Percolation on the Bethe lattice
Scaling function & data collapse for cluster no. density

Critical Phenomena and Percolation Theory: II

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Outline

1. Percolation on the Bethe lattice
   - Onset of percolation: Critical occupation probability
   - Average cluster size
   - Transition to percolation
   - Cluster number density

2. Scaling function & data collapse for cluster no. density
   - General scaling ansatz for cluster no. density
   - Scaling function & data collapse for Bethe lattice
   - Scaling function & data collapse for $d = 1$
A Bethe lattice is a tree where each site has $z$ neighbours:

$$z = 3$$

- Branch
- Sub-branch

$l = 0$, generation number
$l = 1$
$l = 2$
$l = 3$, generation number
Bethe lattice has no loops: Unique path between any sites $i$ & $j$.

- Consider a percolating infinite cluster in the Bethe lattice.
- Perform a walk, where retracing of steps are forbidden.
- Each step has $z - 1$ new sites (sub-branches).
- In average, $p(z - 1)$ sites occupied.

Onset of percolation when

$$p(z - 1) = 1 \iff p_c = \frac{1}{z - 1} = \begin{cases} 1 & \text{for } z = 2; (d = 1) \\ 1/2 & \text{for } z = 3. \end{cases}$$

$p_c$ decreases with increasing coordination number. Sensitive to lattice details. **Non-universal** quantity.
Assume $p < p_c$. Let “center” site be occupied. Average cluster size to which this site belongs:

$$\chi(p) = \text{contribution from center site} + \text{contribution from } z \text{ branches}$$

$$= 1 + zB.$$  \hfill (1)

$B =$ contribution to average cluster size from a given branch

$$= (1 − p) \cdot 0 + p \cdot (1 + (z − 1) B) \quad \text{parent site of branch is empty/occupied}$$

$$= p + p(z − 1)B.$$  

Solve for $B$ and insert in Eq. (1) above:

$$\chi(p) = \frac{1 + p}{1 − p(z − 1)}$$

$$= \frac{p_c(1 + p)}{p_c − p} \quad \text{with } p_c = \frac{1}{z − 1}$$

$$\rightarrow p_c(1 + p_c)(p_c − p)^{-1} \quad \text{for } p \rightarrow p_c^-$$
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$z = 3$
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\[ z = 3 \]
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\( z = 3 \)

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\[ \chi(p) \]

Graph showing the scaling function \( \chi(p) \) as a function of occupation probability \( p \). The graph illustrates the critical occupation probability at which percolation occurs, with a sharp transition indicated by the vertical dashed line.
Exponent characterizing the divergence is independent of coordination number.

Universal
Probability center site belongs to percolating infinite cluster:

\[ P_\infty(p) = p \cdot \text{(prob. at least one branch connects to percolating cluster)} \]
\[ = p \cdot (1 - \text{prob. none of } z \text{ branches connect to percolating cluster}) \]
\[ = p \cdot (1 - Q_\infty^Z(p)). \]

\[ Q_\infty(p) = \text{prob. a branch DOES NOT connect to percolating cluster} \]
\[ = (1 - p) + p \cdot Q_\infty^{Z-1}(p) \text{ parent site of branch is empty/occupied} \]

For \( z = 3 \), solve quadratic equation for \( Q_\infty(p) \):

\[ Q_\infty(p) = \begin{cases} 1 & \text{for } p \leq p_c \\ \frac{1-p}{p} & \text{for } p > p_c. \end{cases} \]
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\[ P_\infty(p) = \begin{cases} 
0 & \text{for } p \leq p_c \\
 p \left[ 1 - \left( \frac{1-p}{p} \right)^3 \right] & \text{for } p > p_c.
\end{cases} \]
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\[ P_\infty(p) = \begin{cases} 
0 & \text{for } p \leq p_c \\
6(p - p_c)^\beta & \text{for } p \to p_c^+; \beta = 1.
\end{cases} \]

Exponent characterising the pick-up is independent of coordination number.
Universal
Cluster number density:
Consider a cluster of size $s$. Define the perimeter $t$ of cluster:
$t = \text{no. of unoccupied nearest-neighbours of cluster.}$

\[
n(s, p) = \sum_{t=1}^{\infty} g(s, t) (1 - p)^t p^s.
\]

Clusters might not have unique geometry or orientation:
$g(s, t) = \text{no. of different } s\text{-clusters with perimeter } t.$

For $d = 1$  
\[
g(s, t) = \begin{cases} 
1 & \text{for } t = 2 \\
0 & \text{otherwise}
\end{cases} \Rightarrow n(s, p) = (1 - p)^2 p^s.
\]
In Bethe lattice there is a unique relationship between $s$ & $t$: $t = 2 + s(z - 2)$; For $z = 3$ we have $t = 2 + s$: 
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\[ n(s, p) = \sum_{t=1}^{\infty} g(s, t) (1 - p)^t p^s = g(s, 2 + s) (1 - p)^{2+s} p^s. \]

Trick to avoid enumerating \( g(s, 2 + s) \) by considering ratio:

\[
\frac{n(s, p)}{n(s, p_c)} = \left[ \frac{1 - p}{1 - p_c} \right]^2 \left[ \frac{(1 - p) p}{(1 - p_c) p_c} \right]^s \\
= \left[ \frac{1 - p}{1 - p_c} \right]^2 \exp \left( s \ln \left[ \frac{(1 - p) p}{(1 - p_c) p_c} \right] \right) \\
= \left[ \frac{1 - p}{1 - p_c} \right]^2 \exp \left( -s / s_\xi \right),
\]

where we have defined the characteristic cluster size

\[
s_\xi(p) = \frac{-1}{\ln \left[ \frac{(1-p) p}{(1-p_c) p_c} \right]} = \frac{-1}{\ln \left[ 1 - 4(p - p_c)^2 \right]} \rightarrow \frac{1}{4} (p - p_c)^{-2} \text{ for } p \rightarrow p_c
\]
Exponent characterising the divergence is independent of coordination number. Universal
Percolation on the Bethe lattice

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\[ P_\infty(p) \text{ prob. site } \in \text{ infinite cluster} \]
\[ \sum_{s=1}^{\infty} sn(s, p) \text{ prob. site } \in \text{ any finite cluster} \]
\[ P_\infty(p) + \sum_{s=1}^{\infty} sn(s, p) = p \]
The cluster no. density & the characteristic cluster size:

\[ n(s, p) = \left[ \frac{1 - p}{1 - p_c} \right]^2 n(s, p_c) \exp \left( -\frac{s}{s_\xi} \right) ; \]

\[ s_\xi(p) = -\frac{1}{\ln \left[ \frac{(1-p)^p}{(1-p_c)^{p_c}} \right]} . \]

\[ \sum_{s=1}^{\infty} s n(s, p) = p - P_\infty(p) \quad \text{finite for all } p, \text{ also at } p = p_c \]

\[ \sum_{s=1}^{\infty} s^2 n(s, p) = \chi(p) \sum_{s=1}^{\infty} s n(s, p) \to \infty \quad \text{for } p \to p_c; \text{ diverges at } p = p_c \]
Ansatz: \( n(s, p_c) \propto s^{-\tau} \) for \( s \gg 1 \):

\[
\sum_{s=1}^{\infty} s n(s, p_c) = \sum_{s=1}^{\infty} s^{1-\tau} \quad \text{finite} \Rightarrow \tau > 2
\]

\[
\sum_{s=1}^{\infty} s^2 n(s, p_c) = \sum_{s=1}^{\infty} s^{2-\tau} \quad \text{infinite} \Rightarrow \tau \leq 3
\]

The Bethe lattice has a cluster number density

\[
n(s, p) \propto s^{-5/2} \exp \left(-s/s_\xi\right) \quad \text{for } s \gg 1, p \to p_c,
\]

\[
s_\xi(p) \propto (p - p_c)^{-2} \quad \text{for } p \to p_c,
\]
At $p = p_c$, $n(s, p_c) \propto s^{-5/2}$ for $s \gg 1$
For $p \neq p_c : n(s, p) = \begin{cases} 
 s^{-\tau} & \text{for } 1 \ll s \ll s_\xi \\
 \text{decays rapidly} & \text{for } s \gg s_\xi 
\end{cases}$
General scaling ansatz for cluster no. density:

\[ n(s, p) \propto s^{-\tau} G(s/s_\xi) \quad \text{for } p \to p_c, \; s \gg 1, \quad (2) \]

\[ s_\xi(p) \propto |p - p_c|^{-1/\sigma} \quad \text{for } p \to p_c, \]

Critical exponents: \( \tau \) and \( \sigma \).

Scaling function \( G \) with dimensionless argument \( s/s_\xi \).

The scaling ansatz Eq. (2) allows a \textbf{data collapse} because

\[ s^\tau n(s, p) \propto G(s/s_\xi) \quad \text{for } p \to p_c, \; s \gg 1. \]

Plotting the transformed cluster no. density \( s^\tau n(s, p) \) vs. the re-scaled cluster size \( s/s_\xi \), all the data fall onto the graph of the scaling function \( G \).
Scaling function & data collapse for Bethe lattice:

\[ n(s, p) \propto s^{-5/2} \exp\left(-s/s_\xi\right) \quad \text{for } p \to p_c, s \gg 1 \quad (3) \]

\[ s_\xi(p) \propto (p_c - p)^{-2} \quad \text{for } p \to p_c \]

The argument \( s/s_\xi \) in the scaling fct is a re-scaled cluster size:

\[ G_{\text{Bethe}}\left(s/s_\xi\right) = \exp\left(-s/s_\xi\right), \]

Eq. (3) allows a data collapse: \( s^{5/2}n(s, p) = G_{\text{Bethe}}\left(s/s_\xi\right) \).

Plotting \( s^{5/2}n(s, p) \) vs. \( s/s_\xi \) the curves collapse onto the graph for the scaling function \( G_{\text{Bethe}} \). Another example of universality.
Data collapse of cluster no. densities for perc. on Bethe lattice:

\[ n(s, p) \]

- \( p = 0.35 \)
- \( p = 0.45 \)
- \( p = 0.4842 \)
- \( p = 0.495 \)
Data collapse of cluster no. densities for perc. on Bethe lattice:
Data collapse of cluster no. densities for perc. on Bethe lattice:

\[ G_{\text{Bethe}}(x) = \exp(-x) \]
Scaling function & data collapse for $d = 1$:

$$n(s, p) = (p_c - p)^2 \exp \left( -\frac{s}{s_\xi} \right)$$

$$= s^{-2} \left[ s(p_c - p) \right]^2 \exp \left( -\frac{s}{s_\xi} \right)$$

$$= s^{-2} \left( \frac{s}{s_\xi} \right)^2 \exp \left( -\frac{s}{s_\xi} \right) \quad \text{for } p \to p_c^-$$

$$= s^{-2} G_{1d}(s/s_\xi) \quad \text{for } p \to p_c^- \quad (4)$$

$$s_\xi(p) = \frac{-1}{\ln p} \to (p_c - p)^{-1} \quad \text{for } p \to p_c^-$$

The argument $s/s_\xi$ in the scaling fct is a re-scaled cluster size:

$$G_{1d} \left( \frac{s}{s_\xi} \right) = \left( \frac{s}{s_\xi} \right)^2 \exp \left( -\frac{s}{s_\xi} \right),$$

Eq. (4) allows a data collapse because $s^2 n(s, p) = G_{1d} \left( \frac{s}{s_\xi} \right)$. Plotting $s^2 n(s, p)$ vs. $s/s_\xi$ the curves collapse onto the graph for the scaling function $G_{1d}$. Another example of universality.
Data collapse of the cluster no. densities for $d = 1$ percolation:

\[ n(s, p) \]

- \( p = 0.4 \)
- \( p = 0.905 \)
- \( p = 0.99 \)
- \( p = 0.999 \)
- \( p = 0.9999 \)
Data collapse of the cluster no. densities for $d = 1$ percolation:
Data collapse of the cluster no. densities for $d = 1$ percolation:

$$G_{1d}(x) = x^2 \exp(-x)$$