Critical Phenomena and Percolation Theory: I

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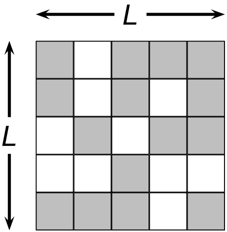
Outline

1. Critical Phenomena & Percolation Theory
   - Aims and objectives
   - Definition and quantities of interest
   - Qualitative behaviour as a function of occupation probability p
2. Percolation in $d = 1$
   - Onset of percolation: Critical occupation probability p_c
   - Cluster number density & characteristic cluster size
   - Average cluster size
3. Summary

Critical Phenomena & Percolation Theory

Aim: Study connections between **macroscopic** quantities and the underlying **microscopic** world in a model displaying a phase transition.

Objective: Gain qualitative and quantitative understanding of critical phenomena and associated concepts such as scale-free behaviour, scaling theory, and universality.



- Each site in a (regular) lattice is occupied randomly and independently with occupation probability p, $0 \leq p \leq 1$.
- A cluster is a group of nearest-neighbour occupied sites.
- The size s of a cluster is the number of sites in the cluster.
- The critical occupation probability p_c is the occupation probability p at which an infinite cluster appears for the first time in an infinite lattice $L = \infty$.

Percolation deals with the

- number of the clusters formed
- properties of the clusters formed

when occupying randomly and independently each site in a lattice with probability p.

Quantities of interest

- Onset of percolation – critical occupation probability, p_c.
- Probability that a site belongs to the infinite cluster, $P_\infty(p)$.
- Geometry of the infinite cluster at $p = p_c$ and $p > p_c$.

Excluding the <u>infinite</u> cluster:

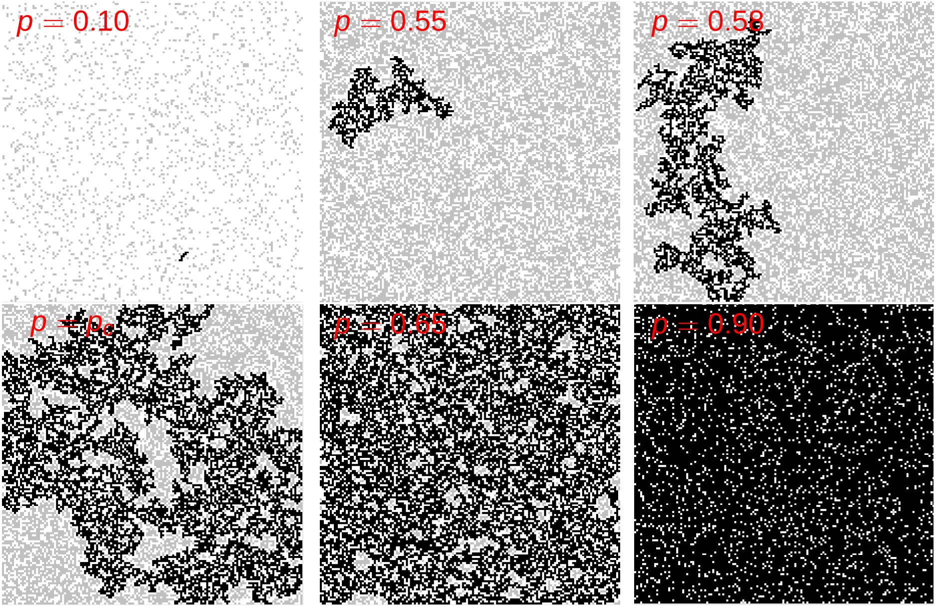
- Average cluster size, $\chi(p)$.
- Typical size of the largest cluster, $s_\xi(p)$.
- Typical radius (linear size) of the largest cluster, $\xi(p)$.

For fixed lattice size L, there is only one parameter, the occupation probability p, $0 \leq p \leq 1$.

- $p = 0$: Empty lattice. No clusters.
- $0 < p < 1$: Percolation is a random process.

  No. of different realisations $2^{L \times L} \approx \begin{cases} 10^{6,773} & \text{for } L = 150; \\ 10^{108,370} & \text{for } L = 600. \end{cases}$
- $p = 1$: Fully occupied lattice. One cluster of size $s = L^2$.

A percolating cluster is one that spans the lattice from left to right, top to bottom, or both.

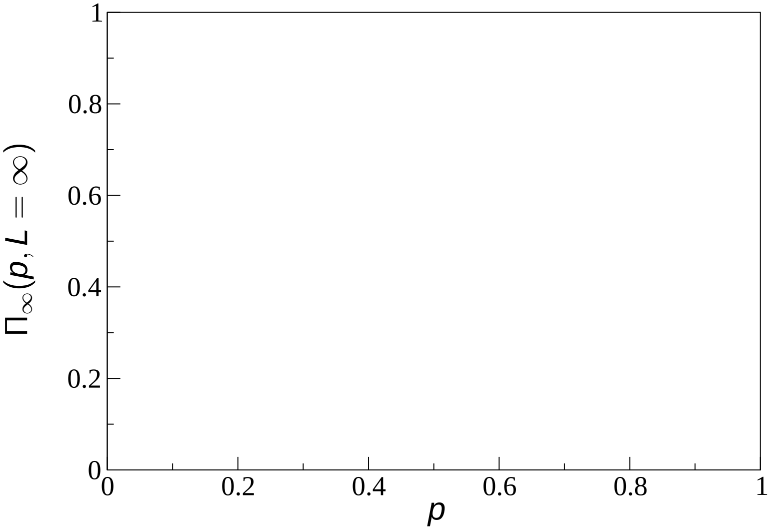

p = 0.10
p = 0.55
p = 0.58
p = pc
p = 0.65
p = 0.90

Typically expects no percolating cluster for “small” p.
Typically expects a percolating cluster for “large” p.

Consider these two probabilities at occupation probability p in a lattice of size L:

- $\Pi_\infty(p; L) =$ prob. that percolating cluster exists.
- $P_\infty(p; L) =$ prob. that a site belongs to a percolating cluster $=$ fraction of volume covered by percolating cluster

Prob. that percolating cluster exists at occupation probability p.



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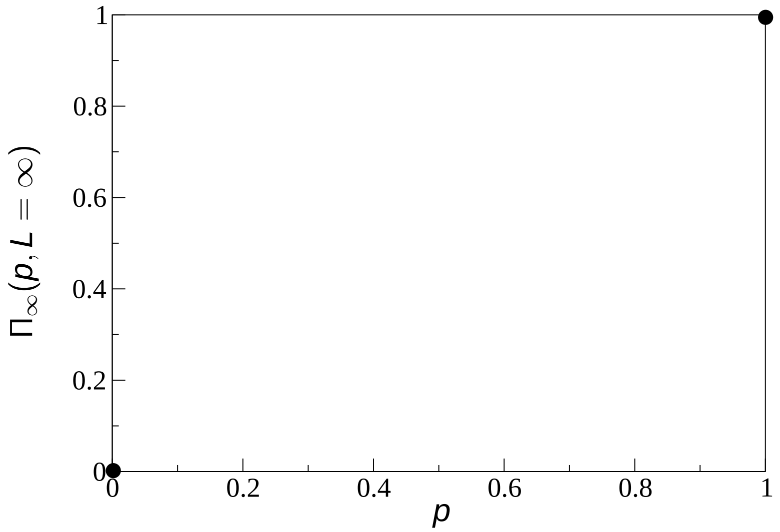

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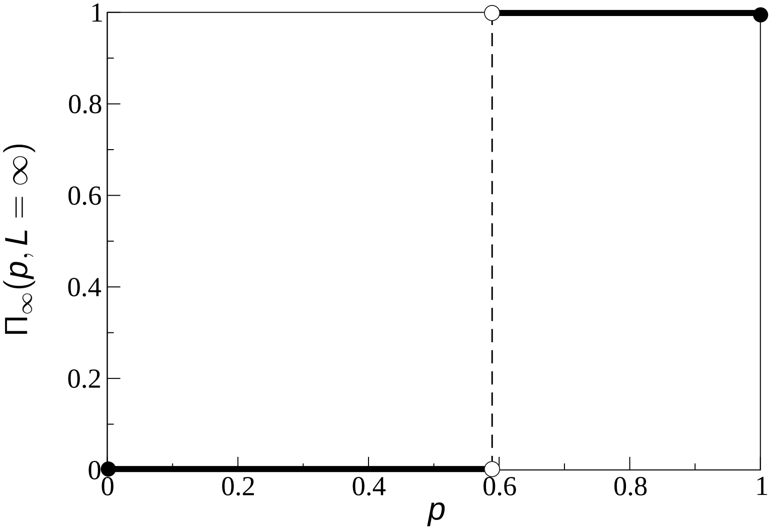
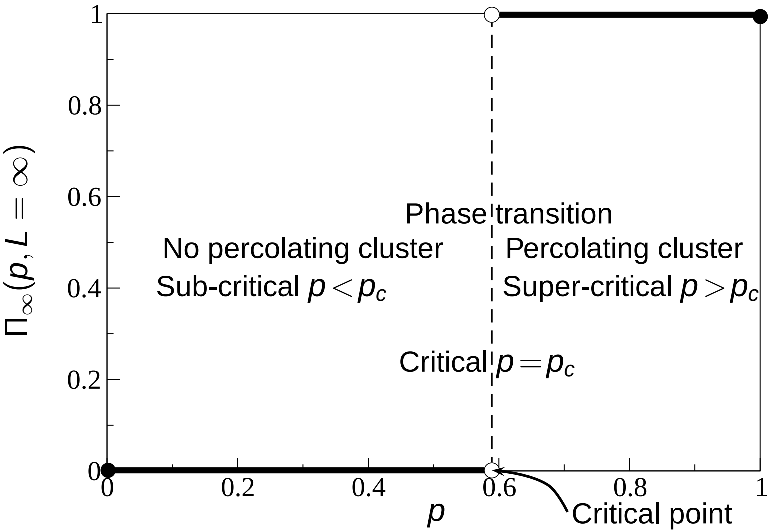



1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
$\Pi_\infty(p, L = \infty)$
p
Phase transition
No percolating cluster
Sub-critical $p < p_c$
Percolating cluster
Super-critical $p > p_c$
Critical $p = p_c$
Critical point

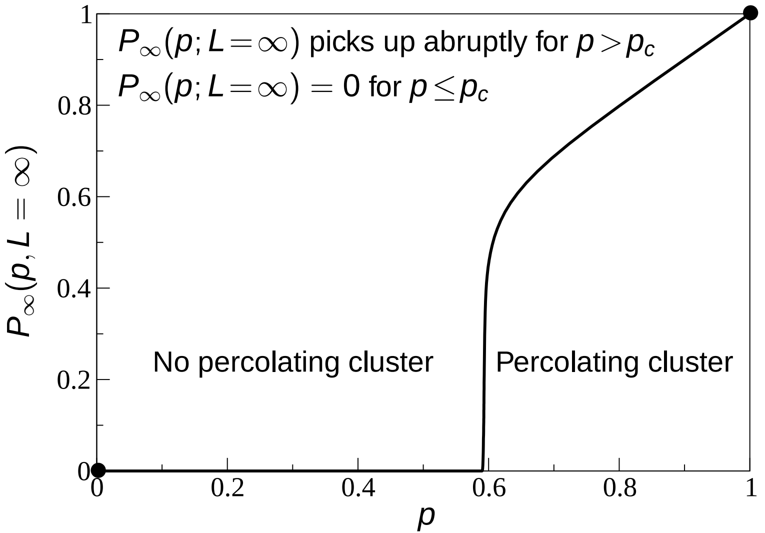
$$\Pi_\infty(p; L = \infty) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c \end{cases}$$

The critical occupation probability p_c is the occupation probability above which a percolating (infinite) cluster appears for the first time in an infinite lattice.

Onset of percolation is a geometrical phase transition:
When increasing p from 0 towards 1, there is a phase transition at $p = p_c$ from a lattice with no percolating infinite cluster for $p < p_c$ to a lattice with a percolating infinite cluster for $p > p_c$.

For two-dimensional square lattice $p_c = 0.59274621\ldots$
For two-dimensional triangular lattice $p_c = 1/2$.
For three-dimensional simple cubic lattice $p_c = 0.311608\ldots\ldots$

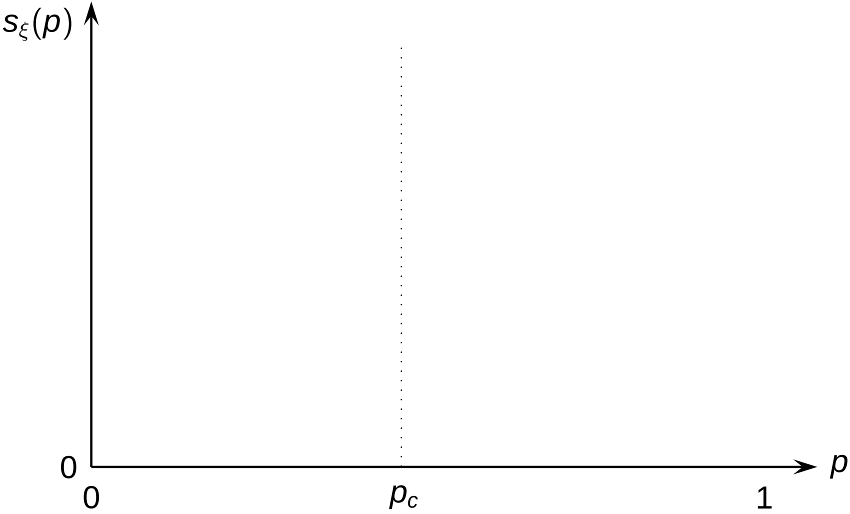
Prob. that site belongs to percolating cluster at occ. prob. p.



$$P_\infty(p; L = \infty) = \begin{cases} 0 & \text{for } p \leq p_c \\ \text{nonzero} & \text{for } p > p_c \end{cases}$$
$$= \begin{cases} 0 & \text{for } p \leq p_c \\ A\,(p - p_c)^\beta & \text{for } p \to p_c^+. \end{cases}$$

The critical exponent β characterises the abrupt pick-up of the order parameter $P_\infty(p)$ for $p \to p_c^+$.

Excluding the percolating (infinite) cluster, what is the typical size of the largest cluster $s_\xi(p)$?



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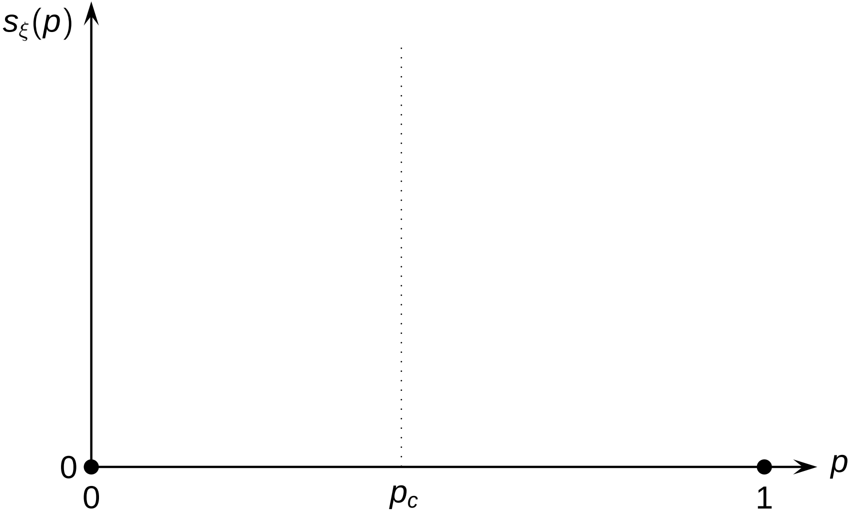

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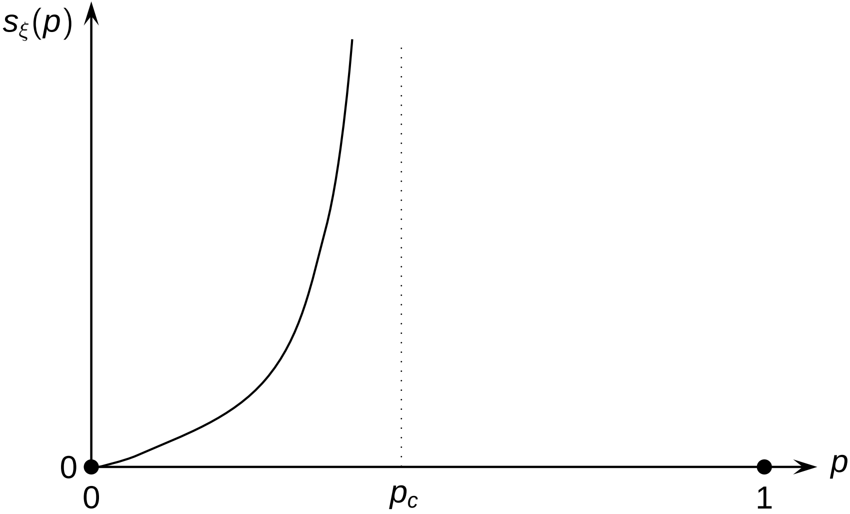

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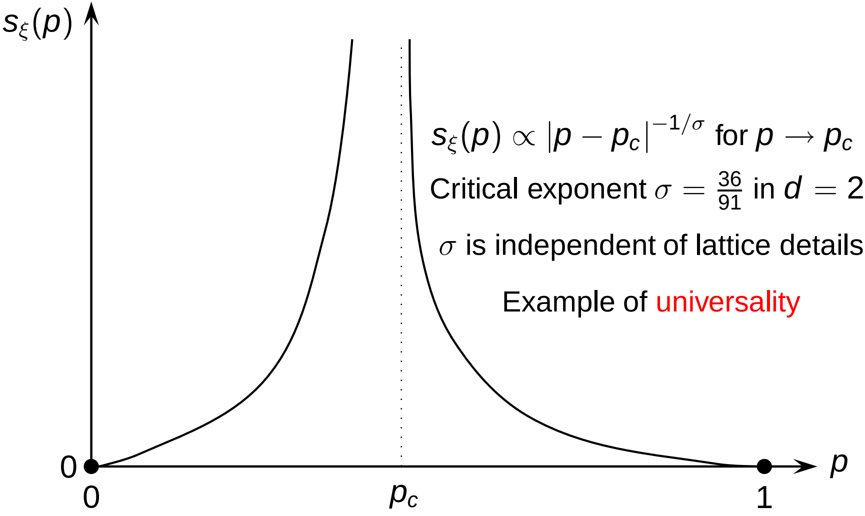


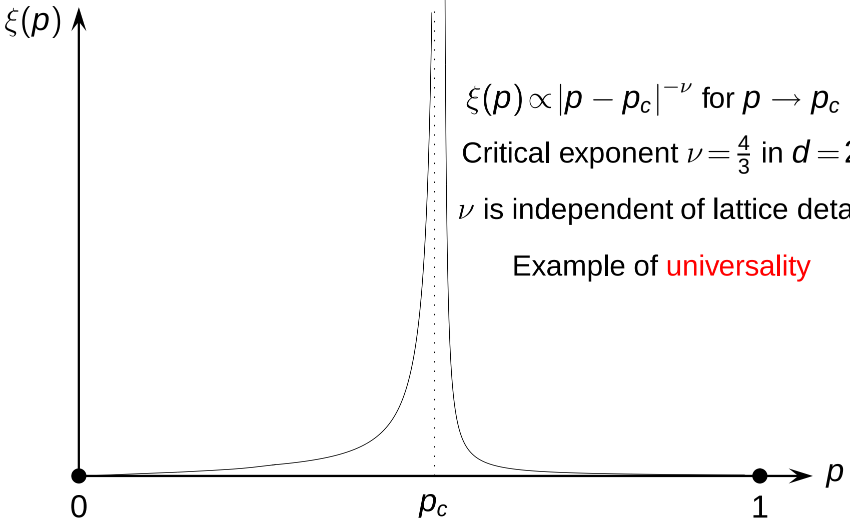
$s_\xi(p) \propto |p - p_c|^{-1/\sigma}$ for $p \to p_c$

Critical exponent $\sigma = \frac{36}{91}$ in $d = 2$.

σ is independent of lattice details

Example of universality

Excluding the percolating (infinite) cluster, what is the typical radius (linear size) of the largest cluster $\xi(p)$?



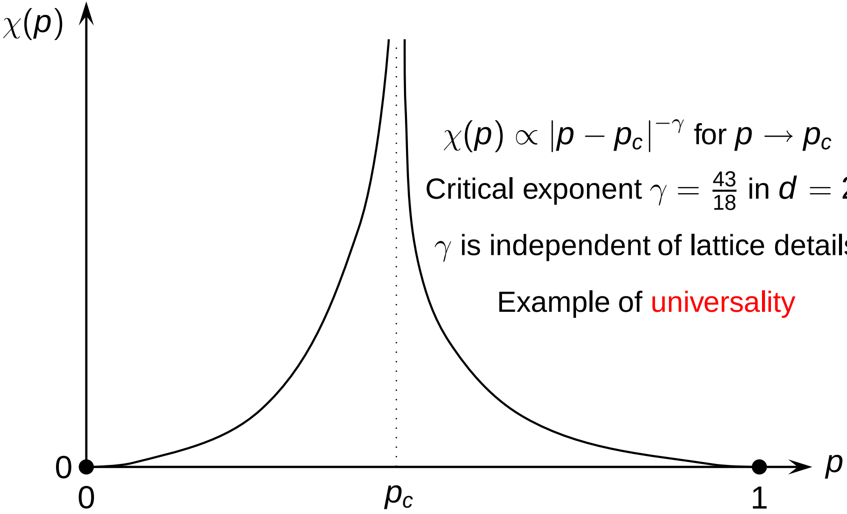
$\xi(p) \propto |p - p_c|^{-\nu}$ for $p \to p_c$

Critical exponent $\nu = \frac{4}{3}$ in $d = 2$

ν is independent of lattice details

Example of universality

Excluding the percolating (infinite) cluster, what is the average cluster size to which an occupied site belongs, $\chi(p)$?

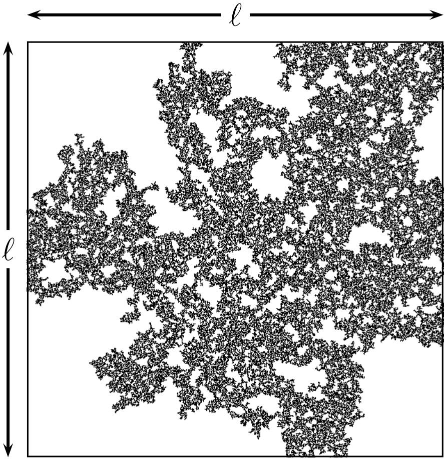


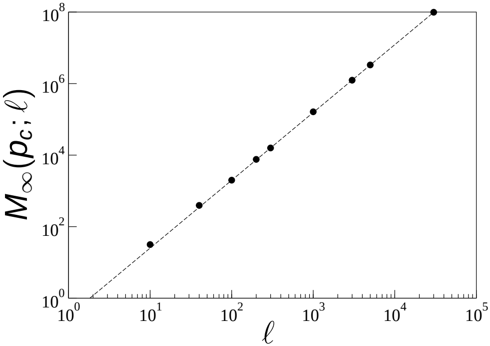
$\chi(p) \propto |p - p_c|^{-\gamma}$ for $p \to p_c$

Critical exponent $\gamma = \frac{43}{18}$ in $d = 2$.

γ is independent of lattice details

Example of universality

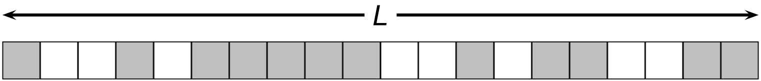




Mass of the percolating cluster at $p = p_c$ increases with window size ℓ:
$M_\infty(p_c;\ell) \propto \ell^D$ for $\ell \gg 1$.

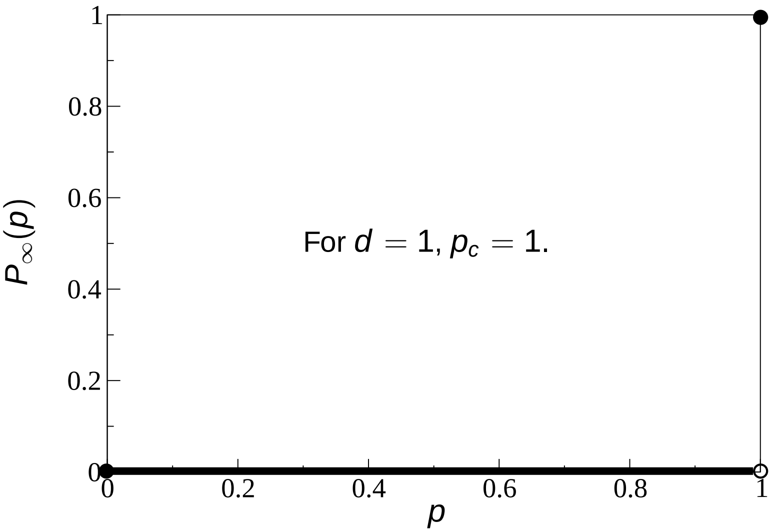
Critical exponent D is the fractal dimension of cluster. $D = \frac{91}{48}$ in $d = 2$.
D is independent of lattice details. Example of universality.

Can be solved analytically. Many of the characteristic features encountered are present for percolation in $d > 1$.

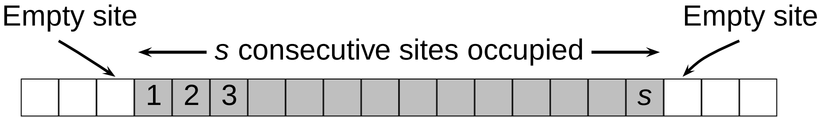


What is the critical occupation probability p_c?

Prob. that site belongs to percolating infinite cluster.

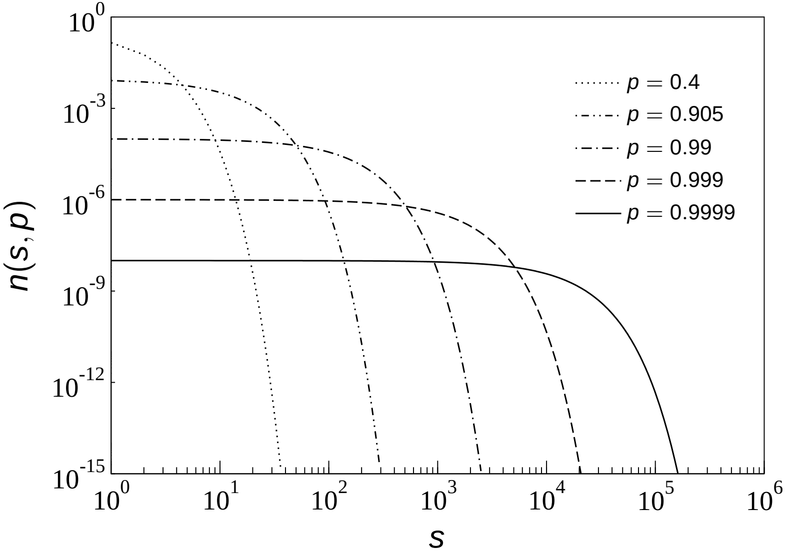


Calculate the cluster size frequency $N(s,p;L)$ probabilistically.



Cluster number density = number of s-clusters per lattice site:

$$
\begin{aligned}
n(s,p) &= \lim_{L\to\infty} \frac{N(s,p{:}L)}{L} \\
&= \text{(prob. empty site)}\cdot\text{(prob. } s \text{ occupied sites)}\cdot\text{(prob. empty site)} \\
&= (1-p)p^s(1-p) \\
&= (1-p)^2 p^s
\end{aligned}
$$

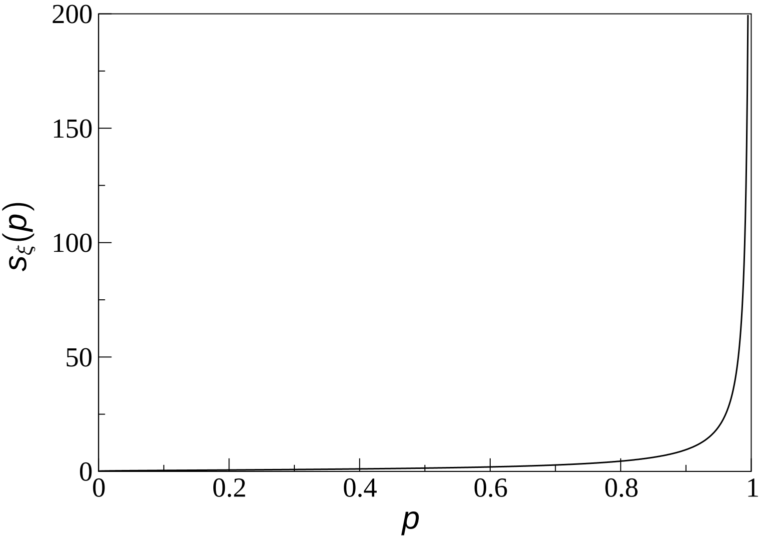

10^0
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
$n(s,p)$
10^0
10^1
10^2
10^3
10^4
10^5
10^6
s
$p = 0.4$
$p = 0.905$
$p = 0.99$
$p = 0.999$
$p = 0.9999$

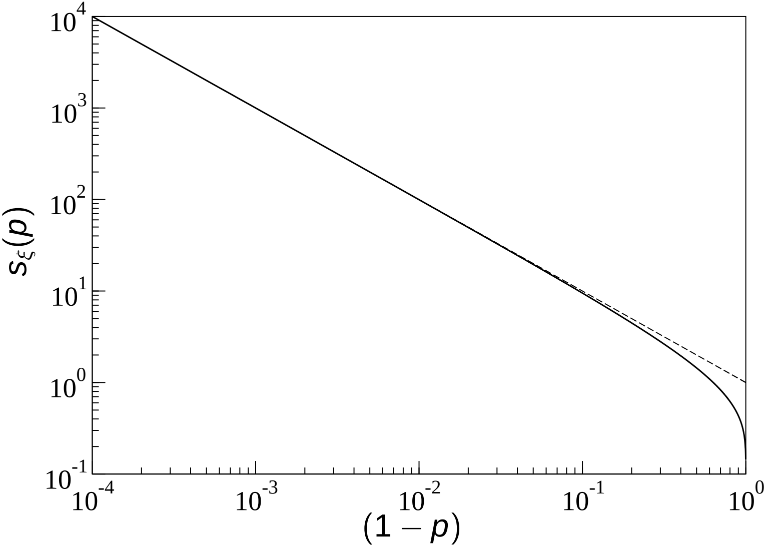
$$
\begin{aligned}
n(s,p) &= (1-p)^2 p^s \\
&= (1-p)^2 \exp\left(\ln p^s\right) \\
&= (1-p)^2 \exp\left(s \ln p\right) \\
&= (1-p)^2 \exp\left(-s/s_\xi\right),
\end{aligned}
$$

with the characteristic cluster size

$$
s_\xi(p) = \frac{-1}{\ln p} = \frac{-1}{ln\,(1-[1-p])} \to (1-p)^{-1} = (p_c - p)^{-1} \text{ for } p \to p_c^- .
$$

For $d = 1$, the critical exponent $\sigma = 1$.


200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
$s_\xi(p)$
p

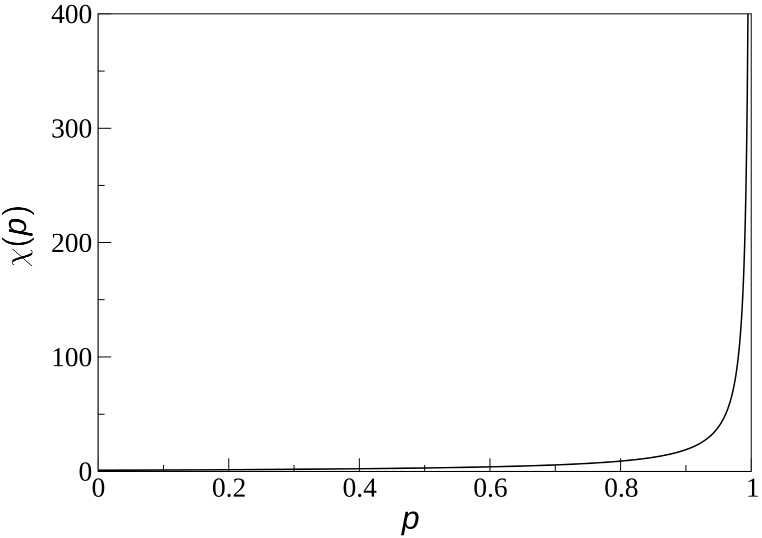

$s_\xi(p)$
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
$(1-p)$

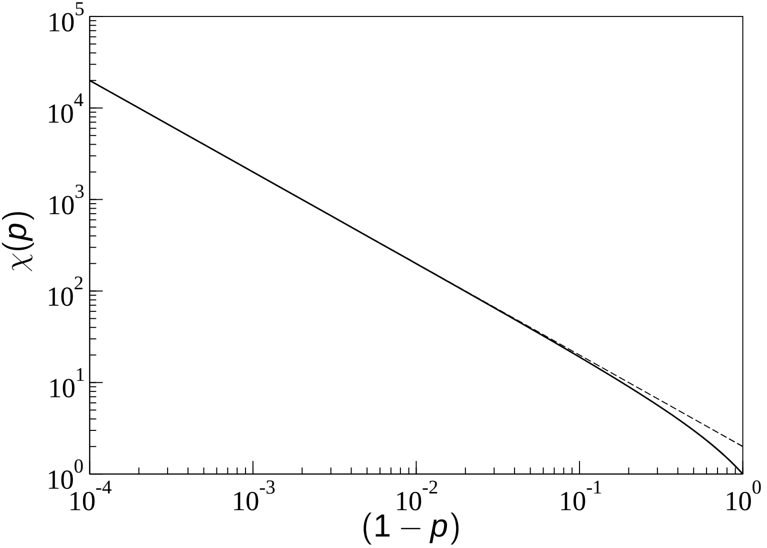
The probability that a site belongs to an s-cluster: $sn(s, p)$.
Given an occupied site - how large, on average, is its cluster?

$$\chi(p) = \frac{\sum_{s=1}^{\infty} s^2 n(s,p)}{\sum_{s=1}^{\infty} sn(s,p)}$$

$$= \frac{1+p}{1-p} \quad \text{(see page 11 in notes)}$$

$$\rightarrow 2(1-p)^{-1} \quad \text{for } p \rightarrow 1^-$$


400
300
200
100
0
$\chi(p)$
0
0.2
0.4
0.6
0.8
1
p


10^5
10^4
10^3
10^2
10^1
10^0
$\chi(p)$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
$(1-p)$

General pattern for the exact solution to $d = 1$ percolation:

- Characteristic cluster size
  $s_\xi(p) = \frac{-1}{\ln p} \to (p_c - p)^{-1} \quad$ for $p \to p_c^-$, $\sigma = 1$.
- Average cluster size
  $\chi(p) = \frac{1+p}{1-p} \to 2\,(p_c - p)^{-1} \quad$ for $p \to p_c^-$, $\gamma = 1$.

Asymptotically close to p_c, the divergence is characterized by a power-law in $(p_c - p)$, the distance away from the critical point.

Special for $d = 1$ is that the phase-transition can only be approached from below, $p \to p_c^-$.

- When increasing occupation probability p from 0 towards 1, there is a phase transition at $p = p_c$ from a lattice with no percolating infinite cluster for $p < p_c$ to a lattice with a percolating infinite cluster for $p > p_c$.
- Sub-critical behaviour for $p < p_c$ where $\xi < \infty$.
- Critical behaviour for $p = p_c$ where $\xi = \infty$.
- Super-critical behaviour for $p > p_c$ where $\xi < \infty$.
- Order parameter picks up abruptly at $p = p_c$:
  $P_\infty(p) \propto (p - p_c)^\beta \quad \text{for } p \to p_c^+$.
- Quantities of interest diverges at $p = p_c$:
  - Characteristic cluster size: $s_\xi(p) \propto |p - p_c|^{-1/\sigma}$ for $p \to p_c$.
  - Average cluster size: $\chi(p) \propto |p - p_c|^{-\gamma}$ for $p \to p_c$.
  - Typical radius of largest cluster: $\xi(p) \propto |p - p_c|^{-\nu}$ for $p \to p_c$.

Thank you for listening!

For a comprehensive introduction to percolation, please see K. Christensen and N.R. Moloney, *Complexity and Criticality*, Imperial College Press (2005), Chapter 1.

Access to animations, please visit **www.complexityandcriticality.com**.