Some unexpected properties of limit cycles of quadratic systems in the plane

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Abstract

We establish that quadratic system can have: (1) a manifold of limit cycles without points of Andronov-Hoff bifurcation; (2) a separatrix cycle around weak focus with two saddles at infinity and critical stability; (3) a complicated curve of separatrix cycles on the plane of two parameters rotating the vector field.

1 Introduction

A quadratic system

$$\frac{dx}{dt} = \sum_{i+j=0}^{2} a_{ij} x^i y^j \equiv P(x,y)$$
$$\frac{dy}{dt} = \sum_{i+j=0}^{2} b_{ij} x^i y^j \equiv Q(x,y) \tag{1}$$

has in generic case twelve parameters, and its limit cycles depend on them [1]. Therefore the computations that appear in the qualitative study of quadratic systems are difficult. They can be overcome by the theory of invariants and comitants [2] or by simplifying system (1) by means of an

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affine transformation of phase variables and a change of the time scale. System (1) can be reduced to a quadratic system with five essential parameters. The parametric families of quadratic system which are obtained doing these simplifications are called canonical. As a rule in every canonical family the parameters have some specific (geometric or another) sense. For instance, some of parameters can rotate the vector field $f = (P, Q)$ associated to system (1).

According to the definition a parameter α rotates the vector field of the system

$$\begin{aligned} \frac{dx}{dt} &= f(x, \alpha)\,, \qquad x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}\,, \\ f &= (f_1, f_2)^T\,, \qquad f(0, \alpha) = 0\,, \end{aligned} \tag{2}$$

in $\mathbb{R}^2$ if the inequality

$$f'_{1\alpha} f_2 - f_1 f'_{2\alpha} \geq 0 \quad (\leq 0)\,, \quad x \in \mathbb{R}^2,\ \alpha \in \mathbb{R} \tag{3}$$

holds, and it is not identity at every limit cycle of system (2).

Condition (3) means that limit cycles of system (2) change their position when the parameter α varies. Let α be a rotation parameter for the planar vector field f. Denote by $L(\alpha)$ the set of the phase plane $\mathbb{R}^2$ formed by the limit cycles of system (2) for a given α. Then $L(\alpha)$, $\alpha \in \mathbb{R}$ fill in some open region Ω and $L(\alpha_1) \cap L(\alpha_2) = \emptyset$, if $\alpha_1 \neq \alpha_2$ [3]. We can define the function of limit cycles $F(x)$, $x \in \Omega$ as the function which is equal to α on the set $L(\alpha)$. If the parameter α does not rotate the vector field we can define the function $F(x)$ in the same way but then it can be multivalued. Equation $\alpha = F(x)$ defines in the space $\mathbb{R} \times \mathbb{R}^2$ an invariant manifold for the system

$$\frac{dx}{dt} = f(x, \alpha)\,, \quad \frac{d\alpha}{dt} = 0\,. \tag{4}$$

Moreover, equation

$$\frac{\partial F}{\partial x_1} f_1(x, F(x)) + \frac{\partial F}{\partial x_2} f_2(x, F(x)) = 0\,, \qquad x \in \Omega$$

is satisfied. The function $F(x)$ gives us the most complete information about the limit cycles of system (2) and their bifurcation when α changes; for instance, their multiplicity. Let $F(x_1, 0) = \varphi(x_1)$ and assume that the point x_1^0 satisfies $\varphi^{(i)}(x_1^0) = 0$, $i = 1, \ldots, k-1$ and $\varphi^{(k)}(x_1^0) \neq 0$. Then the system (2) has a limit cycle of multiplicity κ passing through the point $(x_1^0, 0)$ for $\alpha = \varphi(x_1^0)$. In the case of the Andronov-Hopf bifurcation we can assume that

$$f(x, \alpha) = J(\alpha)x + X(x, \alpha)\,, \quad J(\alpha) = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}\,,$$

where $X(x,\alpha)$ is formed by nonlinear terms of $f(x,\alpha)$ and $F(x)$ is analytic function at $x=0$ if $f(x,\alpha)$ is analytic at $x=0$, $\alpha=0$. It allows us to find the expansion of $F(x)$ into Taylor-series at the point $x=0$. So the function of limit cycles is a nonlocal generalization of Andronov-Hopf manifold.

Our work has several parts. At first we consider those canonical families, which are often used in the investigation of quadratic systems (1). Examples of quadratic systems are given for which a parameter does not rotate the vector field and the corresponding function of limit cycles is multivalued and it presents a turning point.

Furthermore we study the question of the existence of limit cycles around a weak focus in a quadratic system having a critical case of stability for a separatrix cycle with two saddles at infinity. After that we discuss the problem of limit cycles of system (1) by reducing it to a Lienard system.

Finally, we consider the curve of separatrix cycles in the plane of two rotating parameters for the vector field associated to a quadratic system having three finite antisaddles and one finite saddle.

2 Canonical families of quadratic systems

Canonical families of quadratic systems were introduced in the works by Ye Yanqian [1], I. S. Kukles, A. Zegeling, L. M. Perko and others. They are the following:

$$\frac{dx}{dt} = -y(1+ax+My)$$
$$\frac{dy}{dt} = x+\Lambda y + Ax^2 + Bxy + Cy^2\,. \tag{I}$$

$$\frac{dx}{dt} = 1+xy$$
$$\frac{dy}{dt} = a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2\,. \tag{II}$$

$$\frac{dx}{dt} = -(x+1)y+\alpha Q(x,y)$$
$$\frac{dy}{dt} = Q(x,y)\,,\quad Q = x+\lambda y+ax^2+b(x+1)y+cy^2\,. \tag{III}$$

$$\frac{dx}{dt} = P(x,y)\,,\quad P=-y(1+\gamma y)+b_{11}xy+b_{02}y^2$$
$$\frac{dy}{dt} = x(1+\gamma y)-x^2+a_{02}y^2+\beta P(x,y)\,. \tag{IV}$$

$$\frac{dx}{dt} = a_{11}xy + f(x,y)$$
$$\frac{dy}{dt} = b_{11}xy + f(x,y)\,, \quad f = 1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2\,. \tag{V}$$

$$\frac{dx}{dt} = a_1x(1-x) + b_1y(1-y) + (\alpha_1a_1 + \alpha_2b_1)xy\,,$$
$$\frac{dy}{dt} = b_2x(1-x) + a_2y(1-y) + (\alpha_1b_2 + \alpha_2a_2)xy\,, \tag{VI}$$

$$\alpha_1 = \frac{\alpha - 1}{\beta}\,, \quad \alpha_2 = \frac{\beta - 1}{\alpha}\,,$$
$$0 < \alpha < 1\,, \quad 0 < \beta < 1\,, \quad \alpha + \beta < 1\,.$$

$$\frac{dx}{dt} = A_{00} + A_{10}x + A_{01}y + A_{20}x^2 + A_{02}y^2$$
$$\frac{dy}{dt} = B_{00} + B_{10}x + B_{01}y + B_{20}x^2 + B_{02}y^2\,. \tag{VII}$$

All coefficients of systems I-VII are real numbers. It is necessary to note that systems I-V are generic, i.e. every system (1) can be reduced to them except some degenerate systems. Systems VI and VII are not generic. System (1) is reduced to system VI if it has four finite singular points: three saddles and one antisaddle or three antisaddles and one saddle. The next theorem gives an answer to the problem of reduction of system (1) to the VII.

Theorem 1. *In generic case system* (1) *is reduced to system VII if and only if there exists a number* $k \in \mathbb{R}$ *for which the quadratic form*

$$U(x,y) = \sum_{i+j=2} (a_{ij} + kb_{ij})x^iy^5 \tag{5}$$

has a definite sign.

Proof. We write system (1) in matrix form

$$\frac{dx}{dt} = A_0 + A_1x + (x^TAx, x^TBx)^T\,,$$

where $x = (x_1, x_2)^T$, $A_0 = (a_{00}, b_{00})^T$, $A_1 = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}$, $A = \begin{pmatrix} a_{20} & a_{11}/2 \\ a_{11}/2 & a_{02} \end{pmatrix}$, $B = \begin{pmatrix} b_{20} & b_{11}/2 \\ b_{11}/2 & b_{02} \end{pmatrix}$.

Sufficiency. If the condition of the theorem is satisfied, then we consider the next transformation of the phase variables

$$s = S\widetilde{x}\,, \qquad S = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}.$$

In new variables system (1) is the following

$$\frac{d\widetilde{x}}{dt} = S^{-1}A_0 + S^{-1}A_1S\widetilde{x} + S^{-1}\begin{pmatrix} \widetilde{x}^T S^T A S\widetilde{x} \\ \widetilde{x}^T S^T B S\widetilde{x} \end{pmatrix} . \tag{6}$$

Since $S^{-1} = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$\frac{d\widetilde{x}}{dt} = \widetilde{A}_0 + \widetilde{A}_1\widetilde{x} + \begin{pmatrix} \widetilde{x}^T(\widetilde{A} + k\widetilde{B}) \\ \widetilde{x}^T\widetilde{B}\widetilde{x} \end{pmatrix} , \quad \widetilde{A} + k\widetilde{B} = S^T(A + kB)S .$$

Since the quadratic form (5) has a definite sign then $\widetilde{A} + k\widetilde{B}$ also has a definite sign.

Then there exists a transformation $\widetilde{x} = T\widetilde{\widetilde{x}}$ such that the quadratics forms with matrices $\widetilde{A} + k\widetilde{B}$ and $\widetilde{B}$ are reduced to canonical form simultaneously. Taking into account the positivity of the first quadratic form we obtain the system VII:

$$\frac{d\widetilde{\widetilde{x}}}{dt} = \widetilde{\widetilde{A}}_0 + \widetilde{\widetilde{A}}_1\widetilde{\widetilde{x}} + T^{-1}\begin{pmatrix} \widetilde{\widetilde{x}}_1^2 + \widetilde{\widetilde{x}}_2^2 \\ \mu_1\widetilde{\widetilde{x}}_1^2 + \mu_2\widetilde{\widetilde{x}}_2^2 \end{pmatrix} .$$

Necessity. System (1) is reduced to system VII. For instance, for the system

$$\frac{dx}{dt} = A_0 + A_1x + \begin{pmatrix} \lambda_1x_1^2 + \lambda_2x_2^2 \\ \mu_1x_1^2 + \mu_2x^2 \end{pmatrix} , \quad \lambda_1, \lambda_2, \mu_1, \mu_2 \neq 0.$$

We consider the quadratic form

$$W = (\lambda_1 + \rho\mu_1)x_1^2 + (\lambda_2 + \rho\mu_2)x_2^2 .$$

There exists a number $\rho \in \mathbb{R}$ such that

$$(\lambda_1 + \rho\mu_1)(\lambda_2 + \rho\mu_2) > 0 .$$

It means that W has a definite sign.

Since a linear transformation does not change the definiteness of a quadratic form, there exists a number $k \in \mathbb{R}$ such that the quadratic form U has a definite sign. □

From theorem 1 it is easy to prove the next result.

Theorem 2. *Assume that all finite and infinite singular points of system* (1) *are simple. Then it can be transformed to a system VII if and only if its finite singular points are two saddles and two antisaddles, or one saddle and one antisaddle.*

In every system I-VII we can introduce a rotating parameter for the vector field in some domain and consider the corresponding function of limit cycles. Thus the parameter Λ in system I rotates its vector field in every half-plane $1 + ax + by > 0$ (< 0). The parameter a_{11} in system II rotates the vector field in every half-plane $x > 0$ (< 0) if the parameter a_{00} is replaced by $a_{00} + a_{11}$. In system III parameters α and b rotate the vector field in the whole plane and the parameter λ in every half-plane $x > -1$ (< -1) in contrary directions. In system IV the parameters γ and β rotate the vector field in the whole plane (assuming that it has only two finite singular points).

Since systems I-V are generic, they can be transformed into the other one by means of an affine transformation of the phase variables and a change of the time scale. We consider these transformations in more detail.

Transition from III to I. Let $a \neq 0$ then after the transformation

$$x = aX + \alpha aY\,, \qquad y = aY$$

we obtain the next system I

$$\frac{dX}{dt} = -Y(1 + aX + \alpha aY)$$
$$\frac{dY}{dt} = X + (\alpha + \lambda + b)Y + a^2X^2 + a(2a\alpha + b)XY + a(a\alpha^2 + b\alpha + c)Y^2\,.$$

Transition from (1) to III. Assume that system (1) has the finite singular points (x_i, y_i), $i = 1, 2$, $x_1 \neq x_2$. Then the change of variables

$$x = \widetilde{x} + x_1$$
$$y = \widetilde{y} + y_1 + k\widetilde{x}\,, \qquad k = (y_2 - y_1)/(x_2 - x_1)$$

transforms (1) into the system

$$\frac{d\widetilde{x}}{dt} = K_2(\widetilde{x} - \frac{1}{x_2 - x_1}\widetilde{x}^2) + \widetilde{y}(A_0 + A_1\widetilde{x} + A_2\widetilde{y})\,.$$
$$\frac{d\widetilde{y}}{dt} = K_1(\widetilde{x} - \frac{1}{x_2 - x_1}\widetilde{x}^2) + \widetilde{y}(B_0 - kA_0 + (B_1 - kA_1)\widetilde{x} + (B_2 - kA_2)\widetilde{y})\,,$$

where

$$K_1 = -(b_{20} + b_{11}k + b_{02}k^2 - k(a_{20} + a_{11}k + a_{02}k^2))(x_2 - x_1)\,,$$
$$K_2 = -(a_{20} + a_{11}k + a_{02}k^2)(x_2 - x_1)\,,$$
$$A_0 = a_{01} + a_{11}x_1 + 2a_{02}y_1\,, \qquad B_0 = b_{01} + b_{11}x_1 + 2b_{02}y_1\,,$$
$$A_1 = 2a_{02}k + a_{11}\,, \quad A_2 = a_{02}\,, \quad B_1 = 2b_{02}k + b_{11}\,, \quad B_2 = b_{02}\,.$$

The second transformation

$$\tilde{x} = \tilde{\tilde{x}} + (K_2/K_1)\tilde{\tilde{y}}, \qquad \tilde{y} = \tilde{\tilde{y}}$$

writes the system in the form I

$$\begin{aligned} \frac{d\tilde{\tilde{y}}}{dt} &= K_1\tilde{\tilde{x}} + A_{01}\tilde{\tilde{y}} + A_{20}\tilde{\tilde{x}}^2 + A_{11}\tilde{\tilde{x}}\tilde{\tilde{y}} + A_{02}\tilde{\tilde{y}}^2, \\ \frac{d\tilde{\tilde{x}}}{dt} &= \tilde{\tilde{y}}(\widehat{a}\tilde{\tilde{x}} + \widehat{b}\tilde{\tilde{y}} + \widehat{c}), \end{aligned}$$

where

$$\begin{aligned} A_{01} &= K_2 + B_0 - kA_0\,, A_{20} = -K_1/(x_2 - x_1)\,, \\ A_{11} &= -2K_1/(x_2 - x_1) + B_1 - kA_1\,, \\ A_{02} &= -K_2^2/(K_1(x_2 - x_1)) + (B_1 - kA_1)K_2/K_1 + B_2 - kA_2\,, \\ \widehat{a} &= A_1 - (B_1 - kA_1)K_2/K_1\,, \\ \widehat{b} &= A_2 - (B_2 - kA_2)K_2/K_1 + \widehat{a}K_2/K_1 \\ \widehat{c} &= A_0 - (B_0 - kA_0)K_2/K_1\,. \end{aligned}$$

Now after the rescaling of the variables

$$\tilde{\tilde{x}} = pX\,, \qquad \tilde{\tilde{y}} = qY\,,$$

where $p = \widehat{c}/\widehat{a}$, $q = \sqrt{-K_1\,\widehat{c}/\widehat{a}^2}$, $K_1\widehat{c} < 0$ the system becomes

$$\begin{aligned} \frac{dY}{dt} &= X + \Lambda Y + AX^2 + BXY + CY^2\,, \\ \frac{dX}{dt} &= -Y(1 + X + \alpha Y)\,, \end{aligned}$$

where

$$\begin{aligned} &\Lambda = A_{01}q/(K_1p)\,, \quad A = A_{20}p/K_1\,, \\ &B = A_{11}q/K_1\,, \quad C = A_{02}q^2/(K_1p)\,, \quad \alpha = \widehat{b}q/\widehat{c}\,. \end{aligned}$$

Finally the transformation

$$X = x - \alpha y\,, \qquad Y = y$$

leads to canonical system III, where

$$b = B - 2A\alpha\,, \quad a = A\,, \quad c = A\alpha^2 - B\alpha + C\,, \quad \lambda = \Lambda - \alpha - b\,.$$

System IV is a small variation of system I and it can be transformed easily into system III.

Transition from system III to II. In system III make the transformation

$$x = X + \frac{1}{\eta} y$$
$$y = Y \,, \quad \eta \neq 0 \,, \quad (\alpha - \frac{1}{\eta})(\frac{a}{\eta}^2 + \frac{b}{\eta} + c) - \frac{1}{\eta} = 0 \,.$$

Then we obtain the system

$$\frac{dX}{dt} = AX + BY + CX^2 + DXY \,,$$
$$\frac{dY}{dt} = X + \Lambda Y + aX^2 + LXY + MY^2 \,,$$

where

$$A = \alpha - \frac{1}{\eta} \,, \quad \Lambda = \lambda + b + \frac{1}{\eta} \,, \quad L = \frac{2a}{\eta} + b \,,$$
$$M = \frac{a}{\eta^2} + \frac{b}{\eta} + c \,, \quad B = \Lambda A - 1 \,,$$
$$C = aA \,, \quad D = AL - 1 \,.$$

Now doing the second transformation

$$X = \widetilde{x} - \frac{B}{D} \,, \qquad Y = S\widetilde{y} - \frac{C}{D}\widetilde{x} + T \,,$$

where

$$S = \frac{CB^2 - ABD}{D^3} \,, \qquad T = \frac{2BC - AD}{D^2} \,,$$

we obtain the canonical system III

$$\frac{d\widetilde{x}}{dt} = 1 + \widetilde{x}\widetilde{y} \,, \quad \frac{dy}{dt} = \sum_{i`j=0}^{2} a_{ij} \widetilde{x}^i \widetilde{y}^j \,,$$

where

$$a_{00} = \frac{1}{DS^2}\left(-\frac{B}{D} + \Lambda T + a\frac{B^2}{D^2} - \frac{LBT}{D} + MT^2 + CS\right) \,,$$
$$a_{10} = \frac{1}{DS^2}\left(1 - \Lambda\frac{C}{D} - 2a\frac{B}{D} + LT + \frac{LBC}{D^2} - 2\frac{MCT}{D}\right) \,,$$
$$a_{01} = \frac{1}{DS^2}\left(\Lambda S - \frac{LBS}{D} + 2MST\right) \,,$$
$$a_{20} = \frac{1}{DS^2}\left(a - \frac{LC}{D} + \frac{MC^2}{D^2}\right) \,,$$

$$a_{11} = \frac{1}{DS^2}\left(LS - 2\frac{MSC}{D} + CS\right),$$
$$a_{02} = \frac{M}{D}.$$

System V is a small variation of system III and it can be transformed easily to it.

From the analysis of systems I-V we can conclude that the vector field associated to system III has the larger number of rotating parameters. So it is convenient for investigating in the case of four finite singular points. On the contrary system IV is the most convenient for using in the case of two finite singular points. Such system was studied in our work [9]. Therefore we will consider now system III in more detail. In the generic case when the condition $ac > 0$ is satisfied it has four finite singular points: two saddles and two antisadddles, or two ones: one saddle and one antisaddle. If $ac < 0$ then it has one of the following distributions of finite singular points: $3S + 1A$, $3A + 1S$, $2A$, where S means a saddle and A an antisaddle. We consider now the limit cycles of system III around the focus $O(0,0)$, if $\alpha + \lambda + b = 0$ it is a weak focus. In addition if the first Lyapunov value is not equal to zero, then after changing every of parameters α, λ, b a little amplitude limit cycle appears.

So we have three functions of limit cycles $\lambda = F_1(x)$, $b = F_2(x)$, $\alpha = F_3(x)$. It is better to use the two first functions because if we use the third one, then when the parameter α pass through the value zero the invariant line $x = 0$ appears and it originates in some cases structural unstability of system III, see [5]. The eigenvalues of the linearization matrix of system III in the point $O(0,0)$ do not depend on a and c, that is they cannot change the stability of the stronger focus $O(0,0)$ and an Andronov-Hopf bifurcation can not take place. But if we consider system III with a weak focus O, then a and c can change the stability of O and give a bifurcation of a limit cycle generated by changing of nonlinear part of system III. We consider this in more detail. Assume that equality $\alpha + \lambda + b = 0$ holds. Then the singular point O is a weak focus and his stability is defined by the first Lyapunov value

$$\begin{aligned} L_1 &= a^2\alpha(1+\alpha^2) + ab(1+3\alpha^2) + b^2\alpha + a(2\alpha^3 + 2\alpha c) + \\ &\quad + b(2\alpha^2 + c) + 2\alpha c - \alpha = \\ &= 2\alpha(1+\alpha^2)(a + \frac{1}{2\alpha}b + \frac{\alpha^2}{\alpha^2-1})(a + \frac{\alpha}{\alpha^2+1}b + \frac{-2\alpha^2 + c(\alpha^2-1)}{\alpha^2-1}) + \\ &\quad + \frac{\alpha}{(\alpha^2-1)^2}(3\alpha^4 + 2\alpha^2 - 1 - 2c(\alpha^2-1)). \end{aligned} \tag{7}$$

In the generic case the equation $L_1 = 0$ defines a hyperbola on the plane (b, a). From formula (7) it follows that equation $L_1 = 0$ defines a pair of straight lines if

$$\Phi = 3\alpha^4 + 2\alpha^2 - 1 - 2c(\alpha^2 - 1) = 0\,, \qquad \alpha \neq 0\,, \quad \alpha \neq \pm 1\,.$$

If $\alpha > 0$, $\Phi < 0$ then equation $L_1 = 0$ have two different roots with respect to a. If $\alpha > 0$, $\Phi < 0$ then there exists two values of b, b_1, b_2, $b_1 < b_2$, which are defined by equation

$$\begin{aligned} &b^2(\alpha^2 - 1)^2 + b[4\alpha c(\alpha^2 - 1) - 4\alpha^3(\alpha^2 + 3)] + \\ &+ 4\alpha^2 c^2 - 16\alpha^2 c - 8\alpha^4 c + 4\alpha^6 + 8\alpha^4 + 8\alpha^2 = 0\,, \end{aligned} \tag{8}$$

such that by $b_1 < b < b_2$ the equation $L_1 = 0$ has no roots with respect to a. For $b = b_1$ or b_2 it has a multiple root and for $b \notin [b_1, b_2]$ two different roots a_1, a_2. If in the case of different roots the second focus value L_2 is not equal zero, then by changing the parameter a in every neighbourhood of the points a_1, a_2 a small amplitude limit cycle appears around the focus $O(0, 0)$. It means, that in some interval $(0, \delta)$ there exist two functions of limit cycles $a = \varphi_k(x)$, $\varphi_k(0) = a_k$, $k = 1, 2$. As above the function $\varphi_k(x)$ is equal to a value of a for which system III has a limit cycle passing through the point $(x, 0)$. If now we change one of the rotating parameters of the vector field, for instance λ, then a curve of limit cycles appears near the curves $a = \varphi_k(x)$ and the interval $[a_1, a_2]$ of Oa axis. But in this case it should have a turning point, that is the function of limit cycles is not uniquely defined. This is a main difference between a rotating parameter of the vector field from a non rotating one. We consider a concrete example.

Example 1. Let $c = 0.82$; $\alpha = 0.6$; $b = -0.3$; $\lambda = -0.3$. Then $a_1 = 0.03$; $a_2 = 0.516$. By using numerical computations it is easy to obtain the table of values of the functions $a = \varphi_k(x)$, $k = 1, 2$ (Table 1).

x	0.1	0.2	0.3	0.4	0.5
φ_1	0.028	0.023	0.015	0.0005	-0.026
φ_2	-0.517	-0.519	-0.525	-0.535	-0.550

Table 1

We see from Table 1 that $\varphi_1(x)$ changes its sign in the interval $(0.4; 0.5)$. It means that the configuration of singular points changes too.

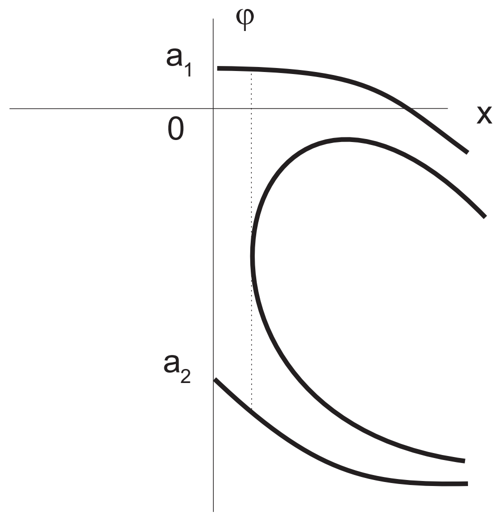

Figure 1

In Figure 1 the curve of limit cycles for some $\lambda < 0$ is given. In the given example it is shown that we can define the curve of limit cycles at every point (x, a) of the plane xOa which corresponds to a limit cycle of system III passing through the point $(x, 0)$ for given value of the parameter a.

The curve can have complicated behaviour, two branches or a turning point, or define a non unique function of limit cycles. The endpoints of the curve of the limit cycles correspond to a bifurcation of an appearance of separatrix cycles, in our case a saddle loop.

We consider now a curve of limit cycles associated with the parameter c. As before we suppose that $O(0,0)$ is a weak focus that is $\alpha + \lambda + b = 0$. A bifurcation value of c is defined for the equation $L_1 = 0$, from which we have

$$c^* = -\frac{(2a\alpha + b)(a\alpha^2 + b\alpha + a + \alpha^2) - \alpha + b\alpha^2}{2\alpha(1+a) + b}, \quad 2\alpha(1+a) + b \neq 0\,.$$

After changing the parameter c ($\Delta c > 0$ or $\Delta c < 0$) a limit cycle appears that is the condition $L_1(c^* + \Delta c)L_2(c^*) < 0$ is necessary for the birth of a limit cycle. If we calculate L_2 taking into account $L_1 = 0$ then

$$L_2(c^*) = \frac{\alpha^3(2a\alpha + b)^2(5\alpha + 2a\alpha + b)}{(-a)(2a\alpha + b + 2\alpha)^2}\left(-b^2(1+a) - 2b\alpha(1 + 2a + 2a^2) - \right.$$
$$\left. -4a^2\alpha^2(1+a) + (1+2a)^2(1-a)\right)\,.$$

From that we have

$$\Delta c(2a\alpha + b + 2\alpha)L_2(c^*) < 0\,.$$

Moreover we know the behaviour of the function of limit cycles $c = \varphi(x)$, $\varphi(0) = c^*$ in a neighborhood of the point $x = 0$: if $\Delta c > 0$ (< 0) then it increases (decreases). Let us consider the second endpoint M^* of the curve Γ: $c = \varphi(x)$. In the simplest case we suppose that a loop of the saddle $S(-\frac{1}{a}, 0)$ corresponds to the point M^*. It takes place for $ac > 0$, $\alpha > 0$ and $a < 0$ or $a > 1$. Then $\operatorname{div} f(S) \neq 0$ because the quadratic system has no limit cycles if $\operatorname{div} f$ is equal zero in its two singular points. If a limit cycle which appears from O turns into the loop, then it should be unstable (stable) by $L_2(c^*) > 0$ (< 0). That is $\operatorname{div} f(S) = \lambda - \alpha + b(1 - \frac{1}{a}) > 0$ (< 0). So the function $\varphi(x)$ has no extrema by

$$L_2(c^*) \operatorname{div} f(S) > 0\,. \tag{9}$$

If we suppose that a semistable limit cycle appears then from condition (9) we have an odd number of extrema of the function $\varphi(x)$. On the contrary if the condition

$$L_2(c^*) \operatorname{div} f(S) < 0 \tag{10}$$

holds, then we have an even number of extrema of $\varphi(x)$. Suppose that the function $\varphi(x)$ has at most one extremum this is the natural conjecture for a quadratic system. Then the condition (10) is the condition for one extremum of $\varphi(x)$. Unfortunately our arguments are not true if the configuration of singular points changes with the parameter c. In this case a separatrix cycle can be a loop of another saddle. This can be observed in a concrete system. It is easy to find a system III for which the function $\varphi(x)$ changes its sign.

The arguments are as follows.

Consider for instance a system III with $b = 0.4$; $\alpha = -0.2$; $\lambda = -0.2$; $c = 0$ and find one of the two functions of limit cycles $a = a_2(x)$ numerically. We obtain Table 2.

x	0.2	0.3	0.4	0.5
a	-0.345013	-0.343085	-0.340627	-0.337729

Table 2

Now put $a = -0.337729$ and find the function of limit cycles $c = \varphi(x)$.

x	0.2	0.3	0.4	0.5	0.6	0.7
c	-0.037	-0.0264	-0.0134	0.0000	0.0134	0.0265

Table 3

From Table 3 we see that the function $\varphi(x)$ changes its sign near the point $x = 0.5$. Consider now a possibility of realisation of condition (10). If we find a system for which the function $c = \varphi(x)$ has one extremum then it means that for some values of the parameter c system III has two limit cycles. It is easy to obtain a system with three limit cycles by changing one of the rotating parameter of the vector field. Assume that system III has $3S + 1A$; that is, it has three finite saddles and one finite antisaddle. Then we can suppose $\alpha > 0$, $a < 0$, $c > 0$, $\alpha + \lambda + b = 0$. For $a = -1$ we have

$$c^* = -(b - \alpha)(b\alpha - 1) \, / \, b \, .$$

For positivity c^* it is necessary that $b \in (-\infty, 0) \cup (1/\alpha, \alpha)$, $(\alpha > 1)$ or $b \in (-\infty, 0) \cup (\alpha, 1/\alpha)$, $(0 < \alpha < 1)$. Then $L_2(c^*) \operatorname{div} f(S) < 0$ is equivalent to $(b < -3\alpha) \cap (b > 1/\alpha)$. A numerical example $\alpha = 0.2$; $b = -0.7$; $\lambda = 0.5$; $a = -1$; $c^* = 1.4657$ shows that the function $c = \varphi(x)$ has one extremum indeed.

Consider a system III with $2A+2S$ and $a < 0$, $c < 0$, $D = \lambda^2 - 4c(a-1) > 0$, $\alpha + \lambda + b = 0$. This is the numerical example with one extremum of $c = \varphi(x)$: $\alpha = 0.03455$; $b = -0.206725$; $\lambda = 0.1721625$; $a = -0.125$; $c^* = -0.04489$.

Let us give one more example of such kind of system with another distribution of singular points on the lines $x = -1$ and $y = 0$ for $a > 1$, $c > 0$, $D > 0$: $\alpha = 0.2$; $b = -7$; $\lambda = 6.8$; $a = 1.1$; $c^* = 0.05$. It is easy to find a system of such kind satisfying conditions $0 < a < 1$, $c > 0$ reducing it to one of the previous systems.

The question arises whether there is a curve of limit cycles associated to the parameter c with at least two branches. It seems that the answer is negative because the bifurcation value c^* is unique. But it is not true. Consider a system III with $a = -0.4$; $b = -0.4$; $\alpha = 0.4$; $\lambda = 0$ and we will find for it a function $c = \varphi(x)$, $\varphi(0) = c^* = 2.776$. It is defined in a close interval $[0, x_1]$, $x_1 \approx 0.6$, and decreases. Its point x_1 corresponds to a loop of the saddle $S_1(-1, -1.2)$. Calculatings show that there are two functions $c = \varphi_i(x)$, $i = 1.2$, $\varphi_1(x_0) = \varphi_2(x_0)$, $x_0 \approx 0.43$. They give one more branch of the curve of the separatrix cycles. We bring out the tables of their values.

x	0.1	0.2	0.3	0.4	0.5	0.6
φ	1.56	1.39	1.25	1.13	1.04	0.97

Table 4

x	0.43	0.5	0.6	0.8	1.0	1.2	1.4	1.8	2.0	2.2	2.4	2.5
φ_1	-5.5	-2.71	-1.67	-0.84	-0.48	-0.27	-0.15	0.006	0.05	0.08	0.10	0.15

Table 5

x	0.43	0.47	0.57	$\cdots$	1.17	1.45
φ_2	-5.5	-7.6	-8.6	$\cdots$	-9.26	-9.17

Table 6

The last point (2.5, 0.15) of Table 5 corresponds to a loop of the saddle $S_2(2.5;0)$. It is difficult to make calculations for $x > 1.45$ in Table 6 but it is easy to prove that the last point of the curve $c = \varphi_2(x)$ for $x = 2.5$ corresponds to another loop of the same saddle S_2. So we have two loops of the saddle S_2 for different values of c. It is the first unexpected property of quadratic system to have a curve of limit cycles associated with the parameter c, and consisting of two branches. One of them begins at the point of a Andronov-Hopf bifurcation and finishes at the point of bifurcation where a saddle loop appears. The second branch has the turning point (0.43; -5.5) and its both ends correspond to different loops of the same saddle $S_2(2.5;0)$. This surprising property is possible to explain only because the parameter c does not rotate the vector field and its change in the same direction gives two different bifurcations of separatrix cycle. It is impossible for a rotating parameter of the vector field.

3 Quadratic system with a weak focus and the condition $\lambda_1\lambda_2 = \mu_1\mu_2$ for the eigenvalues $\lambda_i < 0 < \mu_i$, $i = 1,2$ of the saddles at infinity

If a quadratic system has a weak focus then it cannot have a separatrix loop with a critical stability; that is, a loop of saddle S for which $\operatorname{div} f(S) = 0$. By analogy it seems that a quadratic system having a weak focus does not have a separatrix cycle with two saddles at infinity and a critical stability, i.e. the condition of the title.

For studying of a such systems we use canonical family II. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 1 + xy \\ \frac{dy}{dt} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + ay^2 \,.\end{aligned} \tag{11}$$

By the transformation $x = 1/\xi$, $y = Y/\xi - \xi$, we obtain the system

$$\frac{d\xi}{dt} = -Y \,, \quad \frac{dY}{dt} = \frac{1}{\xi}(P_4(\xi) + P_2(\xi)Y + (a-1)Y^2) \,, \tag{12}$$

where

$$\begin{aligned}P_4(\xi) &= a_{20} + a_{10}\xi + (a_{00} - a_{11})\xi^2 - a_{01}\xi^3 + a\xi^4 \,, \\ P_2(\xi) &= a_{11} + a_{01}\xi - (2a+1)\,\xi^2 \,.\end{aligned}$$

Multiplying the right-hand sides of this system by $-\xi$ we have the next one

$$\frac{d\xi}{dt} = \xi Y \,, \quad \frac{dY}{dt} = -P_4(\xi) - P_2(\xi)Y - (a-1)Y^2 \,. \tag{13}$$

System (13) has an invariant line $\xi = 0$. Therefore we can study it in every half-plane $\xi > 0$, $\xi < 0$ separately. Below we consider it in half-plane $\xi > 0$. By the transformation $x = \alpha X$, $y = \beta Y$, we can make that a singular point A of system (11) has coordinates $x = 1$, $y = -1$. Singular point $A(1,-1)$ of system (11) corresponds to the one $\xi = 1$, $Y = 0$ of system (13). The straight line $x = 0$ is transversal and limit cycles or separatrix cycles of system (11) do not intersect it. Therefore we suppose that system (11) has only two finite singular points (one of them is the weak focus $A(1,-1)$ and the other one is an antisaddle in the half-plane $x < 0$), and two saddles and the node at infinity, the node is in the direction of $x = 0$. The condition $\lambda_1\lambda_2 = \mu_1\mu_2$ is equivalent to the one $a_{11} = 0$. So for the complete description of the given above conditions it is necessary that the next conditions hold

$$\begin{aligned}&a_{11} = 0 \,, \quad a_{01} - 2a - 1 = 0 \,, \quad a_{20} < 0 \,, \quad a > 1 \,, \\ &1 + a_{10} + 2a_{20} < 0 \,, \quad a_{00} + a_{10} + a_{20} - a - 1 = 0 \,.\end{aligned} \tag{14}$$

By a transformation of variables the saddles of system (11) at infinity correspond to the ones of system (13) on the line $\xi = 0$.

Using the transformation $Y = \xi^{1-a}y$ it is easy to get the following Lienard system

$$\begin{aligned}&\frac{d\xi}{dt} = y \,, \quad \frac{dy}{dt} = -P_4(\xi)\xi^{2a-3} - P_2(\xi)\xi^{a-2}y \,, \\ &f(\xi) = P_2(\xi)\xi^{a-2} \,, \quad g(\xi) = P_4(\xi)\xi^{2a-3} \,.\end{aligned} \tag{15}$$

Then we obtain

$$G(x) = \int_1^x g(\xi)d\xi = \widetilde{P}_4(x)x^{2a-2} - \widetilde{P}_4(1)$$
$$F(x) = \int_1^x f(\xi)d\xi = \widetilde{P}_2(x)x^{a-1} - \widetilde{P}_2(1)\,, \qquad (15')$$

where

$$\widetilde{P}_4(\xi) = \frac{a_{20}}{2a-2} + \frac{a_{10}}{2a-1}\xi + \frac{1+a-a_{10}-a_{20}}{2a}\xi^2 - \xi^3 + \frac{a}{2a+2}\xi^4\,,$$
$$\widetilde{P}_2(\xi) = \frac{2a+1}{a}\xi - \frac{2a+1}{a+1}\xi^2\,.$$

System (13) does not have a limit cycle and separatrix cycle if the system

$$G(\xi) = G(\eta)\,, \qquad F(\xi) = F(\eta) \qquad (16)$$

does not have solutions in the region $0 < \eta < 1$, $1 < \xi < +\infty$ [1]. System (16) is equivalent to the next one

$$\begin{aligned} &\frac{a_{20}}{2a-2}(1-p^{2a-2}) + \frac{a_{10}}{2a-1}(1-p^{2a-1})\xi - \frac{a_{10}+a_{20}}{2a}(1-p^{2a})\xi^2 = 0 \\ &\frac{1}{a}(1-p^a) - \frac{1}{a+1}(1-p^{a+1})\xi = 0\,, \\ &p = \eta/\xi\,, \qquad 0 < p < 1\,. \end{aligned} \qquad (17)$$

Elimination of the variable ξ in system (17) gives us only one equation with respect to p, namely

$$\begin{aligned} R(p) = &\frac{a_{20}}{2a-2}(1-p^{2a-2})(1-p^{a+1})^2 + \\ &+ \frac{a_{10}}{2a-1}\frac{a+1}{a}(1-p^{2a-1})(1-p^a)(1-p^{a+1}) - \\ &- \frac{(a_{20}+a_{10})}{2a^3}(1-p^{2a})(1-p^a)^2 = 0\,, \qquad 0 < p < 1\,. \end{aligned} \qquad (18)$$

We transform the function R to the next

$$R = a_{20}R_1 + a_{10}R_2\,, \qquad (19)$$

where

$$R_1(p,a) = \frac{1}{2a-2}(1-p^{2a-2})(1-p^{a+1})^2 - \frac{(a+1)^2}{2a^3}(1-p^{2a})(1-p^a)^2\,,$$
$$\begin{aligned} R_2(p,a) = &\frac{a+1}{a(2a-1)}(1-p^{2a-1})(1-p^a)(1-p^{a+1}) \\ &- \frac{(a+1)^2}{2a^3}(1-p^a)(1-p^{2a})^2\,. \end{aligned}$$

Then equation (18) is

$$-\frac{a_{10}}{a_{20}} = \frac{R_1(p,a)}{R_2(p,a)} \equiv r(p,a)\,. \tag{20}$$

When a is fixed equation (20) has a solution if a_{10}, a_{20} satisfy the conditions (14) and consequently

$$m = \min_{p\in[0,1]} r(p,a) < -\frac{a_{10}}{a_{20}} < \max_{p\in[0,1]} r(p,a) = M\,. \tag{21}$$

The function $r(p,a)$ has a removable singularity for $p=1$. For $a=2$ we have $r(p,a)\equiv 1$. In this case for $a_{20}+a_{10}=0$ the singular point $A(-1,1)$ is a center. It corresponds to the most complicated case Q_4 of Zoladek classification [6], corresponding system (11) is

$$\frac{dx}{dt} = 1 + xy\,, \quad \frac{dy}{dt} = 3 + a_{20}(x^2 - x) + 5y + 2y^2\,. \tag{22}$$

A more careful analysis of the function $r(p,a)$ shows that for every $a>1$, $a\neq 2$ it has a finite minimum and finite maximum in (21) for which $1 < a < 2$, then

$$m = \frac{(a^2-a-1)(2a-1)}{(a-1)^2(a+2)}\,, \tag{23}$$

and it is reached at $p=0$,

$$M = \frac{2a-3}{a-1}\,, \tag{24}$$

and it is reached at $p=1$ and corresponds to the appearance of a zero of the function $R(p)$ by change for example the parameter a_{01} for fixed a_{20}. If $a>2$, then the word minimum it is necessary to replace of the word maximum, that is, M is defined by formula (23) and m by (24). It is known if in the function $R(p)$ a zero appears in interval $(0,1)$ at the point $p=1$ by change the parameter a_{10} then it means a birth of a limit cycle of system (15) from the point (1,0) through an Andronov-Hopf bifurcation. By change of a_{10} from minimum until maximum this zero disappears at the point $p=0$. It means that all limit cycles disappear and at some moment appears a separatrix cycle. Consider the next example:

$$a_{02} = -3\,;\; a = 1.5\,; a_{01} = 4\,; a_{11} = 0\,; a_{00} = 5.5 - a_{10}\,;$$

a_{10} varies in the close interval $[0; 2.4]$. Then zero of the function $R(p)$ appears for $p=1$ and disappears at $p=0$.

Simultaneously from a weak focus $A(1,-1)$ of system (11) appears a limit cycle which turns into a separatrix cycle with two saddle at infinity an a critical stability for $a_{10} \approx 1.5$. So our conjecture is not true and this is the second unexpected property of the limit cycles of a quadratic system.

Theorem 3. *Suppose that quadratic system* (11) *has only a finite focus* $A(1,-1)$ *and finite antisaddle* $B(-\beta, 1/\beta)$, $\beta > 0$; *and at infinity a node in the direction of* Oy *axis and two saddles satisfying the condition* $\lambda_1\lambda_2 = \mu_1\mu_2$ *for their eigenvalues* $\lambda_k < 0 < \mu_k$, $k = 1, 2$, *i.e. the next conditions hold*

1. $a > 1$, $a \neq 2$, $a_{02} < 0$, $a_{01} = 2a+1$, $a_{00} = 1 + a - a_{10} - a_{20}$, $1 + a_{10} + 2a_{20} < 0$, $a_{11} = 0$;

2. *the polynomial*

$$P_4(x) = a_{20} + a_{10}x + a_{00}x^2 - (2a+1)x^3 + ax^4$$

is

$$P_4(x) = a(x-1)(x+\beta)(x^2 + px + q)$$

with $p < 0$, $1 + \frac{1}{a} < |p| < q + 1$, $q > \frac{1}{4}(1 + \frac{1}{a})^2$.

If the next condition holds

3. $m = \frac{2a-3}{a-1} < -\frac{a_{10}}{a_{20}} < \frac{(a^2-a-1)(2a-1)}{(a-1)^2(a+1)} = M$, $1 < a < 2$; $M < -\frac{a_{10}}{a_{20}} < m$, $a > 2$,

then system (12) *can have a limit cycle and a separatrix cycle with two saddles at infinity and a critical stability.*

This is realized by a concrete example. If the condition (3) is not satisfied, then system (11) does not have a limit cycles and a separatrix cycle around the focus $A(1,-1)$. From conditions (1) and (2) system (11) does not have a limit cycle and a separatrix cycle around the second antisaddle.

Now it is known that a quadratic system having $3S + 1A$ can have a separatrix cycle with two saddles at infinity. Then under condition (2) of Theorem 3 the polynomial $x^2 + px + q$ has two roots if $x > 1$ or $x < 0$. In this case our arguments are indeed true if moving the parameter a_{10} gives a separatrix cycle with two singular points at infinity and they are not true if a saddle loop appears. So in this case it is necessary at first to solve the problem of distinguishing between different separatrix cycles.

4 Some conjectures about limit cycles of quadratic systems

Consider a polynomial Lienard system

$$\begin{aligned} \frac{dx}{dt} &= y - F(x) \\ \frac{dy}{dt} &= -x\,, \end{aligned} \tag{25}$$

where $F(x)$ is a polynomial of degree $2k+1$. A well-known conjecture (Smale) is that system (25) has at most k limit cycles around the unique finite singular point $O(0,0)$. If $F(x)$ is a polynomial of seventh degree, then it seems that the number of limit cycles is not greater that the number of positive roots of the odd part of $F(x)$; that is, the function $F(x) - F(-x)$. Quadratic system (11) can be reduced to the Lienard one

$$\begin{aligned} \frac{dx}{dt} &= y - F(x) \\ \frac{dy}{dt} &= -g(x)\,, \end{aligned} \tag{26}$$

where $g(x) = P_4(x)x^{2a-3}$, $f(x) = P_2(x)x^{a-2}$, $x > 0$

$$\begin{aligned} P_4(x) &= a_{20} + a_{10}x + (a_{00} - a_{11})x^2 - a_{01}x^3 + ax^4\,, \\ P_2(x) &= a_{11} + a_{01}x - (2a+1)x^2\,. \end{aligned}$$

We assume that system (11) has the focus $A(1,-1)$ which corresponds to the focus $A_0(1,0)$ of system (26). From system (26) we have

$$\begin{aligned} G(x) &= \int_0^x g(x)dx = \widetilde{P}_4(x)x^{2a-2} - \widetilde{P}_4(1)\,, \\ F(x) &= \int_0^x f(x)dx = \widetilde{P}_2(x)x^{a-1} - \widetilde{P}_2(1)\,, \end{aligned}$$

where

$$\begin{aligned} \widetilde{P}_4(x) &= \frac{a_{20}}{2a-2} + \frac{a_{10}}{2a-1}x + \frac{a_{00} - a_{11}}{2a}x^2 - \frac{a_{01}}{2a+1}x^3 + \frac{a}{2a+2}x^4\,, \\ \widetilde{P}_2(x) &= \frac{a_{11}}{a-1} + \frac{a_{01}}{a}x - \frac{2a+1}{a+1}x^2\,. \end{aligned}$$

All finite singular points of system (26) in $x > 0$ are on Ox axis. We assume that $A_1(x_1,0)$, $A_2(x_2,0)$ are the nearest singular points of (26) to the point

$A_0(1,0)$. If in the interval $(0,1)$ or $(1,+\infty)$ there are no singular points, then we put $x_1=0$ or $x_2=+\infty$.

We introduce the new variable u by formula (7)

$$u=\sqrt{2G(x)}\,\mathrm{sign}(x-1)\,. \tag{27}$$

The function (27) has the inverse one $x=\varphi(u)$, $x_1<x<x_2$, $u_1<u<u_2$, $u_k^2=2G(x_k)$, $k=1,2$. System (26) in the new variable after the corresponding transformation of time t is

$$\begin{aligned}\frac{du}{dt}&=y-\widetilde{F}(u)\,,\\ \frac{dy}{dt}&=-u\,,\end{aligned} \tag{28}$$

where $\widetilde{F}(u)=F(\varphi(u))$, $u\in(u_1,u_2)$. For some quadratic systems the function $\widetilde{F}(u)$ is well approximated by a polynomial of seventh degree and has the same number of limit cycles as system (25) with $k=3$. Then it is expected that the number of limit cycles of system (11) around the focus $A(1,-1)$ is at most the number of positive roots of the function $\widetilde{F}(u)-\widetilde{F}(-u)=F_1(u)$ in the interval $|u|<u_0$, $u_0=\min\{u_1,u_2\}$.

We will show that the number of positive roots of the function $F_1(u)$ is not greater than the number of solutions of the next system

$$\begin{aligned}G(\xi)&=G(\eta)\\ F(\xi)&=F(\eta)\,,\end{aligned} \tag{29}$$

$x_1<\eta<1$, $1<\xi<x_2$.

In order to reduce system (29) to a single equation we introduce a new variable

$$p=\eta/\xi\,,\qquad \frac{x_1}{x_2}<p<1\,. \tag{30}$$

Then system (29) takes the form

$$\sum_{k=0}^{4}A_k\xi^k=0\,,\qquad \sum_{k=2}^{4}B_k\xi^{k-2}=0\,, \tag{31}$$

where

$$A_0=\frac{a_{20}}{2a-2}(p^{2a-2}-1)\,,\qquad A_1=\frac{a_{10}}{2a-1}(p^{2a-1}-1)\,,$$

$$A_2=\frac{a_{01}-a_{10}-a_{20}-a}{2a}(p^{2a}-1)\,,\qquad A_3=-\frac{a_{01}}{2a+1}(p^{2a+1}-1)\,,$$

$$A_4=\frac{a}{2a+2}(p^{2a+2}-1)\,,$$

$$B_2 = \frac{a_{11}}{a-1}(p^{a-1}-1)\,, \quad B_3 = \frac{a_{01}}{a}(p^a - 1)\,, \quad B_4 = -\frac{2a+1}{a+1}(p^{a+1}-1)\,.$$

We do not consider some values of the parameter a, $a = 0, \pm 1, \pm 1/2, \ldots$ for which our formulas lose the sence. System (29) has a solution if and only if the resultant $R(p)$ of the polynomials of system (31) with respect to ξ is equal to zero. The resultant R is

$$R = B_2 D_2^2 - B_3 D_1 D_2 + B_4 D_1^2\,, \tag{32}$$

$$D_1 = [C_2 B_4]\,, \qquad D_2 = [C_3 B_4]\,,$$

$$C_2 = A_0 B_4^2\,, \quad C_3 = A_1 B_4^2 - B_2[A_3 B_4]\,, \quad C_4 = B_4[A_2 B_4] - B_3[A_3 B_4]\,.$$

By definition $[A_k B_l] = A_k B_l - A_l B_k$.

If the variables ξ and η satisfy the equation $G(\xi) = G(\eta)$, then

$$\xi = 1 + \varphi(u)\,, \qquad \eta = 1 + \varphi(-u)\,, \qquad 0 < u < u_0$$

$$u_0 = \min\{\sqrt{2G(x_1)}\,, \sqrt{2G(x_2)}\}\,, \tag{33}$$

$p = \frac{1+\varphi(-u)}{1+\varphi(u)}$, where the function $\varphi(u)$ is defined from equation (27). Then the system (29) is equivalent (perhaps in the smallre domain $1+\varphi(-u_0) < \eta < 1$, $1 < \xi < 1 + \varphi(u_0)$) to the single equation

$$F(1+\varphi(-u)) = F(1+\varphi(u))\,,$$

which is the equation $F_1(u) = 0$. The variables u and p are in one-to-one correspondence but the region $\frac{x_1}{x_2} < p < 1$ of definition of the variable p can be greater, than $p(u_0) < p < 1$. So the number of positive roots of equation $F_1(u) = 0$ is not greater than the number of roots of the equation $R(p)$ in the interval $\frac{x_1}{x_2} < p < 1$. We suggest the next conjecture.

Conjecture. Suppose that a quadratic system (11) has the focus $A(1,-1)$, i.e. the conditions

$$\sum_{i+j=0}^{2} a_{ij}(-1)^j = 0\,, \qquad 2a - a_{01} - a_{10} - 2a_{20} \equiv L > 0\,,$$

$$(a_{11} + a_{01} - 2a - 1)^2 - 4L < 0\,,$$

are satisfied, and that $x_1 < 1$, $x_2 > 1$ are the nearest positive roots of the equation

$$a_{20} + a_{10}x + (a_{00} - a_{11})x^2 - a_{01}x^3 + ax^4 = 0\,, \tag{34}$$

to $x = 1$. If equation (34) does not have roots in the interval $(0,1)$ or $(1,+\infty)$, then we take $x_1 = 0$ or $x_1 = +\infty$.

Then the number of limit cycles of system (11) around the focus $A(1,-1)$ is not greater than number N of roots of the equation $R(p) = 0$ in the interval $(x_1/x_2, 1)$, where $R(p)$ is defined by formulas (32). In the case $N = 0$ the conjecture is true.

Now we consider an example which is not in contradiction with the conjecture.

Example 1. *Suppose that in system* (11) $a_{20} = -3$*;* $a = 1.5$*;* $a_{01} = 3.2$*;* $a_{11} = 0.8$*;* $a_{00} = 5.5 - a_{10}$*, and* a_{10} *varies. Then* $A(1,-1)$ *is a weak focus. If the parameter* a_{10} *changes from value 3.8 to the value 3.233, then two roots of the function* $R(p)$ *in the interval* $(0,1)$ *appear. The first of them appears from the point* $p = 1$*, the second one appears from the point* $p = 0$*. Later they disappear (for* $a_{10} = 3.231$*) into a multiple root.*

In the corresponding system (10) two limit cycles appear. One of them from the focus $A(1,-1)$ the second one from a separatrix cycle with two saddles at infinity. Later they disappear in a multiple limit cycle. But limit cycles disappear earlier than the roots of equation $R(p) = 0$ and appear later than the roots.

Remark. *With the help of the function*

$$\widetilde{R} = B_2 - B_3\frac{D_1}{D_2} + B_4\left(\frac{D_1}{D_2}\right)^2$$

we can calculate the focus values of the focus $A(1,-1)$ *of system* (11)*, because between zeros of* $\widetilde{F}(u)$ *in a neighborhood of the point* $u = 0$ *and the zeros of* $\widetilde{R}(p)$ *in a neighborhood of the point* $p = 1$ *we have one-to-one correspondence. The function* D_1/D_2 *has a removable singularity at* $p = 1$*. After removing it we can expand this function in Taylor series at the point* $p = 1$*. If the focus* $A(1,-1)$ *is not weak, then* $\widetilde{R}'(1) \neq 0$*. If* A *is a weak focus, then* $\widetilde{R}'(1)$*,* $\widetilde{R}''(1) = 0$ *and* $\widetilde{R}'''(1)$ *is equal to the first focus value up to constant not equal to zero.*

A calculation gives for $a_{01} = -a_{11} + 2a + 1$, the condition of weak focus,

$$\widehat{\widehat{v_3}} = \frac{1}{4(-a_{11} - 2a - 1)} \left(-a_{11}^2(1+a) - a_{11}(2a^2 + a - 1) + \right.$$

$$\left. + a_{11}(2a-1)a_{20} + a_{11}aa_{10} + (2a+1)(a_{10}(a-1) + a_{20}(2a-3))\right) =$$

$$= -\frac{1}{4(a_{11} + 2a + 1)} S \, .$$

From [6, 8] we have

$$\widehat{v_3} = \frac{1}{(\sqrt{a_{11} - a_{10} - 2a_{20} - 1})^3} \, Q \, .$$

If $a_{11} = 0$, then for $\widehat{v_3} = 0$ we have the expression

$$-\frac{a_{10}}{a_{20}} = \frac{2a-3}{a-1} \, ,$$

which was used in Theorem 3. For calculating the second focus value it is necessary to calculate $\widetilde{R}^v(1)$ and so on. If $R(p) \equiv 0$ with $p \in (0,1)$, then it means that $A(1,-1)$ is a center.

5 Curves of separatrix cycles. The case of three antisaddles and one saddle

We investigate bifurcations of codimension one and two for system III by changing the rotating parameters λ, b of the vector field. If system III is of kind $3A + 1S$, then without loss of generality we can suppose that

$$0 < a < 1 \, , \quad \alpha > 0 \, , \quad c < 0 \, , \tag{35}$$

$D = \lambda^2 - 4c(a-1) > 0$. Suppose that the singular point $O(0,0)$ is a focus, then $1\alpha + \lambda + b/ < 2$, by analogy $A(-\frac{1}{a}, 0)$ is focus if $|1\alpha + b(\frac{1}{a} - 1) - \lambda| < 2\sqrt{\frac{1}{a} - 1}$. Singular points $S_1(-1, y_1)$, $B(-1, y_2)$, $|y_1| < |y_2|$ are a saddle and an antisaddle respectively.

For finding the singular points at infinity we make the Poincàre transformation $x = 1/\xi$, $y = \eta/\xi$. Then we obtain the next system

$$\begin{aligned} \frac{d\xi}{dt} &= \xi(-\eta(\xi+1) + \alpha Q_1) \\ \frac{d\eta}{dt} &= (\alpha\eta - 1)Q_1 - \eta^2(\xi + 1) \, , \end{aligned} \tag{36}$$

$Q_1 = \xi + (\lambda b)\eta\xi + a + b\eta + c\eta?$

The singular points at infinity are obtained from the equation

$$F(\eta) = \alpha c\eta^3 + (\alpha b - c - 1)\eta^2 + (\alpha a - b)\eta - a = 0\,. \qquad (37)$$

Lemma. *Suppose that conditions (35) are satisfied. Then for the roots of equation (37) one of the following statements holds (If we do not take into account multiple roots):*

(1) equation (37) has a single root $\eta_1 < 0$*:*

(2) equation (37) has three roots η_1, η_2, η_3 *for which one of the following conditions is satisfied:*

(a) $\eta_1 < \eta_2 < \eta_3 < 0$,

(b) $\eta_1 < 0 < \eta_2 < \eta_3 < 1/\alpha$,

(c) $\eta_1 < 0 < 1/\alpha < \eta_2 < \eta_3$.

The proof is obvious.

Theorem 4. *Suppose that the conditions of the lemma are satisfied. Then system III has the next singularities at infinity:*

(1) one saddle with the tangent coefficient $\eta_1 < 0$,

(2) three singular points with tangent coefficients η_1, η_2 η_3 *for which one of the following statements is satisfied:*

(a) $\eta_1 < \eta 2 < \eta_3 < 0$ *and they are a saddle, a node and a saddle respectively,*

(b) $\eta_1 < 0 < \eta_2 < \eta_3 < 1/\alpha$ *and they are a saddle, a saddle and a node respectively,*

(c) $\eta_1 < 0 < 1/\alpha < \eta_2 < \eta_3$ *and they are a saddle, a node and a saddle respectively.*

The proof follows from the calculation of the eigenvalues for every singular point. It should be noted that the straight line $\alpha y = x + 1$ is transversal, and that the limit cycles do not intersect it.

First we will study the behavior of the curve of separatrix cycles in the plane λOb around the focus $O(0,0)$. This curve is located in the strip $|\lambda + b + \alpha| < 2$, otherwise the singular point O becomes a node and there are no limit cycles around it. It should be noted that these properties as continuity and the decreasing when $\lambda \to +\infty(-\infty)$, $b \to -\infty(+\infty)$ follow from the rotating parameters of the vector field and by changing one of them in a generic case only one separatrix cycle appears. Degenerate cases are studied separtely. Now we investigate the behavior of the curve of the separatrix cycles in detail. We begin with the quadratic system having an invariant straight line. Then they can be transformed into the next system

$$\frac{dx}{dt} = xy\,, \qquad \frac{dy}{dt} = Q(x,y)\,, \tag{38}$$

where Q is a polynomial of degree two. It is obvious from system (38) that it has four finite singular points, then two of them belong to this line. Since $O(0,0)$ is a focus, then for system III two cases are possible: (B) a straight line is passing through two antisaddles A and B; (S) a straight line contains the antisaddle A and the saddle S. The straight line $y = kx + p$ is invariant if and only if the next conditions are satisfied

$$\begin{aligned} (-1+\alpha k)(a+bk+ck^2) - k^2 = 0\,, \\ (-1+\alpha k)(\lambda p + bp + cp^2) - kp = 0\,. \end{aligned} \tag{39}$$

The first of equations (39) is equation (37). Therefore k is equal to one of the roots η_i, $i = 1,2,3$. Separatrix cycles with two saddles at infinity are possible only for statement (b) of Theorem 4. And in this case the straight line $y = kx + b$ can be chosen so that it passes through the saddle S_2 at infinity, then $0 < k = \eta_3 < 1/\alpha$. The straight line is passing through the point $A(-\frac{1}{a}, 0)$ therefore $p = k/a$ and equations (39) imply

$$\lambda = \frac{a}{k} + ck(1 - \frac{1}{a})\,, \qquad b = -\frac{a}{k} - ck + \frac{k}{-1+\alpha k}\,. \tag{40}$$

Taking into account conditions (35) we have the inequality $\lambda > 0$. It means that points S and B are in the half-plane $x > 0$. The phase portraits are in Figure 2.

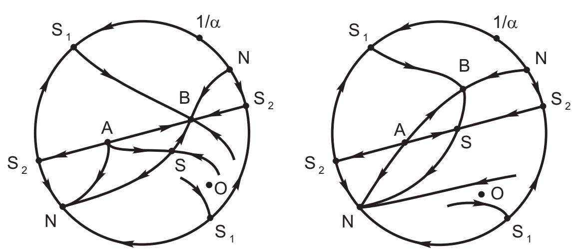

Figure 2

The phase portraits are builded modulo limit cycles around the focus O. The infinity in the half-plane $\alpha y < x + 1$ is stable. Hence, if div $f(0) < 0$, then a single limit cycle exists around the focus $O(0.0)$; and if $\operatorname{div} f(0) > 0$, then there are no limit cycles. A calculation gives

$$\operatorname{div} f(O) = \frac{-c\alpha^2 k^2 + (a\alpha^2 + a + c)k - a\alpha}{a(-1+\alpha k)} = \frac{V(k)}{W(k)}\,.$$

If α, a, c, b are fixed and k varies in $0 < k < 1(\alpha)$, then $W(k) < 0$, $V(0) = -\alpha a < 0$, $V(1/\alpha) = a/\alpha > 0$. So limit cycles are possible. Now if in case S we decrease the parameter λ, the vector field rotates counterclockwise and an upper separatrix of the saddle S_2 comes to the node N and it crosses the Ox-axis.

The right separatrix of the saddle S_1 is crossing the Ox-axis and both points of intersection move towards each other, and there exists a single value $\lambda = \lambda^*$ for which they meet and give a separatrix cycle with two saddles at infinity. If in case B we decrease the parameter λ, then the upper separatrix L of the saddle S_2 cannot go to the saddle S because two saddles always connect through a straight line and system III does not have another invariant line different from $y = kx + p$. This means that it remains in the node B sagging step-by-step. But decreasing λ lead to colliding of the singular points S, B. Thus at some moment they collide and disappear. After that the separatrix L goes to the node N and the proof follows as before.

Now we analyze the bifurcation around the focus $B(-1/a, 0)$. We also begin with system III having an invariant straight line $y = kx$, $k < 0$ for the next conditions

$$\lambda = ck - \frac{1-a}{k}\,, \qquad b = -\frac{a}{k} - ck + \frac{k}{-1+\alpha k}\,, \qquad k = \eta_1\,.$$

Two cases are possible:

(B') an invariant straight line is passing through the antisaddles O, B;

(S') it is passing through O and the saddle S. Since $0 < a < 1$, $c < 0$, $k < 0$, then $\lambda > 0$ and S, B are in the half-plane $y > 0$. The phase portraits are given in Figure 3.

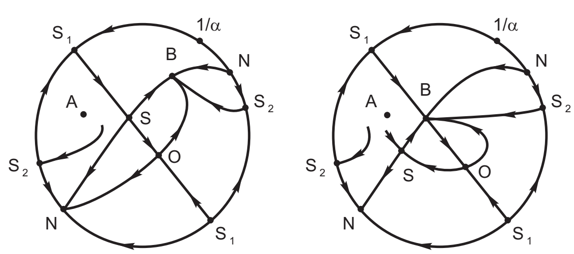

Figure 3

There is the possibility that a limit cycle exists around the focus $A(-1/a, 0)$. In the case S' we decrease the parameter λ, then the vector field rotate clockwise and the separatrix of the saddle S_1 goes to the node N and it moves towards the lower separatrix of the saddle S_2. There exists a single value $\lambda = \lambda^{**}$ for which they meet and give a separatrix cycle with two saddles at infinity.

In case B' we decrease the parameter λ, then the separatrix of the saddle S_1 cannot go to the saddle S. And the arguments follow as before. So we obtain

Theorem 5. *Suppose that system III satisfies the next conditions:*

(1) $0 < a < 1$, $c < 0$, $D = \lambda^2 - 4c(a-1) > 0$, $\alpha > 0$.

(2) the equation

$$c\eta^2 + b\eta + a = \eta^2/(\alpha\eta - 1)$$

has roots $\eta_1 < 0 < \eta_2 < \eta_3 < 1/\alpha$. *Then there exists a single value* $\lambda = \lambda^*$ *for which the system has a separatrix cycle with two saddles at infinity around the focus* $O(0, 0)$*; and there exists a single* $\lambda = \lambda^{**}$ *for which the system has a separatrix cycle of the same kind around the focus* $A(-1/a, 0)$*. In the others cases a separatrix cycle of this kind is impossible.*

There are four forms of separatrix cycles defining a bifurcation of codimension one:

(1) A $D1$-cycle with two saddles at infinity;

(2) A $L1$-loop of saddle S on line $x = -1$;

(3) A $D2$-separatrix cycle defined by a trajectory going from a infinite saddle to a parabolic region of a saddle-node at infinity;

(4) A $L2$-loop of a finite saddle-loop going from a hyperbolic domain to a parabolic one.

There are five forms of separatrix cycles of codimension two:

(1) A $L3$-loop of a finite saddle-node going from a hyperbolic region and dividing hyperbolic and parabolic regions of this saddle-node;

(2) A $D3$-separatrix cycle with a infinite saddle and a infinite saddle-node and dividing the hyperbolic and parabolic regions of this saddle-node;

(3) A $T1$-separatrix cycle with a saddle and a saddle-node at infinity, and finite saddle going strictly in parabolic region of the saddle-node;

(4) A $T2$-separatrix cycle with a saddle and a saddle-node at infinity and a finite saddle-node connecting parabolic regions of the both saddle-nodes;

(5) A $T3$-separatrix cycle with two saddles at infinity and a finite saddle.

In the plane λOb a straight line $b = b_0$ corresponding to a saddle-node at infinity under the condition $0 < \eta_2 < \eta_3 < 1/\alpha$, divides the cases of appearance of a loop with a finite saddle and a cycle with two saddles at infinity. The value b_0 we can give by parameter η, $0 < \eta < 1/\alpha$.

$$c = \frac{a}{\eta^2} - \frac{1}{(\alpha\eta - 1)^2}\,, \qquad b = -\frac{2a}{\eta} + \frac{a\eta^2}{(\alpha\eta - 1)^2}\,.$$

By $c < 0$ we have more little region for the parameter η:

$$\frac{1}{\alpha + 1/\sqrt{a}} < \eta < 1/\alpha\,.$$

In conclusion we describe the structure of a curve L_A of separatrix cycles around the focus $A(-1/a, 0)$. It begins in the point $\lambda = +\infty$, $b = +\infty$ after it decreases strictly and its first part corresponds to a loop of the saddle S. Furthermore all depends of which straight line it is meeting:

(1) vertical line $\lambda = \lambda_0 = 2\sqrt{c(a-1)}$ of finite sadle-nodes on $x = -1$;

(2) horizontal line $b = b_0$ of saddle-nodes at infinity with the condition $0 < \eta_2 = \eta_3 < 1/\alpha$.

In the first case the first part of L_A finishes into a cycle of kind $L3$ for some $b = b^*$, $\lambda = \lambda_0$; furthermore L_0 has a vertical part $b^* < b < b_0$, $\lambda = \lambda_0$ with separatrix cycles of kind $L2$ and finishes in a point $b = b_0$, $\lambda = \lambda_0$ with cycle of kind T_2; furthermore L_A has a horizontal part $b = b_0$ $\lambda_0 < \lambda < \lambda^*$ with cycles of kind $D2$ finishing in a point $\lambda = \lambda^*$, $b = b_0$ with a cycle of kind $D3$, and the last part with a point (λ^*, b_0) until $\lambda = -\infty$, $b = -\infty$ corresponds to cycles of kind $D1$.

In the second case the first part of L_0 finishes at the point $\lambda = \lambda^{**} < \lambda_0$, $b = b_0$ by cycle of kind $T1$, furthermore the behavior is the same.

It is the third unusual property of a curve of separatrix cycles for quadratic systems.

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