Exceptional sets for definition of quasiconformality

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1 Introduction

Let X and Y be metric spaces and $f\colon X \to Y$ a homeomorphism. Then the distortion of f at a point $x \in X$ is

$$H(x) := \limsup_{r\to 0} H_f(x,r), \tag{1}$$

where

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)},$$

and

$$L_f(x,r) := \sup\{|f(x)-f(y)| : |x-y| \le r\},$$
$$l_f(x,r) := \inf\{|f(x)-f(y)| : |x-y| \ge r\}.$$

By $|x-y|$ we denote the distance between x and y in a metric space. We say that f is quasiconformal if there is a constant H so that $H(x) \le H$ for every $x \in X$.

It was recently shown by Heinonen and Koskela [4] that, in the euclidean setting, the upper limit "lim sup" in this metric definition of quasiconformality can be replaced with "lim inf" and this still results in the same class of mappings. For a weaker version of this statement in the more abstract

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case of a so-called Loewner space (cf. [5]) see the paper [1] by Balogh and Koskela. This significant improvement on the definition and its consequences then quickly found applications to complex dynamics in the work of Przytycki and Rohde [7].

It is a fundamental fact that one can allow for an exceptional set in the metric definition of quasiconformality when "lim sup" is used. Indeed, by a result of Gehring [2], it suffices to assume that $H(x) \leq H$ for all points x outside an exceptional set of σ-finite $(n-1)$-dimensional measure. Contrary to this the geometric and discrete nature of the arguments used in [4] required one to assume that "lim inf" be uniformly bounded for all points to deduce quasiconformality. The aim of this note is to show that, quite surprisingly, one has similar exceptional sets even when we use "lim inf". This is the content of the following result.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$ be a domain and suppose that $f\colon \Omega \to f(\Omega) \subset \mathbf{R}^n$ is a homeomorphism. If there is a set E of σ-finite $(n-1)$-measure so that*

$$\liminf_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \leq H$$

for each $x \in \Omega \setminus E$, then f is quasiconformal in Ω.

The work of Gehring allows for a further exceptional set: it suffices that $H(x) \leq H$ almost everywhere and $H(x) < \infty$ everywhere outside an exceptional set of σ-finite $(n-1)$-dimensional measure. We can strengthen Theorem 1.1 to an analog of this but it does not admit as pleasing a formulation as Theorem 1.1.

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^n$ be a domain and suppose that $f\colon \Omega \to f(\Omega) \subset \mathbf{R}^n$ is a homeomorphism. Suppose that there are sets E and E_b so that E has σ-finite $(n-1)$-measure, E_b has zero n-measure, and*

$$\liminf_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \leq H$$

for each $x \in (\Omega \setminus E) \setminus E_b$, and

$$H(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} < \infty$$

for each $x \in E_b$. Then f is quasiconformal in Ω.

We expect that this result will become a useful tool in problems where one attempts to prove that topological conjugacy implies conformal conjugacy. In fact, Graczyk and Smirnov [3] have already communicated us an application of Theorem 1.1 to the complex iteration theory of rational functions.
Our proof of Theorem 1.2 is a combination of ideas from the argument of Heinonen and Koskela in [4] for the uniformly bounded case and from the classical proof of the absolute continuity of a quasiconformal mapping along almost all lines parallel to the coordinate axis (cf. [8]) together with a modification of the usual lower bounds on moduli of curve families. We do not know if one could replace the "lim sup" with "lim inf" in the set E_b in Theorem 1.2 but we doubt this.
The proof of Theorem 1.2 and the arguments in [4] and [5] give a version of Theorem 1.1 in the setting of a Loewner space: a self homeomorphism of a Loewner space that satisfies $H(x) \leq H < \infty$ for all x outside a set E of vanishing capacity is quasiconformal. Here we declare a set E to be of vanishing capacity if the modulus of the family of all non-constant curves that intersect E is zero; each such set is of Hausdorff dimension zero. This result can be somewhat strengthened. For example, the uniform boundedness of $H(x)$ can be replaced with uniform boundedness of the substitute of "lim inf" as in [1]. We conjecture that it suffices to assume the uniform boundedness of $H(x)$ outside a set of vanishing λ-dimensional Hausdorff measure for some $\lambda > 0$ only depending on the data of the Loewner space. This might even hold with the substitute of "lim inf" as in [1] where the removability of certain Cantor-type sets of positive Hausdorff dimension is shown. The examples given in [1] indicate that the number λ above can be much smaller than the Hausdorff dimension of the Loewner space.
We wish to thank Fred Gehring and Stas Smirnov whose inquiries of a potential exceptional set lead us to work on this question.

2 Proof of Theorems 1.1 and 1.2

Notice first that Theorem 1.1 follows from Theorem 1.2 by choosing the empty set for E_b. Thus it suffices to prove Theorem 1.2.
Because the claim of Theorem 1.2 is of local nature, we may assume that Ω

is a proper subset of $\mathbf{R}^n$ and it suffices to prove the following: given $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)/100$,

$$\frac{L_f(x_0,r)}{l_f(x_0,r)} \leq H'$$

for some constant H' independent of x_0, r. In fact, H' will only depend on H and n. Fix x_0 and $0 < r < d(x_0, \partial\Omega)/100$, and write $L = L_f(x_0, r)$ and $l = l_f(x_0, r)$.
Without loss of generality we may assume that $29l \leq L$. For each $x \in A := [f^{-1}(\overline{B}(f(x_0), L)) \setminus f^{-1}(B(f(x_0), l))] \setminus (E \cup E_b)$ we pick a radius $0 < r_x < r/100$ so that $\operatorname{diam}(f(B(x, r_x))) \leq l/10$ and $H_f(x, r_x) \leq 2H$. By the Besicovitch covering theorem we find a countable collection $B_1 = B(x_1, r_{x_1}), B_2, ...$ of these balls so that

$$A \subset \cup_j B_j$$

and $\sum_j \chi_{B_j}(x) \leq C_n$ for each $x \in \mathbf{R}^n$.
We define a function ρ by the formula

$$\rho(x) := \sum_j \left(\log\frac{L}{l}\right)^{-1} \frac{\operatorname{diam} fB_j}{\operatorname{dist}(fB_j, f(x_0))} \frac{1}{\operatorname{diam} B} \chi_{2B_j}(x). \tag{2}$$

Then the estimates at page 73 in [4] show that

$$\int_{\mathbf{R}^n} \rho^n \, dx \ \leq \ C \left(\log\frac{L}{l}\right)^{1-n}. \tag{3}$$

Here the constant C depends only on H, n.
We will obtain a uniform bound on L/l by showing that

$$\int_{\mathbf{R}^n} \rho^n \, dx \geq \delta > 0, \tag{4}$$

where δ depends only on H, n.

Write $F_1 = f^{-1}(S^{n-1}(f(x_0), l))$ and $F_2 = f^{-1}(S^{n-1}(f(x_0), L))$. If we knew that $\int_\gamma \rho ds \geq c > 0$ for some constant only depending on H, n and for each curve γ connecting F_1 and F_2, then the desired lower bound would follow from usual modulus estimates. This need not be the case in our situation and thus more work is needed. We proceed to show that there are sufficiently many curves for which such an estimate holds and begin with a modulus estimate tailored for our needs. This estimate is a modification of a result of Koskela and Rohde in [6].

Lemma 2.1. *Let u be a non-negative Borel function in $\mathbf{R}^n$ such that for each y in a continuum $F \subset \mathbf{R}^n$*

$$\int_{[y,w)} u\, ds \geq 1$$

for each $w \in S_y \subset S^{n-1}(y,1)$, where S_y satisfies $H^{n-1}(S_y) \geq a > 0$. Then

$$\int_{\mathbf{R}^n} u^n\, dx \geq c(n,a)\, diam\,(F).$$

Proof. Fix $y \in F$ and let $0 < r < R < 1$. Then for each $w \in S^{n-1}$ we have by the Hölder inequality

$$\int_r^R u(y+tw)\, dt = \int_r^R u(y+tw) t^{(n-1)/n} t^{(1-n)/n}\, dt$$

$$\leq (\int_r^R u(y+tw)^n t^{n-1}\, dt)^{1/n} (\log \frac{R}{r})^{(n-1)/n}.$$

Set $r_j = 2^{-j}$. For $j = 0, 1, 2, \ldots$ write $A_j(y) = B(y, r_j) \setminus B(y, r_{j+1})$ and set $I_j(w) = (\int_r^R u(y+tw)dt)^n$, where $R = r_j, r = r_{j+1}$. Then integration of the above inequality with respect to w over S^{n-1} (and using $\log(1+x) \leq x$) gives

$$\int_{S^{n-1}} I_j(w) d\sigma \leq \int_{A_j(y)} u^n(x)\, dx.$$

Suppose now that for each j

$$\int_{A_j(y)} u^n(x)\, dx \leq c(n,a) 2^{-j},$$

where the constant $c(n,a)$ will be chosen later. Write $\mathrm{Bad}_j(s) = \{w \in S^{n-1} : I_j(w) \geq s\}$. Then

$$H^{n-1}(\mathrm{Bad}_j(s)) \leq c(n,a) s^{-1} 2^{-j},$$

and

$$H^{n-1}(\cup_j \mathrm{Bad}_j(c_n' 2^{-j/2})) \le \sum_j \frac{c(n,a)}{c_n'}\, 2^{-j/2} \le \frac{c(n,a)}{c_n'}. \tag{5}$$

For each $w \in S^{n-1}$ not in $\cup_j \mathrm{Bad}_j(c_n' 2^{-j/2})$ we have

$$\int_0^1 u(y+tw)\,dt = \sum_j I_j(y)^{1/n} \le \sum_j (c_n'\, 2^{-j/2})^{1/n}.$$

By choosing a suitable c_n' we conclude that

$$\int_0^1 u(y+tw)\,dt \le \frac{1}{2}$$

for each $w \in S^{n-1}$ not in $\cup_j \mathrm{Bad}_j(c_n' 2^{-j/2})$. Define $c(n,a) = ac_n'/2$. By (5) there is some w outside the bad set so that the segment of length 1 in w direction from y intersects S_y. We conclude that there is an index j such that

$$\int_{A_j(y)} u^n\,dx \ge c(n,a)\,2^{-j}$$

and thus

$$\int_{B(y,2^{-j})} u^n\,dx \ge c(n,a)\,2^{-j}.$$

By the Besicovitch covering theorem we may then cover F with balls $B(y_i,r_i)$ of the above type and so that only a bounded number (depending on n) of these balls overlap. Then

$$\mathrm{diam}\,(F) \le \sum_i r_i \le \sum_i \frac{1}{c(n,a)} \int_{B(y_i,r_i)} u^n\,dx \le c'(n,a) \int_{\mathbf{R}^n} u^n\,dx,$$

as desired.

We continue with an estimate on the size of E along almost all radii. This estimate is of standard type (cf. [8, 30.16]) and proved using a projection argument. We leave the details to the reader.

Lemma 2.2. *If $E \subset \mathbf{R}^n$ is a set of σ-finite $(n-1)$-measure, then for $(n-1)$-a.e. $w \in S^{n-1}$ the intersection of E with the radius $[0,w)$ is at most countable.*

In order to simplify the rest of the proof of Theorem 1.2, we make some reductions. By composing f with a preliminary orthogonal transformation and replacing F_1, F_2 with appropriate subcontinua, we may assume that $\operatorname{dist}(F_1, F_2) = 2\operatorname{diam}(F_1) = 2\operatorname{diam}(F_2) = \frac{1}{2}$. Let $x_1 \in F_1$ and $x_2 \in F_2$ be points such that $|x_2 - x_2| = \operatorname{dist}(F_1, F_2)$. Assume without loss of generality that $x_1 = -e_1/4$ and $x_2 = e_1/4$. Let S be the intersection of $B(0, \frac{1}{2})$ and the hyperplane orthogonal to e_1 through the origin. Given $x \in F_1,\ y \in F_2$, and $z \in S$ we write $\gamma(x, y, z)$ for the curve consisting of two line segments, one from x to z and the other one from z to y. The above normalizations are made to guarantee that the length of each such line segment is no more than 1. This is convenient in order to apply Lemma 2.1 later on.

Lemma 2.3. *Let $x \in F_1$ and $y \in F_2$. Then for $(n-1)$-almost every $z \in S$,*

$$\int_{\gamma(x,y,z)} \rho\, ds \geq \delta > 0,$$

where δ depends only on H, n.

Proof. Notice that if B_j intersects a curve γ, then the contribution of B_j to the integral of ρ over γ is at least

$$\frac{1}{2}\left(\log \frac{L}{l}\right)^{-1} \frac{\operatorname{diam} f B_j}{\operatorname{dist}(f B_j, f(x_0))}.$$

For a curve $\gamma(x, y, z)$ we notice that the desired lower bound on the integral follows from the fact that $f(\gamma(x, y, z))$ then has to pass through the annular region $B(f(x_0), L) \setminus B(f(x_0), l)$, provided the balls B_j cover $\gamma(x, y, z)$ up to a set N whose image under f has vanishing one-dimensional Hausdorff measure. By Lemma 2.2 the contribution of the set E to N is at most countable for almost every z and thus we are reduced to showing that, for almost every z,

$$H^1(f(E_b \cap \gamma(x, y, z))) = 0. \tag{6}$$

The lemma thus follows from Lemma 2.4 below by a simple decomposition argument.

The following lemma is distilled from a standard proof of the absolute continuity of a quasiconformal mapping along almost all lines parallel to coordinate axes (cf. [8, 31.2]).

Lemma 2.4. *Let* $f\colon B(0,3) \to f(B(0,3)) \subset \mathbf{R}^n$ *be a homeomorphism. Suppose that* $H_f(x) < \infty$ *for all* $x \in E$*, where the set* E *is of* n*-measure zero. Then*

$$H^1(f([w/2,w) \cap E)) = 0 \tag{7}$$

for almost every $w \in S^{n-1}$.

Proof. We define a Borel measure μ on S^{n-1} by the formula

$$\mu(U) = |f(\pi^{-1}(U))|,$$

where π is the projection of $\overline{B}(0,1) \setminus B(0,\frac{1}{2})$ onto S^{n-1} and $|A|$ is the n-measure of a set A. By the Radon-Nikodym theorem, for almost every $w \in S^{n-1}$,

$$\frac{\mu(B(w,r))}{r^{n-1}} \to \lambda$$

when r tends to zero, for some finite λ that may depend on w. Fix such a w. Because the n-measure of E is zero, we may assume that $H^1([w/2,w)\cap E) = 0$.

Write $E_w = [w/2,w) \cap E$ and

$$E_i = \{x \in E_w :\ H_f(x) < 2^i\},$$

and

$$E_i^k = \{x \in E_i :\ L_f(x,r) \leq 2^i l_f(x,r) \text{ for all } 0 < r < \frac{1}{k}\}$$

when $i = 1,2,...$ Then $f(E_w)$ is a countable union of the sets $f(E_i^k)$ and it suffices to verify that $H^1(f(E_i^k)) = 0$ for each i and all k.

Fix i and k, and let $\epsilon > 0$. Because $H^1(E_i^k) = 0$, we find an open set U (with $H^1(U) < \epsilon/2$) and compact sets F_j such that $E_i^k \subset U = \cup F_j$ and $H^1(F_j) \leq \epsilon 2^{-j}$. Then $f(E_i^k) = \cup_j f(E_i^k \cap F_j)$. Because F_j is compact, for all $0 < r < r(F_j)$, we can find (cf. [8, 31.1]) a finite collection $I_1,...,I_p$ of intervals of length r so that $F_j \subset \cup_{l=1}^p I_l$ and $pr < \epsilon 2^{-j+1}$. Pick such a cover for $r < \frac{1}{4k}$. We construct sets A_l by defining $A_l = \emptyset$ if $I_l \cap F_j^k = \emptyset$ and by selecting $y_l \in I_l \cap F_j^k$ otherwise and setting $A_l = B(y_l, r)$ in this case. It follows that

$f(E_i^k \cap F_j)$ can be covered by balls $B(f(y_1), L_f(y_1,r)), ..., B(f(y_p), L_f(y_p,r))$. By Hölder's inequality and the definition of E_i^k we estimate

$$\begin{aligned}(\sum_1^p L_f(y_l,r))^n &\leq p^{n-1}\sum_1^p L_f(y_l,r)^n \\ &\leq C_n p^{n-1} 2^{in} |f(\pi^{-1}(B(w,4r))|\end{aligned}$$

Assuming r to be sufficiently small, we have

$$|f(\pi^{-1}(B(w,4r))| \leq 8^{n-1}\lambda r^{n-1}.$$

Combining this with the bound $pr \leq \epsilon 2^{-j+1}$ we arrive at

$$(\sum_1^p L_f(y_l,r))^n \leq C(n,i)\lambda\epsilon^{n-1}2^{-j(n-1)}.$$

By using this estimate for each j and the fact that $E_i^k \subset \cup F_j$, and summing this estimate over j, we conclude that $f(E_i^k)$ can be covered by a countable collection of sets V_l so that

$$\sum_l \operatorname{diam}(V_l) \leq C(n,i)\lambda^{1/n}\epsilon^{(n-1)/n}.$$

Letting $\epsilon \to 0$ we deduce that $H^1(f(E_i^k)) = 0$, as desired.

The reason for the appearance of $H(x)$ in Theorem 1.2 comes from the assumptions of Lemma 2.4. We do not know if the assumption $H(x) < \infty$ could be replaced with $\liminf_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)} < \infty$.

Proof of Theorem 1.2. Assume first that, for each $y \in F_1$,

$$\int_{[y,w)} \rho\, ds \geq \delta/2$$

for every $w \in S_y \subset S$, where $H^{n-1}(S_y) \geq \frac{1}{2}H^{n-1}(S)$ and δ is the constant of Lemma 2.3. Then we immediately deduce inequality (4) from Lemma 2.1 and the claim follows.

The proof is now completed by the following simple observation. Suppose that there is $y \in F_1$ such that $\int_{[y,w)} \rho\, ds < \delta/2$ for every $w \in S_y \subset S$ with $H^{n-1}(S_y) \geq \frac{1}{2}H^{n-1}(S)$. Then, Lemma 2.3 shows that for each $x \in F_2$,

$$\int_{[x,w)} \rho\, ds \geq \delta/2$$

for every $w \in S_x \subset S$, where $H^{n-1}(S_x) \geq \frac{1}{2}H^{n-1}(S)$. Thus we are back in a situation analogous to the one in the beginning of the proof. The claim follows.

References

[1] Z. Balogh, P. Koskela, *Quasiconformality, quasisymmetry and removability in Loewner spaces*, Duke Math. J. to appear.

[2] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[3] J. Graczyk, S. Smirnov, *Weak expansion and geometry of Julia sets*, preprint.

[4] J. Heinonen, P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), 61-79.

[5] J. Heinonen, P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.

[6] P. Koskela, S. Rohde, *Hausdorff dimension and mean porosity*, Math. Ann. **309** (1997), 593–609.

[7] F. Przytycki, S. Rohde, *Rigidity of holomorphic Collet-Eckmann repellers*, Ark. Mat. to appear.

[8] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. 229. Springer-Verlag, 1971.

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