TWO-WEIGHT, WEAK-TYPE NORM INEQUALITIES FOR FRACTIONAL INTEGRALS, CALDERÓN-ZYGMUND OPERATORS AND COMMUTATORS

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Abstract. We give A_p-type conditions which are sufficient for the two-weight, weak-type (p,p) inequalities for fractional integral operators, Calderón-Zygmund operators and commutators. For fractional integral operators, this solves a problem posed by Sawyer and Wheeden [27]. At the heart of all of our proofs is an inequality relating the Hardy-Littlewood maximal function and the sharp maximal function which is strongly reminiscent of the good-λ inequality of Fefferman and Stein [12].

1. Introduction

Let M be the Hardy–Littlewood maximal operator. Given a pair of weights (u,v) and p, $1 < p < \infty$, it is well known that the weak-type inequality

$$u(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq \frac{C}{t^p}\int_{\mathbb{R}^n} |f|^p v\, dx \tag{1.1}$$

holds if and only if $(u,v) \in A_p$: there exists a positive constant K such that for all cubes Q,

$$\left(\frac{1}{|Q|}\int_Q u\, dx\right)\left(\frac{1}{|Q|}\int_Q v^{-p'/p}\, dx\right)^{p/p'} \leq K. \tag{1.2}$$

For other classical operators, however, the A_p condition is not sufficient for the weak (p,p) inequality. In fact, of the operators we are interested in,

1991 *Mathematics Subject Classification.* 42B20,42B25.

Key words and phrases. weights, weak-type inequalities, fractional integral operators, Calderón-Zygmund operators, commutators.

We would like to thank E. Sawyer for sharing with the second author an unpublished manuscript which suggested our approach.

The first author is supported by a Ford Foundation fellowship; the second author by DGICYT Grant PB40192, Spain. The second author is also grateful for the invitation and hospitality of the Centre de Recerca Matemàtica, Barcelona, where this work was completed.

a necessary and sufficient condition for the weak (p,p) inequality is known only for fractional integral operators. (See Sawyer [26].) While this result is very interesting, it involves estimating the fractional integral operator on a family of test functions and so is difficult to verify in practice.

Sufficient, A_p-type conditions can also be gotten from sufficient conditions for the strong (p,p) inequality. Neugebauer [17] showed that

$$\left(\frac{1}{|Q|}\int_Q u^r\,dx\right)^{1/rp}\left(\frac{1}{|Q|}\int_Q v^{-rp'/p}\,dx\right)^{1/rp'} \le C, \quad r>1, \tag{1.3}$$

is sufficient for the strong (p,p) inequality for the maximal operator, for Calderón-Zygmund operators and commutators. Sawyer and Wheeden [27] showed that for $0<\alpha<n$,

$$|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q u^r\,dx\right)^{1/rp}\left(\frac{1}{|Q|}\int_Q v^{-rp'/p}\,dx\right)^{1/rp'} \le C \tag{1.4}$$

is sufficient for the strong-type (p,p) inequality for fractional integral operators. (Additional sufficient conditions are found in [19], [20] and [23]. We give precise definitions of these operators in Section 2 below.)

In general, sufficient conditions for the weak (p,p) inequality which are derived from strong (p,p) conditions are not sharp. The purpose of this paper is to show that for the operators we consider, there are conditions which are weaker than (1.3) and (1.4) which are sufficient for the weak-type inequality. Roughly, it suffices to strengthen the A_p condition (1.2) by introducing a "power bump" on the left-hand term alone, rather than on both terms as in (1.3) and (1.4).

Our first result is for fractional integral operators. It solves a problem posed by Sawyer and Wheeden [27].

Theorem 1.1. *Given a pair of weights (u,v), p, $1<p<\infty$, and α, $0<\alpha<n$, suppose that for some $r>1$ and for all cubes Q,*

$$|Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q u^r\,dx\right)^{1/rp}\left(\frac{1}{|Q|}\int_Q v^{-p'/p}\,dx\right)^{1/p'} \le C<\infty. \tag{1.5}$$

Then the fractional integral operator I_α satisfies the weak (p,p) inequality

$$u(\{x\in\mathbb{R}^n : |I_\alpha f(x)|>t\}) \le \frac{C}{t^p}\int_{\mathbb{R}^n}|f|^p v\,dx. \tag{1.6}$$

Our second result is for Calderón-Zygmund operators.

Theorem 1.2. *Let T be a Calderón-Zygmund operator. Given a pair of weights (u,v) and p, $1<p<\infty$, suppose that for some $r>1$ and for all* *cubes Q,*

$$\left(\frac{1}{|Q|}\int_Q u^r\,dx\right)^{1/rp}\left(\frac{1}{|Q|}\int_Q v^{-p'/p}\,dx\right)^{1/p'} \le C < \infty. \tag{1.7}$$

Then T satisfies the weak (p,p) inequality

$$u(\{x\in\mathbb{R}^n : |Tf(x)| > t\}) \le \frac{C}{t^p}\int_{\mathbb{R}^n}|f|^p v\,dx. \tag{1.8}$$

Remark 1.3. Though for clarity we have stated Theorem 1.2 for Calderón-Zygmund operators, it is true for a much larger class of operators. To be precise: if there exists some δ, $0<\delta<1$, and a constant C_δ such that for every $f\in C_0^\infty(\mathbb{R}^n)$ and every x

$$M^{\#}(|Tf|^\delta)(x)^{1/\delta} \le C_\delta\, Mf(x), \tag{1.9}$$

then (1.7) implies (1.8).

Alvarez and Pérez [3] showed that inequality (1.9) holds for Calderón-Zygmund operators. In this case it can be thought of as extending the classical estimate

$$M^{\#}(Tf)(x) \le C_r M(|f|^r)(x)^{1/r}, \tag{1.10}$$

where T is a regular singular integral operator and $r>1$, (see García-Cuerva and Rubio de Francia [13, p. 204].) In some sense, (1.9) contains more information than (1.10) since the latter does not suffice to prove Theorem 1.2.

Alvarez and Pérez also showed that inequality (1.9) (and so Theorem 1.2) holds for the following operators: weakly strongly singular integral operators (see C. Fefferman [11]), some pseudo-differential operators in the Hörmander class (see Hörmander [14]), and a class of oscillatory integral operators related to those introduced by Phong and Stein [24]. They used (1.9) to generalize Coifman's theorem [7] relating the L^p norm of singular integral operators and the maximal function.

Remark 1.4. For Calderón-Zygmund operators we have been able to prove stronger results; these will appear in [10]. By different methods we showed that we may replace the "power bump" in (1.7) by a "bump" in the scale of Orlicz spaces. More precisely, we replace the L^r norm by the $L(\log L)^{p-1+\delta}$ norm with $\delta>0$. However we are unable to extend these results to the broader class of operators discussed in the previous remark.

Remark 1.5. Conditions (1.5) and (1.7) are sufficient for the fractional maximal operator and the Hardy-Littlewood maximal operator to be bounded from $L^{p'}(u^{-p'/p})$ to $L^{p'}(v^{-p'/p})$. (See [20], [21].) We conjecture that the boundedness of the corresponding maximal operator is itself sufficient for inequalities (1.6) and (1.8) to hold. In particular we believe that the Orlicz space conditions given in [20] and [21] are sufficient.

Our last result is about (linear) commutators. These operators are defined by

$$C_b^k f(x) = \int (b(x) - b(y))^k K(x,y) f(y)\, dy,$$

where K is a kernel satisfying the standard estimates and b is a locally integrable function. (See Section 2 for a precise definition.)

Since commutators has a greater degree of "singularity" than the associated Calderón-Zygmund operator, we need a slightly stronger condition. Roughly, we need to "bump" the right-hand term as well, but it suffices to do so in the scale of Orlicz spaces. Recall that if B is an increasing Young function and if Q is any cube, we define the mean Luxembourg norm of a measurable function f with respect to B by

$$\|f\|_{B,Q} = \inf\left\{ \lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f|}{\lambda}\right)\, dx \leq 1 \right\}.$$

(For more information on Orlicz spaces, see Section 2 below.)

Theorem 1.6. *Let T be a Calderón-Zygmund operator and b a function in BMO. Given a pair of weights (u,v), p, $1 < p < \infty$, and $k \geq 0$, suppose that for some $r > 1$ and for all cubes Q,*

$$\left(\frac{1}{|Q|}\int_Q u^r\, dx\right)^{1/rp} \|v^{-1/p}\|_{C_k,Q} \leq C < \infty, \tag{1.11}$$

where $C_k(t) = t^{p'}\log(1+t)^k$. Then the commutator C_b^k satisfies the weak (p,p) inequality

$$u(\{x \in \mathbb{R}^n : |C_b^k f(x)| > t\}) \leq \frac{C}{t^p}\int_{\mathbb{R}^n} |f|^p v\, dx. \tag{1.12}$$

When $k = 0$, $C_b^0 = T$, and so in this case Theorem 1.6 reduces to Theorem 1.2.

Remark 1.7. As a corollary to Theorem 1.6 we get a new proof of the one-weight, strong (p,p) norm inequality for commutators, which was first proved in a more general form by Alvarez, Bagby, Kurtz and Pérez [2] and Segovia and Torrea [28]. If $w \in A_p$ then w and $w^{-p'/p}$ both satisfy the reverse Hölder inequality and so inequality (1.11) holds for some $r > 1$ and for $p \pm \epsilon$. The strong-type inequality follows by interpolation.

The proofs of Theorems 1.1, 1.2 and 1.6 all follow the same outline. Each relies on our so-called principal lemma, Theorem 3.4 below, which relates the Hardy-Littlewood maximal operator and the Fefferman-Stein sharp maximal operator via an inequality strongly reminiscent of a good-λ inequality. To apply Theorem 3.4 we use a series of results which relate the given operator, the sharp maximal operator and the maximal operator. For Calderón-Zygmund operators this is inequality (1.9). Similar inequalities hold for fractional integral operators and commutators: see Lemmas 4.4 and 6.1.

The remainder of this paper is organized as follows: in Section 2 we give a number of definitions and lemmas needed in later sections. The heart of the paper is Section 3, where we prove Theorem 3.4. Finally, in Sections 4, 5 and 6 we prove Theorems 1.1, 1.2 and 1.6.

Throughout this paper all notation is standard or will be defined as needed. All cubes are assumed to have their sides parallel to the co-ordinate axes. Given a cube Q, $l(Q)$ will denote the length of its sides and for any $r > 0$, rQ will denote the cube with the same center as Q and such that $l(rQ) = rl(Q)$. We will denote the collection of all dyadic cubes by Δ and by $\Delta(Q)$ the collection of all dyadic subcubes relative to the (not necessarily dyadic) cube Q. By weights we will always mean non-negative, locally integrable functions which are positive on a set of positive measure. Given a Lebesgue measurable set E and a weight w, $|E|$ will denote the Lebesgue measure of E and $w(E) = \int_E w\,dx$. Given $1 < p < \infty$, $p' = p/(p-1)$ will denote the conjugate exponent of p. Finally, C will denote a positive constant whose value may change at each appearance.

2. Preliminary Ideas

In this section we give a number of definitions and lemmas needed in later sections.

The main operators. First we define the operators in Theorems 1.1, 1.2 and 1.6.

Fractional integral operators: Given α, $0 < \alpha < n$, define the fractional integral operator of order α by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

For more information, see Stein [30, pp. 117-120].

Calderón-Zygmund operators: Given a kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ —i.e. a locally integrable, complex-valued function defined off the diagonal— we say that it satisfies the standard estimates if there exist δ, $0 < \delta \leq 1$, and

C finite such that for all distinct points x and y in $\mathbb{R}^n$, and all z such that $|x-z| < \frac{1}{2}|x-y|$:

1. $|K(x,y)| \leq C|x-y|^{-n}$;
2. $|K(x,y) - K(z,y)| \leq C|x-z|^{\delta}/|x-y|^{n+\delta}$;
3. $|K(y,x) - K(y,z)| \leq C|x-z|^{\delta}/|x-y|^{n+\delta}$.

A bounded linear operator $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ (here $\mathcal{D}'$ is the space of distributions) is said to be associated with a kernel K if

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)g(x)f(y)\,dxdy$$

for all f and g in $C_0^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(f)\cap\mathrm{supp}(g) = \emptyset$. T is said to be a Calderón-Zygmund operator if its associated kernel satisfies the standard estimates and it extends to a bounded linear operator on L^2. For more information, see Coifman and Meyer [8] and Christ [6].

Important examples of such operators are the Calderón-Zygmund singular integral operators:

$$Tf(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^n} k(x-y)f(y)\,dy,$$

where $k \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ and $K(x,y) = k(x-y)$ satisfies the standard estimates. For more information see García-Cuerva and Rubio de Francia [13, p. 192].

Commutators: Given a Calderon-Zygmund operator T and a function b in BMO, let M_b denote multiplication by b. We define the linear operators C_b^k by $C_b^0 = T$, $C_b^1 = [M_b, T] = M_bT - M_bT$, and for $k > 1$, $C_b^k = [M_b, C_b^{k-1}]$. If $f \in C_0^\infty(\mathbb{R}^n)$ then

$$C_b^k f(x) = \int (b(x) - b(y))^k K(x,y)f(y)\,dy, \quad x \notin \mathrm{supp}(f).$$

Commutators were introduced by Coifman, Rochberg and Weiss [9], who showed they were bounded on L^p, $1 < p < \infty$.

Maximal operators. Key to the proofs of our results are a number of maximal operators. For completeness we give their definitions here.

The maximal operator: Given a locally integrable function f and α, $0 \leq \alpha < n$, define

$$M_\alpha f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\alpha/n}}\int_Q |f|\,dy.$$

If $\alpha = 0$ this is the Hardy-Littlewood maximal operator and we write Mf for $M_0 f$; if $0 < \alpha < n$ this is the fractional maximal operator of order α. We use the Hardy-Littlewood maximal operator to control Calderón-Zygmund operators and commutators, and the fractional maximal operator to control fractional integral operators. (See inequality (1.9) and Lemmas 4.4 and 6.1.)

We define the dyadic maximal and fractional maximal operators M^d and M^d_α similarly except the supremums are restricted to dyadic cubes containing x. Given $\delta > 0$ we define the δ-maximal operator by $M_\delta f(x) = M(|f|^\delta)(x)^{1/\delta}$. We define M^d_δ similarly. From the context there should be no confusion between the fractional maximal operator and the δ-maximal operator.

The sharp maximal operator: Given a locally integrable function f and a cube Q, let f_Q denote the average of f over Q:

$$f_Q = \frac{1}{|Q|}\int_Q f\,dx.$$

Define the sharp maximal function of f by

$$M^\# f(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy.$$

The sharp maximal function was introduced by Fefferman and Stein [12]. Again, define the dyadic sharp maximal function $M^{\#,d}$ by restricting the supremum to dyadic cubes. Given $\delta > 0$, define the sharp δ-maximal function by $M^\#_\delta f(x) = M^\#(|f|^\delta)^{1/\delta}$, and define $M^{\#,d}_\delta$ similarly.

Orlicz spaces. In Section 6 we will need the following facts about Orlicz spaces. (For further information see Bennett and Sharpley [4] or Rao and Ren [25].) A function $B : [0,\infty) \to [0,\infty)$ is a Young function if it is convex and increasing, and if $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$.

Given a Young function B, define the mean Luxembourg norm of f on a cube Q by

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f|}{\lambda}\right)\,dy \le 1\right\}.$$

When $B(t) = t^p$, $1 \le p < \infty$,

$$\|f\|_{B,Q} = \left(\frac{1}{|Q|}\int_Q |f|^p\,dx\right)^{1/p},$$

that is, the Luxembourg norm coincides with the (normalized) L^p norm. There is another characterization of the Luxembourg norm, due to Krasnosel'skiĭ and Rutickiĭ [16, p. 92] (also see Rao and Ren [25, p. 69]) which we will need:

$$\|f\|_{B,Q} \le \inf_{\mu>0}\left\{\mu + \frac{\mu}{|Q|}\int_Q B\left(\frac{|f|}{\mu}\right)\,dx\right\} \le 2\|f\|_{B,Q}. \tag{2.1}$$

Given three Young functions A, B and C such that for all $t > 0$,

$$A^{-1}(t)C^{-1}(t) \leq B^{-1}(t), \tag{2.2}$$

then we have the following generalized Hölder's inequality due to O'Neil [18]: for any cube Q and all functions f and g,

$$\|fg\|_{B,Q} \leq 2\|f\|_{A,Q}\|g\|_{C,Q}. \tag{2.3}$$

Define the maximal operator M_B by

$$M_B f(x) = \sup_{Q \ni x} \|f\|_{B,Q}.$$

The dyadic maximal operator M_B^d is defined in similarly, except the supremum is restricted to dyadic cubes containing x. It follows from an inequality due to Stein [29] that for $k \geq 1$, if $B_k(t) = t\log(1+t)^{k-1}$, then $M^k f \approx M_{B_k} f$, where $M^k = M \cdot M \cdots M$ is the k-th iterate of the maximal function. (See Carozza and Passarelli di Napoli [5] and the references given there.)

The Calderón-Zygmund Decomposition. Our proofs depend heavily on the Calderón-Zygmund decomposition and a generalization of it to Orlicz space norms. To be precise and to establish notation, we state the result here. For a proof see [21]; this is an adaptation of the classical proof given in García-Cuerva and Rubio de Francia [13, p. 137].

Lemma 2.1. *Given a Young function B, suppose f is a non-negative function such that $\|f\|_{B,Q}$ tends to zero as $l(Q)$ tends to infinity. Then for each $t > 0$ there exists a disjoint collection of dyadic cubes $\{C_i^t\}$ such that for each i, $t < \|f\|_{B,C_i^t} \leq 2^n t$,*

$$\{x \in \mathbb{R}^n : M_B^d f(x) > t\} = \bigcup_i C_i^t,$$

and

$$\{x \in \mathbb{R}^n : M_B f(x) > 4^n t\} \subset \bigcup_i 3C_i^t.$$

Moreover, the cubes are maximal: if Q is a dyadic cube such that $Q \subset \{M_B^d f(x) > t\}$, then $Q \subset C_i^t$ for some i.

To recapture the classical lemma, let $B(t) = t$ and note that if $f \in \bigcup_{p\geq 1} L^p$ then

$$\|f\|_{B,Q} = \frac{1}{|Q|}\int_Q f\,dx \to 0 \quad \text{as} \quad |Q| \to \infty.$$

More generally, to apply Lemma 2.1 it suffices to assume that f is bounded and has compact support.

3. The Principal Lemma

In this section we prove our principal lemma: an inequality linking the sharp maximal function and the Hardy-Littlewood maximal function. In spirit, though not in detail it resembles the good-λ inequality of Fefferman and Stein [12]. (Also see García-Cuerva and Rubio de Francia [13, pp. 161-3] and Journé [15, p. 41].)

To state the principal lemma we first need a definition and a lemma.

Definition 3.1. Given $r > 1$ and a weight u, define the set function A_u^r on measurable sets $E \subset \mathbb{R}^n$ by

$$A_u^r(E) = |E|^{1/r'} \left(\int_E u^r \, dx \right)^{1/r} = |E| \left(\frac{1}{|E|} \int_E u^r \, dx \right)^{1/r}.$$

(The second equality holds provided $|E| > 0$.)

Lemma 3.2. *For any $r > 1$ and weight u, the set function A_u^r has the following properties:*

1. *If $E \subset F$ then $A_u^r(E) \leq \left(\frac{|E|}{|F|} \right)^{1/r'} A_u^r(F)$;*
2. *$u(E) \leq A_u^r(E)$;*
3. *If $\{E_j\}$ is a sequence of disjoint sets and $\bigcup_j E_j = E$ then*

$$\sum_j A_u^r(E_j) \leq A_u^r(E).$$

Proof. Condition (1) follows immediately from Definition 3.1, and Condition (2) is just Hölder's inequality. Condition (3) also follows from Hölder's inequality:

$$\begin{aligned} \sum_j |E_j|^{1/r'} \left(\int_{E_j} u^r \, dx \right)^{1/r} &\leq \left(\sum_j |E_j| \right)^{1/r'} \left(\sum_j \int_{E_j} u^r \, dx \right)^{1/r} \\ &= |E|^{1/r'} \left(\int_E u^r \, dx \right)^{1/r}. \end{aligned}$$

□

Remark 3.3. The key property is Condition (1), which plays the same role that the A_∞ condition plays in the proof of weighted good-λ inequalities. (See, for example, Journé [15, p. 41].) Given any other set function A which satisfied Conditions (2) and (3) of Lemma 3.2 and satisfied

$$A(E) \leq \phi(|E|/|F|) A(F), \quad \phi(t) \to 0 \text{ as } t \to 0,$$

we could immediately derive corresponding conditions governing weak-type norm inequalities for the operators we are interested in.

Originally, we had hoped to replace the "power bumps" in (1.5), (1.7) and (1.11) by Orlicz space conditions. Intuitively, the appropriate set function would be $A(E) = |E|\|u\|_{B,E}$, where B is some Young function—for example, $B(t) = t\log(1+t)^\epsilon$, $\epsilon > 0$. For such B, Conditions (2) and (3) hold; we will show this in the course of proving Lemma 5.1 below. However, Condition (1) fails.

We can now state and prove our principal lemma.

Theorem 3.4. *Given a non-negative function f in $\bigcup_{p\geq 1} L^p$, $r > 1$, a weight u, $\delta > 0$ and $\epsilon > 0$ sufficiently small, then for each $t > 0$ there exists a subcollection $\{Q_j^t\}$ of dyadic cubes from the Calderón-Zygmund decomposition of f^δ at height t^δ, $\{C_i^{t^\delta}\}$, with the property that*

$$\left(\frac{1}{|Q_j^t|}\int_{Q_j^t} |f^\delta - (f^\delta)_{Q_j^t}|\,dx\right)^{1/\delta} > \epsilon^{1/\delta} t,$$

and such that for all $p > 0$,

$$\sup_{t>0} t^p u(\{x \in \mathbb{R}^n : M_\delta^d f(x) > t\}) \leq C \sup_{t>0} t^p \sum_j A_u^r(Q_j^t), \tag{3.1}$$

where the constant C depends on ϵ, r, p and n.

As a corollary to the proof we have the following stronger inequality.

Corollary 3.5. *With the same hypotheses and notation as Theorem 3.4, we have that*

$$\sup_{t>0} t^p \sum_i A_u^r(C_i^{t^\delta}) \leq C \sup_{t>0} t^p \sum_j A_u^r(Q_j^t). \tag{3.2}$$

Proof. First note that it will suffice to prove this result for $\delta = 1$. For arbitrary $\delta > 0$, $M_\delta^d f(x) > t$ is equivalent to $M^d(f^\delta)(x) > t^\delta$, so the general case follows if we replace f by f^δ and t by t^δ.

Second, we may assume that u is bounded and has compact support. To see that the general case follows, fix a weight u and let $u_k = \min(u,k)\chi_{B(0,k)}$. Since u_k is bounded, inequalities (3.1) and (3.2) hold with u replaced by u_k. Since $\lim_k u_k = \sup_k u_k = u$, if we take the limit as k tends to infinity we may exchange limit and supremum and apply the monotone convergence theorem to get the desired result.

Fix p, $1 < p < \infty$, and fix f. For each $t > 0$, let $\Omega_t = \{x \in \mathbb{R}^n : M^d f(x) > t\}$. Now fix $N = 2^n + 1$ (the reason for this choice will be clear below); by the Calderón-Zygmund decomposition, Lemma 2.1, we can write

$\Omega_{Nt} = \bigcup_k C_k^{Nt}$ and $\Omega_t = \bigcup_i C_i^t$. By maximality, for each k, $C_k^{Nt} \subset C_i^t$ for some i. By Lemma 3.2, Conditions (2) and (3),

$$\begin{aligned} t^p u(\Omega_{Nt}) &= t^p \sum_k u(C_k^{Nt}) \\ &\leq t^p \sum_k A_u^r(C_k^{Nt}) \\ &= t^p \sum_i \sum_{C_k^{Nt} \subset C_i^t} A_u^r(C_k^{Nt}) \\ &\leq t^p \sum_i A_u^r(\Omega_{Nt} \cap C_i^t). \end{aligned}$$

Divide the indices i into two sets: $i \in F$ if

$$\frac{1}{|C_i^t|} \int_{C_i^t} |f - f_{C_i^t}|\, dx \leq \epsilon t,$$

and $i \in G$ if the opposite inequality holds. The cubes $\{C_i^t : i \in G\}$ are the cubes in the conclusion of the theorem, and we relabel them $\{Q_j^t\}$.

If $i \in F$ then we claim that

$$A_u^r(\Omega_{Nt} \cap C_i^t) \leq \epsilon^{1/r'} A_u^r(C_i^t).$$

By Lemma 3.2, Condition (1), it will suffice to show that

$$|\Omega_{Nt} \cap C_i^t| \leq \epsilon |C_i^t|.$$

By the maximality of the Calderon-Zygmund decomposition, if $x \in \Omega_{Nt} \cap C_i^t$ then

$$M^d f(x) = M^d(f\chi_{C_i^t})(x).$$

Hence

$$\begin{aligned} \Omega_{Nt} \cap C_i^t &= \{x \in C_i^t : M^d(f\chi_{C_i^t})(x) > Nt\} \\ &= \{x \in C_i^t : M^d(f\chi_{C_i^t})(x) - f_{C_i^t} > Nt - f_{C_i^t}\} \\ &\subset \{x \in C_i^t : M(|f - f_{C_i^t}|\chi_{C_i^t})(x) > t\}. \end{aligned}$$

Since the dyadic maximal operator is weak-type $(1,1)$ with constant 1, (see Journé [15, p. 10]), and since $i \in F$,

$$|\Omega_{Nt} \cap C_i^t| \leq \frac{1}{t} \int_{C_i^t} |f - f_{C_i^t}|\, dx \leq \epsilon |C_i^t|.$$

Therefore, we have shown that

$$t^p \sum_k A_u^r(C_k^{Nt}) \leq \epsilon^{1/r'} t^p \sum_i A_u^r(C_i^t) + t^p \sum_j A_u^r(Q_j^t). \tag{3.3}$$

Since $f \in \bigcup_{p\geq 1} L^p$, $|\Omega_t| < \infty$. Since u is bounded, it follows that $\sum_i A_u^r(C_i^t) < \infty$. Therefore, for each $M > 0$,

$$\sup_{0<t<M} t^p \sum_i A_u^r(C_i^t)$$

is finite. If we take the supremum of each side of inequality (3.3) we get

$$\sup_{0<t<M/N} t^p \sum_k A_u^r(C_k^{Nt}) \leq \sup_{0<t<M} \epsilon^{1/r'} t^p \sum_i A_u^r(C_i^t) + \sup_{0<t<M} t^p \sum_j A_u^r(Q_j^t);$$

equivalently,

$$\sup_{0<t<M} t^p \sum_i A_u^r(C_i^t) \leq \epsilon^{1/r'} N^p \sup_{0<t<M} t^p \sum_i A_u^r(C_i^t) + N^p \sup_{0<t<M} t^p \sum_j A_u^r(Q_j^t).$$

If we fix ϵ sufficiently small—to be precise, $\epsilon < N^{-pr'}$—then by rearranging terms we get

$$\begin{aligned}\sup_{0<t<M} u(\{x \in \mathbb{R}^n : M^d f(x) > t\}) &\leq \sup_{0<t<M} t^p \sum_i A_u^r(C_i^t) \\ &\leq \frac{N^p}{1-\epsilon^{1/r'}N^p} \sup_{0<t<M} t^p \sum_j A_u^r(Q_j^t).\end{aligned}$$

Since this holds for all $M > 0$, if we take the limit as M tends to infinity we get inequalities (3.1) and (3.2). □

4. Fractional Integral Operators

In this section we prove Theorem 1.1. The proof depends on three lemmas; the first two are due to Sawyer and Wheeden [27].

Lemma 4.1. *Given a non-negative function f and α, $0 < \alpha < n$, there exists a constant C_α, depending only on α and n, such that for any cube Q_0,*

$$\sum_{Q\in\Delta(Q_0)} |Q|^{\alpha/n} \int_Q f\,dx \leq C_\alpha |Q_0|^{\alpha/n} \int_{Q_0} f\,dx.$$

Definition 4.2. Given α, $0 < \alpha < n$, and $z \in \mathbb{R}^n$, define the translated dyadic fractional integral operator $I^d_{\alpha,z}$ by

$$I^d_{\alpha,z} f(x) = \sum_{\substack{Q+z\in\Delta \\ Q\ni x}} |Q|^{\alpha/n-1} \int_Q f\,dy.$$

If $z = 0$ we write I_α^d for $I_{\alpha,0}^d$.

Lemma 4.3. *Given a weight u, α, $0 < \alpha < n$, and p, $1 < p < \infty$, then there exists a constant C such that for every function f,*

$$\sup_{t>0} t^p u(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \leq C \sup_{z\in\mathbb{R}^n} \sup_{t>0} t^p u(\{x \in \mathbb{R}^n : |I_{\alpha,z}^d f(x)| > t\}).$$

Lemma 4.4. *Given α, $0 < \alpha < n$, there exists a constant D_α such that for any function f, dyadic cube Q_0 and $x \in Q_0$,*

$$\frac{1}{|Q_0|} \int_{Q_0} |I_\alpha^d f - (I_\alpha^d f)_{Q_0}|\, dx \leq D_\alpha M_\alpha^d f(x).$$

Proof. By the definition of I_α^d, for $x \in Q_0$,

$$I_\alpha^d f(x) = \sum_{\substack{x\in Q\in\Delta \\ Q\subset Q_0}} |Q|^{\alpha/n-1} \int_Q f\, dx + \sum_{\substack{Q\in\Delta \\ Q_0\subseteq Q}} |Q|^{\alpha/n-1} \int_Q f\, dx;$$

hence,

$$\frac{1}{|Q_0|} \int_{Q_0} I_\alpha^d f\, dx = \frac{1}{|Q_0|} \sum_{\substack{Q\in\Delta \\ Q\subset Q_0}} |Q|^{\alpha/n} \int_Q f\, dx + \sum_{\substack{Q\in\Delta \\ Q_0\subseteq Q}} |Q|^{\alpha/n-1} \int_Q f\, dx.$$

Therefore, by Lemma 4.1,

$$\begin{aligned} \frac{1}{|Q_0|} \int_{Q_0} |I_\alpha^d f - (I_\alpha^d f)_{Q_0}|\, dx &\leq \frac{2}{|Q_0|} \sum_{\substack{Q\in\Delta \\ Q\subset Q_0}} |Q|^{\alpha/n} \int_Q |f|\, dx \\ &\leq 2C_\alpha |Q_0|^{\alpha/n-1} \int_{Q_0} |f|\, dx \\ &\leq 2C_\alpha M_\alpha^d f(x). \end{aligned}$$

□

Proof of Theorem 1.1. By Lemma 4.3 it will suffice to prove inequality (1.6) with I_α replaced by $I_{\alpha,z}^d$ and with a constant independent of z. In the proof that follows it will be clear that all the constants are independent of z, so in fact it will suffice to prove inequality (1.6) for I_α^d.

Since I_α^d is a positive operator, by a standard argument we may assume that f is non-negative, bounded and has compact support. Then $I_\alpha^d f \in \bigcup_{p\geq 1} L^p$, so we can apply Theorem 3.4 to it. Fix p, $1 < p < \infty$, $\delta = 1$

and $\epsilon > 0$ sufficiently small. Then for each $t > 0$ there exists a sequence of disjoint dyadic cubes $\{Q_j^t\}$ such that

$$\frac{1}{|Q_j^t|}\int_{Q_j^t} |I_\alpha^d f - (I_\alpha^d f)_{Q_j^t}|\,dx > \epsilon t \tag{4.1}$$

and (by the Lebesgue differentiation theorem)

$$\begin{aligned}\sup_{t>0} t^p u(\{x\in\mathbb{R}^n : |I_\alpha^d f(x)| > t\}) &\leq \sup_{t>0} t^p u(\{x\in\mathbb{R}^n : M^d(I_\alpha f)(x) > t\}) \\ &\leq C\sup_{t>0} t^p \sum_j A_u^r(Q_j^t).\end{aligned}$$

Fix t; then by Lemma 4.4, for each j,

$$Q_j^t \subset \{x\in\mathbb{R}^n : M_\alpha^d f(x) > \epsilon D_\alpha^{-1} t\}.$$

By an argument analogous to that for the dyadic maximal operator (cf. Lemma 2.1), we can write the right-hand side as the union of disjoint dyadic cubes $\{P_k^t\}$ such that for each k,

$$|P_k^t|^{\alpha/n-1}\int_{P_k^t} f\,dx > \epsilon D_\alpha^{-1} t.$$

Further, the P_k^t's are maximal with this property; in particular, for each j there exists k such that $Q_j^t \subset P_k^t$. Therefore, by Lemma 3.2, Condition (3),

$$\begin{aligned}t^p\sum_j A_u^r(Q_j^t) &= t^p\sum_k\sum_{Q_j^t\subset P_k^t} A_u^r(Q_j^t) \\ &\leq t^p\sum_k A_u^r(P_k^t) \\ &\leq (\epsilon^{-1}D_\alpha)^p\sum_k |P_k^t|\left(\frac{1}{|P_k^t|}\int_{P_k^t} u^r\,dx\right)^{1/r} \\ &\qquad\times\left(|P_k^t|^{\alpha/n-1}\int_{P_k^t} f\,dx\right)^p.\end{aligned}$$

By Hölder's inequality and inequality (1.5),

$$\begin{aligned}
&\leq C \sum_k |P_k^t|^{\alpha p/n} \left(\frac{1}{|P_k^t|} \int_{P_k^t} u^r \, dx \right)^{1/r} \\
&\qquad \times \left(\frac{1}{|P_k^t|} \int_{P_k^t} v^{-p'/p} \, dx \right)^{p/p'} \int_{P_k^t} f^p v \, dx \\
&\leq C \sum_k \int_{P_k^t} f^p v \, dx \\
&\leq C \int_{\mathbb{R}^n} f^p v \, dx.
\end{aligned}$$

The constant is independent of t, so if we take the supremum over all $t > 0$ we get inequality (1.6).

□

Remark 4.5. At the cost of a more complex argument similar to that for Calderón-Zygmund operators (cf. Lemma 5.1 below) we can dispense with the dyadic fractional integral operator and prove Theorem 1.1 directly for I_α. The key inequality is the non-dyadic analogue of Lemma 4.4 due to Adams [1]: $M^\#(I_\alpha f)(x) \leq C M_\alpha f(x)$.

5. Calderón-Zygmund Operators

In this section we prove Theorem 1.2. The proof is similar to that of Theorem 1.1, but is complicated by the fact that we cannot pass to an equivalent dyadic operator. To compensate we need the following lemma which is also needed in the proof of Theorem 1.6.

Lemma 5.1. *Let B be a Young function. Suppose that for some function $f \in \bigcup_{p\geq 1} L^p$ and for some $t > 0$ there exists a constant μ, $0 < \mu \leq 1$, and a collection of dyadic cubes $\{Q_j\}$ such that for each j,*

$$|Q_j \cap \{x \in \mathbb{R}^n : M_B f(x) > t\}| \geq \mu |Q_j|.$$

Then there exists a constant $\nu > 0$, depending on n and μ, and a subcollection $\{P_k\}$ of the Calderón-Zygmund decomposition of f at height νt, $\{C_i^{\nu t}\}$, such that for each j, $Q_j \subset 3P_k$ for some k.

If we replace M_B by M_B^d in the hypothesis then we can strengthen the conclusion by finding P_k's such that $Q_j \subset P_k$ and by letting $\mu = \nu$.

Proof. We first consider the non-dyadic case. By Lemma 2.1,

$$E_t = \{x \in \mathbb{R}^n : M_B f(x) > t\} \subset \bigcup_i 3C_i^{\gamma t},$$

where $\gamma = 4^{-n}$. If we had $Q_j \subset 3C_i^{\gamma t}$ for some i we would be done, but this need not be the case, even if $\mu = 1$.. However, for each j there is a collection of indices A_j such that

$$Q_j \cap E_t \subset \bigcup_{i\in A_j} 3C_i^{\gamma t} \quad \text{and} \quad 3C_i^{\gamma t} \cap Q_j \neq \emptyset, \quad i \in A_j.$$

There are two possibilities: first, there exists $i \in A_j$ such that $l(Q_j) \leq l(3C_i^{\gamma t})$. Then $Q_j \subset 9C_i^{\gamma t}$ and by inequality (2.1),

$$\begin{aligned} 2\|f\|_{B,9C_i^{\gamma t}} &\geq \inf_{\mu>0}\left\{\mu + \frac{\mu}{|9C_i^{\gamma t}|}\int_{9C_i^{\gamma t}} B\left(\frac{|f|}{\mu}\right)\,dx\right\} \\ &\geq 9^{-n}\inf_{\mu>0}\left\{\mu + \frac{\mu}{|C_i^{\gamma t}|}\int_{C_i^{\gamma t}} B\left(\frac{|f|}{\mu}\right)\,dx\right\} \\ &= 9^{-n}\|f\|_{B,C_i^{\gamma t}} \\ &> 9^{-n}\gamma t. \end{aligned}$$

Alternatively, $l(Q_j) > l(3C_i^{\gamma t})$ for all $i \in A_j$. But then for each $i \in A_j$, $3C_i^{\gamma t} \subset 3Q_j$, and so

$$\begin{aligned} 2|3Q_j|\|f\|_{B,3Q_j} &\geq \inf_{\mu>0}\left\{\mu|3Q_j| + \mu\int_{3Q_j} B\left(\frac{|f|}{\mu}\right)\,dx\right\} \\ &\geq \sum_{i\in A_j}\inf_{\mu>0}\left\{\mu|C_i^{\gamma t}| + \mu\int_{C_i^{\gamma t}} B\left(\frac{|f|}{\mu}\right)\,dx\right\} \\ &= \sum_{i\in A_j}|C_i^{\gamma t}|\|f\|_{B,C_i^{\gamma t}} \\ &> 3^{-n}\gamma t\sum_{i\in A_j}|3C_i^{\gamma t}| \\ &\geq 3^{-n}\gamma t|Q_j \cap E_t| \\ &\geq 9^{-n}\mu\gamma t|3Q_j|. \end{aligned}$$

So in either case, for each j there exists a cube $\bar{Q}_j$ containing Q_j such that

$$\|f\|_{B,\bar{Q}_j} > \frac{\mu\gamma t}{2\cdot 9^n}.$$

Now by the same argument that is used to prove the Calderón-Zygmund decomposition, Lemma 2.1, we can show that there exists a subcollection $\{P_k\}$ of $\{C_i^{\nu t}\}$, $\nu = \frac{1}{2}\mu\gamma 36^{-n} = \frac{1}{2}\mu 108^{-n}$, such that for each j, $Q_j \subset \bar{Q}_j \subset 3P_k$ for some k. This completes the proof for M_B.

The proof in the dyadic case is very similar, but is simplified considerably by the fact that if two dyadic cubes intersect then one is contained in the other. □

Proof of Theorem 1.2. By a standard argument, we may assume that $f \in C^\infty(\mathbb{R}^n)$ and has compact support. Then $Tf \in L^2$, so we may apply Theorem 3.4 to it. Fix p, $1 < p < \infty$, $\delta < 1$ and $\epsilon > 0$ sufficiently small. For each $t > 0$ there exists a sequence of disjoint dyadic cubes $\{Q_j^t\}$ such that

$$\left(\frac{1}{|Q_j^t|}\int_{Q_j^t} ||Tf|^\delta - (|Tf|^\delta)_{Q_j^t}|\,dx\right)^{1/\delta} > \epsilon^{1/\delta} t$$

and

$$\begin{aligned}\sup_{t>0} t^p u(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) &\le \sup_{t>0} t^p u(\{x \in \mathbb{R}^n : M_\delta^d(Tf)(x) > t\}) \\ &\le C \sup_{t>0} t^p \sum_j A_u^r(Q_j^t).\end{aligned}$$

As we noted in the Introduction, T satisfies inequality (1.9). Therefore, for each j,

$$Q_j^t \subset \{x \in \mathbb{R}^n : M_\delta^\#(Tf)(x) > \epsilon^{1/\delta} t\} \subset \{x \in \mathbb{R}^n : Mf(x) > \beta t\},$$

where $\beta = C_\delta^{-1}\epsilon^{1/\delta}$.

By Lemma 5.1 (with $\mu = 1$), for each $t > 0$ there exists a sequence of disjoint dyadic cubes $\{P_k^t\}$ such that for each j, $Q_j^t \subset 3P_k^t$ for some k, and such that

$$\frac{1}{|P_k^t|}\int_{P_k^t} |f|\,dx > \rho t,$$

where $\rho > 0$ depends only on β and n. Then by Lemma 3.2, Condition (3), for each $t > 0$,

$$\begin{aligned} t^p \sum_j A_u^r(Q_j^t) &= t^p \sum_k \sum_{Q_j^t \subset 3P_k^t} A_u^r(Q_j^t) \\ &\le t^p \sum_k A_u^r(3P_k^t) \\ &\le \rho^{-p} \sum_k |3P_k^t| \left(\frac{1}{|3P_k^t|}\int_{3P_k^t} u^r\,dx\right)^{1/r} \left(\frac{1}{|P_k^t|}\int_{P_k^t} |f|\,dx\right)^p. \end{aligned}$$

By Hölder's inequality and inequality (1.7),

$$\begin{aligned}
&\leq C \sum_k \left(\frac{1}{|3P_k^t|} \int_{3P_k^t} u^r \, dx \right)^{1/r} \\
&\qquad \times \left(\frac{1}{|3P_k^t|} \int_{3P_k^t} v^{-p'/p} \, dx \right)^{p/p'} \int_{P_k^t} |f|^p v \, dx \\
&\leq C \sum_k \int_{P_k^t} |f|^p v \, dx \\
&\leq C \int_{\mathbb{R}^n} |f|^p v \, dx.
\end{aligned}$$

The constant is independent of t, so if we take the supremum over all $t > 0$ we get inequality (1.8). This completes our proof.

□

6. Commutators

In this section we prove Theorem 1.6. The proof depends on Theorem 1.2 and the following analogue of inequality (1.9) for commutators.

Lemma 6.1. *Given a Calderón-Zygmund operator T, a function b in BMO, constants δ_0 and δ_1, $0 < \delta_0 < \delta_1 < 1$, and $k \geq 1$, there exists a constant K, depending on the BMO norm of b, such that for every function $f \in C_0^\infty(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$,*

$$M_{\delta_0}^{\#,d}(C_b^k f)(x) \leq K \sum_{i=0}^{k-1} M_{\delta_1}^d(C_b^i f)(x) + K M^{k+1} f(x).$$

This result is found in [23]. As given there, the non-dyadic maximal operator appears in the first term on the right-hand side, but it is immediate from the proof that it is still true with the dyadic maximal operator there. Further, the proof for $k > 1$ is a straight-forward extension of the case $k = 1$, which is in [22].

Proof of Theorem 1.6. When $k = 0$, Theorem 1.6 reduces to Theorem 1.2, so we may fix $k \geq 1$. By a standard argument we may assume that $f \in C^\infty(\mathbb{R}^n)$ and has compact support. Then $C_b^k f \in L^2$, so we may apply Theorem 3.4 to it. Fix p, $1 < p < \infty$, δ_0 and δ_1, $0 < \delta_0 < \delta_1 < 1$, and $\epsilon > 0$ sufficiently small. Then for each $t > 0$ there exists a sequence of disjoint dyadic cubes $\{Q_j^t\}$ such that

$$\left(\frac{1}{|Q_j^t|}\int_{Q_j^t} ||C_b^k f|^{\delta_0} - (|C_b^k f|^{\delta_0})_{Q_j^t}|\,dx\right)^{1/\delta_0} > \epsilon^{1/\delta_0} t$$

and

$$\begin{aligned}\sup_{t>0} t^p u(\{x\in\mathbb{R}^n\colon |C_b^k f(x)|>t\}) &\leq \sup_{t>0} t^p u(\{x\in\mathbb{R}^n\colon M_{\delta_0}^d(C_b^k f)(x)>t\})\\ &\leq C\sup_{t>0} t^p \sum_j A_u^r(Q_j^t).\end{aligned}$$

By Lemma 6.1, for each j and t,

$$\begin{aligned} Q_j^t &\subset \bigcup_{i=1}^{k-1}\{x\in\mathbb{R}^n : M_{\delta_1}^d(C_b^i f)(x) > \beta t\}\\ &\quad \cup \{x\in\mathbb{R}^n : M_{\delta_1}^d(Tf)(x)>\beta t\}\\ &\quad \cup \{x\in\mathbb{R}^n : M^{k+1}f(x)>\beta t\}\\ &\equiv \left(\bigcup_{i=1}^{k-1} F_i^{\beta t}\right)\cup F_0^{\beta t}\cup F_k^{\beta t},\end{aligned}$$

where $\beta = \epsilon^{1/\delta_0}K^{-1}(k+1)^{-1}$. For each j and t we cannot have that $|Q_j^t\cap F_i^{\beta t}| < (k+1)^{-1}|Q_j^t|$ for all i. Hence, for some i, $|Q_j^t\cap F_i^{\beta t}| \geq (k+1)^{-1}|Q_j^t|$; if this is the case we write $Q_j^t\in\mathcal{F}_i^{\beta t}$. Thus,

$$\sup_{t>0} t^p\sum_j A_u^r(Q_j^t) \leq \sum_{i=0}^k \sup_{t>0} t^p \sum_{Q_j^t\in\mathcal{F}_i^{\beta t}} A_u^r(Q_j^t).$$

To complete the proof we will show that each term of the outer sum on the right-hand side is dominated by $C\int_{\mathbb{R}^n}|f|^p v\,dx$. There are three cases.

Case 1: cubes in $\mathcal{F}_k^{\beta t}$. As we noted in Section 2, there exists a constant $\beta'>0$ such that

$$\{x\in\mathbb{R}^n : M^{k+1}f(x)>\beta t\}\subset\{x\in\mathbb{R}^n : M_B f(x)>\beta' t\},$$

where $B(t)=t\log(1+t)^k$. Therefore, by Lemma 5.1 (with $\mu=(k+1)^{-1}$), there exists a constant $\nu>0$ such that, for each $t>0$ there exists a collection of disjoint dyadic cubes $\{P_l^t\}$ such that for each j, $Q_j^t\subset 3P_l^t$ for some l and such that $\|f\|_{B,P_l^t}>\nu t$. We now proceed exactly as we did at the end of the proof of Theorem 1.2. Since $C_k(t)=t^{p'}\log(1+t)^k$, $C_k^{-1}(t)\approx t^{1/p'}\log(t)^{-k}$, and so $t^{1/p}C_k^{-1}(t)\leq B^{-1}(t)$. Then, by Lemma 3.2, Conditions (2) and (3), the generalized Hölder's inequality (2.3) and inequality (1.11),

$$\begin{aligned}
\sup_{t>0} t^p \sum_{Q_j^t \in \mathcal{F}_k^{\beta t}} A_u^r(Q_j^t) &\leq \sup_{t>0} t^p \sum_l A_u^r(3P_l^t) \\
&\leq C \sup_{t>0} \sum_l |3P_l^t| \left(\frac{1}{|3P_l^t|} \int_{3P_l^t} u^r\,dx \right)^{1/r} \|f\|_{B,P_l^t}^p \\
&\leq C \sup_{t>0} \sum_l \left(\frac{1}{|3P_l^t|} \int_{3P_l^t} u^r\,dx \right)^{1/r} \\
&\qquad \times \|v^{-1/p}\|_{C_k,P_l^t}^p \int_{P_l^t} |f|^p v\,dx \\
&\leq C \sup_{t>0} \sum_l \left(\frac{1}{|3P_l^t|} \int_{3P_l^t} u^r\,dx \right)^{1/r} \\
&\qquad \times \|v^{-1/p}\|_{C_k,3P_l^t}^p \int_{P_l^t} |f|^p v\,dx \\
&\leq C \sup_{t>0} \sum_l \int_{P_l^t} |f|^p v\,dx \\
&\leq C \int_{\mathbb{R}^n} |f|^p v\,dx.
\end{aligned}$$

Case 2: cubes in $\mathcal{F}_0^{\beta t}$. Given $t > 0$, let $s = (\beta t)^{\delta_1}$. Again by Lemma 5.1 (the dyadic case), if $Q_j^t \in \mathcal{F}_0^{\beta t}$ then for some i, $Q_j^t \subset C_i^s$, where $\{C_i^s\}$ is the Calderón-Zygmund decomposition of $|Tf|^{\delta_1}$ at height s. Hence, by Lemma 3.2, Conditions (2) and (3),

$$\sup_{t>0} t^p \sum_{Q_j^t \in \mathcal{F}_0^{\beta t}} A_u^r(Q_j^t) \leq \sup_{t>0} t^p \sum_i A_u^r(C_i^s).$$

Fix $\epsilon > 0$ sufficiently small. By Corollary 3.5 there exists a subcollection $\{\bar{Q}_j^t\}$ of $\{C_i^s\}$ such that if $x \in \bar{Q}_j^t$ then $M_{\delta_1}^{\#,d}(Tf)(x) > \beta'' t$, where $\beta'' = \epsilon^{1/\delta_1}\beta$, and such that

$$\sup_{t>0} t^p \sum_i A_u^r(C_i^s) \leq C \sup_{t>0} t^p \sum_j A_u^r(\bar{Q}_j^t).$$

We can now argue exactly as we did in the proof of Theorem 1.2 to get

$$\sup_{t>0} t^p \sum_{Q_j^t \in \mathcal{F}_0^{\beta t}} A_u^r(Q_j^t) \leq C \int_{\mathbb{R}^n} |f|^p v\, dx.$$

Case 3: cubes in $\mathcal{F}_i^{\beta t}$, $1 \leq i \leq k-1$. Fix i and $\epsilon > 0$ sufficiently small. Then arguing exactly as we did in Case 2, by Corollary 3.5 there exists a collection of disjoint dyadic cubes $\{\bar{Q}_j^t\}$ such that if $x \in \bar{Q}_j^t$ then $M_{\delta_1}^{\#}(C_b^i f)(x) > \beta'' t$, where $\beta'' = \epsilon^{1/\delta_1}\beta$, and such that

$$\sup_{t>0} t^p \sum_{Q_j^t \in \mathcal{F}_i^{\beta t}} A_u^r(Q_j^t) \leq C \sup_{t>0} t^p \sum_j A_u^r(\bar{Q}_j^t).$$

We now apply Lemma 6.1 and repeat the argument at the beginning of this proof. When we do so we reduce the degree of the highest order commutator appearing from i to $i-1$. Therefore, after repeating our argument a finite number of times, we will reduce to collections of cubes satisfying conditions such as those in Case 1 and Case 2. Repeating those arguments will then give us the desired inequality.

□

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