STABILITY OF HILBERT POINTS OF GENERIC K3 SURFACES

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Abstract. The main result of this paper shows that: *If X is a K3 surface with Picard group* $\mathrm{Pic}(X)$ *of rank 1 and the primitive divisor class on X has degree at least 12, then the Hilbert point of the model given by taking the complete linear system associated to this class is stable.* As a corollary, the open subset of stable points of any component of a Hilbert scheme whose generic member parameterizes a K3 surface embedded by a complete linear series is *non-empty.* The paper also speculates on how this result might be used as a first step in producing modular compactifications of moduli spaces of K3 surfaces.

1. Introduction

This note provides a small first step towards using Mumford's geometric invariant theory [2] to produce a modular compactifications of moduli spaces of K3 surfaces. In this introductory section, we first summarize our results and then discuss possible extensions and applications of these results. The results about linear series in K3 surfaces which we need — all standard — are collected in Section 2 and the geometric invariant theory framework used is reviewed in the Section 3. Readers unfamiliar with this set-up might want to review this section before reading the remainder of this introduction. The proof of the main theorem is given in the Section 4. We work over $\mathbb{C}$, write divisors additively and, to simplify notation slightly, denote $\mathcal{O}_X(D)$ simply by D when writing cohomology groups.

Date: January 30, 1999.

1991 *Mathematics Subject Classification.* Primary: 14J28; Secondary: 14D22, 14D25, 14J28.

Key words and phrases. K3 surface, Hilbert stability, moduli.

This paper was written out while the author was in residence at the Centre de Recerca Matemtica. He wishes to thank Sebastià Xambó for his assistance in making arrangements, Fordham University for providing sabbatical leave and a Faculty Fellowship, the Ministerio de Educación y Cultura of Spain for its support during his stay, and, most of all, the Centre and its staff for their helpfulness and hospitality.

The main technical result of the paper is:

Theorem 1. *Let X be a K3 surface, D be a pseudo-ample divisor on X of degree at least 12, and F be a weighted filtration of $W = H^0(X, kD)$ given by linear subseries V_i such that no curve is contained in the base loci of any V_i. Then there is a constant M depending only on $h^0(X, kD)$ such that for $m > M$, the m^{th}-Hilbert point of the pair of $(X, |kD|)$ (that is, of the projective model of X associated to the complete linear system $|kD|$) is stable with respect to the filtration F for the natural* $\mathrm{PGL}(W)$ *action.*

If X is a K3 surface whose Picard group $\mathrm{Pic}(X)$ has rank 1 and D is a primitive divisor class on X, (that is, one which generates $\mathrm{Pic}(X)$), then no linear subseries V of $|D|$ can have a base locus which contains a curve and the theorem yields the unconditional stability result:

Corollary 2. *Let X be a K3 surface whose Picard group* $\mathrm{Pic}(X)$ *has rank 1 such that the primitive divisor class D on X has degree at least 12 and let $W = H^0(X, D)$. Then there is a constant M depending only on $h^0(X, D)$ such that for $m > M$, the m^{th}-Hilbert point of pair $(X, |D|)$ is stable for the natural* $\mathrm{PGL}(W)$ *action.*

Algebraic K3 surfaces form a countable collection of 19-dimensional families $\mathcal{M}_d$ indexed by an even integer $d \geq 4$ such that each member X of $\mathcal{M}_d$ carries a distinguished pseudo-ample divisor class D with degree $D^2 = d$, $h^0(X, mD) = \frac{d}{2}m^2 + 2$ and $h^1(X, mD) = h^2(X, mD) = 0$. Thus, if we let $r = \frac{d}{2} + 2$, the choice of a member X of $\mathcal{M}_d$ plus a basis of $h^0(X, D)$ determines a point in the Hilbert scheme $\mathcal{H}_d$ of Hilbert points of surfaces in $\mathbb{P}^{r-1}$ with Hilbert polynomial $P(m) = \frac{d}{2}m^2 + 2$. By varying the choice of basis, each X determines a $\mathrm{PGL}(r)$ orbit in $\mathcal{H}_d$ and set of such orbits forms a smooth locally closed subset $\mathcal{K}_d$ of $\mathcal{H}_d$. General results from geometric invariant theory guarantee that the locus of stable orbits in $\mathcal{K}_d$ is open but it might be empty. However, since the generic member of $\mathcal{M}_d$ has Picard group of rank 1 generated by D, Corollary 2 rules this out.

Corollary 3. *The open subset of* $\mathrm{PGL}(r)$ *stable orbits in $\mathcal{K}_d$ is* non-empty.

Remarks.

(1) There is an implicit choice of a large integer m involved in the construction of $\mathcal{H}_d$. All sufficiently large m yield isomorphic schemes (since, in Grothendieck's construction, all the corresponding schemes represent the same functor). While it is not *a priori* clear that the notion of $\mathrm{PGL}(r)$ stability shares this independence of m, the Corollary shows that this is true at least for such generic orbits. In the rest of this introduction, I'll ignore this implicit m.

(2) The restriction that the degree must be at least 12 is almost certainly not essential but additional combinatorics would be required to remove it. For example, the degree 4 case is completely handled by Shah [11] — see also page 51 of [7].

As they stand, these results leave much to desire. There is no reason to suspect that K3 surfaces whose Picard groups have rank 1 (or that models associated to primitive divisor classes) should have special stability properties. Indeed, since Hodge theory constructs analytic moduli in a uniform way for all K3 surfaces, a natural expectation would be that all quasi-ample divisor classes give models which have stable Hilbert points. The major difficulties to be overcome in proving this seem to arise in unwinding the combinatorics of the Hilbert-Mumford Numerical Criterion 6. The natural approach is to verify Gieseker's Criterion 9 (which is what I use to deduce Theorem 1) for general filtrations. Standard results of Saint-Donat [10] and Reider [9] on base-point freeness and vanishing of higher cohomology for linear series on K3 surfaces are sufficiently sharp that any geometrical estimates needed should follow readily from them. However, I have been only to pass from them to the inequality in Gieseker's Criterion for more general divisors on K3 surfaces for other very special classes of filtrations (cf., Remark 13).

However, even lacking a more satisfactory direct proof of stability for Hilbert points of general K3 surfaces, Corollary 3 does have one interesting potential application. It could serve as a first step in the construction of *modular* compactifications for moduli spaces of algebraic K3 surfaces. Following [5], I call a compactification $\overline{\mathcal{M}}$ of a moduli space $\mathcal{M}$ modular if it coarsely represents a moduli functor which contains the moduli functor of $\mathcal{M}$ as an open subfunctor. More informally, to say that $\overline{\mathcal{M}}$ is modular means that its boundary points correspond to well-behaved degenerations of varieties in $\mathcal{M}$ and is crucial in making effective use of projective techniques to study families of varieties in $\mathcal{M}$. degenerations. Moduli spaces constructed via Hodge theory have natural Baily-Borel compactifications but these are almost never modular. The classic example is the Satake compactification of the moduli space of smooth curves but the same remark applies to period spaces for K3 surfaces. Most modular compactifications to date have been constructed by taking geometric invariant theory quotients of suitable parameter spaces: the moduli space $\overline{\mathcal{M}_g}$ of stable curves of genus g is the best known example. So it is tempting to ask: might the quotient of the closure $\overline{\mathcal{K}_d}$ of $\mathcal{K}_d$ in $\mathcal{H}_d$ provide a modular compactification?

Two difficult problems would need to be resolved to obtain a positive answer to this question. Taking the quotient per se is not one of them: the theory guarantees that a projective quotient space exists. But to produce a modular compactification in this way, we must be able to describe exactly the class of projective varieties have Hilbert points lying in the semi-stable and stable loci of $\overline{\mathcal{K}_d}$. We cannot even achieve this for all of $\mathcal{K}_d$ itself where the geometry is certainly simpler. Indeed, a direct approach seems out of reach: even for stable curves this remains an open problem.

However, Gieseker's construction of $\overline{\mathcal{M}_g}$ implements a two step indirect strategy which avoids the need to directly verify the stability of any additional Hilbert points. Step one involves producing conditions which are necessary for the stability of a point of $\overline{\mathcal{K}_d}$. This has both intrinsic and extrinsic components. Intrinsically, restrict the possible stable degenerations to an expected class — for K3 surfaces the Kulikov-Persson-Pinkham degenerations [8] would seem the natural class. Extrinsically, restrict the possible linear series giving a stable embedding especially on reducible degenerations. The key simplification is that necessary conditions can be deduced by proving that varieties which fail to satisfy them have *unstable* Hilbert points and proving instability is vastly easier than proving stability. It suffices to show that a single filtration is destabilizing, the right filtration is usually picked out by the failure of the desired condition, and this in turn makes the combinatorics of the verification straightforward. In fact, this can be turned around and used to iteratively refine and test new necessary conditions. Step two involves studying a one-parameter family of varieties with stable Hilbert points degenerating to a variety in the expected class — those not ruled out by the necessary conditions — and using these conditions to deduce that limiting fiber of the family in the geometric invariant theory compactification must equal the limiting fiber in the original family. It is here that the Corollary 3 is a crucial novelty: this step depends on having proved stability for *some* non-empty open locus. Of course, this second step can only succeed if the necessary conditions in step one are actually sufficient as well: in essence, modulo the work in the two steps, the strategy reduces the construction of a modular compactification to that of identifying the corresponding moduli functor.

However, I should emphasize that carrying out either half of this strategy for K3 surfaces is a substantial problem as a look at Gieseker's paper [4] on the necessary conditions for stability of models of stable curves or Caporaso's [1] on their sufficiency will make clear. At this point, I do not even have a precise conjecture about what such compactifications might look like for K3 surfaces.

The original impetus to look into the problems discussed here was provided some years ago by David Morrison and I wish to thank him for showing me Lemma 5 and for many patient and helpful discussions at that time.

2. Linear series on K3 surfaces

We call a divisor C on a K3 surface X pseudo-ample if the linear series $|C|$ has no fixed components and if the associated map $\phi_C : X \to \mathbb{P}\big(H^0(X,C)\big)$ is birational.

Lemma 4. *If C is a pseudo-ample divisor on a K3 surface, then*

1. $h^0(X,C) = \frac{C^2}{2} + 2$ *and* $h^1(X,C) = 0$.
2. ϕ_C *has no base points.*
3. *The generic member of the linear series $|C|$ is non-singular.*
4. $\phi_C(X)$ *is projectively normal.*
5. *There is a collection of disjoint connected curves E_i of arithmetic genus 0 on X each of which is a platonic configuration of smooth rational components such that ϕ_C blows-down each E_i to a DuVal rational double point and such that away from the E_i the map ϕ_C is an isomorphism.*

All these assertions can be found in Saint-Donat [10]: see 2.6 for 1, 3.1 for 2, and 6.1 for 3-5. Note also that when $\mathrm{Pic}(X)$ has rank 1 then 5 implies that ϕ_C is an isomorphism.

Lemma 5. *Let C be a pseudo-ample divisor on a K3 surface X, let V be a subspace of $H^0(X,C)$ and let $|V|$ denote the associated linear subseries. Choose a blowup $\pi : Y \to X$ such that the pullback under π of the sections in V generate an invertible subsheaf $\mathcal{O}_Y(C_V)$ of $\mathcal{O}_Y(\widetilde{C})$.*

1. *If $C_V^2 = 0$, then there exists a divisor C' and a Lefschetz pencil $|L|$ in $|C'|$ such that $C_V \sim kC'$ for some integer $k \geq \big(\dim(V) - 1\big)$ and such that every divisor in $|V|$ is of the form $F_V + E_1 + E_2 + \cdots + E_k$ where F_V the fixed locus of V and each $E_i \in |L|$.*
2. *If $C_V^2 > 0$, then $C_V^2 \geq 2\big(\dim(V) - 2\big)$ and hence $C^2 - C_V^2 \leq 2\,\mathrm{codim}(V)$.*

Proof. 1. This is a standard exercise which we leave to the reader.
2. By construction $\mathcal{O}_Y(C_V)$ is generated by its sections and so has no base-points. Since X is not rational, we can apply Lemma 3.2 of [3] (essentially a Clifford's theorem argument) to conclude that $C_V^2 \geq 2\big(h^0(Y,C_V) - 2\big)$. Since π^* injects V into $H^0(Y,C_V)$, this proves the inequality. □

3. Background on Hilbert stability

In this section, my main goal is to set up the notation which will be used in Section 4 and to recall several criteria for the stability of Hilbert points which will be used in the last two sections. I will not even try to sketch the proofs which are standard. Readers who wish to see a discussion and proof of the Numerical Criterion for Hilbert Points 6 or the Asymptotic Numerical Criterion 7 may consult pp.206-211 of [5]. For details on Gieseker's Criterion 9 which gives the sufficient condition for the stability of a Hilbert point which we will use, see Gieseker's original paper [3]. Both topics are also covered in Mumford's notes [7].

For keep the notation as simple as possible, I will state all these criteria only for surfaces. All the explicit and implied constants in what follows are *uniform* by which I mean that they depend only on the Hilbert polynomial P of the pair (X, C).

Let X be a surface and C be a pseudo-ample divisor on X with degree $C^2 = c$ and with $h^0(X, C) = r$. We let $[X, C]_m$ denote the m^{th}- Hilbert point of the pair $(X, |C|)$. Let F be a complete *weighted filtration* of $W = H^0(X, C)$ which we write as:

$$\begin{array}{c} W = V_0 \supsetneqq V_1 \supsetneqq \cdots \supsetneqq V_{r-1} \supsetneqq \{0\} \\ w_0 \geq w_1 \geq \cdots \geq w_{r-1} \end{array}$$

and in which we assume for the moment that the *weights* w_i are integers which sum to 0. This data determines a one-parameter subgroup of $\mathrm{PGL}(W)$ by choosing a basis of W compatible with F and using the weights as exponents. Every one parameter subgroup arises in this way although not from a unique F.

For large m, F induces a weighted filtration on $\mathrm{Sym}^m(W)^\vee$ and hence via the surjection

$$\mathrm{Sym}^m(W)^\vee \to H^0(X, mC)$$

a weighted filtration on $H^0(X, mC)$. The weight of a basis of $H^0(X, mC)$ compatible with the weighted filtration induced by F on this space is is denoted by $w_F(m)$. We say that $[X, C]_m$ is stable, strictly semi-stable or unstable with respect to F according to whether $w_F(m)$ is negative, zero or positive.

Numerical Criterion 6. *For large m, $[X, C]_m$ is stable for the natural* $\mathrm{PGL}(r)$ *action if and only only if it is stable with respect to* every *non-trivial F; $[X, C]_m$ is unstable if it is unstable with respect to* any *F.*

The function $w_F(m)$ has a rational *leading coefficient* e_F such that

$$\left| w_F(M) - e_F \frac{m^3}{3!} \right| < 0(m^2)\,.$$

Asymptotic Numerical Criterion 7. *If there is a positive δ such that $e_F < -\delta(w_0 - w_{r-1})$ [respectively: $e_F > \delta(w_0 - w_{r-1})$], then for large m $[X, C]_m$ is stable [respectively: unstable] with respect to F.*

Note that if e_F is merely negative or positive then we can conclude that, for large m, $[X, C]_m$ is stable or unstable with respect to F but that, in order to guarantee a *uniform* lower bound for m, we definitely need the uniform bound for e_F provided by adding the δ term.

To state Gieseker's Criterion 9 we begin by choosing, once and for all, a blow-up $\pi : Y \to X$ on which C has proper transform $\widetilde{C}$ and such that for each $i = 0, 1, \ldots, r-1$, the the pullbacks under π of the sections in V_i generate an invertible subsheaf $\mathcal{O}_Y(C_i)$ of $\mathcal{O}_Y(\widetilde{C})$. Define $d_{ij} = C_i \cdot C_j$: these numbers are independent of the choice of Y since they remain unchanged if we replace any given Y by a further blow-up and since any two such Y are dominated by a third . Write C_i for C_{ii} and note that this is the degree of the projective model of X associated to the linear system V_i.

In Section 2 of [3], Gieseker shows that,

$$(8)\quad e_F \leq w_0 c - \frac{1}{3} \max_{\substack{0=j_0<j_1<\cdots \\ \cdots<j_h=r-1}} \sum_{k=0}^{h-1} \Big(\left(d_{j_k} + d_{j_k j_{k+1}} + d_{j_{k+1}}\right) \left(w_{j_k} - w_{j_{k+1}}\right) \Big)$$

The criterion we will use will be a formal translation of this inequality. First, we want to remove the requirement that the weights w_i sum to 0. We can think of doing this as simply increasing the weight of every section of $H^0(X, C)$ by

$$\frac{1}{r} \sum_{i=0}^{r-1} w_i$$

and hence, in turn, increasing the weight of every section of $H^0(X, mC)$ by m times this amount. The net effect is to increase $w_F(m)$ by

$$\frac{m}{r} \left(\sum_{i=0}^{r-1} w_i \right) h^0(X, mC)$$

and since by 1. of Lemma 4, $h^0(X, mC) = m^2 \frac{c}{2} + 2$ this alters the leading coefficient e_F by

$$\frac{3c}{r} \left(\sum_{i=0}^{r-1} w_i \right) .$$

Second, we'd like to use this new found freedom to translate the weights w_i so that $w_{r-1} = 0$ and then rescale them so that they sum to 1. Thirdly, we want to apply 1. of Lemma 4 to C itself to express

$$r = h^0(X, C) = \frac{C^2}{2} + 2 = \frac{c}{2} + 2 \quad \text{or equivalently} \quad c = 2(r-2)\,.$$

Finally, we want to restate the estimate in terms of the quantities $e_{ij} := d - d_{ij}$ and $e_i := d - d_i$. Geometrically, e_i is the amount by which the degree of X in $\mathbb{P}(W)$ is greater than that of its image under the projection to $\mathbb{P}(V_i)$. In particular, if the linear series $|V_i|$ has no base curve, then e_i is just the number of base points of this series (with multiplicity).

Making all the substitutions outlined above in 8 and using the resulting upper bound for e_F in the Asymptotic Numerical Criterion 7 yields, after some arithmetic manipulations:

Gieseker's Criterion 9. *If there is a δ depending only on the Hilbert polynomial P of the pair (X, C) such that for any weighted filtration F with weights w_i summing to 1 and with smallest weight $w_{r-1} = 0$, we have*

$$\min_{\substack{0=j_0<j_1<\cdots\\ \cdots<j_h=r-1}} \left(\sum_{k=0}^{h-1} \left(e_{j_k} + e_{j_k j_{k+1}} + e_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right)\right) \leq 6\Big(\frac{r-2}{r}\Big) - \delta r_0 \,,$$

then there is an M depending only on P such that for $m \geq M$, $[X, C]_m$ is stable for the natural $\mathrm{PGL}(r)$ *action.*

4. Proof of Theorem 1

We continue to use the notation developed in the preceding section using as the divisor C discussed there the divisor kD of the Theorem 1. The hypothesis of the Theorem — that there is no base curve in any linear subseries $|V_i|$ in the filtration F — means that the number e_i which measures the drop $d - d_i = C^2 - C^2_{V_i}$ in degree on projection to V_i counts the number of base points (with multiplicity) of $|V_i|$. By Lemma 5, this is at most $2i = 2\,\mathrm{codim}(V_i)$. Moreover, if $i > j$ then $V_i \subset V_i$ so the base locus of $|V_i|$ will be contained in that of $|V_j|$. Therefore, the fixed points in the intersection a general curve $|V_i|$ with a general curve in $|V_j|$ will simply be the base points of $|V_i|$. In other words, $e_{ij} = e_i$. Combining these estimates, we find that

Lemma 10. *If $i < j$, then $e_i + e_{ij} + e_j \leq 4i + 2j = 3(i + j) - (j - i)$.*

This suggests that, for each increasing subsequence $J = (j_0, j_1, \ldots, j_h)$ with $j_0 = 0$ and $j_j = r - 1$, we define

$$\begin{aligned} S_J &= \sum_{k=0}^{h-1} \left(e_{j_k} + e_{j_k j_{k+1}} + e_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right) \\ P_J &= \sum_{k=0}^{h-1} \left(i_{j_k} + i_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right) \quad \text{and} \\ N_J &= \sum_{k=0}^{h-1} \left(i_{j_k+1} - i_{j_k}\right)\left(w_{j_k} - w_{j_{k+1}}\right) \end{aligned}$$

The last Lemma then just says that: $S_J \leq (3P_J - N_J)$.

Finally, let's define

$$T = \min_{0=j_0<j_1<\cdots<j_h=r-1} \sum_{k=0}^{h-1} \left(e_{j_k} + e_{j_k j_{k+1}} + e_{j_{k+1}}\right)\left(w_{j_k} - w_{j_{k+1}}\right) = \min_J S_J$$

For fixed weights w_i, raising any e_i or e_{ij} can only increase any of the sums whose minimum defines T and hence increase T. Since our goal is to get an upper bound for T we can assume henceforth that for $0 \leq i < j \leq r-1$, we have $e_i = e_{ij} = 2i$ and that $e_{r-1} = 2(r-2)$.

From here on in all the work is combinatorial. The argument is elementary but a bit involved. We will use the assumptions that the w_i are decreasing, that they sum to 1, that $w_{r-1} = 0$, and that

$$T > 6\left(\frac{r-2}{r}\right) - \delta r_0 \text{ for every positive } \delta\,,$$

or equivalently that,

$$\text{(11)} \qquad \text{For any increasing subsequence } J, \quad S_J \geq T \geq 6\left(\frac{r-2}{r}\right),$$

to reach a contradiction. This will verify Gieseker's Criterion 9 for some positive δ which depends only on r since, now that we have fixed the e's, the elements which enter into the definition of T depend only on r.

The key simplifying trick is one which goes back to Lemma 4.13 of [7] and uses the assumption that $w_{r-1} = 0$.

Proposition 12. (1) *For any increasing subsequence $J = (j_0, j_1, \ldots, j_h)$, the sum P_J is twice the area in the first quadrant bounded by the line segments joining the points (i_{j_k}, w_{j_k}) and (i_{j_k+1}, w_{j_k+1}) for $k = 0, \ldots, h-1$.*

(2) *The minimum over all increasing subsequences $J = (j_0, j_1, \ldots, j_h)$ of the quantity P_J equals twice the area in the first quadrant bounded by the Newton polygon or lower convex hull of the points $(i, w_i), i = 0, 1, \ldots, r-1$.*

We first compute S_J using the subsequence $(0, r-1)$. Since $e_0 = w_{r-1} = 0$ and $e_{r-1} = 2(r-2)$ and $S_{(0,r-1)} <= 2(r-2)w_0$. Plugging this into equation (11) gives the estimate $w_0 > \frac{3}{r}$.

Next apply the Proposition to P_J taking as J the full subsequence $J = (0, 1, \ldots, r-1)$. Here the area is a union of trapezoids whose areas sum to

$$\sum_{i=0}^{r-1} w_i - \frac{1}{2}(w_0 + w_{r-1}) = 1 - \frac{w_0}{2}$$

so $3P_J = 6 - 3w_0$.

On the other hand, for any decreasing set of weights, the sequence which minimizes the sums N_J is also the full subsequence when N_J equals w_0: taking a larger jump just multiplies a sum of several differences of i's by the sum of several differences in the weights where formerly only corresponding terms in the sums were being multiplied. Combining the estimates for the full subsequence with that for w_0, we find that

$$T \le P_J - N_J \le 6 - 4w_0 \le 6 - \left(\frac{12}{r}\right) = 6\left(\frac{r-2}{r}\right).$$

Moreover, this inequality is strict unless $w_0 = \frac{3}{r}$ and the Newton polygon of the points (i, w_i) is convex. (If the polygon is not convex we can reduce the P-term by taking a suitable proper subsequence J and this can only raise the N-term.)

We now use the subsequence $J' = (0, 2, 4, \ldots, r-1)$: to simplify the notation I will assume that $r-1$ is even leaving the reader to make the minor alterations needed if it is odd. The area which the Proposition gives for $P_{J'}$ differs from the one it gives for P_J by replacing two trapezoids one with upper vertices at $(2i, w_{2i})$ and $(2i+1, w_{2i+1})$ and the second with upper vertices at $(2i+1, w_{2i+1})$ and $(2i+2, w_{2i+2})$ with a single trapezoid having upper vertices at $(2i, w_{2i})$ and $(2i+2, w_{2i+2})$. Each such replacement is easily checked to add

$$\frac{w_{2i} + w_{2i+2}}{2} - w_{2i+1}$$

to the area. Doubling these and summing gives

$$\begin{aligned} &P_{J'} - P_J \\ =&\sum_{j=0}^{((r-3)/2)} (w_{2j} + w_{2j+2} - 2w_{2j+1}) \\ =&2\sum_{i \text{ even}} w_i - 2\sum_{i \text{ odd}} w_i - (w_0 - w_{r-1}) \\ =&2\sum_{j=0}^{((r-3)/2)} (w_{2j} - w_{2j+1}) - (w_0) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{r-2} (w_i - w_{i+1}) + (w_0 - w_1) - (w_0) \\
&= (w_0 - w_1) .
\end{aligned}
$$

The inequality in the second last line follows from the inequality $w_{2i-1} - w_{2i} \geq w_{2i} - w_{2i+1}$ which in turn is a consequence of the convexity of the Newton polygon. The equality in the last line then follows by telescoping the sum and using $w_{r-1} = 0$.

On the other hand, $N_{J'}$ is always at least $2w_0$ since now every jump in weights is multiplied by 2. Thus,

$$(3P_{J'} - N_{J'}) = (3P_J + 3(w_0 - w_1)) - (N_J + w_0) = (3P_J - N_J) + (2w_0 - 3w_1) .$$

Since we have a contradiction if $3P_{J'} - N_{J'} < 3P_J - N_J$, we must have $w_1 \leq \frac{2}{3} w_0 = \frac{2}{r}$.

Thus we can write $w_1 = \alpha \frac{2}{r}$ for some α with $0 \leq \alpha \leq 1$. Finally, we let $J'' = (0, 1, r-1)$ and compute $S_{J''}$. Since have $w_0 = \frac{3}{r}$, $w_{r-1} = 0$, $e_0 = e_{01} = 0$, $e_1 = e_{1(r-1)} = 2$ and $e_{r-1} = 2(r-2)$, we obtain

$$S_{J''} = \left(\frac{6 + 4\alpha(r-1)}{r} \right) \leq \left(\frac{4r+2}{r} \right) .$$

This is less than

$$\left(\frac{6(r-2)}{r} \right)$$

and provides the desired contradiction as soon as $r \geq 8$, or equivalently, whenever $d = 2(r-2) \geq 12$.

Remark 13. The reader who knows the proof of stability of smooth curves will find much of this argument familiar. What is the reason for the extra complication here? With our normalizations for the weights, the main term of the right hand side of Gieseker's Criterion is, ignoring dimensional constants the ratio $\frac{d}{r}$. For curves this is of order $\frac{1}{r}$ *larger* than 1; for K3's it is of order $\frac{1}{r}$ *smaller* than 2. This difference in sign of the order $\frac{1}{r}$ term seems to be the source of the difficulty. For example, proving stability for a filtration of $H^0(X, kD)$ in which the only base loci are multiples of D itself ought to be an easy case because Riemann-Roch again gives exact information on all the d_{ij}'s. Once again, however, getting estimates of order 2 is straightforward but it seems to be necessary to play off various subsequences to squeeze down any further. Handling general filtrations seems likely to involve combinatorial intricacies comparable to those in [3].

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