Localizations associated to semidirect products

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Abstract

For homotopical localization with respect to any continuous map, there are results describing the relations among the localization functors associated to the maps of a given fibration. Here we treat an analogous question in a group-theoretical context: we study localization functors associated to a short exact sequence of groups. We further specialize to a split short exact sequence of groups. In particular, we describe explicitely the localization functors associated to a semidirect product of finitely generated abelian groups.

0 Introduction

Localization with respect to a group homomorphism has proved to be a useful tool in understanding homotopical localization with respect to any continuous map, as developed by Bousfield, Dror Farjoun and others ([2], [4], [11]). This algebraic technique was introduced by Casacuberta in [5], [6], and led to some further interesting results in e.g. [1], [7], [8], [10], [12]. Thus, to any given group homomorphism f, there is associated a localization functor L_f on the category of groups, rendering f invertible in a universal way. The collection of all group homomorphisms g yielding a localization functor L_g which is naturally equivalent to L_f, is called the localization class $\langle f\rangle$ of f. In the special case where f is of the form $A \to 1$, the associated localization functor is also denoted by P_A, and the class $\langle f\rangle$ is denoted by $\langle A\rangle$, which is called a nullification class. As we recall in Section 1, the collection *Locs* of all localization classes forms a small-complete lattice for an obvious partial order relation.

In homotopy theory there are results describing the relations between the localization functors associated to the spaces and maps of a given fibration (cf. [11]). Here we study an analogous question in the group-theoretical

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context. More precisely, given a short exact sequence of groups

$$1 \to H \xrightarrow{\phi} N \xrightarrow{\pi} K \to 1, \qquad (0.1)$$

we relate the classes $\langle H\rangle$, $\langle N\rangle$, $\langle K\rangle$, $\langle \phi\rangle$ and $\langle \pi\rangle$ under the partial order relation on *Locs* (cf. Proposition 2.2). In the case where the short exact sequence (0.1) splits, we show in Theorem 2.5 that $\langle \phi\rangle = \langle K\rangle * \langle \varepsilon\rangle$, where $\varepsilon\colon H \to H/[H,K]$ is the natural projection, and $\langle K\rangle * \langle \varepsilon\rangle$ denotes the least upper bound of $\langle K\rangle$ and $\langle \varepsilon\rangle$ in the lattice *Locs*. Moreover, the relations between the localization classes associated to a split short exact sequence (0.1) give rise to the following diagrams of localization functors:

$$\begin{array}{ccccccc} \mathrm{Id} & \to & P_{H/[H,K]} & \to & L_\pi & \to & P_H \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_K & \to & P_{H/[H,K]\times K} & \to & P_N & \to & P_{H\times K} \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{Id} & \to & L_\varepsilon & \to & P_{[H,K]} \\ \downarrow & & \downarrow & & \downarrow \\ P_K & \to & L_\phi & \to & P_{[H,K]\times K}, \end{array}$$

where Id denotes the identity functor.

Finally, in Section 3 we illustrate the above results with explicit calculations in the case of a semidirect product of finitely generated abelian groups. We show in Theorem 3.1 that the localizations associated to a nilpotent semidirect product of these groups are particularly easy to describe. On the other hand, Proposition 3.4 shows that even for a semidirect product of cyclic groups for which the corresponding action is perfect, the situation can be more complicated.

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1 Preliminaries

Recall from [6] the definition of localization with respect to a given group homomorphism $f\colon A \to B$. A group G is called *f-local* if the induced map of sets

$$f^*\colon \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$$

is a bijection. For every group G there is a homomorphism $l_G\colon G \to L_f G$ which is initial among all homomorphisms from G into f-local groups. L_f is called the localization functor with respect to f. A homomorphism ϕ is called an *f-equivalence* if $L_f\phi$ is an isomorphism. A group G is *f-acyclic* if $L_f G = 1$.

For a given homomorphism f, the collection of all homomorphisms g such that L_g is naturally equivalent to L_f, is called the *localization class* of f, and it is denoted by $\langle f\rangle$. There is an obvious partial order relation on the collection *Locs* of all localization classes: we say that $\langle f\rangle \leq \langle g\rangle$ if there exists a natural transformation $L_f \to L_g$ of localization functors. Note that this is equivalent to all f-equivalences being g-equivalences, to every g-local group being f-local, or to f being a g-equivalence. With this partial order, *Locs* forms a small-complete lattice (cf. [12]). Recall that the least upper bound of two classes $\langle f\rangle$ and $\langle g\rangle$, which is denoted by $\langle f\rangle * \langle g\rangle$, has the free product $f * g$ as a representative, i.e. $\langle f\rangle * \langle g\rangle = \langle f * g\rangle$.

Recall further that a class $\langle f\rangle$ is called an *epireflection class* if $G \to L_f G$ is surjective for all groups G. By Theorem 2.1 of [12], this is equivalent to $\langle f\rangle = \langle g\rangle$ for some epimorphism g. In the case where g is of the form $A \to 1$, the functor L_g is usually denoted by P_A, and it is called *A-nullification* (cf. [2], [6]). The localization class $\langle g\rangle$ is then called a *nullification class* (which is also denoted by $\langle A\rangle$), while the g-local groups are often refered to as *A-null*. Observe that a group G is f-acyclic if and only if $\langle G\rangle \leq \langle f\rangle$. For general properties of nullification and epireflection classes, in particular the relation with radicals in group theory, we refer to [9] and [12].

2 Localizations associated to a short exact sequence

We start by a preliminary result, giving another description of the least upper bound of two localization classes.

Lemma 2.1. *Let f and g be any group homomorphisms. Then*

$$\langle f\rangle * \langle g\rangle = \langle f * g\rangle = \langle f \times g\rangle.$$

Proof. The first identity was explained in Section 1. Clearly, $\langle f * g\rangle \leq \langle f \times g\rangle$. Furthermore, $f \times g = (f \times \mathrm{id}) \circ (\mathrm{id} \times g)$, where $(f \times \mathrm{id})$ is an f-equivalence and $(\mathrm{id} \times g)$ is a g-equivalence. This implies that $f \times g$ is an $(f * g)$-equivalence. $\square$

Given a short exact sequence of groups of the form

$$1 \to H \xrightarrow{\phi} N \xrightarrow{\pi} K \to 1, \tag{2.1}$$

we first clarify the relations among the localization functors L_ϕ, L_π, P_H, P_N and P_K. Observe that if $N = H \times K$, then we have a complete description of all involved localization functors, since $\langle H \times K\rangle = \langle H\rangle * \langle K\rangle$, $\langle\phi\rangle = \langle K\rangle$ and $\langle\pi\rangle = \langle H\rangle$. In general, we have the following relations.

Proposition 2.2. *Consider the short exact sequence (2.1). Then the following relations hold:*

$$\langle N\rangle = \langle\pi\rangle * \langle K\rangle, \quad \langle\pi\rangle \le \langle H\rangle \quad \textit{and} \quad \langle K\rangle \le \langle\phi\rangle \le \langle H\rangle * \langle K\rangle.$$

In particular, we have $\langle N\rangle \le \langle H\rangle * \langle K\rangle$. In other words, there is a diagram of localization functors, given by

$$\begin{array}{ccccc} \mathrm{Id} & \to & L_\pi & \to & P_H \\ \downarrow & & \downarrow & & \downarrow \\ P_K & \to & P_N & \to & P_{H\times K} \\ \| & & & & \| \\ P_K & \to & L_\phi & \to & P_{H\times K}, \end{array} \tag{2.2}$$

where Id *denotes the identity functor.*

Proof. Applying the functor $\mathrm{Hom}(-, G)$ to the exact sequence given in (2.1), yields an exact sequence of sets (where π^* is an inclusion)

$$* \to \mathrm{Hom}(K, G) \xrightarrow{\pi^*} \mathrm{Hom}(N, G) \xrightarrow{\phi^*} \mathrm{Hom}(H, G),$$

which implies directly the result. □

An immediate consequence of the previous proposition is the following result (compare with Corollary 4.8 of [2] and Theorem 3.D.1 of [11]).

Corollary 2.3. *Consider the exact sequence (2.1) and let f be any group homomorphism.*

(i) *If H is f-acyclic, then π is an f-equivalence.*

(ii) *If ϕ is an f-equivalence, then K is f-acyclic.* □

However, as the next example shows, the converse of the statements of Corollary 2.3 are false in general.

Example 2.4. (i) Let $f\colon \mathbb{Z}*\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ be the canonical projection. Then $L_f G \cong G_{\mathrm{ab}}$ for any group G, where G_{ab} denotes its abelianization. Consider the short exact sequence $1 \to [N,N] \xrightarrow{\phi} N \xrightarrow{\pi} N/[N,N] \to 1$. Clearly π is an f-equivalence, but the commutator subgroup $[N,N]$ is only f-acyclic if N is soluble of derived length at most two.

(ii) Consider the short exact sequence $1 \to C_3 \xrightarrow{\phi} \Sigma_3 \xrightarrow{\pi} C_2 \to 1$, where Σ_3 denotes the permutation group of a set of three elements, and C_2, C_3 are cyclic groups of order 2, resp. 3. Let $f\colon *_p\mathbb{Z} \xrightarrow{\phi_p} *_p\mathbb{Z}$, where the free product is taken over all primes p which are different from 3, and the homomorphism ϕ_p is just multiplication by p. Then L_f is localization at the prime 3 (cf. [5]). We conclude that C_2 is f-acyclic, but ϕ is not an f-equivalence, since $L_f(C_3) = C_3$ and $L_f(\Sigma_3) = 1$.

We now specialize to the case of a split short exact sequence of groups:

$$1 \to H \xrightarrow{\phi} H \rtimes K \xrightarrow{\pi} K \to 1. \tag{2.3}$$

We denote the action of an element $x \in K$ on an element $a \in H$ by $x \cdot a = s(x)as(x)^{-1}$, where $s\colon K \to H \rtimes K$ is a splitting homomorphism for π. Let $[H,K]$ denote the commutator subgroup, i.e. the normal subgroup of H generated by the elements of the form $(x\cdot a)a^{-1} = xax^{-1}a^{-1}$ (from now on we will omit writing the inclusion s). Denote by $\varepsilon\colon H \to H/[H,K]$ the canonical homomorphism. Observe that, for every group G, the set $\mathrm{Hom}(H \rtimes K, G)$ is in bijective correspondence with the subset of $\mathrm{Hom}(H,G) \times \mathrm{Hom}(K,G)$ consisting of those pairs (ψ,ξ) such that

$$\psi(x\cdot a) = \xi(x)\psi(a)\xi(x)^{-1},$$

for all $a \in H$ and $x \in K$.

Theorem 2.5. *Consider the split short exact sequence (2.3). Then* $\langle\phi\rangle = \langle K\rangle * \langle\varepsilon\rangle$.

Proof. Suppose that G is ϕ-local. By Proposition 2.2, G is K-null. Moreover, we claim that any homomorphism $\psi\colon H \to G$ is trivial on $[H,K]$. Indeed, since $\mathrm{Hom}(K,G)$ is trivial and $\phi^*\colon \mathrm{Hom}(H \rtimes K, G) \cong \mathrm{Hom}(H,G)$, it is clear that ψ satisfies $\psi(x\cdot a) = \psi(a)$ for all $a \in H$, $x \in K$. This is equivalent to ψ being trivial on $[H,K]$, which in turn implies that G is ε-local. Hence, G is both K-null and ε-local. Conversely, suppose that G is K-null and ε-local. Any homomorphism $H \rtimes K \to G$ then corresponds to (ψ, id) for some homomorphism $\psi\colon H \to G$. Hence, ϕ^* is clearly injective. Moreover, since G is ε-local, any homomorphism $\psi\colon H \to G$ is trivial on

$[H,K]$, so that $\psi(x \cdot a) = \psi(a)$ for all $a \in H$, $x \in K$. Hence, (ψ, id) defines a homomorphism from $H \rtimes K$ to G, and clearly $\phi^*((\psi, \mathrm{id})) = \psi$. This shows that ϕ^* is surjective, thereby concluding the proof. $\square$

With the terminology introduced in [12], Theorem 2.5 thus states that $\langle \phi \rangle$ is an epireflection class.

Theorem 2.6. *Consider the split short exact sequence (2.3). Then a group G is π-local if and only if every homomorphism from $H \rtimes K$ to G is trivial on H. Moreover,*

$$\langle H/[H,K] \rangle \leq \langle \pi \rangle \leq \langle H \rangle.$$

Proof. The first claim is obvious, and the fact that $\langle \pi \rangle \leq \langle H \rangle$ was proved in Proposition 2.2. Suppose now that G is π-local, so that any homomorphism $(\psi, \xi)\colon H \rtimes K \to G$ is trivial on H, i.e. $\psi = 1$. Given any homomorphism $f\colon H/[H,K] \to G$, the composition $f \circ \varepsilon$ is trivial on $[H,K]$. This means that $(f \circ \varepsilon, 1)$ is a homomorphism from $H \rtimes K$ to G, so that $f \circ \varepsilon = 1$. Since ε is an epimorphism, we infer that f is trivial, and hence that G is $H/[H,K]$-null. $\square$

Summarizing the previous results, for a split short exact sequence (2.3), we have the following diagrams of localization functors:

$$\begin{array}{ccccccc} \mathrm{Id} & \to & P_{H/[H,K]} & \to & L_\pi & \to & P_H \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_K & \to & P_{H/[H,K] \times K} & \to & P_{H \rtimes K} & \to & P_{H \times K} \end{array} \tag{2.4}$$

and

$$\begin{array}{ccccc} \mathrm{Id} & \to & L_\varepsilon & \to & P_{[H,K]} \\ \downarrow & & \downarrow & & \downarrow \\ P_K & \to & L_\phi & \to & P_{[H,K] \times K}, \end{array} \tag{2.5}$$

where Id denotes the identity functor.

At the beginning of this section, we already dealt with the special case of a semidirect product (2.3) where the action of K on H is trivial. Another "extremal" case is when the action is perfect, i.e. $[H,K] = H$.

Corollary 2.7. *Consider the split short exact sequence (2.3). If $[H,K] = H$, then $\langle \phi \rangle = \langle H \rangle * \langle K \rangle$. Therefore, diagrams (2.4) and (2.5) collapse to*

$$\begin{array}{ccccc} \mathrm{Id} & \to & L_\pi & \to & P_H \\ \downarrow & & \downarrow & & \downarrow \\ P_K & \to & P_{H \rtimes K} & \to & L_\phi \cong P_{H \times K}. \end{array} \tag{2.6}$$ $\square$

Note that the above result is a special case of the general fact that $\langle [H,K] \rangle \leq \langle \varepsilon \rangle$ if and only if $\langle [H,K] \rangle = \langle \varepsilon \rangle$.

We close this section by exhibiting an example showing that in general the arrows in diagram (2.6) are not necessarily natural equivalences.

Example 2.8. Consider the split short exact sequence

$$1 \to C_3 \xrightarrow{\phi} \Sigma_3 \xrightarrow{\pi} C_2 \to 1.$$

The commutator subgroup is $[C_3, C_2] = C_3$. It is easy to check that all localizations in diagram (2.6) are different, by evaluating them on the groups C_2, C_3, $C_3 \times C_2$ and Σ_3.

Other examples, generalizing Example 2.8, and illustrating that the arrows in diagrams (2.4) and (2.5) are not necessarily natural equivalences, will be exhibited in the next section, where we specialize to split short exact sequences of finitely generated abelian groups.

3 Localizations associated to semidirect products of finitely generated abelian groups

From [4, Proposition 7.3] we know that $P_{(H \rtimes K)_{\mathrm{ab}}} = P_{H/[H,K] \times K}$ and $P_{H \rtimes K}$ are naturally equivalent when restricted to the category of nilpotent groups. In particular, if $H \rtimes K$ is itself nilpotent, we infer that $P_{H \rtimes K}$ is equivalent to $P_{H/[H,K] \times K}$ on the category of all groups. Here we show that, under the additional assumption that H and K are finitely generated abelian groups, these nullifications actually are equivalent to $P_{H \times K}$.

Theorem 3.1. *Consider the split short exact sequence (2.3), where H and K are finitely generated abelian groups. If $H \rtimes K$ is nilpotent, then*

$$\langle H/[H,K] \rangle = \langle \pi \rangle = \langle H \rangle \quad \textit{and} \quad \langle H \rtimes K \rangle = \langle H \times K \rangle .$$

Therefore, diagrams (2.4) and (2.5) collapse to

$$\begin{array}{ccccccc} \mathrm{Id} & \to & L_\varepsilon & \to & P_{[H,K]} & \to & P_H \cong L_\pi \cong P_{H/[H,K]} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_K & \to & L_\phi & \to & P_{[H,K] \times K} & \to & P_{H \times K} \cong P_{H \rtimes K} \cong P_{H/[H,K] \times K} . \end{array} \tag{3.1}$$

Proof. To prove the first claim, it suffices to show that $\langle H/[H,K] \rangle = \langle H \rangle$. If H is a finite abelian group, then it follows easily from the fact that

$H \rtimes K$ is nilpotent, that the set $T(H)$ of prime numbers p for which H has p-torsion coincides with the set $T(H/[H,K])$ of prime numbers p for which $H/[H,K]$ has p-torsion. Hence, $\langle H/[H,K] \rangle = \langle H \rangle = \langle \prod_{p \in T(H)} C_p \rangle$, where C_p denotes the cyclic group of order p. If H is not finite, then we can choose generators $a_1, ..., a_m$ for the torsion-free part of H such that the torsion-free part of $[H,K]$ is generated by $t_1 a_1, ..., t_m a_m$ for some integers $t_1, ..., t_m$ with $t_1 | ... | t_m$. Since $H \rtimes K$ is nilpotent, we infer that $t_m = 0$. Hence, $\langle H \rangle = \langle H/[H,K] \rangle = \langle \mathbb{Z} \rangle$. Finally, if $\langle \pi \rangle = \langle H \rangle$, then the second claim follows immediately from Proposition 2.2 and Lemma 2.1. □

In the case of a semidirect product of finitely generated abelian groups, we can even go further in our description of the associated localization functors. We henceforth adopt the convention that the cyclic group of order m is written $\mathbb{Z}_m$ when we want to use additive notation, and that we write C_m when thought of this cyclic group as a multiplicative group. We allow m to be any positive integer or to be ∞, with the convention that $C_\infty = \mathbb{Z}_\infty = \mathbb{Z}$. Let

$$H = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \quad \text{and} \quad K = C_{m_1} \times \cdots \times C_{m_l} \tag{3.2}$$

be finitely generated abelian groups (we do not exclude the case where some of the factors of H and/or K are infinite cyclic groups). Choose generators x_i for C_{m_i} and a_j for $\mathbb{Z}_{n_j}$. Suppose further that, for this choice of generators, the action of x_i on H is given by a $(k \times k)$-matrix P_i (representing the corresponding element of $\mathrm{Aut}(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$). Hence, the commutator subgroup $[H,K]$ is the subgroup of H generated by the images of $(P_i - I)$ for all i, where I denotes the identity matrix, i.e.,

$$[H,K] = \mathrm{grp}\,\{\mathrm{im}\,(P_1 - I), \cdots, \mathrm{im}\,(P_l - I)\}.$$

We restate the results of Theorem 2.5 and Theorem 2.6 in this particular case.

Proposition 3.2. *Consider the split short exact sequence (2.3), where H and K are given by (3.2). Then*

(i) $\langle \phi \rangle = \langle K \rangle * \langle \varepsilon \rangle$, *where* $\varepsilon \colon H \to H/\mathrm{grp}\,\{\mathrm{im}\,(P_1 - I), \cdots, \mathrm{im}\,(P_l - I)\}$.

(ii) $\langle H/\mathrm{grp}\,\{\mathrm{im}\,(P_1 - I), \cdots, \mathrm{im}\,(P_l - I)\} \rangle \leq \langle \pi \rangle \leq \langle H \rangle$,

where P_i is the matrix corresponding to the action of x_i on H. □

For example, if every P_i is a diagonal matrix, i.e. if there exist integers t_{ij} such that $x_i \cdot a_j = t_{ij} a_j$ for all i, j, then $\langle \phi \rangle = \langle K \rangle * \langle \varepsilon \rangle$, where ε is the natural projection from $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ onto $\mathbb{Z}_{n_1}/\operatorname{grp}\{g_1 a_1\} \times \cdots \times \mathbb{Z}_{n_k}/\operatorname{grp}\{g_k a_k\}$, and, furthermore,

$$\langle \mathbb{Z}_{n_1}/\operatorname{grp}\{g_1 a_1\} \times \cdots \times \mathbb{Z}_{n_k}/\operatorname{grp}\{g_k a_k\}\rangle \leq \langle \pi \rangle \leq \langle \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}\rangle,$$

where the integer $g_j = \gcd\{t_{ij} - 1 \mid t_{ij} - 1 \neq 0 \text{ and } i = 1, ..., k\}$.

We now confine our attention to

$$1 \to \mathbb{Z}_n = \langle a \rangle \xrightarrow{\phi} \mathbb{Z}_n \rtimes C_m \xrightarrow{\pi} C_m = \langle x \rangle \to 1, \tag{3.3}$$

where n is a positive integer (and m is a positive integer or ∞).

Proposition 3.3. *Consider the split short exact sequence (3.3). Let the corresponding action be given by $x \cdot a = ta$ with $0 \leq t < n$. Suppose that the prime decomposition of n is given by $n = p_1^{r_1} \cdots p_k^{r_k}$ (where $r_i \geq 1$). Suppose further that $t - 1 = p_1^{s_1} \cdots p_k^{s_k} Q$, where Q is coprime to $p_1, \cdots, p_k$ and $0 \leq s_i \leq r_i$. Then*

(i) $\langle \pi \rangle = \langle \pi_1 \rangle * \cdots * \langle \pi_k \rangle$, *where* $1 \to \mathbb{Z}_{p_j^{r_j}} \to \mathbb{Z}_{p_j^{r_j}} \rtimes C_m \xrightarrow{\pi_j} C_m \to 1$ *represents the induced action;*

(ii) $\langle \varepsilon \rangle = \langle \varepsilon_1 \rangle * \cdots * \langle \varepsilon_k \rangle$, *where the projection* $\varepsilon_j \colon \mathbb{Z}_{p_j^{r_j}} \to \mathbb{Z}_{p_j^{s_j}}$ *is induced by* $\varepsilon \colon \mathbb{Z}_n \to \mathbb{Z}_n/[\mathbb{Z}_n, C_m]$.

Proof. Clearly, $[\mathbb{Z}_n, C_m] = \operatorname{grp}\{(t-1)a\} \cong \mathbb{Z}_{p_1^{r_1 - s_1}} \times \cdots \times \mathbb{Z}_{p_k^{r_k - s_k}}$. Hence $\mathbb{Z}_n/[\mathbb{Z}_n, C_m] \cong \mathbb{Z}_{p_1^{s_1}} \times \cdots \times \mathbb{Z}_{p_k^{s_k}}$. Moreover, it is clear that a group G is π-local if and only if G is π_j-local for all j. □

The above result enables us to reduce the general case of a semidirect product (3.3) to the case where n is a power of a prime number.

Proposition 3.4. *Consider the split short exact sequence (3.3) where $n = p^r$ for a prime p and $r \geq 1$. Suppose that the corresponding action is given by $x \cdot a = ta$, where $0 \leq t < p^r$. Suppose further that $t - 1 = p^s Q$, where $0 \leq s < r$ and Q is coprime to p. Then*

(i) $\langle \phi \rangle = \langle C_m \rangle * \langle \varepsilon \rangle$, *where* $\varepsilon \colon \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^s}$.

(ii) *If* $\mathbb{Z}_{p^r} \rtimes C_m$ *is nilpotent (i.e., $s \geq 1$), then* $\langle \pi \rangle = \langle \mathbb{Z}_p \rangle$.

(iii) *If the action is perfect (i.e., $s = 0$), then* $\langle \pi \rangle \leq \langle \mathbb{Z}_p \rangle$. □

To complete our description, we recall that $P_{\mathbb{Z}_p} G = G/I_p(G)$ for every group G, where $I_p(G)$ denotes the p-radical (or p-isolator) of the group G. Furthermore, if $f \colon \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^s}$ (with $r > s$) is the canonical projection, then the effect of L_f on a group G is to reduce the order of any element of order p^k (for $k \geq s$) to p^s. Finally, L_π kills all the elements g of a group G for which $g^{p^r} = 1$ and for which there exist an element y in G such that $y^m = 1$ (if m is finite) and $[y, g] = g^{t-1}$.

Finally, to see that we may have $\langle \pi \rangle < \langle \mathbb{Z}_p \rangle$ in Proposition 3.5 (iii), it suffices to consider a perfect action of C_q on $\mathbb{Z}_p$, where p and q are distinct primes (cf. Example 2.8). Indeed, $\langle \pi \rangle = \langle \mathbb{Z}_p \rangle$ would imply that $\langle \mathbb{Z}_p \rtimes C_q \rangle = \langle \mathbb{Z}_p \times C_q \rangle$, which is impossible, since $P_{\mathbb{Z}_p \rtimes C_q}(\mathbb{Z}_p \times C_q) \neq 1$.

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