A NOTE ON FOURIER MULTIPLIERS AND SOBOLEV SPACES

Victor Olevskii

Abstract. We construct a continuous function in the Sobolev space $W_{2,\frac{1}{2}}$ on the circle which is not a Fourier multiplier in l_p, $p \neq 2$.

1. Introduction.

An L_∞-function f on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is called *a Fourier multiplier in* l_p, $1 \leq p \leq \infty$, if the linear operator

$$u \mapsto \widehat{f \cdot \check{u}}$$

is bounded in l_p. The signs ˆ and ˇ stand for the Fourier transform on $\mathbb{T}$ and the inverse one. The class of all such multipliers is denoted by $\mathbb{M}_p(\mathbb{T})$. A detailed account on Fourier multipliers (e.g. some statements given here without references) can be found in [EG].

It is well-known that $f \in \mathbb{M}_p(\mathbb{T})$, $1 \leq p \leq \infty$, once $\sum |\hat{f}(n)| < \infty$. By definition, the Sobolev space $W_{2,\alpha}$ consists of all functions on $\mathbb{T}$ such that $\sum |\hat{f}(n)|^2 |n|^{2\alpha} < \infty$. Since

$$\sum |\hat{f}(n)| \leq \sum |\hat{f}(n)|^2 \, |n|^{2\alpha} \sum |n|^{-2\alpha},$$

any function of $W_{2,\alpha}$, $\alpha > \frac{1}{2}$, is a Fourier multiplier in all l_p. For $\alpha \leq \frac{1}{2}$, however, not every function in $W_{2,\alpha}$ is even bounded.

Seeger [S] posed a question whether $f \in W_{2,\frac{1}{2}} \cap L_\infty$ implies $f \in \mathbb{M}_p(\mathbb{T})$. The negative answer is given here. In fact, we show that if boundedness of a function does not follow from a Sobolev type condition itself then such an implication fails.

Given a sequence $\{w_n\}$, $0 < w_0 \leq w_1 \leq ...$, we consider a space $W_{2,w}$ of all functions f on $\mathbb{T}$ with the norm

$$\|f\|_w = \Big(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 w_{|n|} \Big)^{\frac{1}{2}} < \infty \, .$$

Theorem. *Let* $p \neq 2$. *Then* $W_{2,w} \cap \mathrm{C}(\mathbb{T}) \subset \mathbb{M}_p(\mathbb{T}) \iff \sum_{n=0}^{\infty} w_n^{-1} < \infty$.

Due to the above arguments, our claim is the following. *Let* $\sum w_n^{-1} = \infty$. *Then there exists a continuous function* f *in* $W_{2,w}$ *which is not a Fourier multiplier in* l_p *for* $p \neq 2$.

2. Preliminaries.

The usual norm in $\mathrm{C}(\mathbb{T})$ is denoted by $\|\cdot\|_{\mathrm{C}}$. By $c(p)$ and $K(p)$ we will mean positive constants depending only on p.

Lemma 1. *Let* $\sum_{n=0}^{\infty} w_n^{-1} = \infty$. *Then for any* $\epsilon > 0$, $h > 0$, *and* $0 < \delta \leq \frac{1}{2}$ *there exists a trigonometric polynomial* g *such that* i) $g(0) = 1$, ii) $\|g\|_{\mathrm{C}} = 1$, iii) $\|g\|_w < \epsilon$, *and* iv) $|g(t)| < h$ *outside the interval* $(-\delta, \delta)$.

Proof. We put $a_m = (\sum_{n=0}^{m} w_n^{-1})^{-1}$, and define

$$g_m(t) \;=\; a_m \sum_{n=0}^{m} w_n^{-1} e^{2\pi i n t}.$$

Clearly, $g_m(0) = \|g_m\|_{\mathrm{C}} = 1$ and $\|g_m\|_w^2 = a_m$. Let $U_n(t) = \sum_{k=0}^{n} e^{2\pi i k t}$. By the Abel transform and increase of w_n, we have

$$|\sum_{n=0}^{m} w_n^{-1} e^{2\pi i n t}| = |\sum_{n=0}^{m} (w_n^{-1} - w_{n+1}^{-1})\, U_n(t) + w_{m+1}^{-1}\, U_m(t)| \leq w_0^{-1} \max_n |U_n(t)|.$$

For any n, $|U_n(t)| = |(1 - e^{2\pi i (n+1) t})\,(1 - e^{2\pi i t})^{-1}| \leq 2\,|1 - e^{2\pi i t}|^{-1}$. It follows that $|g_m(t)| < a_m w_0^{-1} \pi \delta^{-1}$ for $t \in \mathbb{T} \setminus (-\delta, \delta)$. Observe now that $a_m \to 0$. Hence (iii) and (iv) hold for a sufficiently large m. •

The next statement (see e.g. [O]) is a consequence of classical results of Rudin and Shapiro (see [R]) and de Leeuw [dL].

Lemma 2. *For any natural* N *there exists a sequence* $\{\xi_k\}$, $k = 1, ..., N$, *with* $\xi_k = \pm 1$, *such that for* $1 < p < \infty$

$$\|f\|_{\mathbb{M}_p(\mathbb{T})} \geq c(p)\, N^{|\frac{1}{p} - \frac{1}{2}|} \tag{1}$$

for any $f \in \mathrm{C}(\mathbb{T})$ *taking the values* ξ_k *on an arithmetic progression.*

Finally, we recall the following well-known fact. Let $\|f\|_V = |f(0)| +$ the variation of f on $\mathbb{T}$. Then for $1 < p < \infty$

$$\|f\|_{\mathbb{M}_p(\mathbb{T})} \leq K(p)\, \|f\|_V. \tag{2}$$

3. Proof of Theorem.

For $n = 1, 2, ...$ let

$$f_n(t) \;=\; \alpha_n \sum_{k=1}^{N} \xi_k^{(N)}\, g^{(N)}(t - t_k^{(N)})\,,$$

where $t_k^{(N)} = \frac{k}{N}$, $k = 1, ..., N$, $\xi_k^{(N)}$ are taken from Lemma 2, and $g^{(N)}(t)$ is the polynomial from Lemma 1 for $\epsilon^{(N)} = \delta^{(N)} = \frac{1}{N}$ and $h^{(N)} = \frac{1}{N^2}$. Then $\|f_n\|_w < \alpha_n$ and $\|f_n\|_{\mathrm{C}} < 2\alpha_n$.

The numbers $N = N(n)$ and α_n are defined inductively as follows. Let $p_n = (\frac{1}{2} + \frac{1}{n})^{-1}$ and $M_n = \|\sum_{j=1}^{n} f_j\|_{\mathbb{M}_{p_n}(\mathbb{T})}$. We put $\alpha_1 = N(1) = 1$. For $n > 1$, we set $\alpha_n = \min\{2^{-n}, \frac{1}{2N(n-1)}\,\alpha_{n-1}\}$, and take N such that

$$c(p_n)\, N^{\frac{1}{n}} \;\geq\; 2\,\alpha_n \max\{n, M_{n-1}\} + 9K(p_n)\,, \tag{3}$$

where $c(p)$ and $K(p)$ are the constants from (1) and (2).

Let now $f = \sum_{n=1}^{\infty} f_n$. Clearly, $f \in \mathrm{C}(\mathbb{T})$ and $\|f\|_w < \sum_{n=1}^{\infty} \alpha_n \leq 2$. To estimate its multiplier norm, consider the function $f^{(n)} = \alpha_n^{-1} \sum_{j=n}^{\infty} f_j$. For $k = 1, ..., N = N(n)$ let $b_k^{(n)} = f^{(n)}(t_k^{(N)}) - \xi_k^{(N)}$. We have

$$|b_k^{(n)}| \;\leq\; |\alpha_n^{-1} f_n(t_k^{(N)}) - \xi_k^{(N)}| + \alpha_n^{-1} \sum_{j=n+1}^{\infty} |f_j(t_k^{(N)})| \;\leq$$

$$\sum_{i\neq k} |g^{(N)}(t_k^{(N)} - t_i^{(N)})| + 2\,\alpha_n^{-1} \sum_{j=n+1}^{\infty} \alpha_j \;\leq\; (N-1)\, h^{(N)} + 4\,\alpha_n^{-1}\alpha_{n+1} \;\leq\; \frac{3}{N}.$$

Let us denote the function taking the values $b_k^{(n)}$ at the points $t_k^{(N)}$, and linearly interpolated on the complementary intervals, by $\tilde{f}^{(n)}$. It follows that $\|\tilde{f}^{(n)}\|_V \;\leq\; 9$. Observe that the difference $f^{(n)} - \tilde{f}^{(n)}$ satisfies the conditions of Lemma 2. Hence from (1), (2), and (3) we obtain

$$\|\sum_{j=n}^{\infty} f_j\|_{\mathbb{M}_{p_n}(\mathbb{T})} \;\geq\; 2\max\{n, \|\sum_{j=1}^{n-1} f_j\|_{\mathbb{M}_{p_n}(\mathbb{T})}\},$$

which implies $\|f\|_{\mathbb{M}_{p_n}(\mathbb{T})} \geq n$. Due to the fact that $\|f\|_{\mathbb{M}_p(\mathbb{T})} \geq \|f\|_{\mathbb{M}_{p_n}(\mathbb{T})}$ for p with $|\frac{1}{p} - \frac{1}{2}| \geq \frac{1}{n}$, we finally conclude that $f \notin \mathbb{M}_p(\mathbb{T})\ \ \forall\, p \neq 2$. •

References

[EG] Edwards R. and Gaudry G., *Littlewood-Paley and Multiplier Theory*, Springer-Verlag, Berlin, 1977.

[dL] de Leeuw K., *On L_p multipliers*, Ann. Math. **81** (1965), 364-379.

[O] Olevskii V., *Variation, homeomorphisms and Fourier multipliers*, C. R. Acad. Sci. Paris **325** (1997), Sér. I, 639-644.

[R] Rudin W., *Some theorems on Fourier coefficients*, Proc. Amer. Math. Soc. **10** (1959), 855-859.

[S] Seeger A., *private communication.*

Centre de Recerca Matemtica, Apartat 50, E-08193, Bellaterra, Spain.
E-mail: vit@crm.es