Cellular approximations using Moore spaces

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Abstract

For a two-dimensional Moore space M with fundamental group G, we identify the effect of the cellularization CW_M and the fiber $\overline{P}_M$ of the nullification on an Eilenberg–Mac Lane space $K(N,1)$, where N is any group: both induce on the fundamental group a group theoretical analogue, which can also be described in terms of certain universal extensions. We characterize completely M-cellular and M-acyclic spaces, in the case when $M = M(\mathbf{Z}/p^k, 1)$.

0 Introduction

Let M be a pointed connected CW-complex. The nullification functor P_M and the cellularization functor CW_M have been carefully studied in the last few years (see e.g. [8], [17], [18], [14]). These are generalizations of Postnikov sections and connective covers, where the role of spheres is replaced by a connected CW-complex M and its suspensions. This list of functors also includes plus-constructions and acyclic functors associated with a homology theory, for which M is a universal acyclic space ([2], [13], [21], [23]). Recall that a connected space X is called *M-cellular* if $CW_M X \simeq X$, or, equivalently, if it belongs to the smallest class $\mathcal{C}(M)$ of spaces which contains M and is closed under homotopy equivalences and pointed homotopy colimits. Analogously, X is called *M-acyclic* if $P_M X \simeq *$ or, equivalently, $\overline{P}_M X \simeq X$. It was shown in [14] that the class of M-acyclic spaces is the smallest class $\overline{\mathcal{C}(M)}$ of spaces which contains M and is closed under homotopy equivalences, pointed homotopy colimits, and extensions by fibrations.

Very interesting examples are given by the family of Moore spaces $M(\mathbf{Z}/p, n)$, the homotopy cofiber of the degree p self-map of S^n. For $n \geq 2$, these spaces are the "building blocks" for simply-connected p-torsion spaces.

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More precisely, it is shown in [3] (see also [11]) that the $M(\mathbf{Z}/p,n)$-cellular spaces are exactly the $(n-1)$-connected spaces X such that $p\cdot\pi_n X=0$ and $\pi_k X$ is p-torsion for $k>n$. However the methods used in those papers can not handle the case $n=1$. In this paper we introduce the group theoretical tools that are necessary to deal with this case. They apply to the more general situation when M is a two-dimensional CW-complex with fundamental group G. As we will see in Proposition 2.10 and in the introduction of Section 3, the interesting phenomena occur when $H_2(M;\mathbf{Z})=0$. In that case we say that M is a Moore space of type $M(G,1)$, and we shortly write $M=M(G,1)$.

The *G-socle* of a group N, which we denote by $S_G N$, is the subgroup of N generated by the images of all homomorphisms from G into N. We introduce the class $\mathcal{C}(G)$ for any group G. It is the smallest class of groups containing G which is closed under isomorphisms and colimits. We construct explicitely the right adjoint C_G to the inclusion of $\mathcal{C}(G)$ in the category of groups and show the following (see also Theorem 2.3).

Theorem 2.9 *Let M be a two dimensional CW-complex with fundamental group G. Let $X=K(N,1)$ where N is any group. Then we have a natural isomorphism*

$$\pi_1(CW_M X)\cong C_G N.$$

Moreover, the action of $C_G N$ on the higher homotopy groups of $CW_M X$ is trivial.

We further prove the existence of a central extension

$$0\to A\to C_G N\to S_G N\to 1$$

which is universal in the sense explained in Theorem 2.7.

The proof of such results uses a description of Chachólski, exhibiting $CW_M X$ as the fibre of a map $X\to LX$, where LX is obtained from X by first killing all maps from M, and then applying ΣM-nullification.

This leads us, in the case when $G=\mathbf{Z}/p$, to the following result. We must note that our proof is also valid for $M(\mathbf{Z}/p,n)$ with $n\geq 2$, cases which were previously dealt with in [3] or [11].

Theorem 6.2 *Let $M=M(\mathbf{Z}/p,1)$ be the cofiber of the degree p self-map of S^1 and X be a connected space. Then X is M-cellular if and only if $\pi_1 X$ is generated by elements of order p and $H_n(X;\mathbf{Z})$ is p-torsion for $n\geq 2$.*

In particular, a nilpotent space X is $M(\mathbf{Z}/p,1)$-cellular if and only if $\pi_1 X$ is generated by elements of order p and $\pi_n(X)$ is p-torsion for $n\geq 2$.

Of course, the homotopy groups of non-nilpotent M-cellular spaces need not be p-groups. For instance, the universal cover of $M(\mathbf{Z}/2,1)$ is S^2. Likewise, a space all whose homotopy groups are p-groups need not be M-cellular, as shown by Example 6.4, where we compute the $M(\mathbf{Z}/2,1)$-cellularization of $K(\Sigma_3,1)$.

Nullifications with respect to Moore spaces are better understood. Our aim here is to investigate the homotopy fibre of such nullifications.

Recall that the *G-radical* of a group N, which we denote by $T_G N$ as in [10] or [8], is the smallest subgroup of N such that $\mathrm{Hom}(G, N/T_G N) = 0$. It is known [10] that when M is a two dimensional CW-complex with fundamental group G, then $\pi_1 P_M X \cong \pi_1 X / T_G(\pi_1 X)$. If in addition M is a $M(G,1)$, the space $P_M X$ can be viewed as the fibrewise R-completion, in the sense of Bousfield and Kan [9], of a covering fibration associated to the G-radical subgroup (for a suitable coefficient ring R); see [9], [10], and [12].

We introduce the class $\overline{\mathcal{C}(G)}$, for any group G. It is the smallest class of groups containing G which is closed under isomorphisms, colimits, and extensions. We show in Proposition 3.11 that the fundamental group of any M-acyclic space belongs to this class. We define then $D_G N$ as the fundamental group of $\overline{P}_M K(N,1)$, note that the action of $D_G N$ on the higher homotopy groups of $\overline{P}_M K(N,1)$ is trivial, and prove:

Theorem 3.13 *Let $M = M(G,1)$ be a two-dimensional Moore space. Then D_G is right adjoint to the inclusion of $\overline{\mathcal{C}(G)}$ in the category of groups.*

As in the case of cellularization, there exists a central extension

$$0 \to B \to D_G N \to T_G N \to 1 \tag{0.1}$$

which is universal in the sense explained in Theorem 3.3.

A very enlightening example is given by the acyclic space described by Berrick and Casacuberta in [2, Example 5.3], which turns out to be an $M(G,1)$ for some acyclic group G. In this case $M(G,1)$-nullification is equivalent to Quillen's plus-construction and the G-radical of any group N is its largest perfect subgroup. Thus, in this case, the central extension (0.1) is the usual universal central extension of $T_G N$.

Similar results have been obtained by Mislin and Peschke in [21] in the case when P_M is the plus construction associated to a generalized homology theory. In all these cases CW_M and $\overline{P}_M$ coincide.

Finally, we obtain the following result for $G = \mathbf{Z}/p$ (compare with [3] or [11]).

Theorem 6.1 *Let $M = M(\mathbf{Z}/p, 1)$. Then X is M-acyclic if and only if $\pi_1 X$ coincides with its $\mathbf{Z}/p$-radical and $H_n(X;\mathbf{Z})$ is p-torsion for $n \geq 2$.*

In particular, a nilpotent space X is $M(\mathbf{Z}/p, 1)$-acyclic if and only if $\pi_1 X$ coincides with its $\mathbf{Z}/p$-radical and $\pi_n(X)$ is p-torsion for $n \geq 2$.

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1 Preliminary results

We give here a short review of the terminology involved in the theory of homotopical localization. We also remind the reader some of the results needed in this paper. More details can be found in [17], [18], [14], [7].

By an idempotent augmented functor in the category of spaces, we mean a functor E from the category of pointed spaces to itself. It preserves weak homotopy equivalences and is equipped with a natural transformation $c : E \to \mathrm{Id}$ from E to the identity functor, such that $c_{EX} : EEX \simeq EX$ for all spaces X.

The most important examples for us are CW_M and $\overline{P}_M$. Let M be a connected CW-complex. A map $Y \to X$ is an *M-cellular equivalence* if it induces a weak equivalence on pointed mapping spaces

$$\mathrm{map}_*(M, Y) \xrightarrow{\sim} \mathrm{map}_*(M, X)\,.$$

There exists then, for each connected space X, a map $CW_M X \to X$, called *M-cellular approximation*, which is universal (initial) among all M-cellular equivalences to X. The spaces for which $CW_M X \simeq X$ are called *M-cellular*. The class of M-cellular spaces has been identified as the smallest class $\mathcal{C}(M)$ of spaces containing M and closed under weak equivalences and pointed homotopy colimits; see [14, Theorem 8.2] and [18, 2.D]. A connected space Z is said to be *M-null* if $\mathrm{map}_*(M, Z)$ is weakly contractible, i.e., $[\Sigma^k M, X] = *$ for all $k \geq 0$. There exists a map $X \to P_M X$, called *M-nullification*, which is universal (terminal) among all maps from X to an M-null space. Finally denote by $\overline{P}_M X \to X$ the homotopy fibre of $X \to P_M X$. A connected space X is called *M-acyclic* if $P_M X \simeq *$, i.e., $\overline{P}_M X \simeq X$. The class of M-acyclic spaces has been identified in [14, Theorem 17.3] as the class

$\overline{\mathcal{C}(M)}$. In addition to being closed under weak equivalences and pointed homotopy colimits, it is also closed under extensions by fibrations.

So, every M-cellular space is M-acyclic and furthermore, it is known that each ΣM-acyclic space is M-cellular ([18, 3.B.3]). Hence by universality we have natural maps

$$\overline{P}_{\Sigma M}X \xrightarrow{\alpha} CW_M X \xrightarrow{\beta} \overline{P}_M X. \tag{1.2}$$

Thus $CW_M X$ can be thought of as the fiber of a mixing process between M- and ΣM-nullification. More precisely:

Theorem 1.1. ([14, Theorem 20.5]) *Let M be any connected CW-complex. There is a fibration*

$$CW_M X \longrightarrow X \xrightarrow{\eta} LX$$

where η is the composition of the inclusion $X \to X'$ of X into the homotopy cofibre of the evaluation map $\vee_{[M,X]} M \to X$, followed by $X' \to P_{\Sigma M}(X')$.

Note that the inclusion $X \to X'$ is in fact functorial in the homotopy category, and it is universal (initial) among all maps $X \to Z$ such that $M \to X \to Z$ is homotopically trivial (compare with [2, Corollary 2.2]). Hence, $X \to LX$ is also functorial in the homotopy category.

The fundamental groups of $P_M X$ and LX have a group theoretical meaning in the case when M is a two-dimensional CW-complex, as we next explain. The *G-socle* of a group N, which we denote by $S_G N$, is the subgroup of N generated by the images of all homomorphisms from G into N. The *G-radical* of N, which we denote by $T_G N$ as in [10] or [8], is the smallest subgroup of N such that $\mathrm{Hom}(G, N/T_G N) = 0$. The G-radical of N can be constructed as a (possibly transfinite) direct limit of subgroups T_i where T_1 is the G-socle of N and $T_i/T_{i-1} = S_G(N/T_{i-1})$. In other words, the groups N such that $T_G N = N$ are precisely the groups which have a normal series whose factors coincide with their G-socles. The first link between the topological nullification and their discrete analogues is given by the following result. Its first part is [10, Theorem 3.5] (see also [6, Theorem 5.2]).

Lemma 1.2. *If M is a two-dimensional CW-complex with fundamental group G and X is any space, we have isomorphisms*

$$\pi_1(P_M X) \cong \pi_1 X / T_G(\pi_1 X) \quad \text{and} \quad \pi_1 X' \cong \pi_1 LX \cong \pi_1 X / S_G(\pi_1 X),$$

where X' and LX are as in Theorem 1.1. □

Such a two-dimensional CW-complex M is called a Moore space if $H_2(M;\mathbf{Z}) = 0$. It has type $M(G,1)$ if $\pi_1 M \cong G$. Using Theorem 1.1 it is easy to see that two Moore spaces of type $M(G,1)$ determine the same cellularization and nullification functors.

Bousfield computed in [8, Section 7] the effect of the nullification functor with respect to a two-dimensional Moore space $M(G,1)$ on nilpotent spaces; see also [10, Theorem 4.4] for $G = \mathbf{Z}/p$, and [12, Theorem 2.4] when $G = \mathbf{Z}[1/p]$. Let J be the set of primes p for which G_{ab} is uniquely p-divisible. Define $R = \mathbf{Z}_{(J)}$, the integers localized at J, if G_{ab} is torsion, and $R = \oplus_{p\in J}\mathbf{Z}/p$ otherwise. Let $R_\infty X$ be the Bousfield-Kan R-completion of X; see [9]. Then $P_M X$ can be obtained as the fibrewise R-completion of the covering fibration associated to the G-radical of $\pi_1 X$. That is, we have a diagram of fibrations

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & X & \longrightarrow & K(\pi_1 X/T_G(\pi_1 X),1) \\ \downarrow & & \downarrow & & \downarrow\mathrm{id} \\ R_\infty \tilde{X} & \longrightarrow & P_M X & \longrightarrow & K(\pi_1 X/T_G(\pi_1 X),1), \end{array} \tag{1.3}$$

where $R_\infty \tilde{X}$ is simply connected, as the fundamental group of $\tilde{X}$ is R-perfect, i.e., $H_1(\tilde{X};R) = H_1(T_G(\pi_1 X);R) = 0$. Hence $R_\infty \tilde{X}$ coincides with $\tilde{X}^+_{HR}$, the plus-construction with respect to ordinary homology with coefficients in R; see [9, VII.6] and [12]. The universal cover of $P_M X$ is thus equivalent to the three following spaces:

$$P_M \tilde{X} \simeq R_\infty \tilde{X} \simeq \tilde{X}^+_{HR}.$$

Let $A_R X$ denote the *R-acyclic functor*, that is the homotopy fiber of $X \to X^+_{HR}$ (cf. [9, VII, 6.7]). Then, a connected space X is HR-acyclic, i.e., $\tilde{H}_*(X;R) = 0$, if and only if $A_R X \simeq X$. The above remark immediately implies the following.

Proposition 1.3. *Let $M = M(G,1)$ be a two-dimensional Moore space, and X be any connected space. If $\tilde{X}$ denotes the covering of X corresponding to the subgroup $T_G(\pi_1 X)$, then*

$$\overline{P}_M X \simeq A_R \tilde{X} \simeq \overline{P}_M \tilde{X},$$

where R is the ring associated to G as above. □

Corollary 1.4. *Let $M = M(G,1)$ be a two-dimensional Moore space, and X be any connected space. Then, X is M-acyclic if and only if*

$$T_G(\pi_1 X) = \pi_1 X \quad \textit{and} \quad H_k(X;R) = 0 \quad \textit{for} \quad k \geq 2\,. \qquad \square$$

2 The fundamental group of $M(G,1)$-cellular spaces

In this section we define algebraically a G-cellularization functor C_G in the category of groups. We show that $C_G N$ coincides with the fundamental group of the $M(G,1)$-cellularization of $K(N,1)$. This yields a characterization of $C_G N$ as a certain universal central extension of the G-socle of N. We also prove that the action of $C_G N$ is trivial on the higher homotopy groups of $CW_{M(G,1)}K(N,1)$.

As suggested by Dror-Farjoun, we introduce the closed class of groups $\mathcal{C}(G)$. It is the smallest class of groups containing G, and closed under isomorphisms and taking colimits. In other words, if $F : I \to \mathit{Groups}$ is a diagram with $F(i) \in \mathcal{C}(G)$ for any $i \in I$, then $\mathrm{colim}_I F$ should again belong to $\mathcal{C}(G)$.

The following proposition gives the explicit construction of a G-cellularization functor. The existence of such a functor is also ensured by [5, Corollary 7.5].

Proposition 2.1. *Let G be a group. The inclusion $\mathcal{C}(G) \subset \mathit{Groups}$ has a right adjoint $C_G : \mathit{Groups} \to \mathcal{C}(G)$.*

Proof. For any group N, the map $C_G N \to N$ is constructed by induction as follows. First define $C_0 = *_{h:G\to N} G$, the free product of as many copies of G as there are morphisms from G to N, and let $h_0 : C_0 \to N$ be the evaluation (so that $h_0(C_0) = S_G N$). Now take the free product $*_{(h',h'')}G$, where $h', h'' : G \to C_0$ is any pair of morphisms coequalized by h_0. Define C_1 as the coequalizer of $*_{(h',h'')}G \rightrightarrows C_0$, and repeat this process (maybe transfinitely). Notice that this inductive construction of $C_G N$ shows that we have a natural epimorphism $C_G N \twoheadrightarrow S_G N$ for any group N. The group $C_G N$ is in $\mathcal{C}(G)$ and the morphism $c : C_G N \to N$ is universal (terminal), i.e., c induces a bijection of sets $\mathrm{Hom}(G,c) : \mathrm{Hom}(G, C_G N) \cong \mathrm{Hom}(G,N)$. □

By analogy to the case of spaces, a group N in $\mathcal{C}(G)$ is called *G-cellular.*

Lemma 2.2. *Let M be a two-dimensional CW-complex with fundamental group G. A group homomorphism $N \to N'$ induces an isomorphism $C_G N \cong C_G N'$ if and only if $CW_M K(N,1) \simeq CW_M K(N',1)$.*

Proof. The pointed mapping space $\mathrm{map}_*(M, K(N,1))$ is weakly equivalent to the discrete set $\mathrm{Hom}(G,N)$. □

Since π_1 commutes with homotopy colimits, the fundamental group of any M-cellular space is $\pi_1 M$-cellular, for any M. Furthermore, the following holds (this could also have been taken as definition of C_G):

Theorem 2.3. *Let M be a two-dimensional CW-complex with fundamental group G. Let $X = K(N,1)$ where N is any group. Then we have a natural isomorphism*

$$\pi_1(CW_M X) \cong C_G N.$$

Proof. By the previous observation, $\pi_1(CW_M X)$ is G-cellular. It thus only remains to prove that $c : CW_M X \to X$ induces a bijection of sets $\pi_1(c)_* = \mathrm{Hom}(G, \pi_1(c))$. Consider the following commutative diagram (of sets)

$$\begin{array}{ccc} [M, CW_M X] & \xrightarrow{e} & \mathrm{Hom}(G, \pi_1(CW_M X)) \\ \downarrow_{c_*} & & \downarrow_{\pi_1(c)_*} \\ [M, X] & \xrightarrow{e'} & \mathrm{Hom}(G, N). \end{array}$$

Since M is two-dimensional, e is surjective and X being a $K(N,1)$, e' is bijective. On the other hand, c_* is also bijective by the universal property of CW_M. Thus, $\pi_1(c)_*$ is bijective, as desired. □

Lemma 2.4. *Let $\tilde{X}$ denote the covering of X corresponding to the subgroup $S_G(\pi_1 X)$ and let M be a two-dimensional CW-complex with fundamental group G. Then*

(i) $(\tilde{X})'$ is the universal cover of X',

(ii) $L\tilde{X}$ is the universal cover of LX,

where X' and LX are defined in Theorem 1.1.

Proof. The fibration $\tilde{X} \to X \xrightarrow{p} K(\pi_1 X / S_G(\pi_1 X), 1)$ induces a bijection $[M, \tilde{X}] \cong [M, X]$ because the map $[M, p]$ is trivial. Apply now Mather-Puppe theorem (see [16, Proposition 6.1]) saying that "the fiber of the push-out is the push-out of the fibers" when the base space is fixed, to get a fibration $\tilde{X}' \to X' \to K(\pi_1 X / S_G(\pi_1 X), 1)$. This is the universal cover fibration by Lemma 1.2. So part (i) holds. For part (ii) we note that the previous fibration has a ΣM-null base, so it is preserved under ΣM-nullification. □

The following corollary could also have been proved directly by checking that the covering $\tilde{X} \to X$ is indeed an M-equivalence.

Corollary 2.5. *Let $\tilde{X}$ denote the covering of X corresponding to the subgroup $S_G(\pi_1 X)$ and let M be a two-dimensional CW-complex with fundamental group G. We have a homotopy equivalence $CW_M X \simeq CW_M \tilde{X}$.* □

Lemma 2.6. *Let G be any group, and $0 \to A \to E \to N \to 1$ be a central extension of groups. Then $C_G E \cong C_G N$ if and only if* $\mathrm{Hom}(G_{\mathrm{ab}}, A) = 0$ *and the natural map* $\mathrm{Hom}(G, N) \longrightarrow H^2(G; A)$ *is trivial.*

Proof. Apply $\mathrm{map}_*(K(G,1), -)$ to the fibration

$$K(E,1) \to K(N,1) \to K(A,2)\,.$$

This gives a new fibration, whose homotopy sequence

$$0 \to \mathrm{Hom}(G,A) \to \mathrm{Hom}(G,E) \to \mathrm{Hom}(G,N) \to H^2(G;A)$$

is exact as in [9, IX, 4.1]). The lemma is proved. □

We now know enough to describe our first universal central extension. It is nothing but a universal central G-cellular equivalence.

Theorem 2.7. *Let G be any group. Then, for each group N, there is a central extension*

$$0 \to A \to C_G N \to S_G N \to 1$$

such that $\mathrm{Hom}(G_{\mathrm{ab}}, A) = 0$ *and the natural map* $\mathrm{Hom}(G, N) \longrightarrow H^2(G; A)$ *is trivial. Moreover, this extension is universal with respect to these two properties.*

Proof. Let M be a two-dimensional CW-complex with fundamental group G and let $X = K(S_G N, 1)$. The space LX is 1-connected by Lemma 1.2 and define then

$$A = \pi_2 LX \cong \pi_2 X' / T_{G_{\mathrm{ab}}}(\pi_2 X')$$

(this follows from [8, Theorem 7.5]). The long exact sequence in homotopy of the fibration $CW_M X \to X \to LX$ produces now the desired central extension, where we identify $C_G(S_G N)$ with $C_G N$. This can be deduced from Corollary 2.5. The G-cellularization of $C_G N \to S_G N$ is also an isomorphism since $\mathrm{Hom}(G, C_G N) \cong \mathrm{Hom}(C, S_G N)$. Hence $\mathrm{Hom}(G_{\mathrm{ab}}, A) = 0$ and the natural map $\mathrm{Hom}(G, N) \longrightarrow H^2(G; A)$ is trivial by Lemma 2.6. The universal property is a direct consequence of the same lemma. □

Corollary 2.8. *Let M be a two-dimensional CW-complex with fundamental group G. Then $S_G N = N$ if and only if $LK(N,1)$ is 1-connected, and $C_G N \cong N$ if and only if $LK(N,1)$ is 2-connected. In particular*

(1) $\pi_2 LK(N,1) \cong H_2 LK(S_G N, 1)$*;*

(2) $\pi_3 LK(N,1) \cong H_3 LK(C_G N, 1)$. □

In the next theorem we identify the universal cover of $CW_M K(N,1)$ and remark that the action of the fundamental group is trivial. This could also be seen as a particular case of [21, Corollary 7.7].

Theorem 2.9. *Let M be a two-dimensional CW-complex with fundamental group G. Then $CW_M K(N,1) \simeq CW_M K(C_G N,1)$ and the universal cover fibration is given by*

$$\Omega LK(C_G N,1) \longrightarrow CW_M K(N,1) \longrightarrow K(C_G N,1).$$

Moreover the action of $C_G N$ on $\pi_n CW_M K(N,1)$ for $n \geq 2$ is trivial.

Proof. The first part follows from Lemma 2.2. The proof of the second part has been suggested by Carles Casacuberta. The fibration

$$CW_M K(N,1) \to K(C_G N,1) \to LK(C_G N,1)$$

induces a long exact sequence of $C_G N$-modules in homotopy. But the space $LK(C_G N,1)$ is 2-connected by Corollary 2.8, so that the action of $C_G N$ on the higher homotopy groups of $CW_M K(N,1)$ is trivial. □

Proposition 2.10. *Let M be a two-dimensional CW-complex with fundamental group G. Assume that $H_2(M;\mathbf{Z}) \neq 0$. Then*

$$CW_M K(N,1) \simeq K(C_G N,1)\,.$$

Proof. Choose a presentation $\phi : *\mathbf{Z} \to *\mathbf{Z}$ of G, and realize it as a map f between wedges of circles having M as its homotopy cofiber. Note that a simply connected space Y is ΣM-null if and only if the G-radical of $\pi_2 Y$ is trivial, and $\pi_k Y$ is $\phi_{\rm ab}$-local for any $k \geq 3$, i.e., $\mathrm{Hom}(\phi_{\rm ab}, \pi_k Y)$ is bijective (see [22, Theorem 4.3.6]). When $H_2(M;\mathbf{Z}) \neq 0$, the homomorphism $\phi_{\rm ab}$ is not injective and any $\phi_{\rm ab}$-local group is trivial. So $LK(C_G N,1)$ is the trivial space, as it is already 2-connected (see [22, Corollary 4.3.9]). □

Example 2.11. If $G = C_2$ and N is nilpotent then $C_G N = S_G N$ (by Corollary 6.3 below). However, $C_G N \not\cong S_G N$ in general, as shown by the following example, which was suggested by Alejandro Adem:

$$N = \langle a,b,c,d \mid a^2 = b^2 = c^2 = d^2 = 1, abab = cdcd \rangle\,.$$

In other words N is the push-out of the diagram $C_2 * C_2 \leftarrow \mathbf{Z} \to C_2 * C_2$ where both arrows send the generator of $\mathbf{Z}$ to the commutator. The Mayer–Vietoris sequence shows then that $H_2 N \cong \mathbf{Z} \cong A$. Thus $C_G N$ is an extension of N by $\mathbf{Z}$. This also provides an example of a quotient of a free product of copies of G which is not cellular.

3 The fundamental group of $M(G,1)$-acyclic spaces

We imitate now the preceding section, replacing CW_M by $\overline{P}_M$. First we change our closed class. In addition to being closed under isomorphisms and colimits, the class $\overline{\mathcal{C}(G)}$ is assumed to be closed under taking arbitrary extensions. That is, if $N \hookrightarrow E \twoheadrightarrow Q$ is an extension with $N, Q \in \overline{\mathcal{C}(G)}$, then E belongs to $\overline{\mathcal{C}(G)}$ as well. The right adjoint of the inclusion of $\overline{\mathcal{C}(G)}$ in the category of groups is denoted by D_G and we construct it by topological means, namely as the fundamental group of $\overline{P}_{M(G,1)}K(N,1)$. It could be interesting to have an algebraic description of $D_G N$, similar to that of $C_G N$, in terms of colimits and extension by short exact sequences.

A well known topological proof of the existence of the universal central extension over a perfect group N uses Quillen's plus-construction. We will follow exactly the same line of proof here, the plus-construction being replaced by a nullification functor with respect to a Moore space. This is a true generalization of this old result in light of [2], where it is proven that the plus-construction is indeed the nullification with respect to a Moore space. Another approach is taken in [21], where the plus-construction associated to any homology theory determines a universal central extension.

When M is a two-dimensional CW-complex with fundamental group G which is not a Moore space, i.e., $H_2(M;\mathbf{Z}) \neq 0$, the effect of P_M is drastic. As in the proof of Proposition 2.10, one shows that

$$P_M X \simeq K(\pi_1 X / T_G(\pi_1 X), 1).$$

Hence $\overline{P}_M X$ is the covering of X corresponding to the subgroup $T_G(\pi_1 X)$. From now on, we will therefore only consider Moore spaces.

Define, for a two-dimensional Moore space $M = M(G,1)$,

$$D_G N := \pi_1(\overline{P}_M K(N,1)).$$

This does not depend on the choice of M by the observation made after Lemma 1.2.

Lemma 3.1. *Let M be a two-dimensional Moore space with fundamental group G and let $B = \pi_2 P_M K(T_G N, 1)$. The space $K(B,2)$ is M-null, i.e.* $\mathrm{Hom}(G_{\mathrm{ab}}, B) = 0 = \mathrm{Ext}(G_{\mathrm{ab}}, B)$.

Proof. The space $P_M K(T_G N, 1)$ is 1-connected by Lemma 1.2. The second Postnikov section $K(\pi_2 X, 2)$ of a simply connected M-null space X is M-null as well, since $\pi_2 P_M X$ only depends on $\pi_2 X$ by [8, Theorem 7.5]. □

The groups B satisfying $\mathrm{Hom}(G_{\mathrm{ab}}, B) = 0 = \mathrm{Ext}(G_{\mathrm{ab}}, B)$ can only be of the two following forms ([8, 7.1]):

Fact 3.2. Let J denote the set of primes p such that G_{ab} is uniquely p-divisible, and J' the complementary set of primes. Then, either G_{ab} is J'-torsion and B is J-local, or G_{ab} is uniquely J-divisible and B is Ext-J-complete (in the sense of [9]). In other words, $\mathrm{Hom}(G_{\mathrm{ab}}, B) = 0 = \mathrm{Ext}(G_{\mathrm{ab}}, B)$ if and only if $\mathrm{Hom}(H, B) = 0 = \mathrm{Ext}(H, B)$ where $H = \oplus_{p \in J'} \mathbf{Z}/p$ if G_{ab} is torsion, or $H = \mathbf{Z}[J^{-1}]$ otherwise.

We are now ready to prove the existence of our second universal central extension.

Theorem 3.3. *Let G be the fundamental group of a two-dimensional Moore space, which we denote by M. Then, for each group N, there is a central extension*

$$0 \to B \to D_G N \to T_G N \to 1$$

such that $\mathrm{Hom}(G_{\mathrm{ab}}, B) = 0 = \mathrm{Ext}(G_{\mathrm{ab}}, B)$. *Moreover, this extension is universal (initial) with respect to this property.*

Proof. The idea of the proof is analogous to that of Theorem 2.7. Let $X = K(T_G N, 1)$. The fibration $\overline{P}_M X \to X \to P_M X$ produces the desired extension using Proposition 1.3, where $B = \pi_2 P_M K(T_G N, 1)$ satisfies the property by Lemma 3.1.

We check now the universal property. Let B' be an abelian group having the above property, and $0 \to B' \to E \to T_G N \to 1$ a central extension. Realize it as a fibration $K(E, 1) \longrightarrow X \longrightarrow K(B', 2)$, where the base space is M-null. There exists therefore a map $P_M X \to K(B', 2)$, unique up to homotopy, inducing a map of fibrations. □

Example 3.4. Let $G = C_2$ and N be the group described in Example 2.11. Then $D_G N$ is an extension of N by $\mathbf{Z}[1/2]$.

Remark 3.5. The group $T_G N$ is R-perfect and the central extension of Theorem 3.3 is the universal central extension of $T_G N$ with coefficients in R. By this we mean the central extension induced from the fibration $A_R K(T_G N, 1) \to K(T_G N, 1) \to K(T_G N, 1)^+_{HR}$. These two extensions coincide by Proposition 1.3. Even though this "R universal central extension" seems to be classical, we do not know any other reference than [21].

Example 3.6. Let $G = \mathbf{Z}[1/p]$, so that $R = \mathbf{Z}/p$. Mislin and Peschke computed in [21, Proposition 5.4] that

$$B \cong \mathrm{Ext}(\mathbf{Z}(p^\infty), H_2(T_G N; \mathbf{Z})) \oplus \mathrm{Hom}(\mathbf{Z}(p^\infty), H_1(T_G N; \mathbf{Z})),$$

where $\mathbf{Z}(p^\infty)$ is the p-torsion subgroup of $\mathbf{Q}/\mathbf{Z}$. Let $G = \mathbf{Z}/p$, so that $R = Z[1/p]$. Then $B = H_2(T_G N; R)$.

An interesting consequence of the previous result is that the functor D_G is idempotent. It is worth noting that it seems rather difficult to prove this fact directly from the definition.

Theorem 3.7. *Let M be as above. Then $\overline{P}_M K(D_G N, 1) \simeq \overline{P}_M K(N, 1)$ and in particular the functor D_G is idempotent. The universal cover fibration is given by*

$$\Omega P_M K(D_G N, 1) \longrightarrow \overline{P}_M K(N, 1) \longrightarrow K(D_G N, 1).$$

Moreover the action of $D_G N$ on $\pi_n \overline{P}_M K(N, 1)$ for $n \geq 2$ is trivial.

Proof. The functor P_M preserves the fibration

$$K(D_G N, 1) \to K(T_G N, 1) \to K(B, 2)$$

of Theorem 3.3 since $K(B, 2)$ is M-null by Lemma 3.1. Thus so does the functor $\overline{P}_M$. That is, we have $\overline{P}_M K(D_G N, 1) \simeq \overline{P}_M K(T_G N, 1)$. The later space is equivalent to $\overline{P}_M K(N, 1)$ by Proposition 1.3. The statements about the universal cover follow as in Theorem 2.9. □

Remember that the ring R is determined by the group G as follows: $R = \mathbf{Z}_{(J)}$ if $G_{\rm ab}$ is torsion, and $R = \oplus_{p \in J} \mathbf{Z}/p$ otherwise. We say that a group N is *super R-perfect* if $H_1(N; R) = 0 = H_2(N; R)$.

Proposition 3.8. *Let G be the fundamental group of a two-dimensional Moore space M. The following statements are equivalent:*

(1) $D_G N \cong N$.

(2) The space $P_M K(N, 1)$ is 2-connected.

(3) $H^2(N; B) = 0$ *for any B such that* $\mathrm{Hom}(G_{\rm ab}, B) = 0 = \mathrm{Ext}(G_{\rm ab}, B)$.

(4) $T_G N = N$ *and N is super R-perfect.*

Proof. Theorem 3.3 implies that (1), (2), and (3) are equivalent. We only prove that (4) implies (2). Since N coincides with its G-radical, $P_M K(N, 1) \simeq K(N, 1)^+_{HR}$ (see diagram (1.3)), and it is 1-connected. Thus $K(N, 1)^+_{HR} \simeq K(N, 1)_{HR}$ the HR-homological localization by [21, Proposition 1.6]. Moreover $\pi_2 K(N, 1)_{HR}$ is an HR-local group by [4, Theorem 5.5]. But $H_1(\pi_2 K(N, 1)_{HR}; R) = 0$, so it has to be trivial. □

As a consequence, we obtain the following formulae for the low-dimensional homotopy groups of $P_M K(N, 1)$; cf. [1, Corollary 8.6].

Corollary 3.9. *Let M be a two-dimensional Moore space with fundamental group G. Then*

(1) $\pi_2 P_M K(N,1) \cong H_2 P_M K(T_G N, 1)$*;*

(2) $\pi_3 P_M K(N,1) \cong H_3 P_M K(D_G N, 1)$*.* □

We end this section by proving that this topological construction gives nothing else but the right adjoint of the inclusion of $\overline{\mathcal{C}(G)}$ into the category of groups. We denote the class $\{N \mid D_G N \cong N\}$ by $\mathcal{D}(G)$.

Proposition 3.10. *Let M be a two-dimensional Moore space with fundamental group G. The class $\mathcal{D}(G)$ is closed under arbitrary extensions and colimits.*

Proof. The class of G-radical groups is closed under colimits and extensions, and so is the class of super R-perfect groups: An easy Hochschild–Serre spectral sequence argument shows that an extension of super R-perfect groups is again super R-perfect, and a Mayer–Vietoris argument shows it for a push-out. Since homology commutes with telescopes, the proposition is proved. □

Proposition 3.11. *Let M be a two-dimensional Moore space with fundamental group G. Then $\overline{\mathcal{C}(G)} = \mathcal{D}(G)$.*

Proof. By Proposition 3.10, $\overline{\mathcal{C}(G)} \subset \mathcal{D}(G)$. To show the converse we prove that the fundamental group of any space in $\overline{\mathcal{C}(M)}$ is in $\overline{\mathcal{C}(G)}$. But $\overline{\mathcal{C}(M)}$ is the smallest class containing M which is closed under homotopy colimits and extensions by fibrations. Clearly the fundamental group of the homotopy colimit of a diagram all whose values have π_1 in $\overline{\mathcal{C}(G)}$ is again in $\overline{\mathcal{C}(G)}$. So assume we have a fibration $F \to E \to B$ of connected spaces, where the fundamental groups of F and B are in $\overline{\mathcal{C}(G)}$. We have to prove $\pi_1 E \in \overline{\mathcal{C}(G)}$. The cokernel of the boundary morphism $\pi_2 B \to \pi_1 F$ is isomorphic to the coinvariants $(\pi_1 F)_{\pi_2 B} = colim_{\pi_2 B}(\pi_1 F)$ and thus belongs to $\overline{\mathcal{C}(G)}$. Therefore $\pi_1 E$ is an extension of two groups of $\overline{\mathcal{C}(G)}$. □

Corollary 3.12. *Let G be the fundamental group of a two-dimensional Moore space M. A group N is then in $\mathcal{D}(G)$ if an only if there exists an M-acyclic space X with $\pi_1 X \cong N$.* □

Theorem 3.13. *Let M be a two-dimensional Moore space with fundamental group G. The augmented functor D_G is then right adjoint to the inclusion of $\overline{\mathcal{C}(G)}$ in the category of groups, i.e. for any group $L \in \overline{\mathcal{C}(G)}$, we have an isomorphism* $\mathrm{Hom}(L, D_G N) \cong \mathrm{Hom}(L, N)$*.*

Proof. The map $K(D_G N,1) \to K(N,1)$ induces a weak equivalence

$$\overline{P}_M K(D_G N,1) \simeq \overline{P}_M K(N,1)$$

by Theorem 3.7. Let $L \in \overline{\mathcal{C}(G)}$. Then

$$\mathrm{map}_*(\overline{P}_M K(L,1), K(D_G N,1)) \simeq \mathrm{map}_*(\overline{P}_M K(L,1), K(N,1)),$$

i.e. $\mathrm{Hom}(L, D_G N) \cong \mathrm{Hom}(L,N)$. □

4 Acyclic spaces

We illustrate the preceding sections by the case when the Moore space M is acyclic. We identify the functors $\overline{P}_M$ and CW_M. The motivating example is the universal acyclic group $\mathcal{F}$ of Berrick and Casacuberta [2, Example 5.3]. It satisfies $S_{\mathcal{F}}N = T_{\mathcal{F}}N = \mathcal{P}N$, the maximal perfect subgroup of N. In this case $C_{\mathcal{F}}N = D_{\mathcal{F}}N = \widetilde{\mathcal{P}N}$ the universal central extension of $\mathcal{P}N$ and the two central extensions coincide. If $M = M(\mathcal{F},1)$, the functors $\overline{P}_M$ and CW_M coincide with Dror's acyclic functor A, the fiber of Quillen's plus-construction.

We want to consider now an arbitrary acyclic group G, and an acyclic complex M with fundamental group G. This space M is of course not determined by the group. Since $\Sigma M \simeq *$, the fibration of Theorem 1.1 has the form

$$CW_M X \to X \to X'$$

where X' is the homotopy cofibre of the map $\vee_{[M,X]} M \to X$. Let X_N^+ denote the plus-construction of X with respect to a perfect, normal subgroup N of $\pi_1 X$, and let $A_N X$ be the homotopy fibre of natural map $X \to X_N^+$. The universal property of the plus-construction ensures that $X' \simeq X_S^+$, where $S = S(M,X)$ is the topological socle, i.e. the subgroup generated by the images of all homomorphisms $\pi_1(M) \to \pi_1(X)$ which are induced by maps $M \to X$ (see [2, Section 2]). This subgroup of $\pi_1 X$ is also called the subgroup swept by M. Arguing similarly with $P_M X$ we deduce the following; cf. [2, Corollary 2.2].

Theorem 4.1. *Let M be an acyclic space CW-complex. Then the map $\beta : CW_M X \to \overline{P}_M X$ is equivalent to $A_S X \to A_T X$ where S is the subgroup swept by M and T is such that $\pi_1 P_M X \cong \pi_1 X/T$.* □

When M is two-dimensional, $S = S_G N$ and $T = T_G N$ for all X, and N denotes the fundamental group of X. We also have $S = S_G N$ if M is any CW-complex and $X = K(N,1)$. Therefore, we deduce the following.

Corollary 4.2. *Let X be a space with fundamental group N and M an acyclic CW-complex with fundamental group G. Suppose that M is of dimension two or $X = K(N,1)$. Then $\pi_1(CW_M X) \cong C_G N$ is the universal central extension of $S_G N$.* □

5 Nilpotent spaces

When X is a nilpotent space, the homotopy long exact sequence associated to the fibration $\overline{P}_M X \to X \to P_M X$ yields the homotopy groups of the M-acyclic part of X, as follows:

Proposition 5.1. *Let $n \geq 1$, M be a Moore space $M(G,n)$, and let X be any connected space. Suppose that X is nilpotent if $n = 1$. Let J be the set of primes p such that G is uniquely p-divisible and J' be the complementary set of primes. Then $\overline{P}_M X$ is $(n-1)$-connected and for $k \geq n$*

(I) if G_{ab} is torsion, then

$$\pi_k(\overline{P}_M X) \cong \begin{cases} \Pi_{p\in J'}(\mathbf{Z}(p^\infty) \otimes \pi_{k+1}X \;\oplus\; \mathrm{Tor}(\mathbf{Z}(p^\infty), \pi_k X)) \\ \qquad\qquad\qquad\qquad\qquad \textit{if } k \geq n+1, \\ \Pi_{p\in J'}(\mathbf{Z}(p^\infty) \otimes \pi_{n+1}X \;\oplus\; T_G(\pi_n X)) \\ \qquad\qquad\qquad\qquad\qquad \textit{if } k = n; \end{cases}$$

(II) if G_{ab} is not torsion, then

$$\pi_k(\overline{P}_M X) \cong \begin{cases} \Pi_{p\in J}(\mathrm{Ext}(\mathbf{Z}[1/p], \pi_{k+1}X) \;\oplus\; \mathrm{Hom}(\mathbf{Z}[1/p], \pi_k X)) \\ \qquad\qquad\qquad\qquad\qquad \textit{if } k \geq n+1, \\ \Pi_{p\in J}(\mathrm{Ext}(\mathbf{Z}[1/p], \pi_{n+1}X) \;\oplus\; D_G(\pi_n X) \\ \qquad\qquad\qquad\qquad\qquad \textit{if } k = n. \end{cases}$$

Proof. We use from [8, Theorem 7.5] that in the first case we have

$$\pi_k(P_M X) \cong \begin{cases} \pi_k(X) \otimes \mathbf{Z}_{(J')} & \text{if } k \geq n+1, \\ \pi_n X / T_G(\pi_n X) & \text{if } k = n; \end{cases}$$

In the second case we have:

$$\pi_k(P_M X) \cong \begin{cases} \Pi_{p\in J}(\mathrm{Ext}(\mathbf{Z}(p^\infty), \pi_k X) \oplus \mathrm{Hom}(\mathbf{Z}(p^\infty), \pi_{k-1} X)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } k \geq n+1, \\ \\ \pi_n X / T_G(\pi_n X) \qquad \text{if } k = n. \end{cases}$$

□

Example 5.2. Let G be a rational group of rank 1 of type $(r_2, r_3, r_5, \dots)$. That is, G is the additive subgroup of $\mathbf{Q}$ generated by the fractions $1/p^s$, for $s \leq r_p$ (we write $r_p = \infty$ if G is uniquely p-divisible). Note that if $r_p < \infty$ then the G-radical contains the $\mathbf{Z}/p$-radical. Moreover, in the category of abelian groups, the G-socle coincides with the G-radical if and only if $G = \mathbf{Z}[J^{-1}]$ (see [19]).

This allows us to construct two subgroups of $\mathbf{Q}$ having the same set of primes for which they are uniquely p-divisible, but with distinct radical, and thus distinct acyclic approximation. Fix a prime p and define G by $r_p = \infty$ and $r_q = 1$ when $q \neq p$, so that $H = \mathbf{Z}[1/p]$. Let $M = M(G,1)$ and $M' = M(H,1)$. Then we have $\overline{P}_{M'} K(\mathbf{Z}[1/p], 1) \simeq K(\mathbf{Z}[1/p], 1)$, while $\overline{P}_M K(\mathbf{Z}[1/p], 1) \simeq *$.

We next give a description of the class of nilpotent M-acyclic spaces; compare with Corollary 7.9 in [8], see also [20]. The case when $n = 1$ gives a less general result than Corollary1.4, but gives a characterization of $M(G,1)$-acyclic spaces in terms of their homotopy groups rather than their homology groups.

Proposition 5.3. *Let $n \geq 1$, M be a Moore space $M(G,n)$, and let X be any connected space. Suppose that X is nilpotent if $n = 1$. Then, X is M-acyclic if and only if X is $(n-1)$-connected, $\pi_n X$ coincides with its G-radical and $\pi_k(X)$ is J'-torsion for $k \geq n+1$ in the case when G is torsion, or $\pi_k(X)$ is uniquely J-divisible for $k \geq n+1$ otherwise.* □

6 The torsion case

In this section we only deal with the case of the Moore spaces $M(\mathbf{Z}/p^k, 1)$, for $k \geq 1$. We give a characterization of $M(\mathbf{Z}/p^k, 1)$-cellular spaces, which holds even for non-nilpotent spaces. The $M(\mathbf{Z}/p^k, 1)$-acyclic spaces have been already identified in Corollary 1.4. The following reformulation only makes use of the fact that a group A is p-torsion if and only if $A \otimes \mathbf{Z}[1/p] = 0$.

Theorem 6.1. *Let $M = M(\mathbf{Z}/p^k, 1)$, $k \geq 1$. Then a space X is M-acyclic if and only if $\pi_1 X$ coincides with its $\mathbf{Z}/p$-radical and $H_n(X;\mathbf{Z})$ is p-torsion for $n \geq 2$.* □

Theorem 6.2. *Let $M = M(\mathbf{Z}/p^k, 1)$, $k \geq 1$. Then a space X is M-cellular if and only if $\pi_1 X$ is generated by elements of order p^l for $l \leq k$ and $H_n(X;\mathbf{Z})$ is p-torsion for $n \geq 2$.*

Proof. We use again the fact that $CW_M X$ can be obtained as the fiber of the map $X \to P_{M(\mathbf{Z}/p^k,2)} X'$, where X' is the cofiber of $\vee M \to X$. So we have to find a necessary and sufficient condition for $P_{M(\mathbf{Z}/p,2)} X'$ to be trivial. First X' has to be 1-connected, and this is equivalent to $\pi_1 X$ coinciding with its $\mathbf{Z}/p^k$-socle. Knowing that X' and thus $P_{M(\mathbf{Z}/p,2)} X'$ are 1-connected, the triviality of the latest is equivalent to its acyclicity. By Proposition 5.3 the homotopy, or equivalently the reduced integral homology of X', has to be p-torsion. The long exact sequence in homology of the cofibration sequence defining X' shows that this is equivalent to $\tilde{H}_*(X;\mathbf{Z})$ being p-torsion. □

Corollary 6.3. *Let $M = M(\mathbf{Z}/p^k, 1)$. A nilpotent space X is M-cellular if and only if $\pi_1 X$ is generated by elements of order p^l for $l \leq k$ and $\pi_n(X)$ is p-torsion for $n \geq 2$.* □

The characterization given in [14, 12.5] or [15] of $M(\mathbf{Z}/2, n)$-cellular spaces (π_n is generated by involutions, and the higher homotopy groups are 2-torsion) is true for $n = 1$ if we work in the category of nilpotent spaces. An easy counter-example for non-nilpotent spaces is given by $M(\mathbf{Z}/2, 1)$ itself. It is of course an $M(\mathbf{Z}/2, 1)$-cellular space, even though $\pi_2 M(\mathbf{Z}/2, 1) \cong \mathbf{Z}$. We finally consider the following example.

Example 6.4. The symmetric groups Σ_n are C_2-cellular for $n \geq 2$. Indeed, using the presentation $\Sigma_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i^2 = 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_{i+j}\sigma_i = \sigma_{i+j} \text{ for } j \geq 2 \rangle$, one can obtain Σ_n as a push-out of a family of homomorphisms between free products of C_2. For $n - 1 \geq j \geq 1$, let G_j be the coproduct of $n-1-j$ copies of $C_2 * C_2$, and H_j the coproduct of as many copies of C_2. Define $G_j \to H_j$ to be the coproduct of the fold maps $C_2 * C_2 \to C_2$. Let K be the coproduct of $n - 1$ copies of C_2, and call the generators $x_1, \cdots, x_{n-1}$. Define a map $G_1 \to K$ by sending the $(2i-1)$st generator to $x_i x_{i+1} x_i$ and the $2i$th one to $x_{i+1} x_i x_{i+1}$, where $n - 1 \geq i \geq 1$. For $j \geq 2$, the map $G_j \to K$ is defined by sending the $(2i - 1)$st generator to $x_i x_{i+j} x_i$ and the $2i$th one to x_{i+j}. The push-out of the diagram

$$(*_{j=1}^{n-1} H_j) \leftarrow (*_{j=1}^{n-1} G_j) \to K$$

is then the symmetric group Σ_n.

The spaces $K(\Sigma_n, 1)$ have therefore C_2-cellular fundamental group, and 2-torsion higher homotopy groups. They are however not $M(\mathbf{Z}/2, 1)$-cellular by Theorem 6.2, since the integral homology of $K(\Sigma_n, 1)$ contains 3-torsion. Actually, we can even compute the cellularization of $K(\Sigma_3, 1)$. We know by Theorem 1.1 that it is the fiber of the map $K(\Sigma_3, 1) \to P_{M(\mathbf{Z}/2,2)}K(\Sigma_3, 1)'$. The later space is simply connected, and is 3-complete. Its mod 3 cohomology is that of $K(\Sigma_3, 1)$, so it is by [9, VII, 4.4] the delooping of $S^3\{3\}$, the fiber of the degree 3 self-map of S^3. In other words $CW_{M(\mathbf{Z}/2,1)}K(\Sigma_3, 1)$ is a space whose fundamental group is Σ_3 and whose universal cover is $S^3\{3\}$, i.e., the universal cover fibration is

$$S^3\{3\} \to CW_{M(\mathbf{Z}/2,1)}K(\Sigma_3, 1) \to K(\Sigma_3, 1).$$

The action of Σ_3 on $S^3\{3\}$ is trivial by Theorem 2.9.

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