Dynamical CW-complexes

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Abstract: *We introduce a class of CW-complexes, called* dynamical CW-complexes. *They are essentially* special polyhedra *as defined by Matveev, together with an orientation on their* 1*-cells satisfying two simple combinatorial properties. These complexes are called dynamical because all the complexes in a natural subclass carry some* non-singular semi-flow. *We give a necessary and sufficient criterion for these complexes to be* foliated *by compact leaves, and we study the implications for the topology and the fundamental group of the complex.*

Introduction

At the origin of this work, there was the question of how to associate a notion of dynamics (symbolic or combinatorial) to a finitely presented-group. Since any such group is the fundamental group of a finite 2-complex, I was first interested in introducing a dynamic on a finite 2-complex. Non-singular semi-flows appeared then to be quite natural dynamical objects for this purpose. Indeed, several classes of 2-dimensional complexes carrying non-singular semi-flows had already been studied. These were the *templates* of Williams (see [Wi2] or [BW] for instance) and the *dynamic branched surfaces* of Christy (see [Ch1,3]). Both have their roots in the notion of *branched surface* introduced by Williams in [Wi1]. However, on one hand, the fundamental groups of templates are only free groups. On the other hand, because these semi-flows come from hyperbolic flows, these authors always assume the existence of a *differentiable structure* on the complexes.

The first step of my work was to substitute to this condition of smoothness, not natural from an algebraic point of view, some combinatorial properties in order that a given complex in the class considered carries a non-singular semi-flow.

This paper presents some of the results in this direction obtained during the preparation of my thesis. One of its aspects is thus to introduce a new class of dynamical system (K, σ_t), where K is a n-dimensional CW-complex $(n \geq 2)$ and $(\sigma_t)_{t \in \mathbf{R}^+}$ a non-singular semi-flow on K, whose definition is in some sense combinatorial.

The CW-complexes we consider here have their origins in a particular class introduced by Casler, Ikeda or Matveev (see [Ca], [I], [Ma1,2]), called *standard complexes*, *closed fake surfaces* (in dimension 2) or *special polyhedra*. The difference between a standard complex and a closed fake surface is essentially the assumption, for a standard complex, to admit an embedding in some compact 3-manifold. A 2-dimensional special polyhedron "is" a standard complex. Special polyhedra were introduced by Casler and Matveev for the study of compact 3-manifolds with boundary for the first, and more generally of any piecewise-linear compact n-manifolds with boundary $(n \geq 2)$ for the second. The importance of these special polyhedra is stressed by the fact that any piecewise-linear compact n-manifold with boundary is the "thickening" of a special $(n-1)$-polyhedron. Moreover, any finitely generated group is the fundamental group of a closed fake surface (see [Wr]).

Definitions and basic results about dynamical CW-complexes are stated in section 1.2. Roughly speaking, a dynamical CW-complex is combinatorially a special polyhedron, but is moreover equipped with an orientation on certain of its 1-cells, which satisfies two simple combinatorial properties. This orientation allows to define a non-singular semi-flow by giving, in some sense, the direction of this semi-flow (see section 4). Let us observe that our n-dimensional complexes do not necessarily embed in compact $(n+1)$-manifolds. However, if such an embedding exists, one obtains a classical dynamical system (M^{n+1}, ϕ_t), where ϕ_t is a non-singular flow on the $(n+1)$-manifold M^{n+1}, which is semi-conjugated to a semi-flow on the dynamical complex (section 4).

From what precedes, asides their intrinsic interest as a dynamical system, the dynamical CW-complexes stand at the cross-roads of different branches of mathematics such as combinatorial group theory or 3-dimensional topology. Since the introduction of the dynamic on these complexes only depends of their combinatorics, a lot of questions arise naturally, which can be summarized in the following formulation:

What are the relationships between the combinatorics of the complex, the properties of the dynamical systems (K, σ_t), the fundamental group of the complex and the topology of the underlying manifold, if any?

Many well-known results motivate this kind of question. It is related to one of the trends in dynamical systems which is to try to understand, for a given dynamical system (X, ψ), what are the relationship between the properties of the underlying space X and the dynamical behaviour of ψ.

Considering a dynamical sytem (M^n, ϕ_t) where M^n is a n-manifold and $(\phi_t)_{t\in\mathbf{R}}$ a non-singular flow on M^n, an important case where such a relationship appears, and has been studied under different aspects, is when the flow $(\phi_t)_{t\in\mathbf{R}}$ admits a *cross-section* S, that is a hypersurface embedded in M^n which intersects all the orbits of the flow transversely, in the same direction and in finite time. A topological consequence of this dynamical property is that the manifold M^n fibers over the circle with fiber S or, in other words, admits a transversely orientable non-singular codim 1-foliation with compact leaves all homeomorphic to S. Conversely, if M^n admits a foliation as above, then there is a non-singular flow on M^n with a cross-section. One says that (M^n, ϕ_t) is the *suspension* or *mapping-torus* of (S, h), h being the return-homeomorphism on S of the flow $(\phi_t)_{t\in\mathbf{R}}$ on M^n. Also, a theorem of Thurston (see [Th3]) illustrates, in the case of compact 3-manifolds with boundary which fiber over the circle, the deep interconnection between geometry and dynamics: the interior of such a manifold admits a hyperbolic metric if and only if the monodromy of the fibration is *pseudo-Anosov*. Many authors have been interested in caracterizing flows admitting cross-sections or manifolds which fiber over the circle (see for instance [Fr] or [Ti]).

In section 3, we study a similar situation in our CW-complex context. We first adapt the notion of *foliation* to our complexes (see section 2.3). We then give a necessary and sufficient criterion, of both combinatorial and cohomological nature, for the complex considered to admit a transversely orientable codim 1-foliation whose leaves are compact and have all the same Euler characteristic. This Euler characteristic condition is substituted to the requirement that all the leaves are homeomorphic, which would be a too strong restriction. We so caracterize, by a finite and effective criterion depending only on the combinatorics of the complex (see remark 6.3) the existence on a given simple n-complex of a dynamical system (K, σ_t) which is the "suspension" of a dynamical system (K', h) where K' is a standard $(n-1)$-complex and h a continuous map of K' (the return-map of the semi-flow σ_t on K').

In the classical case of a compact manifold M^n admitting a non-singular transversely oriented codim 1-foliation with compact leaves, the group $\pi_1(M^n)$ is the semi-direct product of the fundamental group of any of the leaves S with $\mathbf{Z}$ over an *automorphism* $\mathcal{O}$. This automorphism is induced by the return-homeomorphism on S of some non-singular flow transverse to the leaves of the foliation. If $< x_1, \cdots, x_n ; R_1, \cdots, R_k >$ is a presentation of $\pi_1(S)$, the group $\pi_1(M_n)$ admits a presentation of the form $< x_1, \cdots, x_n, t ; R_1, \cdots, R_k, tx_it^{-1} = \mathcal{O}(x_i), i = 1, \cdots, n >$ and $\pi_1(M_n)$ is called the *suspension*, or *mapping-torus of the automorphism* $\mathcal{O}$ *of* $\pi_1(S)$. More generally, we will speak of the *suspension* or *mapping-torus of an* endomorphism *of a group*.

We show that, when considering *regular foliations*, the above result remains true, that is some semi-flow $(\sigma_t)_{t\in\mathbf{R}^+}$ transverse to the leaves of the foliation induces an *automorphism* on the fundamental group of a leaf K'. Moreover, this automorphism appears as a composition of elementary moves which we call *generalized WM-moves* (see section 2.4). This name is motivated by the fact that they are a generalization of the *Whitehead moves* in dimension 1 and of the *Matveev moves* (see [Ma2], [BP], [Pi]) in dimension 2. One states below a weak version of our result on regular foliations.

Theorem 0.1. *A dynamical n-complex K, $n \geq 2$, admits a transversely orientable codim 1-regular foliation $\mathcal{F}$, whose all leaves are compact and have the same Euler characteristic, if and only if K admits a* positive cocycle $u \in C^1(K; \mathbf{Z})$.

In addition, if such a foliation $\mathcal{F}$ exists, then all its leaves are homotopically equivalent. If $\mathcal{L}$ is some leaf of $\mathcal{F}$, the complex K is homotopically equivalent to the suspension of a continuous map $\psi : \mathcal{L} \to \mathcal{L}$, which is a homotopy equivalence. In particular, the fundamental group of K is the suspension of an automorphism of the fundamental group of this leaf.

Conversely, for any continuous map ψ of a standard $(n-1)$-complex, given as a composition of generalized WM-moves and of a homeomorphism (in particular ψ induces an automorphism on the fundamental group of the complex), there is a dynamical n-complex K homotopically equivalent to the suspension of ψ, and which admits a positive cocycle.

Let us observe that the proof of the last assertion of the above theorem is constructive.

Regular foliations are not the whole story. In proposition 5.6, we caracterize the existence, on dynamical 2-complexes, of more degenerate foliations, called *special*. The existence of such a foliation with compact leaves on a dynamical 2-complex, which is also related to the existence of certain integer cocycles, implies that the fundamental group of the complex is the suspension of an *injective*, *non-surjective* free group-endomorphism.

Theorem 0.1 above can be considered as an analog, in our CW-complex setting, of the already mentioned result of Tischler on the existence of foliations with compact leaves on manifolds (see [Ti]). Further analogy with the manifold-context, and more precisely with the Thurston norm on the homology of 3-manifolds (see [Th2]), can be found in the fact that the whole set of non-negative cocycles form a cone (see remark 6.3), in the same way that the set of cohomology classes in $H^1(M^3; \mathbf{Z})$ form cones over some faces of a well-defined convex polyhedron. This last analogy perhaps deserves to be examined in detail.

The last assertion of theorem 0.1 allows in particular to prove that the suspension of any free group-automorphism is the fundamental group of a "suspended" dynamical 2-complex (see proposition 3.13 and corollary 3.14). One so obtains a class of topological 2-dimensional objects which represents all the free group-automorphisms. In [G2], we give another proof by means of a different construction. We obtain there in particular a combinatorial construction for the suspension of any pseudo-Anosov homeomorphism of a compact surface with boundary. Finally, an other paper is to come, in which one considers more closely the problem of finding cross-sections to semi-flows on dynamical 2-complexes.

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1 Simple and dynamical CW-complexes

1.1 Preliminaries

We introduce here the various objects and concepts used in the next sections. The main point is the definition of a *dynamical CW-complex* (see definition 1.7).

If U is an open or closed subset of a topological space X, the *boundary of* U is the set $\overline{U} - \overset{\circ}{U}$, where $\overline{U}$ (resp. $\overset{\circ}{U}$) denotes the closure (resp. the interior) of U in X. A *path* (resp. *loop*) in X is a locally injective continuous map from the interval (resp. circle) to X.

We assume that the reader is familiar with basic notions about *CW-complexes* (see for instance [Mu]). All our complexes are piecewise-linear and, unless otherwise stated, connected, finite and compact.

The 0-cells (resp. 1-cells) of a CW-complex are called the *vertices* (resp. *edges*) of the complex. If e is an oriented edge, then e is said to be an *incoming edge* at its *terminal vertex* $t(e)$ and an *outgoing edge* at its *initial vertex* $i(e)$. Observe that an oriented edge e can be both incoming and outgoing at a same vertex, in the case where this edge is a loop.

A *graph* is a 1-dimensional CW-complex. A path or loop in a graph Γ is *positive* (resp. *negative*) if it is oriented such that its orientation agrees (resp. disagrees) at any point with the orientation of the edges that it intersects.

The *j-skeleton* $K^{(j)}$ of a n-dimensional CW-complex K $(0 \leq j \leq n)$ is the union of all the cells in K whose dimension is less or equal to j. Clearly $K^{(n)} = K$.

The *boundary* of a n-dimensional CW-complex K is the closure in K of the union of all the $(n-1)$-cells of K which are contained in the closure of exactly one open n-cell. The points which do not belong to the boundary of K form the *interior* $\overset{\circ}{K}$ of K.

In what follows, we will need to distinguish among the points of a n-dimensional CW-complex K different kinds of *singular points*. A point in the interior of a n-dimensional CW-complex K is *singular* if it has no neighborhood in K homeomorphic to $\mathbf{R}^n$.

Assume that K has an empty boundary. We denote by $K^{(n-1)}_{sing}$ the set of singular points in $K^{(n-1)}$. One defines the sets $K^{(i)}_{sing}$, $i = n-2, \cdots, 0$, by a descending induction as follows:

The set $K^{(i)}_{sing}$ is the set of points in $K^{(i)} \cap K^{(i+1)}_{sing}$ whose no neighborhood in $K^{(i+1)}_{sing}$ is homeomorphic to $\mathbf{R}^{i+1}$.

By convention, we set $K^{(n)}_{sing} = K$. In the case where K has a non-empty boundary, we define the singular sets $K^{(i)}_{sing}$, $i = 0, \cdots, n$, to be the closure in K of the singular sets $\overset{\circ}{K}{}^{(i)}_{sing}$.

The set $K^{(1)}_{sing}$ is called the *singular graph*. The vertices in $K^{(0)}_{sing}$ are called *crossings*. The connected components of $K^{(m+1)}_{sing} - K^{(m)}_{sing}$, $0 \leq m \leq n-1$, are called the $(m+1)$-*components* of the complex (the 0-components are the crossings).

These singular sets $K^{(i)}_{sing}$, and specially the singular graph $K^{(1)}_{sing}$ will play an important role in the sequel of this paper. Observe that they are subcomplexes of the complex K, which are not necessarily connected. Although they might be very complicated in the general case, they will appear to have a particularly simple structure for the class of CW-complexes we will consider later.

Let C be any open j-cell or j-component in K, $1 \leq j \leq n$. Let $N(x)$ be any small neighborhood of any point x in K. A *germ of C at x* is the closure in K of a connected component of $C \cap N(x)$.

Let $g_x(C)$ be a germ of an open k-cell (or component) C at a point x in K. This germ $g_x(C)$ is *incident* to a germ $g_x(C')$ of a j-cell (or component) C', $k > j$, if $g_x(C') \subset g_x(C)$. We will also say that the k-cell (or component) C is *incident at x to the j-cell C'*.

We will denote by $Con(X)$ the cone over a space X, that is the space $X \times [0,1]$, where $X \times \{1\}$ is identified to a single point. Let us recall that the closed *n-simplex* Δ^n, $n \geq 0$, is defined inductively as follows: The 0-simplex is a point, the closed i-simplex Δ^i is the cone over the closed $(i-1)$-simplex Δ^{i-1}, for $i = 1, 2, \cdots, n$. The symbol D^n will mean the open unit-ball in $\mathbf{R}^n$.

Let K be a n-dimensional CW-complex and let $\psi : K \to K$ be a continuous map. We denote by $Susp_\psi(K)$ the $(n+1)$-dimensional complex $K \times [0,1]/((x,1) \sim (\psi(x),0))$. This $(n+1)$-dimensional complex is called the *suspension*, or *mapping-torus*, of the map ψ of K.

If K is a n-manifold and ψ is a homeomorphism of K, then this construction gives rise to a $(n+1)$-manifold. Whereas the i^{th}-homotopy groups, $i \geq 2$, of a suspended manifold $Susp_\psi(K)$ are isomorphic to the i^{th}-homotopy groups $\pi_i(K)$, the fundamental group $\pi_1(Susp_\psi(K))$ of K is the semi-direct product of $\pi_1(K)$ with $\mathbf{Z}$ over an automorphism $\psi_\#$ of $\pi_1(K)$ induced by ψ (for more details see [S] for instance). If $< x_1, \cdots, x_n \,;\, R^K_1, \cdots, R^K_m >$ is a presentation of $\pi_1(K)$, this implies that the fundamental group of the mapping-torus $Susp_\psi(K)$ admits a presentation of the form $< x_1, \cdots, x_n, t;$ $R^K_1, \cdots, R^K_m, tx_it^{-1} = \psi_\#(x_i), i = 1, \cdots, n >$.

More generally, if $\psi : K \to K$ is any continuous map of any CW-complex K and $\psi_\# : K \to K$ is an endomorphism induced by ψ on $\pi_1(K)$, then the fundamental group of $Susp_\psi(K)$ admits a presentation of the form above. One says that the group $\pi_1(Susp_\psi(K))$ is the *suspension*, or *mapping-torus*, *of the endomorphism $\mathcal{O}$ of* $\pi_1(K)$.

1.2 Basic definitions

The following definition is derived from the notion of *standard complex* introduced by Casler (see [Ca]), in the 2-dimensional case (see also [Ma1,2], [BP], [Pi], [Wr]). In [Ma1], Matveev generalized, under the name of *special polyhedra*, the notion of standard complexes to the n-dimensional case. These special polyhedra are n-dimensional CW-complexes satisfying the following local property: each point of the polyhedron has a neighborhood homeomorphic to a neighborhood of some point in the interior of $Con((\partial\Delta^{n+1})^{(n-1)})$, i.e. the cone over the $(n-1)$-skeleton of the boundary of the $(n+1)$-dimensional simplex. Our simple n-complexes satisfy, in addition of this local property, a global topological condition.

Definition 1.1. A *simple* n-complex ($n \geq 1$) is a n-dimensional CW-complex K such that:

- For any point $x \in K$, there is a neighborhood $N(x)$ of x in K, a neighborhood $N(y)$ of a point y in the interior of $Con((\partial\Delta^{n+1})^{(n-1)})$, and a homeomorphism $h : N(x) \to N(y)$ such that $h(x) = y$.
- For any integer $1 \leq j \leq n-1$, the $(j+1)$-components are either $(j+1)$-cells or are homeomorphic to one of the two $(j+1)$-manifolds $Susp_{Id}(D^j)$ $(= D^j \times \mathbf{S}^1)$ and $Susp_{H_R}(D^j)$, where $H_R : D^j \to D^j$ is the reflexion relative to a diameter $D^{j-1} \subset D^j$ (D^0 is a single point).

A *standard* n-complex is a simple n-complex whose all j-components are j-cells, for any $j \leq n$.

Remark 1.2. Let us observe that the definition of a simple or standard n-complex implies that it has an empty boundary. If one had allowed the point y in item (1) of definition 1.1 to belong to the boundary of $Con((\partial\Delta^{n+1})^{(n-1)})$, one would have obtained a notion of *simple n-complex with boundary*. The j-components, $2 \leq j \leq n$, of a simple n-complex with boundary are required to be cells.

Example: A simple 1-complex is a *trivalent graph*, that is a graph such that exactly *three* (possibly not distinct) edges are incident to each crossing. Such a graph is also a standard 1-complex. The number *three* above is due to the fact that a 2-dimensional simplex, i.e. a triangle, has three vertices.

As illustrated by figure 1, there are three types of points in a simple 2-complex. Two are the types of singular points. The 2-components of such a complex are discs, annuli and Moebius-bands. For a more detailed description of simple 2-complexes, see remark 5.1.

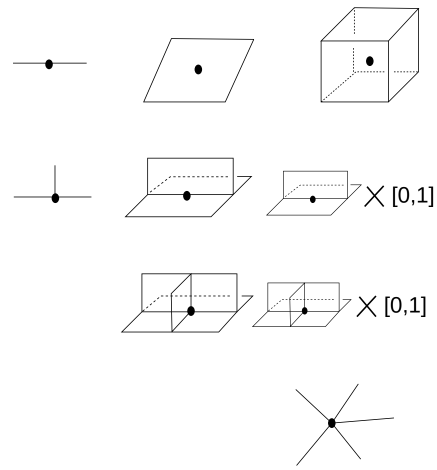

Figure 1: Non-singular and singular points in simple n-complexes

The following two lemmas are about the structure of the singular sets of a simple n-complex. The first one (lemma 1.3) is easy to check.

Lemma 1.3. *Let K be a simple n-complex ($n \geq 1$).*

1. *The sets $K_{sing}^{(i)}$, $i = 0, \cdots, n$, do not depend on the structure of CW-complex of K. Moreover: $K_{sing}^{(0)} \subset K_{sing}^{(1)} \subset \cdots \subset K_{sing}^{(n)} = K$, and, for any $i = 0, \cdots, n$, $K_{sing}^{(i)} \subset K^{(i)}$.*
2. *For any point $x \in K$, there is a smallest integer $0 \leq j \leq n$ such that $x \in K_{sing}^{(j)}$. The point x is non-singular in K if and only if this smallest integer equals n.*

If n, p are two non-negative integers, we pose $C_n^p = \frac{n!}{p!(n-p)!}$ if $n \geq p$ and $C_n^p = 0$ otherwise.

Lemma 1.4. *Let K be a simple n-complex ($n \geq 1$).*

1. *Let x be any point in K, distinct from a crossing. Let $n \geq j \geq 1$ be the smallest integer such that x belongs to $K_{sing}^{(j)}$. Then there is a neighborhood $N(x)$ of x in K, a point y in a $(j-1)$-cell in the interior of $K_0 = Con((\partial \Delta^n)^{(n-2)})$ and a neighborhood $N(y)$ of y in K_0 such that $N(x)$ is homeomorphic to $N(y) \times [0,1]$ (see figure 1).*

2. *The number of germs of k-components at a crossing v which are incident to a given germ of j-component ($k > j$) is equal to C_{n-j+2}^{n-k+2}. The number of germs of j-components at v which are contained in a germ of a k-component at v is C_k^{k-j}. Each set of C_k^{k-j} distinct germs of j-components at v is contained in a germ of k-component ($n-1 \geq j \geq 0$, $n \geq k \geq 1$).*

3. *In particular, the singular graph of a simple n-complex, $n \geq 2$, is a compact graph, possibly not connected, with $(n+2)$ edges incident to each crossing.*

Proof of lemma 1.4: By definition of a simple n-complex, each crossing is the vertex of a cone over the $(n-1)$-skeleton of the boundary of a closed $(n+1)$-dimensional simplex Δ^{n+1}. Thus, a germ of a $(i+1)$-cell at a crossing v is the cone over a i-simplex in $(\partial\Delta^{n+1})^{(n-1)}$ $(n-1 \geq i \geq 0)$. This implies item (1).

From what precedes, each i-simplex in the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ gives rise to a germ of $(i+1)$-cell at the corresponding crossing. Each set of i distinct vertices, $1 \leq i \leq n$, among the $(n+2)$ vertices of $\partial\Delta^{n+1}$ defines a unique $(i-1)$-simplex in $(\partial\Delta^{n+1})^{(n-1)}$. Moreover, any j-simplex σ^j of $(\partial\Delta^{n+1})^{(n-1)}$ contains any i-simplex, $0 \leq i \leq j-1$, whose $i+1$ vertices are among the $j+1$ vertices of σ^j. Similarly, any i-simplex σ^i, $0 \leq i \leq n-2$, is contained in any j-simplex, $i < j \leq n-1$, whose set of vertices contains the i vertices of σ^i. Item (2) follows by a counting argument, and item (3) is then clear. $\Box$

We will also need in the course of this article to consider CW-complexes which are "almost" simple complexes (see sections 2.2 and 2.4). This is the subject of definition 1.5 below.

Let us recall that *collapsing* a closed j-cell ($1 \leq j \leq n$) of a CW-complex K means identifying this closed j-cell to a single point, the resulting space being again a CW-complex.

Definition 1.5. A *simple degenerate n-complex* is a n-dimensional CW-complex K' such that there exists some simple n-complex K and a sequence, possibly empty, of collapses of closed j_i-cells Δ_i of K, $i = 1, \cdots, m$, $j_i \in \{1, \cdots, n\}$, satisfying the following three properties:

- Each closed cell Δ_i is a closed j_i-simplex, $i = 1, \cdots, m$.
- Each closed cell Δ_i is disjoint from any other closed cell Δ_j, $i, j \in \{1, \cdots, m\}$, $j \neq i$.

- The complex obtained by the collapsing of these closed cells Δ_i is isomorphic to the complex K'.

By definition, a simple n-complex is obviously a simple degenerate n-complex.

Example: A simple degenerate 1-complex is a graph whose each vertex is contained in at most four germs of edges.

Remark 1.6. A *simple degenerate n-complex with boundary* is a complex obtained by collapsing a finite number of closed j-cells Δ_i, $j \in \{1, \cdots, n\}$, in the *interior* of a *simple n-complex with boundary* (see remark 1.2), which are required to satisfy the properties of definition 1.5 above.

Important: Let K be any simple n-complex ($n \geq 2$). The singular graph $K^{(1)}_{sing}$ of K admits a canonical structure of CW-complex defined as follows: The vertices are the crossings of the complex K, together with a set of valency 2-vertices, one for each connected component of $K^{(1)}_{sing}$ which is a loop without any crossing. The edges are the 1-components of the complex. We will always assume that the singular graph $K^{(1)}_{sing}$ is equipped with this canonical structure of CW-complex. One easily proves that there always exists a structure of CW-complex for K such that the sets of vertices and edges of K contained in $K^{(1)}_{sing}$ are as defined above. We will always assume K to be equipped with such a structure. This causes no loss of generality for our purpose.

Let K be a simple n-complex, together with an orientation on the edges of the singular graph. Let C be any i-component or i-cell of K ($2 \leq i \leq n$). We will say that C contains an *attractor* (resp. a *repellor*) in its boundary if there is a crossing v of K and a germ $g_v(C)$ of C at v such that all the germs of edges of $K^{(1)}_{sing}$ at v contained in $g_v(C)$ are incoming (resp. outgoing) at v. If C is i-dimensional, there are exactly i such germs of edges (see lemma 1.4, item (2)). We will say that the crossing v above *is* or *gives rise to* an attractor (resp. a repellor) for C (and for the given orientation). Observe that a same crossing v can give rise to $k > 1$ attractors or repellors in the boundary of a same component or cell C.

All the tools needed for the definition of a *dynamical n-complex* have been given.

Definition 1.7. A *simple dynamical n-complex* is a simple n-complex K, $n \geq 2$, together with an orientation on the edges of the singular graph $K^{(1)}_{sing}$ satisfying the following two properties:

1. Each crossing of K is the initial and terminal crossing of at most n edges of $K^{(1)}_{sing}$.
2. Any j-component, $2 \leq j \leq n$, which is a j-cell has exactly one attractor and one repellor for this orientation in its boundary. The other components have no attractor and no repellor in their boundary.

A *standard dynamical n-complex* is a simple dynamical n-complex which is also a standard n-complex ($n \geq 2$).

Remark 1.8. The definition of dynamical n-complex above is stated for $n \geq 2$. A dynamical 1-complex would be a circle so that we will consider only higher dimensional dynamical complexes.

Remark 1.9. Since the number of edges incident to a given crossing is equal to $n+2$ (see lemma 1.4 item (2)), condition (1) in definition 1.7 is equivalent to the fact that each crossing in a simple dynamical n-complex is the initial and terminal crossing of at least 2 edges. For $n=2$, this condition is equivalent to require that each crossing is the terminal crossing of exactly 2 edges. For a more detailed description of simple dynamical 2-complexes, see remark 5.1.

A crossing in a simple n-complex which is the terminal crossing of exactly j edges, $0 \leq j \leq n+2$, will be called a *type j-crossing*. In a simple dynamical n-complex, by definition, the only integer j for which there are type j-crossings belong to the interval $[2, n]$.

Lemma 1.10. *Let K be a simple n-complex, together with an orientation on the edges of its singular graph. For each type j-crossing in K ($0 \leq j \leq n+2$), there are exactly C^k_j attractors (resp. C^k_{n+2-j} repellors) in the set of the boundaries of the k-components in K ($2 \leq k \leq n$).*

Proof of lemma 1.10: Let v be a type j-crossing of K. By definition, a k-component C ($2 \leq k \leq n$) contains v as attractor in its boundary if and only if a germ of C at v contains only germs of incoming edges of $K^{(1)}_{sing}$ at v. Thus counting the number of attractors in the boundaries of the k-components given by the crossing v is equivalent to count the number of germs of k-components at v which contain only germs of incoming edges of $K^{(1)}_{sing}$ at v. By lemma 1.4, item (2), a germ of a k-component at a crossing contains exactly k germs of edges of $K^{(1)}_{sing}$ at this crossing. Moreover, from this same lemma, any set of k germs of 1-components at a crossing is contained in a germ of k-component ($1 \leq k \leq n$). This implies that there are C_j^k germs of k-components containing only germs of incoming edges of $K_{sing}^{(1)}$ at v. The counting is similar for the repellors. Lemma 1.10 is proved. □

It is not true that any simple n-complex admits an orientation which makes it a dynamical n-complex. The following proposition gives a property satisfied by any simple dynamical n-complex. An alternative proof of this proposition could be given using further results of the paper, namely corollary 2.17 and lemma 4.3. However, one prefered to state it at this step, together with a more elementary proof.

Proposition 1.11. *The Euler characteristic of a simple dynamical n-complex is zero.*

Proof of proposition 1.11: The proof consists of a counting argument, which relies on the following lemma:

Lemma 1.12. *Let K be a simple dynamical n-complex ($n \geq 2$). For any $2 \leq j \leq n$, the number of type j-crossings is the same than the number of type $(n+2-j)$-crossings.*

Proof of lemma 1.12: We proceed by a descending induction on j, starting from $j = n$. Assume that there is a type n-crossing in K. By lemma 1.10, this crossing gives rise to exactly one attractor and no repellor in the set of boundaries of n-components. Since there must be one attractor and one repellor in the boundary of each n-component which is a cell, and none in the boundary of the other n-components, there exists a crossing in K which gives rise to a repellor in the boundary of some n-component. The only possible crossing is a crossing which has at least n outgoing edges, thus the only possible crossing is a type 2-crossing. Therefore, for $j = n$, the numbers of type j- and type $(n+2-j)$-crossings in K are the same.

We assume now that, for $j = n, n-1, \cdots, n-m$, $m > 0$, the numbers of type j- and type $(n+2-j)$-crossings are the same. One wants to check this property for $j = n-m-1$. Let v be a type $(n-m-1)$-crossing in K. By lemma 1.10, this crossing gives rise to an attractor in the boundary of a $(n-m-1)$-component. Since K is a simple dynamical n-complex, there exists a crossing which gives rise to a repellor in the boundary of this $(n-m-1)$-component. The only possible crossings are crossings which have at least $n-m-1$ outgoing edges of $K_{sing}^{(1)}$, these are type k-crossings with $k \leq m+3$. For $k \leq m+2$, the hypothesis says that for each type k-crossing, there is also a type $(n+2-k)$-crossing. Therefore, the existence of the type $(n-m-1)$-crossing v implies the existence of a type $(m+3)$-crossing, that is a crossing with exactly $n-m-1$ outgoing edges of $K_{sing}^{(1)}$.

Therefore, for each $2 \leq j \leq n$, there is the same number of type j- and type $(n+2-j)$-crossing. Lemma 1.12 is proved. □

Let K be a simple dynamical n-complex. One denotes by K' the complex obtained from K by supressing all the i-cells which do not belong to $K_{sing}^{(i)}$, $0 \leq i \leq n-1$. We do not supress the n-cells, however the only one remaining are n-components of K. We denote by N_i the number of i-cells of K', $0 \leq i \leq n$ and we set $S(K') = \sum_{i=0}^{n} (-1)^i N_i$. We also pose $N_0 = \sum_{j=2}^{j=n} N_0^j$, where N_0^j is the number of type j-crossings.

Claim: For any $2 \leq k \leq n$ and for each pair (type j-crossing, type $(n+2-j)$-crossing), there are $C_j^k + C_{n+2-j}^k$ k-cells in K'. If n is even, then for each type $\frac{n+2}{2}$-crossing, there are $C_{\frac{n+2}{2}}^k$ k-cells in K' $(2 \leq k \leq n)$.

This claim is a consequence of lemma 1.10 and of the definition of a dynamical n-complex.

By lemma 1.4, item (2), $(n+2)$ germs of 1-cells are incident to each crossing. Thus, $N_1 = \frac{n+2}{2} N_0$. Moreover, lemma 1.12 implies, for any fixed j, the equality $N_0^j + N_0^{n+2-j} = 2N_0^j$. Therefore:

If n is odd, $\quad N_0 - N_1 = -n \sum_{j=2}^{\frac{n+1}{2}} N_0^j$

and otherwise $\quad N_0 - N_1 = -n \sum_{j=2}^{\frac{n}{2}} N_0^j - \frac{n}{2} N_0^{\frac{n+2}{2}}$.

Thus, by the claim above, one has:

If n is odd,
$$S(K') = \sum_{j=2}^{\frac{n+1}{2}} \sum_{k=2}^{n} (-1)^k [(C_j^k + C_j^{n+2-k}) N_0^j] - n \sum_{j=2}^{\frac{n+1}{2}} N_0^j .$$

If n is even,
$$S(K') = \sum_{j=2}^{\frac{n}{2}} \sum_{k=2}^{n} (-1)^k [(C_j^k + C_j^{n+2-k}) N_0^j] - n \sum_{j=2}^{\frac{n}{2}} N_0^j + \sum_{k=2}^{n} (-1)^k C_{\frac{n+2}{2}}^k N_0^{\frac{n+2}{2}} - \frac{n}{2} N_0^{\frac{n+2}{2}} .$$

The expression $\sum_{k=2}^{n}(-1)^k C_i^k$ is equal to $-(\chi(\Delta^i)-i)$ for any integer $i \geq 1$. Moreover, $\chi(\Delta^i)=1$. This holds in particular for $i=j$, $i=n+2-j$ and, if n is even, for $i=\frac{n+2}{2}$. Therefore, one has

$$-nN_0^j+\sum_{k=2}^{n}(-1)^k[(C_j^k+C_j^{n+2-k})N_0^j]$$
$$=(-n-1+j-1+n+2-j)N_0^j=0$$

for any j, and, if n is even,

$$\sum_{k=2}^{n}(-1)^k C_{\frac{n+2}{2}}^k N_0^{\frac{n+2}{2}}-\frac{n}{2}N_0^{\frac{n+2}{2}}=-(1-\frac{n+2}{2})-\frac{n}{2}$$
$$=-1+\frac{n}{2}+1-\frac{n}{2}=0\,.$$

By substituting these results in the expression above for $S(K')$ in both cases, one obtains $S(K')=0$.

Let us now prove that $\chi(K)$ is zero. One obtains from K' a structure of CW-complex for the dynamical n-complex K by:

- subdividing any j-component which is not a cell by a $(j-1)$-cell $\overline{D}^{j-1}\times\{1\}$,
- subdividing the complex K' at $\partial D^{j-1}\times\{1\}$.

Since the operation of subdividing does not change the alternated sum $S(K')$, one has then, for K a simple dynamical n-complex, $\chi(K)=S(K')=0$ for any integer $n\geq 2$. This completes the proof of proposition 1.11. □

2 Cocycles, embeddings and foliations

2.1 Homology of simple complexes

We assume that the reader is familiar with the homology of CW-complexes (see for instance [Mu]). We just recall and introduce here some notions which will be needed later.

Let us first remind that the singular graph $K_{sing}^{(1)}$ of a complex K is always assumed to be equipped with a structure of CW-complex whose 0-cells are the crossings of K, together with a set of valency 2-vertices in the loops containing no crossing, and whose 1-cells are the 1-components of K. Furthermore, these edges and vertices of $K^{(1)}_{sing}$ are the only edges and vertices of K contained in $K^{(1)}_{sing}$.

Let K be a simple n-complex ($n \geq 2$). An *integer cocycle* will denote a cocycle in $C^1(K;\mathbf{Z})$, that is a finite collection of integer weights on the edges of the complex whose algebraic sum along the boundary circles of the 2-cells of K is zero.

The following definitions are stated for any simple n-complex ($n \geq 2$), but they will be considered only in the case of dynamical n-complexes. Let us recall that, in this case, the edges of the singular graph of the complex are equipped with a well-defined orientation (see definition 1.7).

Definition 2.1. Let K be a simple n-complex ($n \geq 2$), together with an orientation on the edges of its singular graph.

1. A *non-negative* cocycle in $C^1(K;\mathbf{Z})$ is an integer cocycle u such that $u(e) \geq 0$ holds for any edge e in the singular graph $K^{(1)}_{sing}$ and there is at least one such edge e with $u(e) > 0$.
2. A *positive* cocycle is a non-negative cocycle which is positive on all the positive embedded loops in $K^{(1)}_{sing}$.
3. A cohomology class $c \in H^1(K;\mathbf{Z})$ is *non-negative* (resp. *positive*) if $c(l) \geq 0$ (resp. $c(l) > 0$) for any positive embedded loop l in $K^{(1)}_{sing}$.

The δ_v*-moves* defined below consist of pushing an integer cocycle u through a crossing v, by removing 1 from the value of u on all the incoming edges at v and adding 1 to its value on the outgoing one. More precisely:

Definition 2.2. Let K be a simple n-complex. Let $u \in C^1(K;\mathbf{Z})$ be a cocycle and let v be any crossing of K.

A δ_v*-move* is the map $\delta_v : C^1(K;\mathbf{Z}) \to C^1(K;\mathbf{Z})$ defined by:

$(\delta_v(u))(e_i) = u(e_i) - 1$ for all the incoming, non-outgoing 1-cells e_i of K at v,

$(\delta_v(u))(f_j) = u(f_j) + 1$, for all the outgoing, non-incoming 1-cells f_j of K at v,

$(\delta_v(u))(x) = u(x)$ for all the other 1-cells x in $K^{(1)}$.

Remark 2.3. One denotes by $\delta_v^{-1} : C^1(K;\mathbf{Z}) \to C^1(K;\mathbf{Z})$ an *inverse* δ_v*-move*, that is $\delta_v \circ \delta_v^{-1} = \delta_v^{-1} \circ \delta_v = Id_{C^1(K;\mathbf{Z})}$. One denotes δ_v^r (resp. δ_v^{-r}) a δ_v-move (resp. a δ_v^{-1}-move) applied consecutively r times at the same crossing v.

Remark 2.4. The image of a cocycle by a $\delta_v^{\pm 1}$-move is a cocycle. Furthermore, two integer cocycles u' and u of a simple n-complex K which are obtained one from the other by a finite sequence of δ_v-moves are cohomologous.

Definition 2.5. With the assumptions and notations of definition 2.2,

A *non-negative δ_v-move* on a non-negative cocycle u is a δ_v-move on u such that $\delta_v(u)$ is non-negative.

Any non-negative integer cocycle defines a non-negative cohomology class. The following lemma shows that the converse is true.

Lemma 2.6. *Let K be a simple dynamical n-complex ($n \geq 2$). Any non-negative cohomology class in $H^1(K; \mathbf{Z})$ is represented by a non-negative cocycle in $C^1(K; \mathbf{Z})$.*

Proof of lemma 2.6: We need first to introduce some terminology. Let $\mathcal{T}$ be a tree together with an orientation on its edges. $\mathcal{T}$ is a *rooted tree* if there is exactly one vertex v in $\mathcal{T}$ whose all incident edges are outgoing edges. This vertex v is the *root* of $\mathcal{T}$. The *ends* of a rooted tree $\mathcal{T}$ are the vertices with exactly one incident edge. These edges are the *terminal edges* of $\mathcal{T}$ (since $\mathcal{T}$ is a rooted tree, each terminal edge is an incoming edge at the corresponding end).

Let $\mathcal{T} = \pi^{-1}(K^{(1)}_{sing})$ be the universal covering of the singular graph of K (π is the associated covering-map). $\mathcal{T}$ is an infinite tree. The edges of $\mathcal{T}$ inherit an orientation from the orientation of the edges of $K^{(1)}_{sing}$.

Let $u \in C^1(K; \mathbf{Z})$ be any integer cocycle in a non-negative cohomology class. If u is a non-negative cocycle, there is nothing to prove. Assume then that u is not a non-negative cocycle. If $K^{(1)}_{sing}$ is not connected, one will repeat the process described below for each connected component. Furthermore, any cocycle representing a non-negative cohomology-class is non-negative on the loops of $K^{(1)}_{sing}$ which do not contain any crossing. This is why, in what follows, one only considers the case where $K^{(1)}_{sing}$ is connected and contains at least one crossing.

Let e be an edge of the singular graph such that $u(e) < 0$.

Let v_0 be a vertex of $\mathcal{T}$ in $\pi^{-1}(i(e))$ and let e_0 be the edge of $\mathcal{T}$ incident to v_0 such that $\pi(e_0) = e$.

One defines inductively a sequence $\mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_i \subset \cdots$ of rooted trees $\mathcal{T}_i \subset \mathcal{T}$ with root v_0, and a sequence of integer cocycles $u_0, u_1, \cdots, u_i, \cdots$ of $C^1(K; \mathbf{Z})$ in the following way:

$\mathcal{T}_0 = e_0$, $u_0 = u$.

i^{th} `step:` Let $v_1^i, \cdots, v_k^i$ be a maximal (in the sense of the inclusion) set of ends of $\mathcal{T}_{i-1}$ such that:

- $\pi(v_j^i) \neq \pi(v_k^i)$ if $j \neq k$.
- If x_j is the terminal edge of $\mathcal{T}_{i-1}$ incident to v_j^i, then $m_j = |u_{i-1}(\pi(x_j))| = max\{|u_{i-1}(\pi(x))|$, x is a terminal edge of $\mathcal{T}_{i-1}$, $\pi(t(x)) = \pi(v_j^i)\}$.

$u_i = (\delta_{v_1^i}^{-m_1} \circ \cdots \circ \delta_{v_k^i}^{-m_k})(u_{i-1})$.

$\mathcal{T}_i$ is the union of $\mathcal{T}_{i-1}$ with the edges x of $\mathcal{T}$ whose initial vertex is one of the ends $v_1^i, \cdots, v_k^i$ of $\mathcal{T}_{i-1}$ and such that $u_i(\pi(x)) < 0$.

An easy induction allows to check that each tree in the above sequence is a rooted tree with root v_0 (and each is contained in $\mathcal{T}$).

Claim 1: If there is an integer $n > 0$ such that for any $j \geq n$, $\mathcal{T}_j = \mathcal{T}_n$, then:

1. The cocycle u_n is in the same cohomology class than u.
2. If x is any edge in $\mathcal{T}_n$, then u_n is non-negative on the edge $\pi(x)$ of $K_{sing}^{(1)}$. In particular, u_n is non-negative on the edge e chosen at the beginning.
3. If $u_n(x) < 0$ for some edge x of $K_{sing}^{(1)}$, then $u(x) < 0$.

Item (1) of claim 1 comes from remark 2.4. Item (2) comes from the construction. For proving item (3), let us first observe that, from item (2), it suffices to look at the edges x of $K_{sing}^{(1)}$ whose no lift under π^{-1} belongs to $\mathcal{T}_n$. Three cases occur, the two last one being not exclusive one from the other. For each of these cases, we give below the only possible changes which might have been applied to u with respect to x during the construction of $\mathcal{T}_n$ and u_n:

`Case 1:` No vertex of x has a lift under π^{-1} which belongs to $\mathcal{T}_n$. Then $u_n(x) = u(x)$.

`Case 2:` $\mathcal{T}_n$ contains a vertex in the lift under π^{-1} of the initial vertex of x. Then, the value of u on x might have been lowered, remaining non-negative, at some step(s) of the construction.

Case 3: $\mathcal{T}_n$ contains a vertex in the lift under π^{-1} of the terminal vertex of x. Then, the value of u on x might have been increased at some step(s) of the construction.

Item (3) is then easily deduced. □

Claim 2: There exists an integer $n > 0$ such that for any $j \geq n$, $\mathcal{T}_j = \mathcal{T}_n$.

Assume that some $\mathcal{T}_j$ contains a positive path from v_0 whose length is strictly greater than the number of crossings of the singular graph. Then this path contains two vertices v_i, v_j of $\mathcal{T}$ such that $\pi(v_i) = \pi(v_j)$. These two vertices are connected by a positive path in $\mathcal{T}_j$. This positive path projects, under π, to a positive loop in the singular graph. By construction, this implies that the cohomology class of u is negative on this loop, which contradicts our assumption. Therefore, the number of crossings of K is a uniform superior bound M (it does not depend on j) on the length of the positive paths with initial vertex v_0 in the trees $\mathcal{T}_j$, $j = 0, 1, \cdots$.

Assume now that, for any integer $n > 0$, there exists $j \geq n$ such that $\mathcal{T}_n$ is strictly contained in $\mathcal{T}_j$. This implies that one has an infinite subsequence of rooted trees $\mathcal{T}_{i_1}, \cdots, \mathcal{T}_{i_k}, \cdots$ such that for any $k \geq 1$, $\mathcal{T}_{i_k}$ is strictly contained in $\mathcal{T}_{i_{k+1}}$. This is a contradiction with the existence of the uniform superior bound M proved above. One so proved claim 2. □

Claim 1 and Claim 2 together imply that, by applying the above construction finitely many times, at most one time for each edge e of the singular graph with $u(e) < 0$, one obtains a non-negative cocycle in the cohomology class of u. □

2.2 Embeddings

Important: From now on, we will sometimes consider non-finite, and non-compact, complexes. In any case, the complexes K considered will always be required to satisfy the following property: The closure in K of any component is compact.

We define below a particular class of embeddings of simple degenerate $(n-1)$-dimensional CW-complexes (see definition 1.5) in simple n-complexes, which we call *r-embeddings* (the letter r stands for *regular*). The r-embedded simple complexes are natural from a cohomological point of view, as shown by lemma 2.13, which gives the relationship between integer cocycles and r-embeddings of simple complexes.

The conditions we add to the definition of an embedding for defining these r-embeddings are conditions of position with respect to the singular set of the complex.

We will say that a $(n-1)$-dimensional CW-complex K_1 embedded in a n-dimensional one K is *c-transverse* to an embedded i-dimensional CW-complex K_2, $1 \leq i \leq n-1$, if, for any x in $K_1 \cap K_2$, for any small neighborhood $N(x)$ of x in K, for any isotopy j_t^1 (resp. j_t^2), $t \in [0,1]$, of K_1 (resp. of K_2) in K with support in $N(x)$, $j_t^1(K_1) \cap j_t^2(K_2)$ is non-empty.

If K_1 and K_2 are not c-transverse at some point $x \in K_1 \cap K_2$, they are *c-tangent* (at x).

Before stating the definition of a r-embedding, let us recall that any point in a simple n-complex K, which is distinct from a crossing of K, admits a neighborhood in K of the form $X \times [0,1]$, where X is as given by lemma 1.4. Let us also observe that $K_{sing}^{(j)}$ is embedded in K for any $j = 0, \cdots, n-1$.

Definition 2.7. Let K be a simple n-complex and let K' be a possibly non finite simple degenerate $(n-1)$-complex $(n \geq 2)$.

An embedding $p : K' \to K$ is a *r-embedding* if $p(K')$ is c-transverse to $K_{sing}^{(j)}$, $j = 1, \cdots, n-1$ and satisfies the following properties:

1. Let x be any point in K' such that $p(x)$ is distinct from a crossing of K. If X is such that a neighborhood of $p(x)$ in K is homeomorphic to $X \times [0,1]$ (see lemma 1.4), then there is a neighborhood $N(x)$ of x in K' such that $p_{|_{N(x)}}$ is a homeomorphism from $N(x)$ onto $X \times \{t\}$, $t \in [0,1]$.

2. For any x in K' such that $p(x)$ is a crossing of K, any germ of j-cell at $p(x)$ in K contains the image under p of at most 1 germ of $(j-1)$-cell at x in K', $j = 2, \cdots, n$.

A r-embedding p of K' in K such that $p(K')$ contains a crossing of K is called a *degenerate* r-embedding.

Example: Let us recall that a simple degenerate 1-complex Γ is a graph whose no vertex is contained in more than four distinct germs of edges. If Γ is r-embedded in a simple 2-complex K, then the vertices of Γ which are in the interior of the edges of the singular graph of K are contained in exactly three germs of edges of Γ. There is one such germ in each germ of 2-cell of K at the point considered (item (1) of definition 2.7). Indeed, a neighborhood of a point interior to an edge of the singular graph of a simple 2-complex is homeomorphic to the cartesian-product of a triod with the interval.

The condition we impose for degenerate embeddings (item (2) of definition 2.7) avoids to have too degenerate situations at the crossings. It will be important when defining and studying foliations of simple n-complexes (see subsection 2.3).

Remark 2.8. We define in the same way the r-embedding of a simple degenerate $(n-1)$-complex in a simple n-complex with boundary, with the additional restriction that the embedding either is disjoint from the boundary, or is contained in a connected component of this boundary.

Remark 2.9. Definition 2.7 above implies that, if $p(K')$ is a simple degenerate $(n-1)$-complex which is r-embedded in K in a non-degenerate way, then K' is a simple $(n-1)$-complex.

Remark 2.10. Let K' be a simple degenerate $(n-1)$-complex r-embedded in a simple n-complex K. The assumption on the compactness of the closure in K' of its components implies that the components of K' are properly embedded in the components of K. This means that their interior is embedded in the interior of the components of K, and their boundary in K' is in the boundary in K of these components.

We take some care below in the definition of being 2-*sided* in a simple n-complex because one has to deal with degenerate embeddings. Moreover, this notion of being 2-sided will be important in the sequel of the paper, when studying foliations of simple complexes or also when constructing semi-flows on dynamical complexes.

Let K be a simple n-complex and let K' be a possibly non finite simple degenerate $(n-1)$-complex r-embedded in K under p.

A *transversal* τ_y to $p(K')$ in K at $y \in p(K')$ is a tree embedded in a small neighborhood of y in K and c-transverse to $p(K')$ at y. Furthermore, one requires that there is exactly one edge of τ_y in each germ of n-cell of K at y if y is not a crossing of K, and at most one edge otherwise.

An *oriented transversal* to $p(K')$ in K at y is a transversal together with an orientation on its edges so that y has exactly one outgoing edge.

Definition 2.11. Let K be a simple n-complex and let $p : K' \to K$ be a r-embedding of a possibly non finite simple degenerate $(n-1)$-complex $(n \geq 2)$.

1. A compact subset U of $p(K')$ is *two-sided* in K if there is a choice of an oriented transversal τ_x at each point $x \in U$ such that a small compact neighborhood $\mathcal{N}(U)$ of U in K satisfies the following property:

   $\mathcal{N}(U) = C_- \cup C_+$, where C_- and C_+ are two compact subsets of $\mathcal{N}(U)$ with $C_- \cap C_+ = U$ and such that each oriented transversal τ_x

at $x \in U$ is oriented from C_- to C_+, i.e. the initial points of the oriented transversal are in C_- and the terminal point is in C_+.

2. With the notations above,

   - A *transverse orientation* to a compact subset $U \subset p(K')$, which is 2-sided in K, is a choice of an oriented transversal at each point $x \in U$, satisfying the above property.
   - One will say that $U \subset p(K')$ is *transversely oriented* in K if it is 2-sided in K and equipped with some transverse orientation.
   - Let $U \subset p(K')$ be transversely oriented in K. Let x be any point in $K^{(1)} \cap U$. The transverse orientation of U *agrees at x* with the orientation of the edges of K incident to x if any such edge is oriented either from C_- to U or from U to C_+.

3. The r-embedded $(n-1)$-complex $p(K')$ is two-sided in K if there is a choice of an oriented transversal at each point $x \in p(K')$ which makes any compact subset of $p(K')$ transversely oriented.

The definitions of a transverse orientation to $p(K')$ and of $p(K')$ being transversely oriented are then straightforward.

Remark 2.12. With the assumptions and notations of definition 2.11 above, assume that K' is compact. Then this definition 2.11 implies that $p(K')$ is 2-sided in K if and only if there is a neighborhood $\mathcal{N}(p(K'))$ of $p(K')$ in K such that:

1. $\mathcal{N}(p(K')) - p(K')$ has two connected components C_- and C_+ which are trivial I-bundles respectively over K'_- and K'_+,

   where K'_- and K'_+ are two simple degenerate $(n-1)$-complexes r-embedded in K (in a non-degenerate way).

2. At each point of $p(K')$, there is a transversal oriented from C_- to C_+.

If p is a non-degenerate r-embedding, being 2-sided is equivalent to require that there is a neighborhood of $p(K')$ in K which is the trivial I-bundle over $p(K')$. One thus has the usual notion of being 2-sided for a hypersurface in a manifold.

In what follows, we will often omit the embedding map p and speak of a *simple degenerate $(n-1)$-complex K' embedded in K*.

Lemma 2.13. *Let K be a simple n-complex ($n \geq 2$). Any cocycle $u \in C^1(K;\mathbf{Z})$ defines a transversely oriented simple $(n-1)$-complex K_u which is r-embedded in K. Furthermore, at each point $x \in K_u \cap e$, where e is any 1-cell of K, the transverse orientation to K_u agrees with the orientation of e if $u(e) > 0$ and disagrees otherwise.*

Conversely, any transversely oriented simple $(n-1)$-complex K_u r-embedded in K in a non-degenerate way defines a unique cocycle in $u \in C^1(K;\mathbf{Z})$.

Proof of lemma 2.13: Let e be any edge of the 1-skeleton of K with $u(e) = \pm k$, $k > 0$. Let $x_1, \cdots, x_k$ be k points in e, each with a weight of $+1$ or -1 according to whether $u(e) = +k$ or $u(e) = -k$. Let j be the smallest integer such that the points x_i belong to $K^{(j)}_{sing}$. By lemma 1.4, the neighborhood of any x_i is homeomorphic to $N(y_i) \times [0,1]$, where $N(y_i)$ is a neighborhood of some point y_i in a $(j-1)$-cell of the cone over the $(n-2)$-skeleton of $\partial\Delta^n$. One considers now around each x_i an embedded piece $N(y_i) \times \{t_i\}$, $t_i \in [0,1]$. One does the same thing for any edge in $K^{(1)}$. By definition of a cocycle with coefficients in $\mathbf{Z}$, the germs of k-cells in these $N(x_i)$, $k = 1, \cdots, n-1$, can be connected by k-cells embedded in the corresponding $(k+1)$-cells of K and such that:

- The $(n-1)$-cells are transversely oriented.
- Their transverse orientation agrees at x_i with the orientation of e if the weights are positive and disagrees otherwise.

If all the $(n-1)$-cells so embedded in a same n-cell of K are not disjointly embedded, then classical cut-and-paste technics respecting the transverse orientation allow to obtain an embedded complex in K. By construction, this complex is the image in K of a r-embedding of a simple $(n-1)$-complex. It is also transversely oriented, the values $u(e)$ being its intersection numbers with the oriented edges e of the singular graph. This last remark allows to prove the converse assertion. □

2.3 Foliations

We introduce here the notion of *regular* foliation of a simple n-complex and state the first consequences of the definition.

Definition 2.14. Let K be any simple n-complex with boundary ($n \geq 2$).

1. A *codim* 1*-regular foliation* $\mathcal{F}$ of K is a union of possibly non-finite simple degenerate $(n-1)$-complexes, called the *leaves of* $\mathcal{F}$, which are

r-embedded in K such that each point of K belongs to exactly one leaf.

2. A *transversely orientable codim* 1*-regular foliation* $\mathcal{F}$ of K is a codim 1-regular foliation of K whose leaves admit a *coherent transverse orientation*, that is a transverse orientation such that:

   Let U be any compact subset of any leaf $\mathcal{L}$ of $\mathcal{F}$. Let $\mathcal{N}(U)$ be a small neighborhood of U in K. The transverse orientation to U in K agrees with the transverse orientation to any connected compact set of $\mathcal{F} \cap \mathcal{N}(U)$.

   A codim 1-regular foliation $\mathcal{F}$ of K is *transversely oriented* if all its leaves are equipped with a coherent transverse orientation.

In what follows, we state some generalities on the position of a codim 1-regular foliation of a simple n-complex K ($n \geq 2$) with respect to the components of K. One will need these results in section 3. The "ultimate" result of this subsection is corollary 2.17.

Since the leaves of a regular foliation $\mathcal{F}$ of a simple n-complex K are r-embedded, item (1) of definition 2.7, together with lemma 1.4, imply the following assertion:

If $\mathcal{L} = p(K')$ is any leaf of $\mathcal{F}$, then each point $x \in K'$ such that $p(x)$ belongs to an open j-component of K, $2 \leq j \leq n$, belongs to the interior of a $(j-1)$-component of K'.

One thus has lemma 2.15 below.

Lemma 2.15. *Let K be a simple n-complex ($n \geq 2$). Let $\mathcal{F}$ be a codim 1-regular foliation of K. Then, for any open j-component $\mathcal{C}$ ($n \geq j \geq 2$) of K, $\mathcal{F} \cap \mathcal{C}$ is a non-singular foliation (in the usual sense) of $\mathcal{C}$.*

We now study the position of a foliation $\mathcal{F}$ as given by lemma 2.15 with respect to the boundary of the cells and components of K.

One needs first to define what means for a codim 1-regular foliation $\mathcal{F}$ of K to have a *point of tangency with the boundary of a cell*. The assumptions and notations used are those of lemma 2.15 above.

By definition of a CW-complex, for any j-cell C ($1 \leq j \leq n$) of K, there is a continuous map $h_C : \overline{D^j} \to K$, which is a homeomorphism from D^j onto C and which sends the boundary of D^j in $\mathbf{R}^j$ to the boundary of C in K. The non-singular foliation of C in K given by lemma 2.15 lifts by h_C^{-1} to a non-singular codim 1-foliation of D^j and the foliation $\mathcal{F} \cap \overline{C}$ of $\overline{C}$ to a (singular) foliation of $\overline{D^j}$. This foliation of $\overline{D^j}$ might have tangency points with the boundary, and also some leaves reduced to a single point in this boundary. Since the Euler characteristic of $\overline{D^j}$ is positive, this last type of leaves always exists. The images under h_C of these tangency and singular points are called *tangency points of $\mathcal{F}$ with the boundary of C.*

We will call *external points of tangency* the points in the boundary of C in K, which are the images under h_C of the singular points of the foliation of $\overline{D^j}$. The other points of tangency of $\mathcal{F}$ with the boundary of C are called *internal points of tangency* (see figure 2).

The above definitions easily extend to the case where C is a component of the complex.

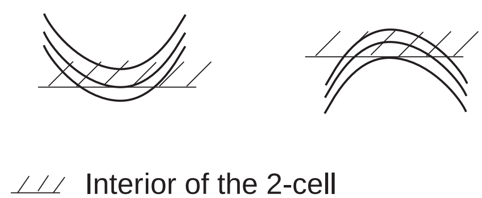

Figure 2: Internal and external points of tangency

Lemma 2.16. *With the assumptions and notations of lemma 2.15,*

1. *The codim* 1*-regular foliation $\mathcal{F}$ has no point of internal tangency with the boundaries of the components of K.*
2. *Any point of external tangency is a crossing of K.*

Proof of lemma 2.16: Let $y = p(x)$ be a point of internal tangency of a leaf $\mathcal{L} = p(K')$ of $\mathcal{F}$ with the boundary of some component. Assume that y is not a crossing of K. Then, item (1) of definition 2.7 implies that y is a non-singular point of this leaf, i.e. x has a neighborhood in K' homeomorphic to $\mathbf{R}^{n-1}$. Indeed, otherwise the embedding p in restriction to a neighborhood of x in K' is not a homeomorphism onto the space $X \times \{t\}$, where X is as given by lemma 1.4. This implies that y is a point of c-tangency between the leaf and the component containing y. This is a contradiction with the fact that the leaf is r-embedded and thus, by definition, c-transverse to the sets $K_{sing}^{(j)}$. Assume now that y is a crossing of K. Then, item (2) implies that y is a point of c-tangency of the leaf with $K_{sing}^{(n-1)}$. This completes the proof of item (1) of lemma 2.16.

By definition of an external point of tangency, such a point x is a leaf of $\mathcal{L} \cap \partial C$ for some leaf $\mathcal{L}$ of $\mathcal{F}$, where ∂C denotes the boundary of C in K.

Since the leaves of a regular foliation are c-transverse to ∂C and satisfy item (1) of definition 2.7, this point is a crossing of the complex. This completes the proof of lemma 2.16. □

Corollary 2.17 below concludes this subsection on foliations of simple n-complexes.

Corollary 2.17. *With the assumptions and notations of lemma 2.16, assume furthermore that $\mathcal{F}$ is transversely oriented. If the edges of the singular graph of K are oriented according to the transverse orientation of $\mathcal{F}$, then:*

- *There is exactly one attractor and one repellor for this orientation in the boundary of any j-component of K which is a j-cell ($n \geq j \geq 2$).*
- *There are no attractors and no repellors in the boundary of the other components.*

Proof of corollary 2.17: By lemma 2.16, the only points of tangency of $\mathcal{F}$ with the boundary of any component occur at the crossings in this boundary, and they are external points of tangency. By definition of the orientation of the edges of K, they are attractors and repellors. Each of these points has index $+1/2$. The Euler characteristic of a cell component is 1, whereas the other components have Euler characteristic 0. Moreover, since the intersection of $\mathcal{F}$ with any component is a foliation of this component (see lemma 2.15), and $\mathcal{F}$ is transversely oriented, the existence of an attractor in the boundary of any component forces the existence of a repellor in this same boundary, and conversely. These three last assertions, together with the Euler-Poincare relation, imply corollary 2.17. □

Remark 2.18. One easily proves that the non-degenerate leaves of $\mathcal{F}$ as given in corollary 2.17 are (possibly non finite) *standard* $(n-1)$-complexes, that is all their components are cells. This is not essential for our purpose.

2.4 WM-moves

The *generalized WM-moves* we introduce here are a generalization of the classical *Whitehead-moves* on graphs and of the *Matveev-moves* (see [Ma2], [Pi], [BP]) on special 2-polyhedra. They consist of applying a collapsing, followed by a *splitting*, to a simple n-complex. The motivation for defining these generalized WM-moves is explained by lemma 2.23 below. This lemma establishes the relationship between the generalized WM-moves and the non-negative δ_v-moves on non-negative cocycles of a dynamical n-complex. The importance of the WM-moves is also stressed by proposition 3.13.

Definition 2.19. A simple n-complex K_1 is obtained from a simple n-complex K_0 ($n \geq 1$) by a *generalized WM-move* if there exist:

- a collapsing C_{Δ_0} of a closed j-cell Δ_0 ($n \geq j \geq 1$) of K_0, which is a j-simplex,
- a collapsing C_{Δ_1} of a closed $(n+1-j)$-cell Δ_1 of K_1, which is a $(n+1-j)$-simplex,

such that there is an isomorphism

$$\alpha\colon C_{\Delta_0}(K_0) \to C_{\Delta_1}(K_1) \quad \text{with} \quad \alpha(C_{\Delta_0}(\Delta_0)) = C_{\Delta_1}(\Delta_1).$$

Remark 2.20. For a simple 1-complex, i.e. a trivalent graph, a generalized WM-move is simply, in an obvious way, a classical Whitehead move.

For a simple 2-complex, a generalized WM-move is a priori more general than a Matveev move (see [Ma2], or also [BP], [Pi]). Indeed, whereas generalized WM-moves are defined "algebraically", Matveev's moves are topological moves defined on special polyhedra which are embedded in compact 3-manifolds. This embedding induces a cyclic ordering on the germs of 2-cells around the edges of the singular graph. Roughly speaking, a Matveev move is a generalized WM-move applied on a cell whose some neighborhood in the complex admits an embedding in $\mathbf{R}^3$, and which preserves the cyclic ordering of the germs of 2-cells around the edges of the singular graph. However, one can see by inspection that any generalized WM-move can be realized by a Matveev move.

A collapsing from a CW-complex K to a CW-complex K' is a homotopy equivalence, and thus admits a (non-unique) inverse-map, up to homotopy. We will call *splitting* from K' to K such maps.

Since a collapsing is a homotopy equivalence, the following lemma is an immediate consequence of the definition of a generalized WM-move:

Lemma 2.21. *Any generalized WM-move from a simple n-complex K to a simple n-complex K' defines a continuous map $\sigma : K \to K'$ which is a homotopy equivalence from K to K'.*

Lemma 2.22 below shows that any WM-move between two simple n-complexes defines a particular kind of $(n+1)$-dimensional CW-complex, called *elementary foliated complex*, satisfying some nice properties. These elementary foliated complexes will be the basic pieces in the construction of a transversely orientable codim 1-regular foliation of a dynamical $(n+1)$-complex, starting from a positive cocycle of the complex (see lemma 2.23 and proposition 3.11). They are also used in proposition 3.13.

Lemma 2.22. *With the assumptions and notations of definition 2.19, let* $\mathcal{C}_{K_0} = (K_0 \times [-1,0])/\sim_0$, $\mathcal{C}_{K_1} = (K_1 \times [0,1])/\sim_1$, *where* $(x,t) \sim_i (x',t')$ *if and only if* $t = t' = 0$ *and* $x \in \Delta_i$.

Let $\mathcal{C}_{K_0K_1} = (\mathcal{C}_{K_0} \cup \mathcal{C}_{K_1})/\sim_\alpha$, *where* $(x,t) \sim_\alpha (x',t')$ *if and only if* $t = t' = 0$ *and* $\alpha(x) = x'$.

Then $\mathcal{C}_{K_0K_1}$ *is a simple* $(n+1)$*-complex with two boundary components* $K_0 \times \{-1\}$ *and* $K_1 \times \{1\}$ *which admits a transversely orientable codim* 1*-regular foliation with compact leaves* $\mathcal{F}$. *Moreover, this foliation* $\mathcal{F}$ *satisfies the following properties:*

1. *All the non-degenerate leaves are homeomorphic either to* K_0 *or* K_1.
2. *There is exactly one degenerate leaf, containing the only crossing of* $\mathcal{C}_{K_0K_1}$, *which is homeomorphic to the complex* $C_{\Delta_0}(K_0) = C_{\Delta_1}(K_1)$ *(see definition 2.19).*

The simple $(n+1)$*-complex with boundary* $\mathcal{C}_{K_0K_1}$ *will be called an* elementary foliated complex*.*

See figure 3.

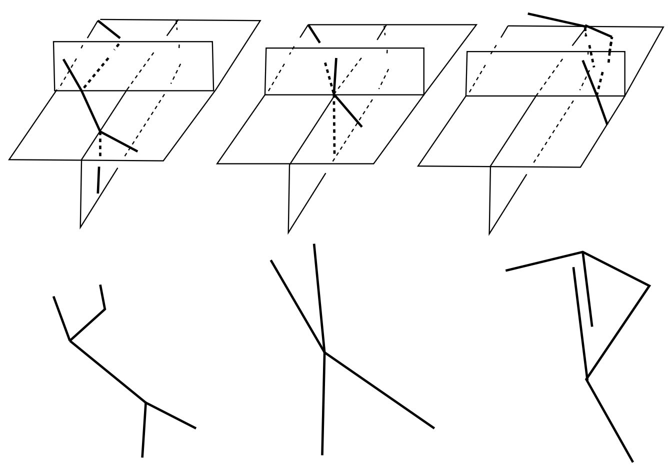

Figure 3: A generalized WM-move

Proof of lemma 2.22: We use the notations of definition 2.19. It is clear that $\mathcal{C}_{K_0K_1}$ is a $(n+1)$-dimensional CW-complex with two boundary components $K_0 \times \{-1\}$ and $K_1 \times \{1\}$. Let us observe that, since K_0 and K_1 are simple n-complexes, they satisfy the local topological property given by condition (1) of definition 1.7. From the construction of $\mathcal{C}_{K_0K_1}$, the following two properties are then clear:

1. There exists exactly one crossing v, resulting from the collapses of the cells Δ_i of K_i, which is the vertex of a cone over Δ_0, and also of a cone over Δ_1.

2. The neighborhood in $\mathcal{C}_{K_0K_1}$ of any point distinct from v is homeomorphic to a neighborhood of some point in the cone over the $(n-1)$-skeleton of the boundary of the closed $(n+1)$-simplex, crossed with the interval.

The second property above implies that any point of $\mathcal{C}_{K_0K_1}$ distinct from a crossing admits a neighborhood homeomorphic to a neighborhood of some point in the cone over the n-skeleton of the boundary of the closed $(n+2)$-dimensional simplex.

Let us now prove that some neighborhood of v in the complex is homeomorphic to $Con((\partial\Delta^{n+2})^{(n)})$. The germs of i-cells incident to some cell in the closed j-simplex Δ_0 are in bijection with the germs of i-cells incident to some cell in the closed $(n+1-j)$-simplex Δ_1. Each pair gives rise to a germ of $(i+1)$-component incident to the crossing v. Together with the first property given above for v, this implies that a neighborhood of v in $\mathcal{C}_{K_0K_1}$ is as announced.

Thus, $\mathcal{C}_{K_0K_1}$ is a simple $(n+1)$-complex with boundary $K_0 \times \{-1\} \cup K_1 \times \{1\}$. The other assertions are then straightforward. □

As already announced, lemma 2.23 below allows to understand the relationship between a non-negative δ_v-move on a non-negative cocycle u in a dynamical n-complex K and a generalized WM-move as defined above. The idea is the following: Lemma 2.13 associates a transversely oriented simple $(n-1)$-complex K_u (resp. $K_{\delta_v(u)}$) to the cocycle u (resp. $\delta_v(u)$). The transverse orientation to K_u and $K_{\delta_v(u)}$ agrees with the orientation of the edges of the singular graph. There is a continuous deformation in K from K_u to $K_{\delta_v(u)}$. The cocycles u and $\delta_v(u)$ only differ on the edges of $K^{(1)}_{sing}$ which are incident to v. Then, by definition of the orientation of the edges of the singular graph for a dynamical n-complex, this deformation realizes a generalized WM-move.

Lemma 2.23. *Let K be a simple dynamical n-complex ($n \geq 2$) which admits a non-negative cocycle $u \in C^1(K;\mathbf{Z})$. If v is a crossing of K such that $\delta_v(u)$ is a non-negative cocycle, let K_u and $K_{\delta_v(u)}$ be as given by lemma 2.13. Then:*

There is a subcomplex K_1 of K containing v which is homeomorphic to an elementary foliated complex $\mathcal{C}_{K_u K_{\delta_v(u)}}$.

In particular, the simple $(n-1)$-complex $K_{\delta_v(u)}$ is obtained from the complex K_u by a generalized WM-move. Moreover, if v is a type j-crossing $(2 \leq j \leq n)$ then this WM-move consists of the collapsing of a $(j-1)$-simplex followed by the splitting of a $(n-j+1)$-simplex.

Proof of lemma 2.23: By lemma 2.13, any r-embedded complex K_u is 2-sided in K. Moreover, the transverse orientation to K_u agrees with the orientation of the edges of the singular graph that it intersects. Since $\delta_v(u)$ is a non-negative cocycle, the cocycle u is positive on the incoming edges of $K^{(1)}_{sing}$ at v. Therefore, any r-embedded complex K_u intersects them positively. Since v is a type j-crossing, there are j such incoming edges at v. By lemma 1.10, there is exactly one j-component C of K whose germ at v contains the germs of these j incoming edges (or, in other words, which has v as attractor in its boundary). The above assertions imply the existence of a closed $(j-1)$-component Δ_0 of K_u contained in this j-component C of K. Clearly, Δ_0 cuts C in two connected components. Moreover, one can choose a neighborhood $N(v)$ of the crossing v in K which contains Δ_0. Thus, from definition 1.1, item (1), one gets that Δ_0 is a $(j-1)$-simplex and one of the two connected components of the complement of Δ_0 in C is a cone over Δ_0 based at the crossing v.

Since K_u is 2-sided, one can then define a continuous deformation

$$\begin{array}{ccccccc} H_0: & K_u & \times & [-1,0] & \to & K \\ & (x & , & t) & \to & i_t(x) \end{array} \text{ such that:}$$

- For $t \in [-1,0[$, $i_t : K_u \to K$ is a non-degenerate r-embedding.
- The map i_0 is such that $i_0(K_u)$ contains v and is the image under a degenerate r-embedding of the CW-complex K' obtained from K_u by the collapsing of Δ_0.
- For any t, t' in $[-1,0]$ such that $t \neq t'$, $i_t(K_u)$ is disjoint from $i_{t'}(K_u)$.

Let $\mathcal{C}_{K_u}$ be the subcomplex of K equal to $\displaystyle\bigcup_{(x,t)\in K_u\times[-1,0]} H_0(x,t)$.

The r-embedded CW-complex K' is 2-sided and admits a neighborhood $N(K')$ in K such that $N(K') - K'$ has two connected components homeomorphic to $K_u \times [-1,0[$ and $K_{\delta_v(u)} \times]0,1]$. Thus there is a r-embedding of $K_{\delta_v(u)}$ which is disjoint from $\mathcal{C}_{K_u}$. Moreover one can choose this r-embedding of $K_{\delta_v(u)}$ such that the neighborhood $N(v)$ above contains a closed $(n+1-j)$-component Δ_1 of $K_{\delta_v(u)}$ which is a simplex. This component Δ_1 is contained in the component of K which has v as repellor in its boundary. Thus, in the same way than above, one has a continuous deformation $\begin{array}{ccccc} H_1 : & K_{\delta_v(u)} & \times & [0,1] & \rightarrow & K \\ & (x & , & t) & \rightarrow & j_t(x) \end{array}$

such that:

- For $t \in]0,1]$, $j_t : K_{\delta_v(u)} \rightarrow K$ is a non-degenerate r-embedding.
- The map j_0 is such that $j_0(K_{\delta_v(u)})$ contains v and is the image under a degenerate r-embedding of the CW-complex K' obtained from $K_{\delta_v(u)}$ by the collapsing of Δ_1.
- For any t, t' in $[-1,0]$ such that $t \neq t'$, $j_t(K_{\delta_v(u)})$ is disjoint from $j_{t'}(K_{\delta_v(u)})$.

Let $\mathcal{C}_{K_{\delta_v(u)}}$ be the subcomplex of K equal to $\bigcup_{(x,t) \in K_{\delta_v(u)} \times [-1,0]} H_1(x,t)$.

By construction, the subcomplex $\mathcal{C}_{K_u} \cup \mathcal{C}_{K_{\delta_v(u)}}$ is homeomorphic, by a "fiber-preserving" homeomorphism, to the elementary foliated complex $\mathcal{C}_{K_u K_{\delta_v(u)}}$. The proof of lemma 2.23 is then easily completed. □

3 Foliations with compacts leaves

In this section, we prove our main theorem.

Theorem 3.1. *A simple n-complex K, $n \geq 2$, admits a transversely orientable codim 1-regular foliation $\mathcal{F}$, whose all leaves are compact and have the same Euler characteristic, if and only if there exists an orientation of the edges of the singular graph such that:*

1. *The complex K together with this orientation is a simple dynamical n-complex.*
2. *There is a positive cocycle in $C^1(K;\mathbf{Z})$, for K equipped with this orientation.*

In addition, if such a foliation $\mathcal{F}$ exists, then all its leaves are homotopically equivalent. If $\mathcal{L}$ is any leaf of $\mathcal{F}$, there is a homotopy equivalence $\psi : \mathcal{L} \rightarrow \mathcal{L}$ (which in particular induces an automorphism on the fundamental group of $\mathcal{L}$) such that the complex K is homotopically equivalent to the suspension $Susp_\psi(\mathcal{L})$.

Let us recall that a positive cocycle of K is a cocycle which is non-negative on the edges of the singular graph $K^{(1)}_{sing}$, and positive on all the positive loops embedded in $K^{(1)}_{sing}$. Let us also recall that, by edges of the singular graph, we mean the 1-components of the complex K. These are the only 1-cells of K contained in $K^{(1)}_{sing}$. The only 0-cells of K contained in $K^{(1)}_{sing}$ are the crossings, together with a set of valency 2-vertices, one in each loop of $K^{(1)}_{sing}$ which does not contain any crossing. All this comes from the chosen structures of CW-complex for our complexes.

3.1 From a regular foliation to a positive cocycle of a dynamical complex

We assume here that one is given a codim 1-regular foliation $\mathcal{F}$ of a simple n-complex K ($n \geq 2$) as in theorem 3.1. The proof of this implication decomposes in two steps. First, one proves that there is an orientation on the edges of the singular graph, induced by some transverse orientation to $\mathcal{F}$, which makes K a simple dynamical n-complex (subsection 3.1.1). Then one proves that this simple dynamical n-complex K admits a positive cocycle (subsection 3.1.2).

3.1.1 An orientation making K a simple dynamical n-complex

We begin the proof of this part by the following proposition:

Proposition 3.2. *Let K be a simple n-complex. If there exists a transversely orientable codim 1-regular foliation $\mathcal{F}$ of K whose leaves are compact and have the same Euler characteristic, then there is an orientation of the edges of the singular graph of K such that K together with this orientation is a simple dynamical n-complex.*

Proof of proposition 3.2: By definition, the leaves of a codim 1-regular foliation of a simple n-complex are c-transverse to the singular graph (see definition 2.7). Thus, any transverse orientation of such a foliation $\mathcal{F}$ induces an orientation on the edges of the singular graph. One has to check that such an orientation satisfies the following two properties (see definition 1.7):

1. Each crossing of the singular graph is the initial and the terminal crossing of at most n edges of $K^{(1)}_{sing}$.
2. There is exactly one attractor and one repellor in the boundary of each j-component which is a j-cell. The other components have no attractor and no repellor in their boundary.

We are first going to prove that condition (1) above is satisfied. This will essentially relie on the fact that all the leaves of $\mathcal{F}$ have the same Euler characteristic. Condition (1) is equivalent to the fact that, if $i \in \{0, 1, n+1, n+2\}$, there is no type i-crossing in K.

Lemma 3.3. *With the notations and assumptions of proposition 3.2,*

There is no type 0- *nor type* $(n+2)$*-crossings in* K *for the orientation induced by any transverse orientation to* $\mathcal{F}$*.*

Proof of lemma 3.3: Assume that a type 0- or type $(n+2)$-crossing exists for the orientation induced by some transverse orientation to $\mathcal{F}$. Let us recall that the leaves of a regular foliation are r-embedded in K (see definition 2.7). In particular, they are c-transverse to $K_{sing}^{(j)}$ for any $j = 1, \cdots, n-1$. Since the orientation of the edges of the singular graph agrees with the chosen transverse orientation of $\mathcal{F}$, item (2) in the definition of a r-embedding and the transversality condition above imply the existence of a leaf reduced to this crossing. This is a contradiction with $\mathcal{F}$ being a codim 1-regular foliation. $\square$

For proving that there are no type 1- nor type $(n+1)$-crossing, one needs first to prove lemma 3.4 below. The main ingredient of this lemma is the following observation:

Let v be a crossing of K and let $\mathcal{L}$ be the degenerate leaf of $\mathcal{F}$ containing the crossing v. Then the germs of cells at v, which contain v either as attractor or as repellor in their boundary, contain also a cell of any leaf in a sufficiently small neighborhood of $\mathcal{L}$ in K.

Lemma 3.4. *With the assumptions and notations of proposition 3.2, assume that the edges of the singular graph of* K *are equipped with some transverse orientation to* $\mathcal{F}$*.*

Let $\mathcal{L}$ *be a degenerate leaf of* $\mathcal{F}$*, and let* $\{v_1, \cdots, v_r\}$ *be the type* j_i*-crossings* $(i = 1, \cdots, r)$ *of* K *contained in* $\mathcal{L}$*. Then, there are two leaves* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *in some neighborhood* $\mathcal{N}(\mathcal{L})$ *of* $\mathcal{L}$ *in* K *such that:*

1. *There are* $C_{j_i}^{k+1}$ *(resp.* $C_{n+2-j_i}^{k+1}$*)* k*-components of* $\mathcal{L}_1$ *(resp.* $\mathcal{L}_2$*) in the intersection of* $\mathcal{L}_1$ *(resp. of* $\mathcal{L}_2$*) with some small neighborhood* $\mathcal{N}(v_i)$ $(i = 1, \cdots, r$ *and* $k = 0, \cdots, n-1)$ *of* v_i *in* K*. These* k*-components are* k*-cells.*

2. *The* k*-cells,* $k = 0, \cdots, n-1$*, of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *which are not contained in the union of the neighborhoods* $\mathcal{N}(v_i)$ *are in bijection with the* k*-cells of* $\mathcal{L}$ *distinct from the crossings* $\{v_1, \cdots, v_r\}$*.*

Moreover, if K is a simple dynamical n-complex, then these two leaves $\mathcal{L}_1$ and $\mathcal{L}_2$ can be chosen so that:

The intersection of $\mathcal{L}_1$ (resp. of $\mathcal{L}_2$) with the above neighborhood $\mathcal{N}(v_i)$ of v_i in K ($i = 1, \cdots, r$) contains exactly one closed $(j_i - 1)$-component (resp. $(n+1-j_i)$-component), which is a $(j_i - 1)$-simplex (resp. $(n+1-j_i)$-simplex).

Proof of lemma 3.4: By definition of a transversely orientable foliation, each leaf is 2-sided in K. By definition of a 2-sided embedding in a simple n-complex (see definition 2.11 and remark 2.12), there is a neighborhood $\mathcal{N}(\mathcal{L})$ of $\mathcal{L}$ in K such that

$\mathcal{N}(\mathcal{L}) - \mathcal{L}$ has two connected components which are homeomorphic to $\mathcal{L}_- \times [-1, 0[$ and $\mathcal{L}_+ \times]0, 1]$, where $\mathcal{L}_- \times \{-t\}$ and $\mathcal{L}_+ \times \{t\}$ are r-embedded in K for any $t \in]0, 1]$.

All the leaves $\mathcal{L}_- \times \{-t\}$ (resp. $\mathcal{L}_+ \times \{t\}$) have crossings along the incoming (resp. outgoing) edges of $K^{(1)}_{sing}$ at the crossings v_i of K. These crossings are ordered along the edges of the singular graph containing them. One chooses $\epsilon > 0$ sufficiently small and a small neighborhood in K of each v_i so that only the last (resp. first) crossings of $\mathcal{L}_1 = \mathcal{L}_- \times \{-\epsilon\}$ (resp. $\mathcal{L}_2 = \mathcal{L}_+ \times \{\epsilon\}$) along the incoming (resp. outgoing) edges at each crossing v_i are contained in this small neighborhood.

By definition of the orientation of the edges of the singular graph, the only cells of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) which might be contained in these neighborhoods of the crossing v_i are the one intersecting the germs at v_i of components in K which contain only incoming (resp. outgoing) germs of edges at v_i. Item (2) of definition 2.7 and the transversality of $\mathcal{F}$ with $K^{(j)}_{sing}$, $j = 1, \cdots, n-1$, imply that any such germ at v_i of any component in K contains a cell of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$), if the ϵ above is chosen sufficiently small. Lemma 1.4 allows to complete the proof of items (1) and (2) of lemma 3.4.

The local topological property satisfied by a simple n-complex (item (1) of definition 1.1) implies that if a k-component of $\mathcal{L}_i$, $i = 1, 2$, is contained in a germ of $(k+1)$-component of K at v, then this k-component of $\mathcal{L}_i$ is a k-simplex.

Together with the property given by definition 1.7, item (1) for the orientation of the edges of the singular graph, this implies the last statement of lemma 3.4. □

One can now prove the following

Lemma 3.5. *With the notations and assumptions of proposition 3.2,*

There are no type 1- nor type $(n+1)$-crossings in K for the orientation induced by any transverse orientation to $\mathcal{F}$.

Proof of lemma 3.5: One chooses any transverse orientation to $\mathcal{F}$. One orients the edges of the singular graph by this transverse orientation. One considers three leaves $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$, as given in lemma 3.4 and uses the notations of this lemma. Let us recall that, by convention, $C_j^k = 0$ for $k > j$. Lemma 3.4 implies the following equalities:

(1) $\chi(\mathcal{L}_1) = X + \sum_{i=1}^{r} \sum_{k=1}^{n} (-1)^{k-1} C_{j_i}^k$,

(2) $\chi(\mathcal{L}) = X + r$,

(3) $\chi(\mathcal{L}_2) = X + \sum_{i=1}^{r} \sum_{k=1}^{n} (-1)^{k-1} C_{n+2-j_i}^k$,

where X is the alternated sum of the number of k-cells, $k = 0, \cdots, n-1$, of $\mathcal{L}_1$ and $\mathcal{L}_2$ given by item (2) of lemma 3.4.

Moreover, $\sum_{k=1}^{n} (-1)^{k-1} C_m^k$ is the Euler characteristic of the closed simplex Δ^{m-1} if $n \geq m$. If $m = n+1$, the factor $(-1)^n$ is missing in the above expression in order to have the Euler characteristic of Δ^n. Thus, $\sum_{k=1}^{n} (-1)^{k-1} C_m^k$ is equal to 1 if $n \geq m$ and to $1 + (-1)^{n+1}$ if $m = n+1$.

Now, since $(2 \leq j_i \leq n) \Leftrightarrow (2 \leq n+2-j_i \leq n)$, $(j_i = 1) \Rightarrow (n+2-j_i = n+1)$ and $(n+2-j_i = 1) \Rightarrow (j_i = n+1)$, the equalities (1), (2) and (3) can be rewritten as follows:

(1) $\chi(\mathcal{L}_1) = X + (1 + (-1)^{n+1}) N_0^{n+1} + (r - N_0^{n+1})$,

(2) $\chi(\mathcal{L}) = X + r$,

(3) $\chi(\mathcal{L}_2) = X + (1 + (-1)^{n+1}) N_0^1 + (r - N_0^1)$,

where N_0^j denotes the number of type j-crossings among the crossings v_i contained in $\mathcal{L}$.

By assumption, all the leaves in $\mathcal{F}$ have the same Euler characteristic. This is in particular true for $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$. This implies that $N_0^{n+1} = 0$ and that $N_0^1 = 0$. Therefore, there are no type 1- nor type $(n+1)$-crossings in K for the orientation on the edges of the singular graph induced by a transverse orientation to $\mathcal{F}$. This completes the proof of lemma 3.5. □

The following corollary is a consequence of all what precedes.

Corollary 3.6. *With the assumptions and notations of proposition 3.2,*

Each crossing of the singular graph of K is the terminal and initial crossing of at most n edges of $K_{sing}^{(1)}$ for the orientation induced by any transverse orientation to $\mathcal{F}$.

For proving that K, equipped with the above orientation, is a simple dynamical n-complex, it remains to prove that there is exactly one attractor and one repellor in the boundary of each j-component which is a j-cell and no attractor nor repellor in the boundary of the other j-components ($2 \leq j \leq n$). This comes from corollary 2.17, which completes the proof of proposition 3.2. □

3.1.2 Existence of a positive cocycle

Let K be a simple n-complex which satisfies the assumptions of proposition 3.2. The edges of the singular graph of K are oriented, as precedingly, by some transverse orientation to $\mathcal{F}$, thus making K a simple dynamical n-complex. One wants to prove that K, equipped with this orientation on the edges of the singular graph, admits a positive cocycle.

Proposition 3.7. *Let K be a simple dynamical n-complex ($n \geq 2$). Assume that K admits a transversely oriented codim 1-regular foliation $\mathcal{F}$, whose leaves are compact, and such that their transverse orientation agrees with the orientation of the edges of the singular graph. Then the cocycle $u \in C^1(K; \mathbf{Z})$ defined by any non-degenerate leaf of $\mathcal{F}$ (see lemma 2.13) is a positive cocycle.*

Proof of proposition 3.7: The proof relies on the following lemma:

Lemma 3.8. *Let K be a simple n-complex which admits a transversely orientable codim 1-regular foliation with compact leaves. Then all the leaves of $\mathcal{F}$ are homotopic in K.*

Proof of lemma 3.8: By definition of a transversely orientable codim 1-regular foliation, all the leaves are 2-sided. By definition of a 2-sided embedding and remark 2.12, all the leaves in a neighborhood of any leaf $\mathcal{L}$ are homotopic to $\mathcal{L}$ in K. The conclusion follows by compactness of K. □

The union of all the non-degenerate leaves of the foliation $\mathcal{F}$ given by proposition 3.7 intersects all the positive embedded loops of the singular graph $K^{(1)}_{sing}$. Since $K^{(1)}_{sing}$ is finite, there is a finite family of leaves of $\mathcal{F}$ whose union intersects all the positive embedded loops of $K^{(1)}_{sing}$. Since, by lemma 3.8, all the leaves of $\mathcal{F}$ are homotopic in K, the cocycles associated to the non-degenerate leaves by lemma 2.13 are cohomologous. Thus, *each* leaf in the above finite family intersects all the positive loops of $K^{(1)}_{sing}$. This is equivalent, for the associated cocycles, to be positive cocycles. This completes the proof of proposition 3.7. □

At this point, we have proved that, if a simple n-complex K, $n \geq 2$, admits a transversely orientable codim 1-regular foliation $\mathcal{F}$, whose leaves are compact and have the same Euler characteristic then:

- The complex K, together with the orientation on the edges of the singular graph induced by any transverse orientation to $\mathcal{F}$, is a simple dynamical n-complex.
- Any non-degenerate leaf of $\mathcal{F}$ defines a positive cocycle $u \in C^1(K; \mathbf{Z})$.

The proof of one implication of theorem 3.1 is thus completed.

3.2 The reverse implication

The goal now is to prove that, if a simple dynamical n-complex ($n \geq 2$) admits a positive cocycle $u \in C^1(K; \mathbf{Z})$, then there is a transversely oriented codim 1-regular foliation of K, whose leaves are compact and are all homotopically equivalent, and such that their transverse orientation agrees with the orientation of the edges of the singular graph.

Lemma 3.9. *Let K be a simple dynamical n-complex ($n \geq 2$) which admits a positive cocycle $u \in C^1(K; \mathbf{Z})$. Then either K has no crossing, or there is a crossing v of K such that $\delta_v(u)$ is a positive cocycle.*

The assumption that K is a dynamical n-complex allows to assure, in the proofs of lemma 3.9 and corollary 3.10, that there is at least one incoming edge of $K^{(1)}_{sing}$ at each crossing of K.

Proof of lemma 3.9: This lemma relies on the following claim, which is easy to check from the definition of a δ_v-move (see definitions 2.2 and 2.5).

Claim: Let $u \in C^1(K; \mathbf{Z})$ be a non-negative cocycle and let v be any crossing of K. A non-negative δ_v-move can be applied to u if and only if $u(e) > 0$ for all the incoming edges e of $K^{(1)}_{sing}$ at v.

Assume that K has at least one crossing and that a crossing as given by lemma 3.9 does not exist. Equivalently, by the claim above, there is at least one incoming edge e of $K^{(1)}_{sing}$ at each crossing v of K such that $u(e) = 0$. Let v_1 be any crossing and let e_{i_1} be an incoming edge at v_1, with $u(e_{i_1}) = 0$. There is also, by assumption, an incoming edge e_{i_2} of $K^{(1)}_{sing}$ at the initial crossing $i(e_{i_1})$ with $u(e_{i_2}) = 0$. The path $e_{i_2} e_{i_1}$ is a positive path in the singular graph $K^{(1)}_{sing}$, and u is zero on this path. By induction, from the finiteness of $K^{(1)}_{sing}$, one so constructs a positive loop l in $K^{(1)}_{sing}$ with $u(l) = 0$. This is a contradiction with u being a positive cocycle and completes the proof of lemma 3.9. □

Corollary 3.10. *With the assumptions and notations of lemma 3.9, if K has at least one crossing, then there is a (non-unique) ordered sequence of N distinct positive cocycles $u = u_0 \to u_1 \to \cdots \to u_N = u$ such that u_i is obtained from u_{i-1} by a δ_{v_i}-move, $i = 1, \cdots, N$. Furthermore, each crossing of K occurs exactly once in the set $\{v_1, \cdots, v_N\}$.*

Proof of corollary 3.10: By lemma 3.9, if $u \in C^1(K; \mathbf{Z})$ is a positive cocycle, there is at least one crossing v_1 of K such that $u_1 = \delta_{v_1}(u)$ is a positive cocycle. By induction, one obtains an ordered sequence of positive cocycles $u = u_0, u_1, \cdots, u_N$ such that:

- For $1 \leq i \leq N$, u_i is obtained from u_{i-1} by a δ_{v_i}-move.
- All the crossings $v_1, \cdots, v_N$ are distinct.
- There is no crossing v distinct from $v_1, \cdots, v_N$ so that a non-negative δ_v-move can be applied to u_N.

One has then two cases:

1. The set $\{v_1, \cdots, v_N\}$ is the set of crossings of K.
2. There is a crossing w of K which does not belong to $\{v_1, \cdots, v_N\}$.

In the case (2), by the claim used in the proof of lemma 3.9, the cocycle u_N is zero on an incoming edge e at w.

This implies that u_N is also zero on some incoming edge of $K^{(1)}_{sing}$ at the crossing $i(e)$. Otherwise, a non-negative $\delta_{i(e)}$-move might be applied to u_N. This would imply that the crossing $i(e)$ is one of the crossings $\{v_1, \cdots, v_N\}$. Since a non-negative δ_v-move has been applied to each of this crossing to obtain the cocycle u_N, and no non-negative δ_w-move, $w = t(e)$, has been applied, this would imply that u_N is positive on e. This is a contradiction with our assumption.

By induction, as in lemma 3.9, one constructs a positive loop l in the singular graph such that $u_N(l) = 0$. Since u_N is cohomologous to u (see remark 2.4), this is a contradiction with u positive.

Therefore, case (1) above is satisfied. It remains to prove that $u_N = u$. For any edge e of the singular graph, exactly one $\delta_{t(e)}$-move has been applied, which lowers the value on e of the corresponding cocycle u_j ($j \leq N$) by 1, and exactly one $\delta_{i(e)}$-move has been applied, which increases the value on e of the corresponding cocycle u_k ($k \leq N$) by 1. The other δ_v-move do not change the value of the corresponding cocycles on e. Without loss of generality, one can assume $j < k$. These remarks imply $u(e) = u_1(e) = u_2(e) = \cdots = u_{j-1}(e)$, $u_j(e) = u(e) - 1 = u_{j+1}(e) = u_{j+2}(e) = \cdots = u_{k-1}(e)$, $u_k(e) = u_{k+1}(e) = \cdots = u_N(e) = u_j(e) + 1 = u(e)$. Since this holds for any edge e, one has $u_N = u$. This completes the proof of corollary 3.10. □

Proposition 3.11. *Let K be a simple dynamical n-complex ($n \geq 2$) which admits a positive cocycle $u \in C^1(K; \mathbf{Z})$.*

This cocycle defines a non-unique ordered sequence of $N+1$ simple $(n-1)$-complexes $\mathcal{L}_0, \cdots, \mathcal{L}_N$, where N is the number of crossings in K and such that:

1. *They are disjointly r-embedded in K in a non-degenerate way, and $\mathcal{L}_0$ is the simple $(n-1)$-complex associated to u by lemma 2.13.*
2. *Each complex $\mathcal{L}_i$ is obtained from $\mathcal{L}_{i-1}$ by a generalized WM-move for $i = 1, \cdots, N$ and $\mathcal{L}_N$ is homeomorphic to $\mathcal{L}_0$.*

The simple $(n-1)$-complexes $\mathcal{L}_0, \cdots, \mathcal{L}_N$ are the leaves of a transversely oriented codim 1-regular foliation $\mathcal{F}$ of K whose leaves are compact and such that:

- *Their transverse orientation agrees with the orientation of the edges of the singular graph.*
- *All the non-degenerate leaves of $\mathcal{F}$ are homeomorphic to one of the simple $(n-1)$-complexes $\mathcal{L}_1, \cdots, \mathcal{L}_N$ and there are exactly N degenerate leaves.*

Proof of proposition 3.11: Let us first assume that K has at least one crossing. From corollary 3.10, one has a sequence of positive cocycles $u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_N = u$ such that u_i is obtained from u_{i-1} by a δ_{v_i}-move, and $\{v_1, \cdots, v_N\}$ is the set of crossings of K. Lemma 2.23 allows to obtain N disjoint subcomplexes $\mathcal{C}_{uu_1}, \mathcal{C}_{u_1u_2}, \cdots, \mathcal{C}_{u_{N-1}u_N}$ of K, which are homeomorphic to elementary foliated complexes $\mathcal{C}_{K_uK_{u_1}}, \mathcal{C}_{K_{u_1}K_{u_2}}, \cdots, \mathcal{C}_{K_{u_{N-1}}K_{u_N}}$. These subcomplexes contain all the crossings of K, and each one contains exactly one crossing. Therefore, the complement in K of $\mathcal{C}_{uu_1} \cup \mathcal{C}_{u_1u_2} \cup \cdots \cup \mathcal{C}_{u_{N-1}u_N}$ has N connected components $B_1, \cdots, B_N$. Each connected component B_i has two boundary components which are homeomorphic to K_{u_i}, $i = 1, \cdots, N$ and is foliated by $K_{u_i} \times [0,1]$. The union

$\mathcal{C}_{uu_1} \cup (K_{u_1} \times [0,1]) \cup \mathcal{C}_{u_1u_2} \cup (K_{u_2} \times [0,1]) \cup \cdots \cup \mathcal{C}_{u_{N-1}u_N} \cup (K_{u_N} \times [0,1])$ gives a transversely oriented codim 1-regular foliation with compact leaves which, by definition of an elementary foliated complex, satisfies all the properties given by proposition 3.11. If K has no crossing, the $(j+1)$-components of K are homeomorphic either to $Susp_{Id}(D^j)$ or $Susp_{H_R}(D^j)$, $j = 1, \cdots, n-1$ (see definitions 1.1 and 1.7). The conclusion in this case is then straightforward, one just has to push an r-embedded complex K_u along the positive loops of the singular graph to obtain a foliation of K by complexes all homeomorphic to K_u. Proposition 3.11 is proved. □

Proposition 3.11 together with lemma 2.21 imply that if a simple dynamical n-complex ($n \geq 2$) admits a positive cocycle, then there is a codim 1-regular foliation $\mathcal{F}$, transversely oriented by the edges of the singular graph, whose leaves are compact and are all homotopically equivalent (in particular, they have the same Euler characteristic). The equivalence of theorem 3.1 is thus proved.

3.3 Conclusion

One now completes the proof of theorem 3.1. One first proves that all the leaves of the foliation $\mathcal{F}$ given by theorem 3.1 are homotopically equivalent.

Lemma 3.12. *Let K be a simple n-complex ($n \geq 2$). If there is a transversely orientable codim 1-regular foliation $\mathcal{F}$ of K, whose leaves are compact and have the same Euler characteristic, then all the leaves are homotopically equivalent.*

Proof of lemma 3.12: From what precedes, we know that K, equipped with the orientation on the edges of the singular graph induced by a transverse orientation to $\mathcal{F}$, is a simple dynamical n-complex. If $\mathcal{L}$ is a non-degenerate leaf of $\mathcal{F}$, by definition of a transversely orientable foliation, there is a neighborhood of $\mathcal{L}$ which is homeomorphic to $\mathcal{L} \times [0,1]$. Thus all the leaves in a neighborhood of $\mathcal{L}$ are homeomorphic to $\mathcal{L}$, and therefore homotopically equivalent to $\mathcal{L}$. Let us now consider some degenerate leaf $\mathcal{L}_D$. Let $v_1, \cdots, v_k$ be the type j_1-,j_2-,$\cdots$,j_k-crossings contained in $\mathcal{L}_D$. Lemma 3.4 gives two leaves $\mathcal{L}_-$ and $\mathcal{L}_+$ in a neighborhood $N(\mathcal{L}_D)$ of $\mathcal{L}_D$ such that:

- The leaf $\mathcal{L}_D$ is obtained from $\mathcal{L}_-$ by the collapses of a (j_1-1)-, (j_2-1)-, $\cdots$,(j_k-1)-cell,
- The leaf $\mathcal{L}_D$ is obtained from $\mathcal{L}_+$ by the collapses of a $(n+1-j_1)$-, $(n+1-j_2)$-, $\cdots$, $(n+1-j_k)$-cell,

  and the closure in $\mathcal{L}_-$ and $\mathcal{L}_+$ of all these cells are distinct.

- All the non-degenerate leaves of $\mathcal{F}$ in $N(\mathcal{L}_D)$ are homeomorphic either to $\mathcal{L}_-$ or to $\mathcal{L}_+$.

Thus $\mathcal{L}_D$, $\mathcal{L}_-$ and $\mathcal{L}_+$ and all the leaves of $\mathcal{F}$ which belong to $N(\mathcal{L}_D)$ are homotopically equivalent. Therefore, by compactness of K, all the leaves of the foliation $\mathcal{F}$ are homotopically equivalent. □

Let $\mathcal{L}$ be a non-degenerate leaf of a foliation $\mathcal{F}$ as above. By cutting K along $\mathcal{L}$, one obtains a CW-complex which is homotopically equivalent to $\mathcal{L} \times [0, 1]$. Thus the simple n-complex K is homotopically equivalent to $Susp_\psi(\mathcal{L})$ (see section 1.1), where $\psi : \mathcal{L} \to \mathcal{L}$ is a continuous map which is a composition of maps induced by the sequence of generalized WM-moves from $\mathcal{L}$ to $\mathcal{L}$ (see proposition 3.11). Since these maps are homotopy equivalences (see lemma 2.21), the map ψ is an homotopy equivalence and thus it induces an automorphism on the fundamental group of $\mathcal{L}$. This argument is easily generalized to the case where $\mathcal{L}$ is a degenerate leaf. The proof of theorem 3.1 is completed. □

Theorem 3.1 and remark 2.18 together imply that a positive cocycle of a simple dynamical n-complex K $(n \geq 2)$ defines a continuous map of a *standard* $(n-1)$-complex which is r-embedded in K. This map appears as a composition of generalized WM-moves. The proposition below gives a kind of converse to this result.

Proposition 3.13. *Let $\psi : K \to K$ be a continuous map of a standard n-complex K $(n \geq 1)$, such that $\psi = \alpha \circ \sigma_r \circ \cdots \circ \sigma_1$, where $\sigma_i : K_{i-1} \to K_i$ $(i = 1, \cdots, r)$, and $\alpha : K_r \to K_0$ are such that:*

1. *Each K_i $(i = 1, \cdots, r)$ is a simple n-complex obtained from K_{i-1} by a generalized WM-move, $K_0 = K$ and α is an homeomorphism from K_r onto K_0.*
2. *The maps σ_i from K_{i-1} to K_i $(i = 1, \cdots, r)$ are induced by the corresponding generalized WM-moves (see lemma 2.21).*

Then there is a simple dynamical $(n+1)$-complex K_S which is homotopically equivalent to the $(n+1)$-dimensional CW-complex $Susp_\psi(K)$. Moreover, this simple dynamical $(n+1)$-complex K_S admits a positive cocycle u such that:

- *A simple n-complex K_u associated to u by lemma 2.13 is homeomorphic to K.*

- *There is an ordered sequence of generalized WM-moves defined by u (see proposition 3.11) which is the sequence given for the definition of ψ.*

Proof of proposition 3.13: Each generalized WM-move from K_{i-1} to K_i, $i = 1, \cdots, r$, defines an elementary foliated $(n+1)$-complex $\mathcal{C}_{K_{i-1}K_i}$ (see definition 2.19). The edges of its singular graph are oriented from K_{i-1} to K_i. One glues $\mathcal{C}_{K_{i-1}K_i}$ to $\mathcal{C}_{K_iK_{i+1}}$ by the identity of K_i, $i = 1, \cdots, r-1$. The $(n+1)$-dimensional CW-complex obtained, denoted by K', has two boundary components K_0 and K_r. One identifies K_r to K_0 by α. The $(n+1)$-dimensional CW-complex obtained is denoted by K_S.

One has to prove that K_S satisfies all the properties of a simple dynamical $(n+1)$-complex.

By definition of an elementary foliated complex, each point in its interior has a neighborhood homeomorphic to the neighborhood of some point in a simple $(n+1)$-complex. Thus, this is also true for any point in the interior of K'. Since α is an homeomorphism, this is still satisfied for any point in the complex K_S.

Let us now prove that the components of K_S are so that K_S is a simple $(n+1)$-complex. Let C be a i-component of K, $1 \leq i \leq n$. Since K is a standard n-complex, C is a i-cell. We are first interested in the components of K'.

If C is not collapsed by any WM-move in the given sequence, then C gives rise to a $D^i \times [0,1]$ component in K'.

Otherwise, C gives rise to a finite set of j-components $S_1, \cdots, S_k$ of K' such that, $\overline{S}_i$ denoting the closure in K' of S_i:

- For each $m = 1, \cdots, k-1$, $\overline{S}_m \cap \overline{S}_{m+1}$ is a crossing of K'.
- The component S_1 is a $(i+1)$-cell which contains exactly one attractor in its boundary at $\overline{S}_1 \cap K_1$ and $\overline{S}_1 \cap K_0 = C$.
- For $p = 1$ to p such that $k = 2p$ or $k = 2p+1$, S_{2p} (resp. S_{2p+1}) is a $(n+1-i)$-cell (resp. a $(i+1)$-cell) of K'.
- The components $S_2, \cdots, S_{k-1}$ contain each exactly one attractor and one repellor in their boundary.
- The component S_k contains exactly one repellor in its boundary at $\overline{S}_k \cap K_{r-1}$ and $\overline{S}_k \cap K_r$ is a $(n-i)$-cell if k is even and a i-cell otherwise.

Since K_S is the quotient of K' under $\alpha : K_r \to K_0$ and α is an homeomorphism, all the assertions above imply easily the following two properties:

- The i-components of K_S are either i-cells or homeomorphic to $Susp_{Id}(D^{i-1})$ or $Susp_{H_R}(D^{i-1})$.
- Each component which is a i-cell has exactly one attractor and one repellor in its boundary and the other components have none.

Thus K_S is a simple dynamical $(n+1)$-complex. Moreover, by construction, the simple n-complex K_0 is r-embedded in K_S. The construction and the chosen orientation of the edges of the singular graph assure that, equipped with the good transverse orientation, it defines a positive cocycle and that this cocycle defines the given sequence of generalized WM-moves. This completes the proof of proposition 3.13. $\square$

The construction we use for proving the above proposition was suggested, for the case of 2-dimensional complexes, by G.Levitt. For this particular case, one presents another construction in [G2].

The following corollary comes straightforward from proposition 3.13, using the well-known result that any free group-automorphism can be expressed as a composition of Whitehead moves.

Corollary 3.14. *Let $\mathcal{O}$ be any automorphism of the free group F_n $(n \geq 1)$.*
Then there is a simple dynamical 2-complex K which admits a positive cocycle $u \in C^1(K; \mathbf{Z})$ such that:

- *The fundamental group of K is the suspension of the automorphism $\mathcal{O}$ of F_n.*
- *If K_u is as given by lemma 2.13, then any sequence of Whitehead moves defined by u on K_u induces the automorphism $\mathcal{O}$ on $\pi_1(K_u) \equiv F_n$, up to conjugacy in the group of outer automorphisms of F_n.*

In particular, from this corollary, any free group-automorphism can be represented by a simple dynamical 2-complex.

4 Dynamical complexes, semi-flows and flows

In this section, we justify the name of *dynamical* n-complex by proving that any such standard n-dimensional CW-complex carries a *non-singular semi-flow.* This allows to define non-singular flows on compact $(n+1)$-manifolds when these complexes are the *spines* of such manifolds.

Let us recall that a *non-singular flow* $(\phi_t)_{t\in\mathbf{R}}$ on a n-manifold ($n \geq 1$) is a group with one parameter $t \in \mathbf{R}$ of diffeomorphisms of the manifold without fixed points.

A *transverse hypersurface* to a non-singular flow on a compact n-manifold M^n ($n \geq 2$) is a properly embedded hypersurface in M^n which intersects transversely and in the same direction the orbits of the flow that it intersects.

A *cross-section* to a non-singular flow on a compact n-manifold M^n ($n \geq 2$) is a transverse hypersurface S in M^n which intersects all the orbits in finite time

- both in the future and the past if M^n has empty boundary or if the flow is tangent to ∂M^n,
- and only in the future in the other cases.

Definition 4.1. A *non-singular semi-flow* on a topological space X is a semi-group with one parameter $t \in \mathbf{R}^+$ of continuous maps of X without fixed points, i.e.:

- For all $x \in X$, $\sigma_0(x) = x$.
- For all t, t' in $\mathbf{R}^+$, $\sigma_{t+t'}(x) = \sigma_t(\sigma_{t'}(x))$.
- The set $\{x \in X \text{ ; for all } t \text{ in } \mathbf{R}^+ \sigma_t(x) = x\}$ is empty.

If K is a simple n-complex, one further requires that a non-singular semi-flow on K restricts to a non-singular flow on each open n-component of K.

A simple degenerate $(n-1)$-complex K' which is embedded and 2-sided in K is *transverse to the semi-flow* if all its intersection-points with the orbits of the semi-flow are transverse, and with the same intersection-sign, once a transverse orientation to K' has been chosen.

A *cross-section* to a non-singular semi-flow on a simple n-complex K is a transverse simple $(n-1)$-complex K' which is r-embedded in K and which intersects all the orbits in finite time. As in the usual case of a flow on a manifold, when a non-singular semi-flow admits a cross-section, it induces a continuous map on this cross-section, which is called the *return-map* of the semi-flow on the cross-section.

In what follows, we will assume that a transverse hypersurface (resp. a $(n-1)$-dimensional CW-complex) to a flow (resp. a semi-flow) is transversely oriented so that the orbits of the flow (resp. semi-flow) intersect it *positively*.

Proposition 4.2. *Any standard dynamical n-complex K ($n \geq 2$) carries a non-singular semi-flow.*

Proof of proposition 4.2: This proposition relies mainly on lemma 4.3 below.

Lemma 4.3. *Let K be a standard dynamical n-complex ($n \geq 2$). Then there is a transversely oriented codim 1-regular foliation defined on K, whose transverse orientation agrees with the orientation of the edges of the singular graph.*

Proof of lemma 4.3: Since K is standard, each n-component of K is an open n-cell. Since K is a dynamical n-complex, each n-component has one attractor and one repellor in its boundary. For each component C of K, there is a continuous map $h_C : \overline{D}^n \to \overline{C}$ which is a homeomorphism from D^n onto C (see section 2.3). Since our complexes are assumed to admit a piecewise-linear structure, the boundary of D^n in $\mathbf{R}^{n+1}$ contains a collection of 0- and 1-simplices whose images under h_C are the crossings and edges of the singular graph contained in $\overline{C}$. These 1-simplices inherit an orientation from the orientation of the edges of $K^{(1)}_{sing}$. The image of one of the 0-simplices is the repellor of C and the image of another one is its attractor. These are the 0-simplices whose all incident 1-simplices are respectively outgoing and incoming. The disc $\overline{D}^n$ obviously admits a foliation by closed $(n-1)$-discs such that:

- Two of these discs are reduced to the above two 0-simplices.
- All the other discs intersect transversely the boundary of $\overline{D}^n$, and are transversely oriented by the 1-simplices of $\partial\overline{D}^n$.
- The intersection of these closed discs with D^n defines a non-singular foliation of D^n.

For each component C, one considers the image under h_C of such a foliation. This clearly defines a regular codim 1-foliation $\mathcal{F}$ of the complex K. The fact that, in each closed n-disc, the $(n-1)$-disc of the foliation constructed are transversely oriented by the 1-simplices of the boundary assures that this foliation $\mathcal{F}$ is transversely oriented by the edges of the singular graph. This completes the proof of lemma 4.3.□

Lemma 4.4. *Let K be a simple dynamical n-complex ($n \geq 2$) which admits a transversely orientable codim 1-regular foliation $\mathcal{F}$. Then, K carries a non-singular semi-flow transverse to $\mathcal{F}$.*

Proof of lemma 4.4: Since K is compact, there are a finite number of leaves $\mathcal{L}_1, \cdots, \mathcal{L}_r$ whose neighborhoods $N(\mathcal{L}_1), \cdots, N(\mathcal{L}_r)$ cover K. Since the leaves can be transversely oriented in a coherent way, one can choose the oriented transversals (see definition in section 2.2) in these neighborhoods $N(\mathcal{L}_i)$ to agree at the intersections $N(\mathcal{L}_i) \cap N(\mathcal{L}_j)$, $i, j = 1, \cdots, r$. These oriented transversals glue together to give the orbits of a non-singular semi-flow on K. This comes from the definition of "2-sidedness" and of "oriented transversal" (see section 2.2 and definition 2.11). $\square$

Lemmas 4.3 and 4.4 imply proposition 4.2. $\square$

Proposition 4.5. *If a simple dynamical n-complex K ($n \geq 2$) admits a positive cocycle $u \in C^1(K; \mathbf{Z})$, then there is a non-singular semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ on K such that some r-embedded complex K_u as given by lemma 2.13 is a cross-section to $(\sigma_t)_{t \in \mathbf{R}^+}$.*

Proof of proposition 4.5: Observe that a positive cocycle of a simple dynamical n-complex defines a transversely orientable codim 1-regular foliation with compact leaves (see section 3.2). Lemma 4.4 allows to define a non-singular semi-flow on K which is transverse to this foliation. The following lemma, classical in its principle, allows to conclude:

Lemma 4.6. *Let K be a simple dynamical n-complex. If a non-singular semi-flow on K admits a transverse codim 1-regular foliation with compact leaves, then any r-embedded leaf is a cross-section to this semi-flow.*

Thus, the non-singular semi-flow on K, constructed above transversely to the foliation defined by the positive cocycle u, admits a simple $(n-1)$-complex K_u associated to this cocycle as a cross-section. $\square$

Remark 4.7. In the 2-dimensional case, we see in [G1] that *any* dynamical 2-complex (that is a not necessarily standard dynamical 2-complex) carries a non-singular semi-flow. The orientation of the edges of the singular graph gives, in some sense, the orientation of the semi-flow. The problem in the general case is to extend a non-singular semi-flow defined in a neighborhood of the boundary of a component which is not a cell to a non-singular flow on the component.

We give below a definition of a spine of a manifold, which is a rather classical notion (see for instance [BP], [Ca], [Ch3], [Ma1,2]).

Definition 4.8. A simple n-complex K ($n \geq 1$) is the *spine* of a compact $(n+1)$-manifold with boundary M_K if there is an embedding $i : K \to M_K$ and a retraction $r_K : M_K \to i(K)$, which is a homotopy equivalence, such that the manifold M_K is homeomorphic to $\partial M_K \times [0,1]$ quotiented by the equivalence relation $(x,t) \sim (x',t')$ if and only if $t = t' = 0$ and $r_K(x) = r_K(x')$. The fibers $r_K^{-1}(x)$ are $(n+2-j)$-ods centered at x, where j is the smallest integer for which $x \in K_{sing}^{(j)}$.

A simple dynamical n-complex which is the spine of some compact $(n+1)$-manifold will be called a *simple dynamical n-spine*.

Remark 4.9. Let us recall, for outlining the importance of this notion of spine, that any piecewise-linear compact n-manifold with boundary, and thus in particular any compact 3-manifold with boundary, admits a $(n-1)$-spine (see [Ca] for the case of 3-manifolds and [Ma1] for the general case). However, since, at the difference of the standard spines of Casler or the special spines of Matveev, we admit n-components which are not n-cells, two non-homeomorphic $(n+1)$-manifolds can admit homeomorphic simple n-spines.

Proposition 4.10 below treats the problem of reconstructing a non-singular flow on the manifold M_K from some non-singular semi-flow on a dynamical 2-spine K. It is rather classical in its principle, but in different settings. The interested reader may refer to the works of Williams or Christy (see [Ch1] or [Wi2]). However, in these works, the authors assume their complexes to admit a smooth structure at each point, this assumptiom being due to the fact that they were interested in the special case of *hyperbolic flows* on 3-manifolds.

Proposition 4.10. *Let K be a simple dynamical n-spine of a $(n+1)$-manifold M_K. Then, for any non-singular semi-flow $(\sigma_t)_{t\in\mathbf{R}^+}$ on K, there is a non-singular flow $(\phi_t)_{t\in\mathbf{R}}$ on M_K, transverse and pointing inward with respect to ∂M_K, such that the retraction $r_K : M_K \to K$ given by definition 4.8 defines a semi-conjugacy between $(\phi_t)_{t\in\mathbf{R}^+}$ and $(\sigma_t)_{t\in\mathbf{R}^+}$.*

Proof of proposition 4.10: Let x be any point of K, and let $I_x = \sigma_\epsilon(x)$, $\epsilon > 0$ small. These small orbit-segments I_x lift, by r_K^{-1}, to oriented intervals I_y in M_K, which are defined to be transverse to ∂M_K, and pointing inward, for any $y \in r_K^{-1}(x) \cap \partial M_K$. All these oriented intervals in M_K glue together to form the orbits of a non-singular flow on M_K, which are transverse to ∂M_K and pointing inward. By construction, the retraction r_K defines a semi-conjugacy between this flow and $(\sigma_t)_{t\in\mathbf{R}^+}$. $\square$

Remark 4.11. Proposition 4.10 above and remark 4.9 imply that one can find two non-singular flows on two non-homeomorphic $(n+1)$-manifolds which are semi-conjugated to a same semi-flow on a dynamical n-spine, but which are not topologically conjugated since their ambient manifolds are not homeomorphic.

Proposition 4.12. *With the assumptions and notations of proposition 4.10,*

1. *If $(\sigma_t)_{t\in\mathbf{R}^+}$ admits a transverse codim 1-regular foliation, then this foliation lifts by r_K^{-1} to a non-singular codim 1-foliation of M_K, which is transverse to ∂M_K and transverse to $(\phi_t)_{t\in\mathbf{R}}$.*

2. *Any positive cocycle $u \in C^1(K;\mathbf{Z})$ defines a cross-section S_u to some non-singular flow $(\phi_t)_{t\in\mathbf{R}}$ on M_K such that $K_u = r_K(S_u)$ is a cross-section in K to the semi-flow $(\sigma_t)_{t\in\mathbf{R}^+}$ semi-conjugated to $(\phi_t)_{t\in\mathbf{R}^+}$ by r_K.*

   In particular, the return-map of $(\sigma_t)_{t\in\mathbf{R}^+}$ on K_u is induced by the return-homeomorphism of $(\phi_t)_{t\in\mathbf{R}}$ on S_u.

Proof of proposition 4.12: This proposition is a consequence of the following lemma, whose proof is easy and left to the reader.

Lemma 4.13. *Let K be a simple dynamical n-complex. Any simple degenerate $(n-1)$-complex, possibly non-finite, r-embedded in K, lifts, by r_K^{-1}, to a properly embedded hypersurface in M_K. In particular, any cocycle $u \in C^1(K;\mathbf{Z})$ defines a hypersurface S_u properly embedded in M_K, which retracts by r_K to K_u.*

Furthermore, a codim 1-regular foliation of K lifts to a non-singular codim 1-foliation of M_K, transverse to ∂M_K.

This lemma and the construction of the flow $(\phi_t)_{t\in\mathbf{R}^+}$ imply that any codim 1-regular foliation transverse to a non-singular semi-flow on K lifts under r_K to a non-singular codim 1-foliation of M_K, transverse to ∂M_K and to $(\phi_t)_{t\in\mathbf{R}^+}$. Item (2) comes from proposition 4.5 and this lemma. □

5 Dynamical 2-complexes

This section treats the particular case of simple dynamical 2-complexes. As already announced in the introduction, simple 2-complexes play an important role in combinatorial group theory and topology of 3-manifolds. One defines *special foliations* of dynamical 2-complexes. At the difference of a regular foliation, a special foliation might have tangency points with the singular graph. We give, as for regular foliations in section 3, a cohomological criterion for the existence of special foliations with compact leaves on dynamical 2-complexes. These special foliations allow us to obtain suspension of *injective free group-endomorphisms*, and not merely of *free group-automorphisms* as in the case of the regular foliations.

Remark 5.1. We gather here some easy observations on simple and dynamical 2-complexes.

- Let K be a simple 2-complex.
  1. The set of singular points of K is a 4-*valent graph*, that is a graph with four edges incident to each crossing. The 2-components are discs, annuli or Moebius-bands.
  2. The simple degenerate 1-complexes r-embedded in K (and thus the leaves of any regular foliation) are graphs with at most four germs of edges incident to each vertex. If the embedding is non-degenerate, the graph is trivalent, i.e. three germs of edges are incident to each crossing.
- Assume now that K is a simple dynamical 2-complex.
  1. At each crossing, there are exactly two incoming and two outgoing edges of $K^{(1)}_{sing}$.
  2. The boundary circles of annuli and Moebius-band components are positive loops in $K^{(1)}_{sing}$. The boundary circle of a disc component decomposes as pq^{-1} where p and q are two positive paths in $K^{(1)}_{sing}$.
  3. If $u \in C^1(K; \mathbf{Z})$ is a non-negative cocycle, then the r-embedded graph Γ_u given by lemma 2.13 is unique up to isotopy in the 2-components, and twists around the cores of the annuli.

5.1 Special foliations of dynamical 2-complexes

Definition 5.2. Let K be a simple 2-complex and let Γ be a simple degenerate 1-complex. A *special embedding* of Γ in K is an embedding $p : \Gamma \to K$ satisfying the following properties:

1. For each point $x \in \Gamma$ with $p(x)$ distinct from the crossings of K and satisfying moreover that
   - either $p(\Gamma)$ is c-transverse to the singular graph at $p(x)$,
   - or $p(x)$ is a non-singular point in K,

   there is a neighborhood $N(x)$ of x in Γ such that $p_{|N(x)}$ is a homeomorphism from $N(x)$ onto $X \times \{t\}$, $t \in [0, 1]$, where X is as given by lemma 1.4.

2. Any point x such that $p(\Gamma)$ is c-tangent to $K^{(1)}_{sing}$ at $p(x)$ is a non-singular point in Γ.

3. If $p(x)$ is a crossing of K, then any germ of 2-cell of K at $p(x)$ contains the images under p of at most two germs of edges of Γ at x. Furthermore, the interior of any germ of 2-cell intersects the image under p of at most one germ of edge of Γ at x.

Definition 5.3. A *codim* 1-*special foliation* $\mathcal{F}$ of a simple 2-complex K is a union of simple degenerate 1-complexes, called the *leaves* of the foliation, which are specially embedded in K and such that each point of K belongs to exactly one of these leaves.

By definition, a r-embedding in a simple 2-complex K is also obviously a special embedding. Thus, a regular foliation of K is a special foliation of this complex. Let us also observe that, conversely, a special embedding which is c-transverse to the singular graph of K is a r-embedding.

Remark 5.4. All the definitions, stated for regular embeddings and foliations, of being 2-sided, admitting a transverse orientation, being transversely oriented,... are adapted in a straightforward way to special embeddings and foliations. Beware however that remark 2.12 is no more true for special embeddings. When writing that the *transverse orientation to a special foliation agrees with the orientation of the edges at the singular graph*, we will mean *at the points where the foliation is c-transverse to these edges.*

Lemmas 4.4 and 4.6 remain clearly true for special embeddings and foliations.

In section 3, we proved a theorem on the existence of a *regular* foliation with compact leaves of a simple n-complex ($n \geq 2$). Here, we assume that we are given some dynamical 2-complex and we prove a result about the existence of a *special* foliation with compact leaves of this complex. The cocycles involved are non-negative cocycles satisfying some additional properties, but which are not necessarily positive cocycles.

We speak in definition 5.5 below of 2-*sided loops*. Here, a loop must be considered as a simple 1-complex without crossings, which is specially embedded in the given dynamical 2-complex. The notion of being 2-sided given in definition 2.11 applies then to a loop.

Definition 5.5. Let K be any simple dynamical 2-complex. A *nice* non-negative integer cocycle $u \in C^1(K; \mathbf{Z})$ is a non-negative integer cocycle such that:

The only positive loops l in the singular graph for which $u(l) = 0$ satisfy the following properties:

1. They are disjointly embedded, 2-sided loops in K whose some transverse orientation agrees with the orientation of the edges of the singular graph.
2. They do not belong to the boundary of any 2-component of K.

Let us recall that a positive cocycle is a non-negative cocycle which is positive on all the positive embedded loops of the singular graph. Thus, a positive cocycle is also a nice non-negative cocycle.

We will call *non-trivial* a simple n-complex which has at least one crossing. We prove in a first step the following proposition:

Proposition 5.6. *A non-trivial dynamical* 2*-complex* K *admits a transversely orientable codim* 1*-special foliation* $\mathcal{F}$ *by compact graphs, whose some transverse orientation agrees with the orientation of the edges of the singular graph, if and only if there exists a nice non-negative cocycle* $u \in C^1(K; \mathbf{Z})$.

Proposition 5.6 treats only the case of non-trivial simple dynamical 2-complexes. This is equivalent to throw away the dynamical 2-complexes which have only annuli or Moebius-band components. Without this slight restriction, the statement of this proposition would have been more complicated. Moreover, with respect to special foliations, the case of trivial complexes is not of great interest. Lemma 5.7 below treats this case.

Lemma 5.7. *Let* K *be a trivial simple dynamical* 2*-complex.*

1. *Then, either* K *admits a positive cocycle or* K *admits no non-negative cocycle.*
2. *Furthermore, if* K *admits a transversely oriented codim* 1*-special foliation by compact leaves, whose transverse orientation agrees with the orientation of the edges of the singular graph, and* K *does not admit any positive cocycle, then this foliation is a foliation by circles.*
3. *In the situation of item (2), the complex* K *is homotopically equivalent to the suspension of a degree* 2^k*-continuous map of the circle, where* k *is the number of loops of the singular graph.*

Proof of lemma 5.7: Assume that there is a non-negative cocycle $u \in C^1(K; \mathbf{Z})$. Then, by definition of a non-negative cocycle, u is positive on at least one loop l in $K^{(1)}_{sing}$. Furthermore, u is also positive on all the loops which are in the boundary of all the annuli components attached along l.

By induction, and since our complexes are assumed to be connected, this implies that u is a positive cocycle.

Assume now that K is as announced in item (2). If some leaf of this foliation $\mathcal{F}$ intersects transversely some loop of the singular graph, then one easily proves the existence of a non-degenerate r-embedded leaf in $\mathcal{F}$. By assumption on the transverse orientation to $\mathcal{F}$, this leaf defines a non-negative cocycle, and thus a positive cocycle by which precedes. This is a contradiction with our assumption. Thus all the loops of the singular graph are leaves of $\mathcal{F}$. All the other leaves of $\mathcal{F}$ are embedded in the interior of the components of K. Item (1) of definition 5.2 implies that they have no crossings. Since circles are the only compact simple 1-degenerate complexes without any crossings, this proves item (2) of lemma 5.7.

Let us now prove the last assertion of this lemma. Let us first observe that K has no Moebius-band components, because of the transversal orientability of $\mathcal{F}$. Consider any loop of the singular graph. Three germs of 2-components are incident to this loop. Since it is a leaf of a transversely oriented foliation, it is 2-sided in K. Thus, two germs are on the same side, say the "- side". Since $\mathcal{F}$ is transversely oriented, the other boundary loop of $\mathcal{F}$ is on the + side. This allows to construct a transversely oriented foliation $\mathcal{C}$ of K by circles which comes from the foliation by circles $\mathbf{S}^1 \times \{t\}$ of each annulus $\mathbf{S}^1 \times [0,1]$. Let us consider the semi-flow on K whose orbits are obtained by the gluing of the intervals $\{z\} \times [0,1]$, $z \in \mathbf{S}^1$. This semi-flow is transverse to the leaves of $\mathcal{C}$ and any leaf is a cross-section to this semi-flow (see lemmas 4.4, 4.6 and remark 5.4). Let us consider two leaves l_- and l_+ of $\mathcal{C}$ which are respectively on the $-$ side and $+$ side of a loop of the singular graph. The map induced by the semi-flow from l_- to l_+ is a degree 2-map of the circle. This easily implies that K is the suspension of a degree 2^k-map of the circle, where k is the number of loops in the singular graph. This completes the proof of lemma 5.7. □

5.2 From a special foliation to a nice cocycle

In a first step, we assume that one has a special foliation $\mathcal{F}$ as given by proposition 5.6. The strategy to prove the existence of a nice non-negative cocycle is to examine what are the possible sets of points of c-tangency of $\mathcal{F}$ with the singular graph $K^{(1)}_{sing}$. The positive loops contained in this set are all the positive loops of the singular graph on which the cocycle associated to a non-degenerate leaf takes the value 0. Once checked that these positive loops are disjointly embedded and transversely oriented by the edges of the singular graph in K, one is done.

Lemma 5.8. *Let $\mathcal{F}$ be a codim 1-special foliation with compact leaves of a simple 2-complex K.*

Let $Tang_{\mathcal{F}}$ be the set of points $x \in K^{(1)}_{sing}$ such that the leaf of $\mathcal{F}$ containing x is c-tangent to $K^{(1)}_{sing}$ at x. We denote by $\mathcal{T}$ the closure in K of $Tang_{\mathcal{F}}$.

Then each connected component of $\mathcal{T}$ is contained in exactly one leaf of $\mathcal{F}$. These connected components are isolated points, embedded compact intervals and embedded loops.

Proof of lemma 5.8: By definition, any connected component of $Tang_{\mathcal{F}}$ is contained in exactly one leaf of $\mathcal{F}$. The set $\mathcal{T}$ is obviously equal to the union of the closure in K of the connected components of $Tang_{\mathcal{F}}$. Moreover, the leaves of $\mathcal{F}$ are disjointly embedded. Thus, for proving the first assertion of lemma 5.8, it suffices to check that the closure of any non-compact subset of a leaf $\mathcal{L}$ of $\mathcal{F}$ does not contain any point in a leaf $\mathcal{L}' \neq \mathcal{L}$ of $\mathcal{F}$. This comes from the compacity of the leaves.

Let us now prove the second point of lemma 5.8. If a connected component of $\mathcal{T}$ is contained in an open edge of $K^{(1)}_{sing}$, then this connected component is either an isolated point or a compact embedded interval. Thus, one only has to look at what happens at the crossings of K. Item (3) of definition 5.2 implies that any leaf Γ of $\mathcal{F}$ contains at most two germs of edges at any crossing v of K. One proved above that each connected component of $\mathcal{T}$ is contained in exactly one leaf of $\mathcal{F}$. Moreover, the leaves are disjointly embedded. These three last assertions imply that each connected component of $\mathcal{T}$ contains at most two germs of edges of $K^{(1)}_{sing}$ at each crossing. This allows to complete the proof of lemma 5.8. □

Lemma 5.9. *With the assumptions and notations of lemma 5.8, let us assume furthermore that:*

- *K is a non-trivial simple dynamical 2-complex.*
- *$\mathcal{F}$ is transversely oriented, such that its transverse orientation agrees with the orientation of the edges of the singular graph.*

Then all the positive loops in $\mathcal{T}$ are 2-sided and transversely oriented by the edges of the singular graph. No positive loop in $\mathcal{T}$ is a boundary loop of a 2-component of K.

Proof of lemma 5.9: We denote by l any positive loop in $\mathcal{T}$. Let us recall that the notion of being 2-sided, and all the subsequent definitions, have been given in the case of r-embeddings (see definition 2.11) but are extended to the case of special embeddings in a straightforward way. The existence of the transverse orientation announced for the above positive loop comes from the fact that it is contained in some leaf of $\mathcal{F}$, which by assumption is transversely oriented by the edges of the singular graph. This allows to define a set of oriented transversals to each of these loops as desired (see definition 2.11).

No Moebius-band component contains the loop l in its boundary because of the transversal orientability of $\mathcal{F}$. Assume now that some annulus component contains l in its boundary. Three germs of 2-components are incident to each point in l. Since l is 2-sided, two germs are on a same side, say the $-$ side. Assume that the above annulus component is on the $-$ side of l. Since $\mathcal{F}$ is transversely oriented, this implies that it is on the $+$ side of its other boundary circle. One easily check that there cannot be an annulus component on the $+$ side of a loop containing a crossing of the complex. This implies that the loop of the singular graph corresponding to the above boundary circle contains no crossing. By induction, we obtain that K is a trivial simple n-complex. This is a contradiction with our assumption.

Therefore, no positive loop in $\mathcal{T}$ is a boundary loop of an annulus or Moebius-band component. Since K is dynamical, a positive loop cannot be the boundary of a disc component (see definition 1.7 and remark 5.1). One so proved lemma 5.9. □

Lemma 5.10. *With the assumptions and notations of lemma 5.9,*

Let u_{sum} be the cocycle equal to the sum of all the cocycles associated to the non-degenerate r-embedded leaves of $\mathcal{F}$ (see lemma 2.13). Then u_{sum} is a non-negative cocycle which is positive on all the positive loops of the singular graph not contained in $\mathcal{T}$, and which is null on the positive loops in $\mathcal{T}$.

Proof of lemma 5.10: Let us recall that the non-degenerate r-embedded leaves are the leaves which contain no crossing of K and which are c-transverse to $K^{(1)}_{sing}$. From the first point of lemma 5.8, each connected component of $\mathcal{T}$ is contained in exactly one leaf of $\mathcal{F}$. By definition of $\mathcal{T}$, this leaf cannot be a non-degenerate r-embedded leaf. Thus, since the leaves are disjoint, none of the non-degenerate r-embedded leaves intersect some edge in $\mathcal{T}$.

Since K is non-trivial, some non-degenerate r-embedded leaf intersects the singular graph and thus defines a non-negative cocycle by the assumption on the transverse orientation to $\mathcal{F}$.

Therefore u_{sum} is a non-negative cocycle. From what precedes, u_{sum} is null on any edge contained in $\mathcal{T}$, and in particular on the positive loops in $\mathcal{T}$. The set of all non-degenerate r-embedded leaves intersect all the other edges. Thus, u_{sum} is positive on these edges. In particular, u_{sum} is positive on all the positive loops of the singular graph which are not contained in $\mathcal{T}$. This completes the proof of lemma 5.10. □

Let K be a non-trivial dynamical 2-complex which admits a transversely orientable codim 1-special foliation by compact graphs, whose some transverse orientation agrees with the orientation of the edges of the singular graph.

Lemma 5.10 gives a non-negative cocycle which is positive on the positive loops of thee singular graph which are not in $\mathcal{T}$ (see definition in lemma 5.8) and null on the positive loops in $\mathcal{T}$. By lemma 5.8, the positive loops in $\mathcal{T}$ are disjointly embedded. By lemma 5.9, they are transversely oriented by the edges of the singular graph and do not belong to the boundary of any 2-component. All this proves one implication of proposition 5.18.

5.3 From a nice cocycle to a special foliation

We assume here that one is given a nice non-negative cocycle of a simple dynamical 2-complex K. The proof of the existence of a special foliation consists of a construction similar to the one used in the proof of the existence of a transversely orientable regular codim 1-foliation of a simple n-complex K (see proposition 3.11). We assumed there the existence of a positive cocycle. The difference here is that the possible existence of loops on which the cocycle takes the value 0 forbids to foliate all the complex.

Before the construction of the foliation, let us notice in lemma 5.11 some consequences of the properties satisfied by the positive loops l_i on which a nice cocycle takes value 0. By *incoming edge at* a loop l_i, we mean the edges of the singular graph which are incoming at some crossing of l_i, and which are not contained in l_i. *Outgoing edges at* l_i are defined in the same way. Observe that, a priori, a same edge can be both an incoming and an outgoing edge at a same loop l_i.

Lemma 5.11. *Let K be a simple dynamical 2-complex, which admits a nice non-negative integer cocycle u. Let l be any positive loop in the singular graph such that $u(l) = 0$. Then:*

1. *The union of all the germs of 2-cells containing some germs of edges in l decomposes as the union of two subsets of disjoint interiors as follows:*

    - *One is an annulus which contains exactly once each edge and crossing of l, and which also contains the germs of outgoing edges at l.*

This is the + side of l.

- *The other consists either of one or two annuli, also of disjoint interior. It contains exactly twice each edge and crossing of l and it also contains the germs of incoming edges at l.*
  This is the − side of l.

2. *If e is any incoming (resp. outgoing) edge at l, then e is not an outgoing (resp. incoming) edge at l.*

3. *The loop l has at least one crossing, and thus at least one incoming and one outgoing edge.*

Proof of lemma 5.11: Item (1) comes from the fact that l is two-sided and transversely oriented by the edges of the singular graph of K. Item (2) is a consequence of item (1). Let us prove item (3). By assumption, the loops l_i on which a nice cocycle takes the value 0 bound no component in K. Since a loop of the singular graph which has no crossing is a boundary loop of an annulus or Moebius-band component, any loop l_i as above has at least one crossing, and thus at least one incoming edge and one outgoing edge. This completes the proof of lemma 5.11. □

Remark 5.12. With the terminology of Christy (see [Ch1]) or Williams (see [Wi1]), item (1) of lemma 5.11 above is equivalent to say on one hand that the complex K admits a *smoothing* along the loop l and on the other hand that the edges of the singular graph are oriented from the *locally* 2-*sheeted side* of l to the *locally* 1-*sheeted side* of l. One says that K admits a *compatible structure of dynamic branched surface along l.*

`Inductive process:`

One considers a nice non-negative cocycle $u \in C^1(K;\mathbf{Z})$. One applies non-negative δ_v-moves, as such as this can be done, by respecting the rule that each δ_v-move is applied at most once.

`End of the inductive process`

Lemma 5.13. *Let K be a simple dynamical 2-complex which admits a nice non-negative integer cocycle $u \in C^1(K;\mathbf{Z})$ (see definition 5.5). Let us assume that u is not a positive cocycle.*

Let $\{l_1, \cdots, l_n\}$ be the set of positive loops of the singular graph with $u(l_i) = 0$, $i = 1, \cdots, n$. Let u_- be the cocycle obtained at the end of the above inductive process.

Then there is a loop l_- in $\{l_1, \cdots, l_n\}$ such that u_- is positive on all the incoming edges at l_-.

Proof of lemma 5.13: By lemma 5.11, item (3), any loop l_i, contains at least one crossing.

Let us observe that, by definition of the inductive process, any crossing v of K satisfies one of the following two properties:

- Either no non-negative δ_v-move can be applied to u_-.
- Or a non-negative δ_v has already been applied in the inductive process from u to u_-.

One considers the loop l_1. If u_- is positive on all the incoming edges at l_1, one is done. Otherwise, there exists an incoming edge e at l_1, with $u_-(e) = 0$.

Since $t(e)$ belongs to the loop l_- which satisfies $u(l_-) = u_-(l_-) = 0$, no non-negative $\delta_{t(e)}$-move can be applied to u or any cocycle obtained from u by a sequence of non-negative δ_v-moves. Therefore, if a non-negative $\delta_{i(e)}$-move has been applied during the inductive process, $u_-(e) > 0$. But $u_-(e) > 0$ is a contradiction with our assumption on e. Thus no non-negative $\delta_{i(e)}$-move has been applied during the inductive process.

This implies, from our observation above, that no non-negative $\delta_{i(e)}$-move can be applied to u_-. Therefore, there also exists an incoming edge at $i(e)$ on which u_- takes the value 0. By induction, one constructs a positive path p_1 from another positive loop in $\{l_1, \cdots, l_n\}$, say l_2, to the loop l_1, with $u_-(p_1) = 0$. One iterates the process. Either one eventually obtains a positive loop l_k, $k \in \{1, \cdots, n\}$, such that u_- is positive on all the incoming edges at l_k. Or one obtains a sequence of positive paths $p_1, \cdots, p_n$, with p_i going from l_{i+1} to l_i, $i = 1, \cdots, n$, $l_{n+1} = l_1$, and $u_-(p_i) = 0$. This implies that u_- is null on a positive loop $L = \gamma_1 p_n \gamma_n p_{n-1} \cdots \gamma_2 p_1$ which intersects the l_i, where γ_j is a positive path contained in l_j. Since u_- is in the same cohomology class than u (see remark 2.4), this is a contradiction with u being a nice non-negative integer cocycle. Therefore, there exists a loop l_- in $\{l_1, \cdots, l_n\}$ as announced. □

Lemma 5.14. *With the assumptions and notations of lemma 5.13, we set $l_- = e_1 \cdots e_r$ and $v_i = t(e_i)$ for $i = 1, \cdots, r$.*

Then, $u_+ = (\delta_{v_1} \circ \delta_{v_2} \circ \cdots \circ \delta_{v_r})(u_-)$ is a non-negative integer cocycle in the same cohomology class than u_-. In particular, $u_+(e_i) = 0$ for any edge e_i, $i = 1, \cdots, r$, in l_-. Furthermore,

- *For any incoming (resp. outgoing) edge e at l_-, $u_+(e) = u_-(e) - 1$ (resp. $u_+(e) = u_-(e) + 1$). In particular, u_+ is positive on all the outgoing edges at l_-.*

- *For any edge $e \in K^{(1)}_{sing}$ which is neither incoming nor outgoing to some crossing in l_-, $u_+(e) = u_-(e)$.*

We will further set $u_+ = \delta_l(u_-)$ and write that u_+ is obtained from u_- by a *non-negative δ_l-move.*

Let us recall that, by lemma 5.11, item (3), the loop l_- contains at least one crossing.

Proof of lemma 5.14: By remark 2.4, u_+ is cohomologous to u_-. If l_- contains only one edge, then this edge is both incoming and outgoing at the unique crossing of l_- and thus clearly u_+ is null on all the edges in l_-.

Assume now that l_- contains more than one crossing. Since l_- is embedded, all the crossings $v_1, \cdots, v_r$ are distinct. Therefore, all the outgoing edges at $v_1, \cdots, v_r$ are distinct. The same is true for all the incoming edges at these crossings.

At each crossing v_i, $i = 1, \cdots, r$, there is exactly one outgoing edge in l which is incoming at v_{i+1}, this is the edge e_{i+1} (we set $v_{r+1} = v_1$ and $e_{r+1} = e_1$). By definition of a δ_v-move, one has $(\delta_{v_{i+1}} \circ \delta_{v_i})(u)(e_{i+1}) = (u(e_{i+1}) - 1) + 1 = 0$. Therefore, u_+ is null on l_-.

Each δ_{v_i}-move increases by 1 the value of u_- on the outgoing edge at v_i which is not in l_-. By lemma 5.11, item (2), an edge which is incoming (resp. outgoing) at some l_i cannot be an outgoing (resp. incoming) edge at this same l_i. Therefore, the value of the cocycle u_+ on any outgoing edge e at l_- is equal to $u_-(e) + 1$, and therefore positive.

For the same reason than above, the value of u_+ on each of the incoming edges e at l_- is equal to $u_-(e) - 1$. The cocycle u_- is assumed to be positive on all these edges. Thus u_+ is non-negative on the incoming edges at l_-.

Finally, by definition of a δ_v-move, $u_+(e) = u(e)$ for any edge $e \in K^{(1)}_{sing}$ which is not incident to some v_i, $i = 1, \cdots, r$. Therefore, u_+ satisfies the announced properties. □

Lemma 5.15. *With the assumptions and notations of lemma 5.13,*

There is a (non-unique) ordered sequence of non-negative integer cocycles $u = u_0 \to u_1 \to \cdots \to u_N$ such that:

1. *Each cocycle u_i, $i = 1, \cdots, N$, is obtained from u_{i-1} either by a non-negative δ_v-move, $v \in K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$, or δ_l-move, $l \in \{l_1, \cdots, l_n\}$.*
2. *There is exactly one non-negative δ_l-move (resp. δ_v-move) for each loop $l \in \{l_1, \cdots, l_n\}$ (resp. for each crossing $v \in K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$). In particular, $u_N = u$ and the sum of the cocycles u_i, $i = 1, \cdots, N$ is positive on all the edges in $K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$.*

Proof of lemma 5.15: The inductive process, together with lemmas 5.13 and 5.14 allow to obtain a sequence of non-negative integer cocycles satisfying item (1) of lemma 5.15. The cocycle u_+ obtained at the end of this sequence is a nice non-negative integer cocycle. Thus, one can iterate this process until it is no more possible to apply a non-negative δ_v-move, $v \in K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$ or δ_l-move, $l \in \{l_1, \cdots, l_n\}$ at a crossing v or a loop l at which such a move has not already been applied. One thus obtains a sequence of non-negative integer cocycles satisfying item (1) and such that each crossing $v \in K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$ or loop $l \in \{l_1, \cdots, l_n\}$ appears at most once in the associated sequence of non-negative δ_v- and δ_l-moves. We denote by u_{sum} the sum of all the cocycles obtained during this process.

Assume that item (2) is not satisfied. If some δ_l-move has not been applied, then u_{sum} is null on some incoming edge at l. If some δ_v has not been applied, $v \in K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$, then u_{sum} is null on some incoming edge at v. Otherwise, there exists a cocycle in the above sequence which is positive on all the incoming edges at l in the first case, and on all the incoming edges at v in the second case. This implies that a non-negative δ_l-move, or δ_v-move, can be applied to this cocycle. This is a contradiction with our assumption.

By induction, one constructs a positive loop L with intersects some l_i, $i = 1, \cdots, n$ and such that $u_{sum}(L) = 0$. By definition $u_{sum}(e) \geq u(e)$ for any edge $e \in K^{(1)}_{sing}$. Thus, $u(L) = 0$. This is a contradiction with u being a nice non-negative integer cocycle.

An easy computation allows then to prove the last two assertions. □

Lemma 5.16. *With the assumptions and notations of lemma 5.14,*

There are two r-embedded graphs Γ_{u_-} and Γ_{u_+} associated to the cocycles u_- and u_+ (see lemma 2.13) such that, if $K_- = \Gamma_{u_-} \times [-1, 0[$ and $K_+ = \Gamma_{u_+} \times]0, 1]$, then:

1. *$\Gamma = \overline{K_-} - K_- = \overline{K_+} - K_+$ is a graph specially embedded in K which contains the loop l_-.*
2. *The only crossings of K contained in Γ are the crossings in l_-.*
3. *Let e be any edge of Γ_{u_-} (resp. Γ_{u_+}). The closure in K of $e \times [-1, 0[$ (resp. $e \times]0, 1]$) is a non-trivial path in Γ between two crossings of Γ (resp. an edge of Γ). This path (resp. edge) belongs to the closure in K of the 2-component containing e.*
4. *Each crossing of Γ belongs to the closure in K of exactly one interval $\{v\} \times [-1, 0[$ (resp. $\{v\} \times]0, 1]$), $v \in \Gamma_{u_-} \cap K^{(1)}_{sing}$ (resp. $v \in \Gamma_{u_+} \cap K^{(1)}_{sing}$). Furthermore, Γ is a trivalent graph.*

See figures 4 and 5.

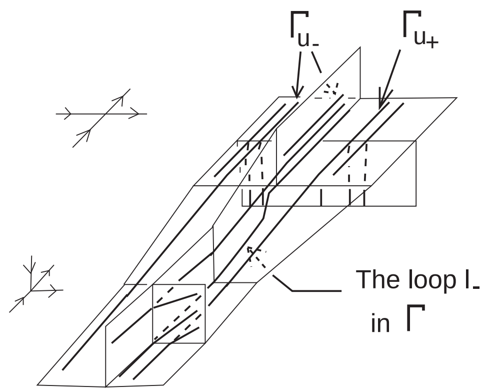

Figure 4: Constructing a u-foliation

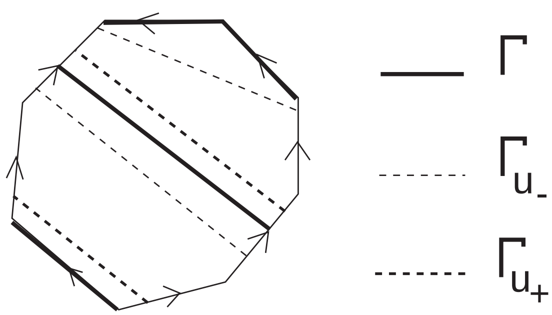

Figure 5: The graphs Γ, Γ_{u_-} and Γ_{u_+} in a disc component

Proof of lemma 5.16: By lemma 2.13, any r-embedded graph Γ_{u_-} or Γ_{u_+} is 2-sided in K. By lemma 5.14, the values of u_+ and u_- on the edges of the singular graph differ of at most one. Moreover, by definition of the cocycles u_- and u_+, the only edges e of the singular graph for which $u_-(e) \neq u_+(e)$ are the incoming and outgoing edges at l_-. The number of vertices of Γ_{u_+} (resp. Γ_{u_-}) along any edge e is equal to the value of u_+ (resp. u_-) on e (see proof of lemma 2.13).

This allows to choose the r-embeddings of Γ_{u_-} and Γ_{u_+} such that:

- The vertices of Γ_{u_-} preceed, along the edges of $K^{(1)}_{sing}$ which are not outgoing edges at l_-, the vertices of Γ_{u_+}.

  This means that the first vertex of $\Gamma_{u_-} \cup \Gamma_{u_+}$ along any edge of $K^{(1)}_{sing}$ which is not an outgoing edge at l_- is a vertex of Γ_{u_-}, and the vertices alternate along these edges.

- The vertices of Γ_{u_+} preceed the vertices of Γ_{u_-} along the outgoing edges at l_-.

One easily checks that two such r-embeddings can be made disjoint.

By these choices of r-embeddings, the last (resp. first) vertex of $\Gamma_{u_-} \cup \Gamma_{u_+}$ along any incoming (resp. outgoing) edge at l_- is a vertex of Γ_{u_-} (resp. Γ_{u_+}).

This allows to consider two disjoint non-compact subcomplexes $K_- = \Gamma_{u_-} \times [-1, 0[$ and $K_+ = \Gamma_{u_+} \times]0, 1]$ of K such that the closure of each interval $\{x\} \times [-1, 0[$, $x \in \Gamma_{u_-}$ intersects the closure of some interval $\{y\} \times]0, 1]$, $y \in \Gamma_{u_+}$, in a unique point which belongs neither to $\{x\} \times [-1, 0[$ nor to $\{y\} \times]0, 1]$. For x or y in $K^{(1)}_{sing}$, these intervals are contained in $K^{(1)}_{sing}$.

Clearly, $\Gamma = \overline{K_-} - K_- = \overline{K_+} - K_+$ is a graph embedded in K. One has to check that this embedding is a special embedding.

Any two intervals $\{x_1\} \times [-1, 0[$, $\{x_2\} \times [-1, 0[$, x_1 and x_2 in Γ_{u_-} with $x_1 \neq x_2$, are disjoint. Thus, if e is any edge of Γ_{u_-}, the closure in K of $e \times [-1, 0[$ is a path in Γ which belongs to the closure in K of the 2-component containing e. Therefore, k distinct germs of edges at a crossing v in Γ_{u_-} define k distinct germs of paths at the point w in Γ which belongs to the closure of $\{v\} \times [-1, 0[$. This implies that this point w is a crossing of Γ. Since Γ_{u_-} is r-embedded in a non-degenerate way, all its crossings are trivalent crossings which belong to the interior of some edges of the singular graph. From our assumption on the intervals $\{v\} \times [-1, 0[$, v a crossing of Γ_{u_-}, the crossings w of Γ as above are thus trivalent crossings which belong to the singular graph. There are no other crossings in Γ because two intervals $\{x_1\} \times [-1, 0[$, $\{x_2\} \times [-1, 0[$, with x_1 and x_2 distinct and both in Γ_{u_-}, are disjoint. Moreover, let us notice that if w is a crossing of Γ interior to an edge of the singular graph, the three edges of Γ incident to w have their germs in three distinct germs of 2-cells of K at w.

Therefore, the embedding of Γ in K satisfies item (1) of definition 5.2 of a special embedding.

One proved also above that, for any edge e of Γ_{u_-} or Γ_{u_+}, the closure of $e \times [-1, 0[$ or $e \times]0, 1]$ is a non-trivial path in Γ between two crossings of Γ.

From the choices of the r-embeddings of Γ_{u_-} and Γ_{u_+}, the graph Γ contains the loop l_-. The union of the open edges in l_- form the set of points of c-tangency of Γ with $K^{(1)}_{sing}$. From the assertions of the preceding paragraph, these points are non-singular points for Γ. Therefore, the embedding of Γ in K satisfies item (2) of definition 5.2.

From lemma 5.14, the values of u_+ and u_- on any edge of $K^{(1)}_{sing}$ which is not an incoming or outgoing edge at l_- are the same. This implies that the only crossings of K contained in Γ are the crossings of l_-. They are also crossings of Γ. Two germs of edges of Γ at each of these crossings are contained in the two germs of edges of l_-. The third germ of edge of Γ at a crossing of l_- is contained in the germ of 2-cell of K which is incident to none of the two germs of edges of l_-. These assertions are easy to check. This implies that Γ satisfies item (3) of definition 5.2.

All what precedes implies therefore in particular that Γ is a graph specially embedded in K. Moreover, all the arguments clearly apply when considering Γ_{u_+} instead of Γ_{u_-}. Thus, among the properties listed in lemma 5.16, it remains only to check that, for any edge e of Γ_{u_+}, the closure in K of $e \times]0, 1]$ is an edge of Γ. This comes from lemma 5.11, item (1). Indeed, the edge e of Γ_{u_+} can be assumed to be contained in the $+$ side of l_-. □

Corollary 5.17. *With the assumptions and notations of lemma 5.16,*

The subcomplex $K_s = \overline{K_-} \cup \overline{K_+}$ *of* K *admits a transversely oriented codim* 1*-special foliation with compact leaves whose transverse orientation agrees with the orientation of the edges of the singular graph. Moreover, all the leaves of* K_s *are homotopically equivalent.*

Proof of corollary 5.17: The existence of a codim 1-special foliation as anounced is straightforward from lemma 5.16. The leaves of such a foliation are $\Gamma_{u_-} \times \{t\}$, $t \in [-1, 0[$, $\Gamma_{u_+} \times \{t\}$, $t \in]0, 1]$ and Γ. Let us check that they are all homotopically equivalent. From lemma 5.16, Γ is a trivalent graph which has the same number of vertices than Γ_{u_-} or Γ_{u_+}. The graphs Γ_{u_-} and Γ_{u_+} are r-embedded in K in a non-degenerate way and thus trivalent (see remark 5.1). Therefore, Γ, Γ_{u_-} and Γ_{u_+} have the same Euler characteristic and thus are homotopically equivalent. This completes the proof of corollary 5.17. □

Let us now complete the proof of proposition 5.6. We assume given a nice non-negative integer cocycle u of a simple dynamical 2-complex K. We denote by $\{l_1, \cdots, l_n\}$ the loops in $K^{(1)}_{sing}$ with $u(l_i) = 0$.

Lemmas 5.15 and 2.23 allow to build a transversely oriented codim 1-regular foliation $\mathcal{F}_r$ with compact leaves of a subcomplex of K, whose transverse orientation agrees with the orientation of the edges of the singular graph. Moreover, the leaves of $\mathcal{F}_r$ intersect all the edges of $K^{(1)}_{sing} - \{l_1, \cdots, l_n\}$.

Corollary 5.17 allows to complete this foliation to a codim 1-transversely oriented special foliation which contains all the crossings of K. Since no loop l_i is the boundary loop of an annulus or Moebius-band component, this foliation is a foliation of K. This completes the proof of proposition 5.6.

A foliation as constructed above, that is by lemma 5.15, together with lemma 2.23 and corollary 5.17, will be called a *u-foliation*.

5.4 Conclusion

In this subsection, we prove the following proposition, which is a refinement of the reverse implication of proposition 5.6:

Proposition 5.18. *Let K be a simple dynamical 2-complex which admits a nice non-negative cocycle $u \in C^1(K;\mathbf{Z})$.*

Then u defines a transversely oriented codim 1-special foliation $\mathcal{F}_u$ of K satisfying the following properties:

1. *All the leaves of $\mathcal{F}_u$ are homotopically equivalent.*
2. *If Γ is any leaf of $\mathcal{F}_u$, there is a map $\psi : \Gamma \to \Gamma$, which induces an injective endomorphism $\mathcal{O}$ on the fundamental group of Γ, such that K is homotopically equivalent to $Susp_\psi(K)$.*
3. *The above endomorphism is surjective if and only if u is a positive cocycle.*

Lemma 5.19. *Let K be a standard dynamical 2-complex which admits a nice non-negative integer cocycle $u \in C^1(K;\mathbf{Z})$. All the leaves of a u-foliation $\mathcal{F}_u$ are homotopically equivalent.*

Proof of lemma 5.19: Since K is compact, there are a finite number of leaves of $\mathcal{F}_u$ whose neighborhoods cover K. By definition of a u-foliation, these neighborhoods are of two types according to the leaf $\mathcal{L}$ considered being a regular or a non-regular leaf of $\mathcal{F}_u$. If $\mathcal{L}$ is a regular leaf, then all the leaves in a neighborhood of $\mathcal{L}$ are homotopically equivalent (see proof of lemma 3.12). If $\mathcal{L}$ is a non-regular leaf, then the same is true by lemma 5.16. Therefore, all the leaves of $\mathcal{F}_u$ are homotopically equivalent. $\square$

Item (1) of proposition 5.18 is so proved.

Definition 5.20. Let $\psi : \Gamma \to \Gamma'$ be a continuous map between two graphs Γ and Γ'. The map ψ is *immersive* if it is surjective and locally-injective.

Lemma 5.21. *With the assumptions and notations of lemma 5.19,*

Let Γ be any leaf of a u-foliation $\mathcal{F}_u$. There exists a map $\psi : \Gamma \to \Gamma$ such that K is homotopically equivalent to $Susp_\psi(\Gamma)$, and satisfying moreover the following properties:

1. *The map ψ is a composition of Whitehead moves, a possibly empty set $\mathcal{S}$ of non-injective immersive graph-maps, and homeomorphisms.*
2. *The set $\mathcal{S}$ above is empty if and only if the cocycle $u \in C^1(K;\mathbf{Z})$ considered is a positive cocycle.*

Proof of lemma 5.21: One considers any non-degenerate r-embedded leaf Γ of $\mathcal{F}_u$. By definition, if $\{l_1, \cdots, l_n\}$ are the positive loops of $K^{(1)}_{sing}$ with $u(l_i) = 0$, then a u-foliation has exactly n specially embedded leaves $\Gamma_1, \cdots, \Gamma_n$ which are not r-embedded. Each such leaf Γ_i contains the loop l_i, $i = 1, \cdots, n$. Lemma 5.15 implies the existence of a finite number of non-degenerate r-embedded leaves $\mathcal{L}_1, \cdots, \mathcal{L}_r$ whose union with these specially embedded leaves satisfies the following property:

The cocycle associated to Γ defines a continuous map $\psi : \Gamma \to \Gamma$ which is a composition of

1. continuous maps $\sigma_i : \mathcal{L}_i \to \mathcal{L}_{i+1}$ induced by the Whitehead moves at the crossings of the singular graph,
2. homeomorphisms $h_i : \mathcal{L}_i \to \mathcal{L}_{i+1}$ when both leaves $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ represent the same cocycle,
3. continuous maps $g_{ij} : \mathcal{L}_i \to \Gamma_j$ and $g'_{ji} : \Gamma_j \to \mathcal{L}_i$,

for some i, j respectively in $\{1, \cdots, r\}$ and $\{1, \cdots, n\}$.

Let us check this assertion. Item (1) is the case where the cocycles associated to $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ (see lemma 2.13) are obtained one from the other by a non-negative δ_v-move.

Item (2) is clear. It remains to check item (3). By lemma 5.16, the neighborhood $N(\Gamma_j)$ of any non-regular leaf Γ_j of $\mathcal{F}_u$ is such that $N(\Gamma_j) - \Gamma_j = \mathcal{L}_i \times [-1, 0[\cup \mathcal{L}_{i+1} \times]0, 1]$ for some index $i \in \{1, \cdots, r-1\}$ and a well-chosen numerotation.

One defines a continuous map $g_{ij} : \mathcal{L}_i \to \Gamma_j$ in the following way:

For any point $x \in \mathcal{L}_i \times \{-1\}$, $g_{ij}(x)$ is the point of Γ_j which belongs to the closure in K of the interval $\{x\} \times [-1, 0[$.

One now proves that the maps g_{ij} above can be chosen to satisfy the announced properties. From lemma 5.16, the map g_{ij} is such that:

1. The image under g_{ij} of any crossing of $\mathcal{L}_i$ is a crossing of Γ_j (item (4) of lemma 5.16).
2. The image under g_{ij} of any edge of $\mathcal{L}_i$ is a non-empty path in the graph Γ_j (item (3) of lemma 5.16).
3. The images under g_{ij} of any two distinct germs of edges of $\mathcal{L}_i$ at any of its crossings v are two distinct germs of edges of Γ_j at the crossing $w = g_{ij}(v)$ (consequence of item (3), lemma 5.16).

Moreover, by definition, the map g_{ij} is surjective. Thus, the map g_{ij} is an immersive map. Let us prove that g_{ij} is non-injective.

Let us consider any leaf Γ_j. By definition, it contains the loop l_j of the singular graph. There are three germs of 2-cells incident to each edge in this loop. Since it is 2-sided in K, two of this germs are on a same side of l_j (see figure 4). Thus, the map g_{ij} above is 2-to-1 over l_j.

Let us now define $g''_{i+1j} : \mathcal{L}_{i+1} \to \Gamma_j$ by declaring that, for any point $x \in \mathcal{L}_{i+1}$, $g''_{i+1j}(x)$ is the point in the closure in K of the interval $\{x\} \times]0, 1]$.

In the same way than for the maps g_{ij}, lemma 5.16 implies that the maps g''_{i+1j} are immersive maps. By item (3) of lemma 5.16, the closure of $e \times]0, 1]$, for any edge e of Γ_{u+}, is an edge of Γ. This implies that these maps g''_{i+1j} are homeomorphisms. Thus, by setting $g'_{ji+1} = g''^{-1}_{i+1j}$, one gets the desired maps.

By cutting K along Γ, one obtains a complex K' homotopically equivalent to $\Gamma \times [0, 1]$. The complex K is obtained from K' by gluing $\Gamma \times \{1\}$ with $\Gamma \times \{0\}$ by the map ψ. Thus, K is homotopically equivalent to $Susp_\psi(\Gamma)$.

One easily generalizes the above proof to the case where Γ is any leaf of $\mathcal{F}_u$.

Since the collection of specially embedded leaves $\Gamma_1, \cdots, \Gamma_n$ is empty if and only if the cocycle u considered is positive, this completes the proof of lemma 5.21. $\square$

Let us now complete the proof of proposition 5.18. We need first a lemma whose proof can be found in [G1,2]. It is a more or less straightforward consequence of [Sta], and an analog, in the context of graphs, to an old result of Whitehead or Nielsen on free groups (see [Wh1,2], [Ni]).

Lemma 5.22. *Let $f: \Gamma \to \Gamma$ be a continuous map of a graph Γ.*

If f is an immersive map, then f induces an injective endomorphism on the fundamental group of Γ. This endomorphism is surjective if and only if f is a homeomorphism.

In the case where u is a positive cocycle, item (2) of proposition 5.18 follows from theorem 3.1. Assume now that u is not a positive cocycle. Then there are specially embedded leaves Γ_j containing loops l_j of the singular graph. Lemmas 5.21 and 5.22 imply that K is homotopically equivalent to a complex $Susp_\psi(\Gamma)$, where Γ is any leaf of $\mathcal{F}_u$ and ψ induces an injective, non surjective endomorphism on the fundamental group of the leaf. This completes the proof of item (2) of proposition 5.18.

Theorem 3.1 proves one implication of item (3) of proposition 5.18. Let us prove the other implication. Assume that there is a map ψ of a leaf Γ of $\mathcal{F}_u$ which induces a surjective endomorphism, and thus an automorphism, on the fundamental group of Γ and such that K is homotopically equivalent to $Susp_\psi(\Gamma)$. Lemmas 5.21 and 5.22 imply that u is a positive cocycle. This completes the proof of proposition 5.18.

6 Remarks

6.1 Connectedness of the leaves

From our assumption on the connectedness of the complexes considered in this paper, when one is given a leaf of a foliation, this leaf is assumed to be connected. However, when one is given a positive cocycle u of a dynamical n-complex K ($n \geq 2$), an associated r-embedded complex K_u (see lemma 2.13) is not necessarily connected. The fact that u defines a regular foliation with compact leaves of K allows to prove that a positive cocycle u defines a connected r-embedded complex K_u if and only if the cohomology class of u is *indivisible*, that is there exists a loop l in K with $u(l) = 1$.

As in the case of positive cocycles, one easily shows that a nice non-negative integer cocycle defines a connected r-embedded complex if and only if its cohomology-class is indivisible.

6.2 Mapping-tori of surface homeomorphisms

In the context of spines of 3-manifolds, one can prove the following topological analog to corollary 3.14 of proposition 3.13:

Any compact 3-manifold with boundary M^3 which admits a fibration f over the circle admits a dynamical 2-spine K with a positive cocycle $u \in C^1(K; \mathbf{Z})$ such that $i_\#([u])$ is in the cohomology class of $H^1(M^3; \mathbf{Z})$ defined by f, where $[u]$ denotes the cohomology-class of u in $H^1(K; \mathbf{Z})$ and $i_\# : H^1(K; \mathbf{Z}) \to H^1(M^3; \mathbf{Z})$ the isomorphism induced by the inclusion i of K in M^3.

Indeed, any such 3-manifold is the mapping-torus of a homeomorphism h of a compact surface with boundary S. Up to isotopy, this homeomorphism can be given by a composition of Whitehead moves and of a homeomorphism ϕ applied on a spine Γ of S, in such a way that each Whitehead move and the homeomorphism ϕ preserve the embedding in S. From such a decomposition, one easily checks that proposition 3.13 gives a dynamical 2-spine K of M^3. The properties of K listed in proposition 3.13 assure that K is as announced above.

6.3 Effectivity of the cocycle-criteria

The criteria given by theorem 3.1 and proposition 5.6 for the existence of a foliation with compact leaves of a simple n-complex are effective, that is one can easily check, by hand or by a computer, if a given simple n-complex admits a positive cocycle. Let us briefly describe the process for the search of positive cocycles. Assume that you are given some simple 2-complex K. Check if the edges of its singular graph admit an orientation which makes it a dynamical 2-complex. Since the singular graph is finite, this is a finite process. In a second step, consider the integral matrix whose lines are the images of the 2-components of the complex by the second boundary-operator, and the columns are the edges of the singular graph. Supress from this matrix the lines corresponding to the Moebius-band components. Let us denote by M_K the resulting matrix. Search for the non-negative integer solutions to the system $M_K X = 0$. A classical result asserts that these non-negative solutions are generated by a finite number of them, i.e. there are n such solutions $S_1, \cdots, S_n$ of this system such that, if S is any non-negative integer solution, then $S = \sum_{i=1}^{n} \lambda_i S_i$, $\lambda_i \geq 0$. Each solution S_i does not necessarily define a cocycle of the 2-complex K, but, in any case, one easily shows that $2 * S_i$ will define a cocycle. Thus, for testing if there is a positive cocycle, it suffices to consider the sum $2 * \sum_{i=1}^{n} S_i$. There is a positive cocycle in $C^1(K; \mathbf{Z})$ if and only if this sum is one.

In the case where K is n-dimensional ($n \geq 2$), one has to check moreover whether the solutions $2 * S_i$ define cocycles of the complex. This is handled by an easy computation, choosing first a structure of CW-complex for K and then considering the matrix of the second boundary-operator for this structure. Each solution $2 * S_i$ defines integer weights on some edges of the 1-skeleton of this structure. One has just to check if some integer weights can be defined on the other edges to be in the kernel of the above matrix.

The search of nice non-negative integer cocycles is done in the same way.

Let us observe that all what precedes implies in particular that, on a given simple dynamical n-complex K, there might be an infinite number of indivisible non-negative cohomology classes (it is necessary that $rk(H_1(K; \mathbf{Z})) > 1$).

Examples

Example 1: In this example, we assume given a trivalent graph with two vertices, together with a cyclic ordering at each of these vertices. We consider a sequence of two Whitehead moves from this graph to an homeomorphic one. Moreover, the two Whitehead moves and the homeomorphism preserve the above cyclic orderings. Up to homeomorphism, there is a unique orientable compact surface with boundary which admits this graph, *equipped with these cyclic orderings at its vertices*, as a spine. We leave the reader check that this surface S is the torus with one boundary component. The above sequence of Whitehead moves and homeomorphism defines a continuous map of the graph, induced by a homeomorphism h of S. In fact, this homeomorphism is induced by the classical automorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ of the torus.

We apply proposition 3.13 for constructing a suspended dynamical 2-complex of the induced automorphism on the fundamental group. Since the

Whitehead moves and the homeomorphism preserve the cyclic orderings of the edges at the vertices, one gets a simple dynamical 2-spine of the 3-manifold which is the suspension of the homeomorphism h of S. This manifold is the complement in $\mathbf{S}^3$ of the figure eight-knot.

Let us describe this construction. The edges of the graph are labelled with 1, 2 and 3. When one edge is collapsed by a Whitehead move, the symbol attached to the new edge created is the same than the original one, with a prime. The homeomorphism α is defined by $\alpha(2') = 3$, $\alpha(3') = 1$ and $\alpha(1) = 2$.

Since there are two Whitehead moves, the singular graph of the suspended dynamical 2-complex is a 4-valent graph with two crossings.

The second picture in figure 6 illustrates what happens to each edge along the process, the graph Γ having been cut at its vertices. It allows to find the 2-components of the complex, together with a decomposition of their boundary in 1-simplices.

Since the edge 1 is collapsed by no Whitehead move, it gives rise to a rectangle. The two other edges are collapsed. Thus, each of them gives rise to two triangles: these are the cones over the edges 2, $2' = \alpha^{-1}(3)$ and 3, $3' = \alpha^{-1}(1)$. The vertex common to the two triangles with bases 2 and $2'$ is a crossing of the complex, and the same is true for the vertex common to the two triangles with bases 3 and $3'$. For obtaining the desired suspended 2-complex, one identifies the top and the bottom of two "bands" when this top and bottom carry the same letters.

The suspended 2-complex has two 2-components. The boundary of each 2-cell inherits a decomposition in 1-simplices which are copies of the edges of the singular graph along which this 2-cell is attached. For finding this decomposition of the boundary of each 2-cell, let us notice that one took care of not putting the two crossings at the same level in this second picture of figure 6. Since all the edges of Γ are incident to both vertices of Γ, it suffices then to subdivide the vertical boundaries of the "bands" at the level of the crossings to obtain the desired decomposition of the boundary of the 2-cells in 1-simplices. These 1-simplices, copies of the edges of the singular graph, are oriented from top to bottom in this picture.

The dynamical 2-complex K is showed in the third picture of figure 6. We also drawed a r-embedding of our original graph Γ into K. The interested reader can easily check that the associated cocycle is a positive one. Let us notice that K admits a compatible structure of dynamic branched surface (along its singular graph) (see remark 5.12).

This dynamic branched surface first appeared in [Ch1]. It is dual to the decomposition in two tetrahedra of the complement in $\mathbf{S}^3$ of the figure-eight knot given by Thurston in its notes (see [Th1]).

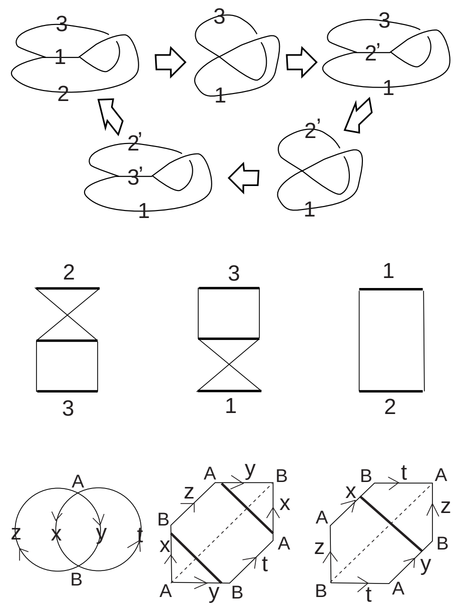

Figure 6: Example 1

Example 2: Figure 7 presents a simple dynamical 2-complex which admits a compatible structure of dynamic branched surface (see remark 5.12). This complex admits a positive cocycle and we show a r-embedded graph associated to such a cocycle. One easily checks, using criterions of embeddability of Christy or Benedetti-Petronio (see [Ch2], [BP]), that this complex does not embed in any compact 3-manifold.

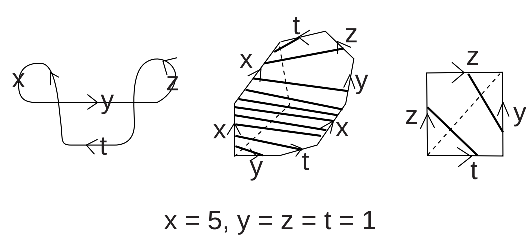

Figure 7: Example 2

Example 3: Figure 8 gives a dynamical 2-complex which does not admit any positive cocycle. There are only non negative cocycles which are not positive. For instance, the solution $X_2 = 2$, $X_7 = 2$, $X_4 = X_9 = 1$ gives such a cocycle. It remains, in the complement of the edges with positive weight, the positive loop $X_3X_{10}X_5$. As the example 2, this 2-complex does not admit any embedding in a compact 3-manifold.

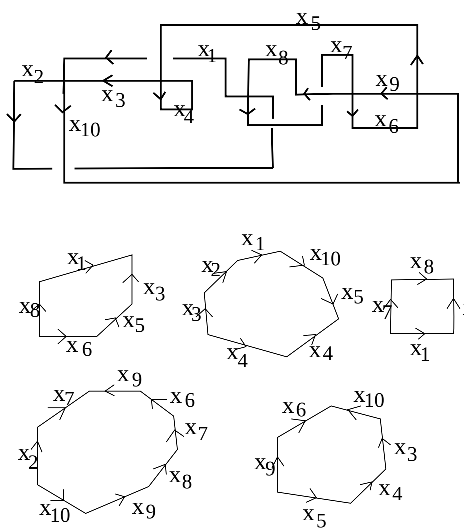

Figure 8: Example 3

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