynomials of degree k.

From (11), (12) and (18), we obtain

$$
M(h) = \sum_{k=0}^{1} m_k(h)J_{k-3} + \sum_{k=0}^{1} \overline{m}_k(h)I_{k-3} = (h+1)P_2(h) , \quad \text{for} \quad n \le 2 ,
\tag{19}
$$

$$
\begin{aligned}
M(h) &= \sum_{k=0}^{1} m_k(h)J_{k-3} + \sum_{k=0}^{2} \overline{m}_k(h)I_{k-3} \qquad (20)\\
&= \frac{(h+1)}{P_1(\sqrt{-h})}(P_1(\sqrt{-h})P_2(h) + \overline{P}_2(h))\,, \quad \text{for} \quad n=3\,,\\
M(h) &= \sum_{k=0}^{2} m_k(h)J_{k-3} + \sum_{k=0}^{2} \overline{m}_k(h)I_{k-3} \qquad (21)\\
&= \frac{(h+1)}{\sqrt{-h}(\sqrt{-h}+1)}[P_3(h) + P_2(h)\sqrt{-h}]\\
&= \frac{1}{h}(hP_3(h) + \widetilde{P}_3(h)\sqrt{-h})\,, \qquad \text{for} \quad n=4\,.
\end{aligned}
$$

For $n \geq 5$, from (12) and (13) we obtain

$$
\begin{aligned}
\sum_{k=3}^{\left[\frac{n+1}{2}\right]} \overline{m}_k(h)I_{k-3} &= 2\sqrt{-h}(v_2-v_1)^2 \sum_{k=3}^{\left[\frac{n+1}{2}\right]} h^{6-2k}P_{2k-6}(h)\overline{m}_k(h)\\
&= \sqrt{-h}(h+1)h^{4-2\left[\frac{n+1}{2}\right]}P_{3\left[\frac{n+1}{2}\right]-5}(h)\,.
\end{aligned} \qquad (22)
$$

From (12) and (14), it follows that

$$
\begin{aligned}
\sum_{k=3}^{\left[\frac{n}{2}\right]} m_k(h)J_{k-3} &= 2\sqrt{-h}(v_2-v_1)^2 \sum_{k=3}^{\left[\frac{n}{2}\right]} h^{5-2k}P_{2k-5}(h)m_k(h)\\
&= \sqrt{-h}(h+1)h^{3-2\left[\frac{n}{2}\right]}P_{3\left[\frac{n}{2}\right]-4}(h)\,.
\end{aligned} \qquad (23)
$$

From (21), (22) and (23), we obtain

$$
M(h) = h^{3-n}[P_3(h)h^{n-3} + \sqrt{-h}P_{\frac{3}{2}n-\frac{5}{2}}(h)]\,, \quad \text{for} \quad n \geq 5 \quad \text{odd}\,, \qquad (24)
$$

$$
M(h) = h^{3-n}[P_3(h)h^{n-3} + \sqrt{-h}P_{\frac{3}{2}n-4}(h)]\,, \quad \text{for} \quad n \geq 6 \quad \text{even}\,. \qquad (25)
$$

From (19),

$$
\#\{-1 < h < 0 \,|\, M(h) = 0\} \leq 2\,, \qquad \text{for} \quad n \leq 2\,.
$$

From (20),

$$
\#\{-1 < h < 0 \,|\, M(h) = 0\} \leq 5\,, \qquad \text{for} \quad n = 3\,.
$$

From (21),

$$\#\{-1 < h < 0 \,|\, M(h) = 0\} \leq 6 \,, \qquad \text{for} \quad n = 4 \,.$$

From (24),

$$\#\{h < 0 \,|\, M(h) = 0\} \leq \frac{3}{2}n - \frac{5}{2} + 1 + 3 + 1 - 1 = \frac{3}{2}n + \frac{3}{2} \,.$$

On the other hand, $M(-1) = 0$, therefore,

$$\#\{-1 < h < 0 \,|\, M(h) = 0\} \leq \frac{3}{2}n + \frac{1}{2} \,, \qquad \text{for} \quad n \geq 5 \quad \text{odd} \,.$$

From (25)

$$\#\{h < 0 \,|\, M(h) = 0\} \leq \frac{3}{2}n - 4 + 1 + 3 + 1 - 1 = \frac{3}{2}n \,.$$

On the other hand, $M(-1) = 0$, therefore

$$\#\{-1 < h < 0 \,|\, M(h) = 0\} \leq \frac{3}{2}n - 1 \,, \qquad \text{for} \quad n \geq 6 \quad \text{even} \,.$$

The proof of the conclusion of Theorem 2 for system A is completed.

4 Linear estimation of the number of zeros of Abelian integral for system B

The system B has two parallel invariant lines: $x = a \pm \sqrt{a^2 + 1}$, if $b = 0$, $a \neq 0$; $y = b \pm \sqrt{b^2 - 1}$, if $a = 0$, $|b| > 1$. By taking a suitable rotation and a rescaling of coordinates (if necessary) such that in the new coordinates these two invariant lines are $x = \pm 1$, system B can be written into the form

$$\dot{x} = -2y(x^2 - 1) \,, \quad \dot{y} = -2x^2 - 2dx - 1 - 2xy^2 \,, \quad d > 1 \,.$$

Consider the following polynomial perturbation of the above system:

$$\begin{aligned} \dot{x} &= -2y(x^2 - 1) + \varepsilon P(x, y) \,, \\ \dot{y} &= -x^2 - 2dx - 1 - 2xy^2 + \varepsilon Q(x, y) \,, \qquad d > 1 \,, \end{aligned} \tag{26}$$

where P, Q are polynomials of degree n. For $\varepsilon = 0$, system (26) has a first integral $H = (y^2 + x + d)/(x^2 - 1)$, such that $H = d_\pm = (-d \pm \sqrt{d^2 - 1})/2$ correspond to two centers. There are two families of periodic orbits $\Gamma_h\colon H = h$, $h \in (d_+, 0) \cup (-\infty, d_-)$, which surround the centers Γ_{d_+} and Γ_{d_-} respectively. Multiplying by the integrating factor $(x^2-1)^{-2}$, system (26) becomes

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial y} + \varepsilon \frac{P}{(x^2-y^2)}\,, \\ \dot{y} &= \frac{\partial H}{\partial x} + \varepsilon \frac{Q}{(x^2-1)^2}\,. \end{aligned} \tag{27}$$

The Abelian integral of system (27) is defined as

$$M(h) = \oint_{H=h} \left(\frac{P}{(x^2-1)^2}\,dy - \frac{Q}{(x^2-1)^2}\,dx \right)\,, \quad h \in [d_+, 0) \cup (-\infty, d_-]\,.$$

Denote by D_h the simple connected region with boundary Γ_h, by Green's formula,

$$\begin{aligned} M(h) &= \iint_{D_h} \left[\frac{\partial}{\partial x} \left(\frac{P}{(x^2-1)^2} \right) + \frac{\partial}{\partial y} \left(\frac{Q}{(x^2-1)^2} \right) \right] dx\,dy \\ &= \iint_{D_h} \left[\frac{P'_x + Q'_y}{(x^2-1)^2} - \frac{4xP}{(x^2-1)^3} \right] dx\,dy\,. \end{aligned}$$

If

$$\begin{aligned} P'_x + Q'_y &= \sum_{i+j \le n-1} a_{ij} x^i y^j\,, \\ P &= \sum_{i+j \le n} b_{ij} x^i y^j\,, \end{aligned}$$

and x_1, x_2 denote the two roots of the equation $hx^2 - x - h - d = 0$, then

$$x_1 x_2 = -1 - \frac{d}{h}\,, \qquad x_1 + x_2 = \frac{1}{h}\,, \tag{28}$$

and

$$\begin{aligned} M(h) &= \sum_{i+2j \le n-1} \int_{x_1}^{x_2} \frac{2a_{i\,2j} x^i}{(2j+1)(x^2-1)^2} (hx^2 - x - h - d)^j \\ &\qquad \sqrt{hx^2 - x - h - d}\,dx \\ &\quad - \sum_{i+2j \le n} \int_{x_1}^{x_2} \frac{8b_{i\,2j} x^i}{(2j+1)(x^2-1)^3} (hx^2 - x - h - d)^j \\ &\qquad \sqrt{hx^2 - x - h - d}\,dx \\ &= \sum_{k=0}^{\left[\frac{n}{2}\right]} m_k(h) J_{k-3} + \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \overline{m}_k(h) I_{k-3}\,, \end{aligned}$$

where m_k, $\overline{m}_k$ are polynomials of h with

$$\deg m_k \le k\,, \qquad \deg \overline{m}_k \le k\,, \tag{29}$$

$$\begin{aligned} I_k &= \int_{x_1}^{x_2} (x^2-1)\sqrt{hx^2-x-h-d}\,dx\,, \\ J_k &= \int_{x_1}^{x_2} x(x^2-1)^k\sqrt{hx^2-x-h-d}\,dx\,. \end{aligned}$$

If $\sqrt{(x_2-x)(x-x_1)} = t(x-x_1)$, then

$$\begin{aligned} I_k &= 2\sqrt{-h}(x_2-x_1)^2 U_k\,, \\ U_k &= \int_0^\infty \frac{t^2[(x_2+x_1t^2)^2-(1+t^2)^2]^k}{(1+t^2)^{2k+3}}\,dt\,, \end{aligned} \tag{30}$$

$$\begin{aligned} J_k &= 2\sqrt{-h}(x_2-x_1)^2 V_k\,, \\ V_k &= \int_0^\infty \frac{t^2(x_2+x_1t^2)[(x_2+x_1t^2)^2-(1+t^2)^2]^k}{(1+t^2)^{2k+4}}\,dt\,. \end{aligned} \tag{31}$$

Changing the variable t by $1/u$ in the integrals (30) and (31), we obtain

$$U_k(x_1,x_2) = U_k(x_2,x_1)\,, \quad V_k(x_1,x_2) = V_k(x_2,x_1)\,.$$

Therefore for $k \ge 0$, U_k and V_k are symmetrical polynomials of (x_1, x_2) of degree $2k$ and $2k+1$, respectively:

$$U_k = \sum_{i+2j\le 2k} c_{ij}(x_1+x_2)^i(x_1x_2)^j\,, \tag{32}$$

$$V_k = \sum_{i+2j\le 2k+1} d_{ij}(x_1+x_2)^i(x_1x_2)^j\,. \tag{33}$$

Substituting (28) into (32) and (33), we obtain

$$U_k = h^{-2k}P_{2k}(h)\,, \tag{34}$$

$$V_k = h^{-2k-1}P_{2k+1}(h)\,. \tag{35}$$

From (30), (31), (34) and (35), we get

$$I_k = \sqrt{-h}h^{-2k-2}(1+4h^2+4dh)P_{2k}(h)\,, \tag{36}$$

$$J_k = \sqrt{-h}h^{-2k-3}(1+4h^2+4dh)P_{2k+1}(h)\,. \tag{37}$$

Next we calculate I_k, J_k for $k<0$. Denote by

$$I_k^{\pm}=\int_{x_1}^{x_2}(x\pm 1)^k\sqrt{(x-x_1)(x_2-x)}\,dx\,.$$

Let $x\pm 1=y$, $y_i=x_i\pm 1$, then

$$I_k^{\pm}=\int_{y_1}^{y_2}y^k\sqrt{(y-y_1)(y_2-y)}\,dy\,.$$

Let $\sqrt{(y-y_1)(y_2-y)}=t(y-y_1)$, we have

$$I_k^{\pm}=2(y_2-y_1)^2\int_0^{\infty}\frac{t^2(y_2+y_1t^2)^k}{(1+t^2)^{k+3}}\,dt\,.$$

Computing we obtain

$$I_{-1}^{-}=\frac{\pi}{2}\left[\frac{1}{h}-2+\frac{2\sqrt{d+1}}{\sqrt{-h}}\right]\,,\qquad h\in[d_+,0)\cup(-\infty,d_-]\,,$$

$$I_{-2}^{-}=\frac{\pi}{2}\left[\frac{1}{\sqrt{d+1}}(-h)^{-\frac{1}{2}}-2+\frac{2}{\sqrt{d+1}}(-h)^{\frac{1}{2}}\right]\,,\qquad h\in[d_+,0)\cup(-\infty,d_-]\,,$$

$$I_{-3}^{-}=-\frac{\pi}{8(d+1)^{\frac{3}{2}}}(-h)^{\frac{3}{2}}\left(\frac{1}{h^2}+4+\frac{4d}{h}\right)\,,\qquad h\in[d_+,0)\cup(-\infty,d_-]\,,$$

$$I_{-1}^{+}=\begin{cases}\frac{\pi}{2}\left(\frac{1}{h}+2+\frac{2\sqrt{d-1}}{\sqrt{-h}}\right)\,, & h\in[d_+,0)\,,\\ \frac{\pi}{2}\left(\frac{1}{h}+2-\frac{2\sqrt{d-1}}{\sqrt{-h}}\right) & h\in(-\infty,d_-]\,,\end{cases}$$

$$I_{-2}^{+}=\begin{cases}\frac{\pi}{2}\left(\frac{1}{\sqrt{d-1}}(-h)^{-\frac{1}{2}}-\frac{2(-h)^{\frac{1}{2}}}{\sqrt{d-1}}-2\right)\,, & h\in[d_+,0)\,,\\ \frac{\pi}{2}\left(-\frac{1}{\sqrt{d-1}}(-h)^{-\frac{1}{2}}+\frac{2(-h)^{\frac{1}{2}}}{\sqrt{d-1}}-2\right)\,, & h\in(-\infty,d_-]\,,\end{cases}$$

$$I_{-3}^{+}=\begin{cases}\frac{\pi}{8(d-1)^{\frac{3}{2}}}\left(-(-h)^{-\frac{1}{2}}-4(-h)^{\frac{3}{2}}+4d\sqrt{-h}\right)\,, & h\in[d_+,0)\,,\\ \frac{\pi}{8(d-1)^{\frac{3}{2}}}\left((-h)^{-\frac{1}{2}}+4(-h)^{\frac{3}{2}}-4d\sqrt{-h}\right)\,, & h\in(-\infty,d_-]\,.\end{cases}$$

Then

$$
\begin{aligned}
I_{-1} &= \int_{x_1}^{x_2} \frac{1}{x^2-1}\sqrt{hx^2-x-h-d}\,dx \\
&= \sqrt{-h}\int_{x_1}^{x_2} \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{2}(I_1^- - I_1^+) \\
&= \begin{cases} \frac{\pi}{2}(\sqrt{d+1}-\sqrt{d-1}-2\sqrt{-h})\,, & h\in[d_+,0)\,, \\ \frac{\pi}{2}(\sqrt{d+1}+\sqrt{d-1}-2\sqrt{-h})\,, & h\in(-\infty,d_-]\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
I_{-2} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{1}{(x^2-1)^2}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{4}\int_{x_1}^{x_2}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{2}{x^2-1}\right]\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{4}(I_{-2}^- + I_{-2}^+) - \frac{1}{2}I_{-1} \\
&= \begin{cases} \frac{\pi}{8}\left[\frac{1}{\sqrt{d+1}}+\frac{1}{\sqrt{d-1}}+2\sqrt{d-1}-2\sqrt{d+1}+\left(\frac{2}{\sqrt{d-1}}-\frac{2}{\sqrt{d+1}}\right)h\right], \\ \qquad h\in[d_+,0)\,, \\ \frac{\pi}{8}\left[\frac{1}{\sqrt{d+1}}-\frac{1}{\sqrt{d-1}}-2\sqrt{d+1}-2\sqrt{d-1}-2\left(\frac{1}{\sqrt{d+1}}+\frac{1}{\sqrt{d-1}}\right)h\right], \\ \qquad h\in(-\infty,d_-]\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
I_{-3} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{1}{(x^2-1)^3}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{8}(I_{-3}^- - I_{-3}^+) - \frac{3}{4}I_{-2} \\
&= \begin{cases} \frac{\pi}{64}\left[(d-1)^{-\frac{3}{2}}-(d+1)^{-\frac{3}{2}}\right](1+4dh+4h^2)-\frac{3}{4}I_{-2} \\ \qquad h\in[d_+,0)\,, \\ -\frac{\pi}{64}\left[(d-1)^{-\frac{3}{2}}+(d+1)^{-\frac{3}{2}}\right](1+4dh+4h^2)-\frac{3}{4}I_{-2} \\ \qquad h\in(-\infty,d_-]\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
J_{-1} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{x}{x^2-1}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \frac{\sqrt{-h}}{2}(I_{-1}^{-}+I_{-1}^{+}) \\
&= \begin{cases} \frac{\pi}{2}(\sqrt{d+1}+\sqrt{d-1}-(-h)^{-\frac{1}{2}})\,, & h\in[d_+,0)\,, \\ \frac{\pi}{2}(\sqrt{d+1}-\sqrt{d-1}-(-h)^{-\frac{1}{2}})\,, & h\in(-\infty,d_-]\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
J_{-2} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{x}{(x^2-1)^2}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}\int_{x_1}^{x_2}\left[\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x+1)^2}+\frac{1}{(x^2-1)^2}\right] \\
&\qquad\qquad \sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}(\frac{1}{4}I_{-1}^{-}-\frac{1}{4}I_{-1}^{+}-\frac{1}{2}I_{-2}^{+})+I_{-2} \\
&= \frac{1}{2}I_{-1}-\frac{\sqrt{-h}}{2}I_{-2}^{+}+I_{-2} \\
&= \begin{cases} \frac{\pi}{8}\left(\frac{1}{\sqrt{d+1}}-\frac{1}{\sqrt{d-1}}-\left(\frac{2}{\sqrt{d-1}}+\frac{2}{\sqrt{d+1}}\right)h\right)\,, & h\in[d_+,0)\,, \\ \frac{\pi}{8}\left(\frac{1}{\sqrt{d+1}}+\frac{1}{\sqrt{d-1}}+\left(\frac{2}{\sqrt{d-1}}-\frac{2}{\sqrt{d+1}}\right)h\right)\,, & h\in(-\infty,d_-]\,. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
J_{-3} &= \sqrt{-h}\int_{x_1}^{x_2} \frac{x}{(x^2-1)^3}\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}\int_{x_1}^{x_2}\left[\frac{1}{8(x-1)^2}-\frac{3}{16(x-1)}+\frac{3}{16(x+1)}\right. \\
&\quad \left.+\frac{1}{4(x+1)^2}+\frac{1}{4(x+1)^3}+\frac{1}{(x^2-1)^3}\right]\sqrt{(x-x_1)(x_2-x)}\,dx \\
&= \sqrt{-h}\left[\frac{1}{8}I_{-2}^{-}-\frac{3}{16}I_{-1}^{-}+\frac{3}{16}I_{-1}^{+}+\frac{1}{4}I_{-2}^{+}+\frac{1}{4}I_{-3}^{+}\right]+I_{-3} \\
&= \begin{cases} -\frac{\pi}{64}\left[(d-1)^{-\frac{3}{2}}+(d+1)^{-\frac{3}{2}}\right](1+4dh+4h^2)+ \\ \quad +\frac{\pi}{32}\left(\frac{1}{\sqrt{d-1}}-\frac{1}{\sqrt{d+1}}\right)+\frac{\pi}{16}\left(\frac{1}{\sqrt{d-1}}+\frac{1}{\sqrt{d+1}}\right)h\,, & h\in[d_+,0)\,, \\ \frac{\pi}{64}\left[(d-1)^{-\frac{3}{2}}-(d+1)^{-\frac{3}{2}}\right](1+4dh+4h^2)+ \\ \quad -\frac{\pi}{32}\left(\frac{1}{\sqrt{d+1}}+\frac{1}{\sqrt{d-1}}\right)+\frac{\pi}{16}\left(\frac{1}{\sqrt{d+1}}-\frac{1}{\sqrt{d-1}}\right)h\,, & h\in(-\infty,d_-]\,. \end{cases}
\end{aligned}
$$

For $n \leq 2$,

$$M(h) = P_2(h)\,. \tag{38}$$

For $n = 3$,

$$M(h) = P_2(h) + \overline{m}_2(h) I_{-1} = \sum_{i=0}^{5} c_i (-h)^{\frac{i}{2}}\,. \tag{39}$$

For $n = 4$,

$$M(h) = \sum_{i=0}^{5} c_i (-h)^{\frac{i}{2}} + m_2(h) J_{-1} = \sum_{i=-1}^{5} d_i (-h)^{\frac{i}{2}}\,. \tag{40}$$

For $n \geq 5$ odd,

$$\begin{aligned}
M(h) &= \sum_{k=0}^{2} (m_k(h) J_{k-3} + \overline{m}_k(h) I_{k-3}) + \sum_{k=3}^{\frac{n-1}{2}} m_k(h) J_{k-3} + \sum_{k=3}^{\frac{n+1}{2}} \overline{m}_k(h) I_{k-3} \\
&= \sum_{i=-1}^{5} d_i (-h)^{\frac{i}{2}} + \sqrt{-h}(1 + 4dh + 4h^2) \\
&\quad \left[\sum_{k=3}^{\frac{n-1}{2}} m_k(h) h^{3-2k} P_{2k-5}(h) + \sum_{k=3}^{\frac{n+1}{2}} \overline{m}_k(h) h^{4-2k} P_{2k-6}(h) \right] \\
&= \sum_{i=-1}^{5} d_i (-h)^{\frac{i}{2}} + \sqrt{-h}(1 + 4h^2 + 4dh) \\
&\quad \left[\sum_{k=3}^{\frac{n-1}{2}} h^{3-2k} P_{3k-5}(h) + \sum_{k=3}^{\frac{n+1}{2}} h^{4-2k} P_{3k-6}(h) \right] \\
&= \sum_{i=-1}^{5} d_i (-h)^{\frac{i}{2}} + \sqrt{-h}(1 + 4h^2 + 4dh) \sum_{i=3-n}^{\frac{n-5}{2}} c_i h^i \\
&= \sum_{i=-1}^{5} d_i (-h)^{\frac{i}{2}} + \sum_{i=3-n}^{\frac{n-1}{2}} \widetilde{c}_i (-h)^{i+\frac{1}{2}} \\
&= a_0 + a_1 h + a_2 h^2 + \sum_{i=3-n}^{\frac{n-1}{2}} a_{i+\frac{1}{2}} (-h)^{i+\frac{1}{2}}\,.
\end{aligned} \tag{41}$$

For $n \geq 6$ even,

$$\begin{aligned} M(h) &= \sum_{k=0}^{2}(m_k(h)J_{k-3} + \overline{m}_k(h)I_{k-3}) + \sum_{k=3}^{\frac{n}{2}} m_k(h)J_{k-3} + \sum_{k=3}^{\frac{n}{2}} \overline{m}_k(h)I_{k-3} \\ &= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h}(1+4h^2+4dh)\sum_{k=3}^{\frac{n}{2}} \big(m_k(h)h^{3-2k}P_{2k-5}(h) + \\ &\quad + \overline{m}_k(h)h^{4-2k}P_{2k-6}(h)\big) \\ &= \sum_{i=-1}^{5} d_i(-h)^{\frac{i}{2}} + \sqrt{-h}\sum_{i=3-n}^{\frac{n}{2}} c_i h^i \\ &= a_0 + a_1 h + a_2 h^2 + \sum_{i=3-n}^{\frac{n}{2}} a_{i+\frac{1}{2}}(-h)^{i+\frac{1}{2}}\,. \end{aligned} \tag{42}$$

Denote by

$$N = \max\{\#\{h \in (d_+, 0) \,|\, M(h) = 0\}\,,\ \#\{h \in (-\infty, d_-) \,|\, M(h) = 0\}\}\,.$$

From (38)–(42), we obtain

$$\begin{aligned} N &\leq 1\,, && \text{if} \quad n \leq 2\,; \\ N &\leq 4\,, && \text{if} \quad n = 3\,; \\ N &\leq 5\,, && \text{if} \quad n = 4\,; \\ N &\leq \frac{3}{2}(n-1)\,, && \text{if} \quad n \geq 5 \quad \text{odd}\,; \\ N &\leq \frac{3}{2}n - 1\,, && \text{if} \quad n \geq 6 \quad \text{even}\,. \end{aligned}$$

Therefore the proof the the conclusion of Theorem 2 for system B is completed.

5 Linear estimation of the number of zeros of Abelian integral for system C

Consider the polynomial perturbation of system C:

$$\begin{aligned} \dot{x} &= -y + yx^2 + \varepsilon P(x,y)\,, \\ \dot{y} &= x + xy^2 + \varepsilon Q(x,y)\,, \end{aligned} \tag{43}$$

where $P = \frac{1}{2}\sum_{i+j\le n} a_{ij}x^iy^j$, $Q = \frac{1}{2}\sum_{i+j\le n} b_{ij}x^iy^j$. For $\varepsilon = 0$, system (43) has a first integral $H = (y^2+1)(x^2-1)$ with integrating factor $2/(x^2-1)$. For $h < -1$, the level curves $\Gamma_h : H = h$ are periodic orbits surrounding the center $\Gamma_{-1} : (0,0)$ (see Figure 1). The Abelian integral for system (43) is defined as

$$M(h) = \oint_{H=h} \left(\frac{2P}{(x^2-1)^2}\, dy - \frac{2Q}{(x^2-1)^2}\, dx \right), \qquad -\infty < h \le -1\,.$$

Denote by D_h the simple connected region with boundary Γ_h. By Green's formula, we have

$$\begin{aligned} M(h) &= 2\iint_{D_h} \left(\frac{\partial(P/(x^2-1)^2)}{\partial x} + \frac{\partial(Q/(x^2-1)^2)}{\partial y} \right) dx\, dy \\ &= 2\iint_{D_h} \left(\frac{P'_x + Q'_y}{(x^2-1)^2} - \frac{4xP}{(x^2-1)^3} \right) dx\, dy \\ &= \iint_{D_h} \left[\frac{1}{(x^2-1)^2} \left(\sum_{i+j\le n} i a_{ij} x^{i-1} y^j + \sum_{i+j\le n} j b_{ij} x^i y^{j-1} \right) \right. \\ &\quad \left. - \frac{4}{(x^2-1)^3} \sum_{i+j\le n} a_{ij} x^{i+1} y^j \right] dx\, dy \qquad \text{(by symmetry)} \\ &= \iint_{D_h} \left[\frac{1}{(x^2-1)^2} \left(\sum_{2i+2j\le n-1} (2i+1) a_{2i+1\ 2j} x^{2i} y^{2j} \right. \right. \\ &\qquad \left. + \sum_{2i+2j\le n-1} (2j+1) b_{2i\ 2j+1} x^{2i} y^{2j} \right) \\ &\quad \left. - \frac{4}{(x^2-1)^3} \sum_{2i+2j\le n-1} a_{2i+1\ 2j} x^{2i+2} y^{2j} \right] dx\, dy\,. \end{aligned}$$

Denote by $\overline{x} = \sqrt{1+\frac{1}{h}}$ the positive root of equation $hx^2 - h - 1 = 0$, then

$$\begin{aligned} M(h) = \int_{-\overline{x}}^{\overline{x}} \frac{1}{(x^2-1)^2} \sum_{2i+2j\le n-1} \frac{(2i+1)}{(2j+1)} a_{2i+1\ 2j} x^{2i} (hx^2-h-1)^j \\ \sqrt{hx^2-h-1}\, dx \end{aligned}$$

$$+ \int_{-\overline{x}}^{\overline{x}} \frac{1}{(x^2-1)^2} \sum_{2i+2j\le n-1} b_{2i\ 2j+1} x^{2i} (hx^2-h-1)^j \sqrt{hx^2-h-1}\, dx$$

$$- \int_{-\overline{x}}^{\overline{x}} \frac{4}{(x^2-1)^3} \sum_{2i+2j\le n-1} \frac{a_{2i+1\ 2j}}{2j+1} x^{2i+2} (hx^2-h-1)^j \sqrt{hx^2-h-1}\, dx$$

$$= \int_{-\overline{x}}^{\overline{x}} \sum_{k=0}^{\left[\frac{n+1}{2}\right]} m_k(h)(x^2-1)^{k-3} \sqrt{hx^2-h-1}\, dx$$

$$= \sum_{k=0}^{\left[\frac{n+1}{2}\right]} m_k(h) I_{k-3}(h)\,, \tag{44}$$

where $m_k(h)$ is a polynomial of degree $\le k$ and

$$I_k = \int_{-\overline{x}}^{\overline{x}} (x^2-1)^k \sqrt{hx^2-h-1}\, dx\,.$$

If $\sqrt{\overline{x}^2 - x^2} = t(x+\overline{x})$, then

$$I_k = 8\sqrt{-h}\,\overline{x}^2 \int_0^\infty \frac{t^2[\overline{x}^2(1-t^2)^2 - (1+t^2)^2]^k}{(1+t^2)^{2k+3}}\, dt\,.$$

Computing we obtain

$$\begin{aligned} I_{-1} &= -\pi(\sqrt{-h}-1)\,, \\ I_{-2} &= -\frac{\pi}{2}(1+h)\,, \\ I_{-3} &= \frac{\pi}{8}(1+h)(3-h)\,. \end{aligned} \tag{45}$$

For $k \ge 0$,

$$\begin{aligned} I_k &= 8\sqrt{-h}\,\overline{x}^2 \int_0^\infty \frac{t^2 \sum_{i=0}^k C_k^i (-1)^{k-i} (1+t^2)^{2k-2i} (1-t^2)^{2i} \overline{x}^{2i}}{(1+t^2)^{2k+3}}\, dt \\ &= 8\sqrt{-h}\,\overline{x}^2 \sum_{i=0}^k d_{k,i} \overline{x}^{2i} \\ &= 8\sqrt{-h}(1+\frac{1}{h}) \sum_{i=0}^k d_{k,i} (1+\frac{1}{h})^i \\ &= 8\sqrt{-h}(1+h)h^{-k-1} P_k(h)\,, \end{aligned} \tag{46}$$

where

$$d_{k,i} = (-1)^{k-i} C_k^i \int_0^\infty \frac{t^2(1+t^2)^{2k-2i}(1-t^2)^{2i}}{(1+t^2)^{2k+3}}\, dt\,,$$

and P_k is a polynomial of degree k.

From (44) and (45), we obtain

$$M(h) = m_0(h) I_{-3} = (1+h)(3-h)P_0(h)\,, \quad \text{if} \quad n = 0\,; \tag{47}$$

$$M(h) = m_0(h) I_{-3} + m_1(h) I_{-2} = (1+h)P_1(h)\,, \quad \text{if} \quad n = 1, 2\,; \tag{48}$$

$$M(h) = \sum_{k=0}^{2} m_k(h) I_{k-3} = (\sqrt{-h}-1)(P_2(h) + P_1(h)\sqrt{-h})\,, \quad \text{if} \quad n = 3, 4\,. \tag{49}$$

From (44), (45) and (46), for $n \geq 5$ we have

$$\begin{aligned} M(h) &= \sum_{k=0}^{2} m_k I_{k-3} + \sum_{k=3}^{\left[\frac{n+1}{2}\right]} m_k I_{k-3} \\ &= (\sqrt{-h}-1)(P_2(h) + P_1(h)\sqrt{-h}) \\ &\quad + \sum_{k=3}^{\left[\frac{n+1}{2}\right]} 8 m_k(h)\sqrt{-h}(1+h)h^{2-k} P_{k-3}(h) \\ &= P_2(h) + \widetilde{P}_2(h)\sqrt{-h} + \sqrt{-h}\sum_{k=3}^{\left[\frac{n+1}{2}\right]} h^{2-k} P_{2k-2}(h) \\ &= P_2(h) + \widetilde{P}_2(h)\sqrt{-h} + h^{2-\left[\frac{n+1}{2}\right]}\sqrt{-h}\sum_{k=3}^{\left[\frac{n+1}{2}\right]} P_{\left[\frac{n+1}{2}\right]+k-2} \\ &= P_2(h) + \widetilde{P}_2(h)\sqrt{-h} + h^{2-\left[\frac{n+1}{2}\right]}\sqrt{-h} P_{2\left[\frac{n+1}{2}\right]-2} \\ &= P_2(h) + h^{2-\left[\frac{n+1}{2}\right]}\sqrt{-h} P_{2\left[\frac{n+1}{2}\right]-2}\,. \end{aligned} \tag{50}$$

Denote by N the number of isolated zeros of $M(h)$ in $(-\infty, -1)$, then the following hold: from (47),

$$N = 0\,, \qquad \text{if} \quad n = 0\,;$$

from (48),

$$N \leq 1\,, \qquad \text{if} \quad n = 1, 2\,;$$

from (49),

$$N \leq 4\,, \qquad \text{if} \quad n = 3, 4\,;$$

from (50),

$$N \leq 2\left[\frac{n+1}{2}\right]\,, \qquad \text{if} \quad n \geq 5\,.$$

Hence, the proof of the conclusion for system C of Theorem 2 is completed.

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