Critical Phenomena and Percolation Theory: II

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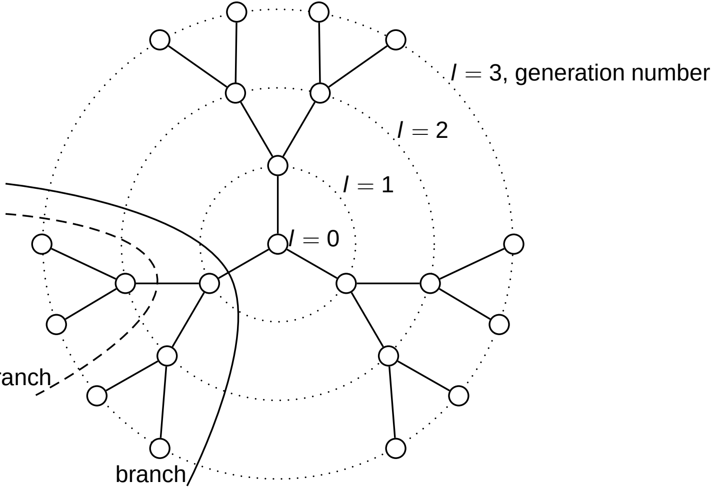
Outline

1. Percolation on the Bethe lattice
   - Onset of percolation: Critical occupation probability
   - Average cluster size
   - Transition to percolation
   - Cluster number density
2. Scaling function & data collapse for cluster no. density
   - General scaling ansatz for cluster no. density
   - Scaling function & data collapse for Bethe lattice
   - Scaling function & data collapse for $d = 1$

A Bethe lattice is a tree where each site has z neighbours:

$z = 3$



Bethe lattice has no loops: Unique path between any sites i & j.

- Consider a percolating infinite cluster in the Bethe lattice.
- Perform a walk, where retracing of steps are forbidden.
- Each step has $z - 1$ new sites (sub-branches).
- In average, $p(z - 1)$ sites occupied.

Onset of percolation when

$$p(z-1) = 1 \Leftrightarrow p_c = \frac{1}{z-1} = \begin{cases} 1 & \text{for } z = 2; (d = 1) \\ 1/2 & \text{for } z = 3. \end{cases}$$

p_c decreases with increasing coordination number.
Sensitive to lattice details. **Non-universal** quantity.

Assume $p < p_c$. Let "center" site be occupied.
Average cluster size to which this site belongs:

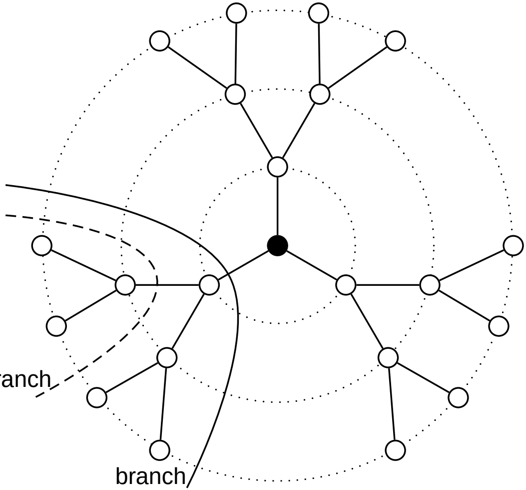
$$\chi(p) = \text{contribution from center site} + \text{contribution from } z \text{ branches}$$
$$= 1 + zB. \tag{1}$$

$$B = \text{contribution to average cluster size from a given branch}$$
$$= (1-p)\cdot 0 + p\cdot(1+(z-1)B) \quad \text{parent site of branch is empty/occupied}$$
$$= p + p(z-1)B.$$

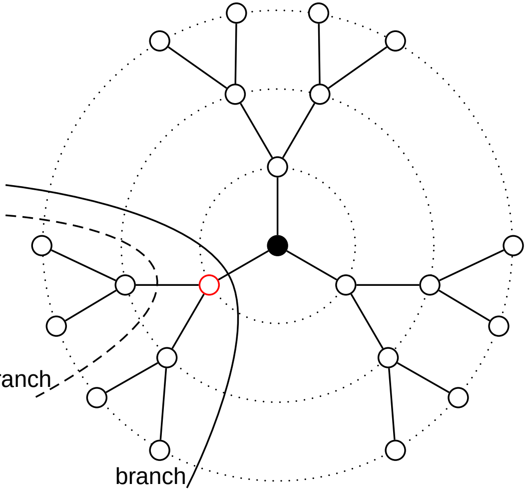
Solve for B and insert in Eq. (1) above:

$$\chi(p) = \frac{1+p}{1-p(z-1)}$$
$$= \frac{p_c(1+p)}{p_c - p} \quad \text{with } p_c = \frac{1}{z-1}$$
$$\to p_c(1+p_c)(p_c - p)^{-1} \quad \text{for } p \to p_c^-$$

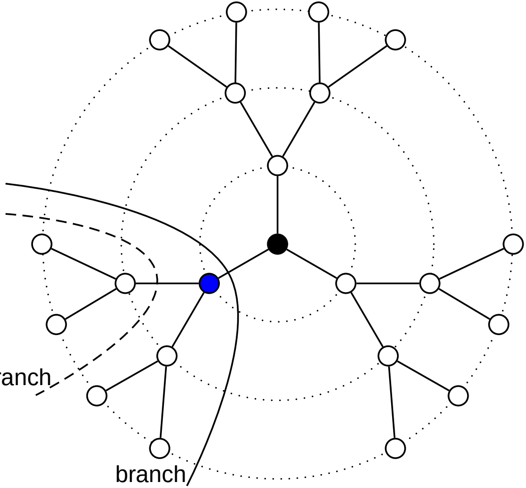
$z = 3$

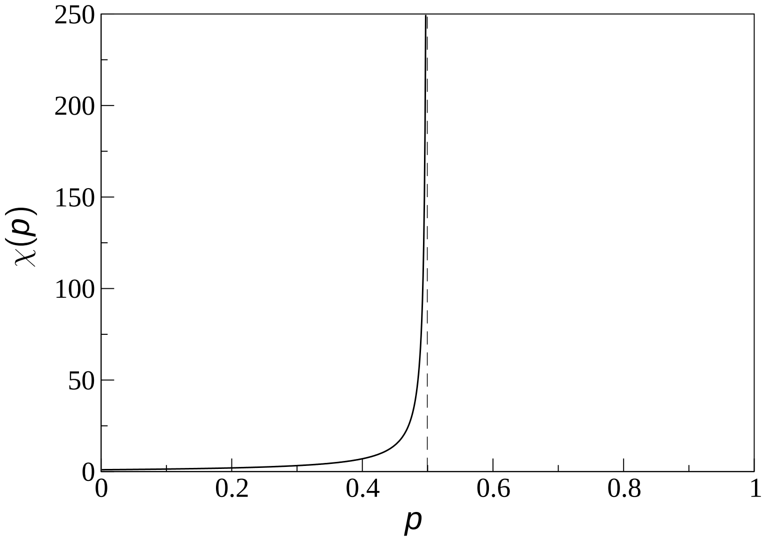


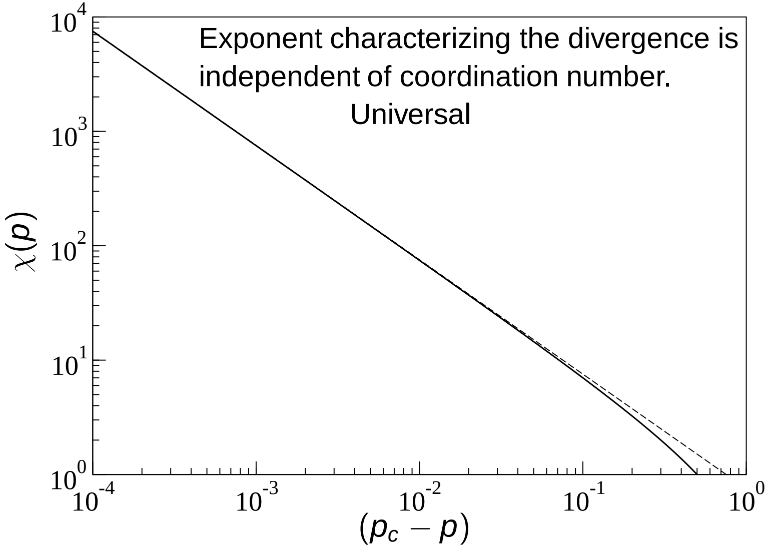
$z = 3$



$z = 3$




250
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
$\chi(p)$
p


Exponent characterizing the divergence is
independent of coordination number.
Universal
10^4
10^3
10^2
10^1
10^0
$\chi(p)$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
$(p_c - p)$

Probability center site belongs to percolating infinite cluster:

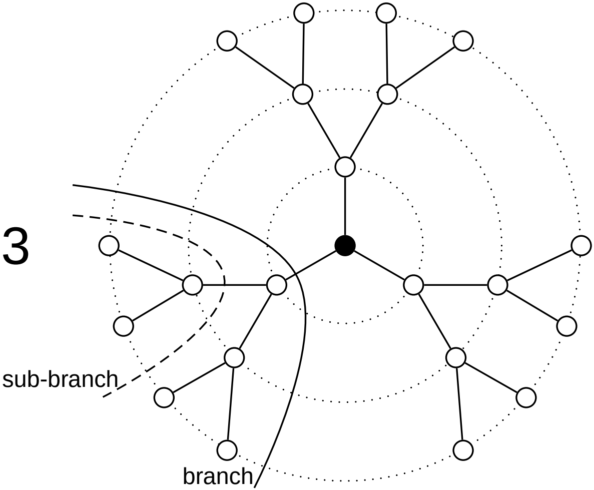
$$\begin{aligned} P_\infty(p) &= p \cdot \text{(prob. at least one branch connects to percolating cluster)} \\ &= p \cdot \text{(1 - prob. none of } z \text{ branches connect to percolating cluster)} \\ &= p \cdot (1 - Q_\infty^z(p)) \,. \end{aligned}$$

$$\begin{aligned} Q_\infty(p) &= \text{prob. a branch DOES NOT connect to percolating cluster} \\ &= (1-p) + p \cdot Q_\infty^{z-1}(p) \quad \text{parent site of branch is empty/occupied} \end{aligned}$$

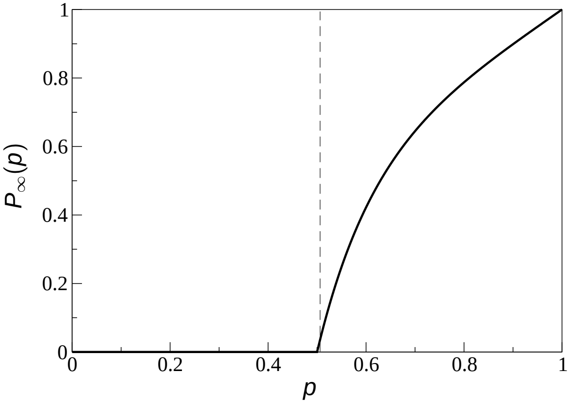
For $z = 3$, solve quadratic equation for $Q_\infty(p)$:

$$Q_\infty(p) = \begin{cases} 1 & \text{for } p \leq p_c \\ \frac{1-p}{p} & \text{for } p > p_c \,. \end{cases}$$

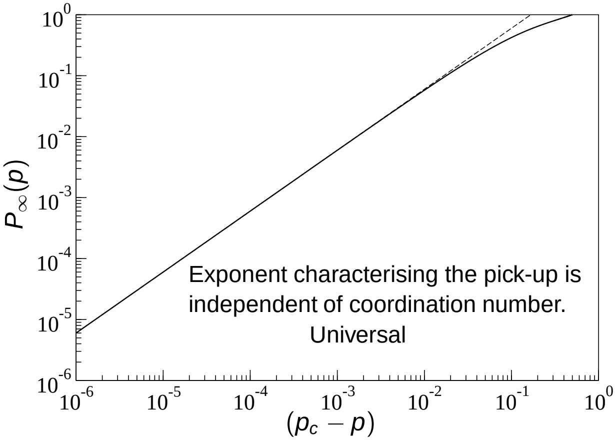
$z = 3$



$$P_\infty(p) = \begin{cases} 0 & \text{for } p \le p_c \\ p\left[1 - \left(\frac{1-p}{p}\right)^3\right] & \text{for } p > p_c. \end{cases}$$



$$P_\infty(p) = \begin{cases} 0 & \text{for } p \le p_c \\ 6\,(p - p_c)^\beta & \text{for } p \to p_c^+;\ \beta = 1. \end{cases}$$



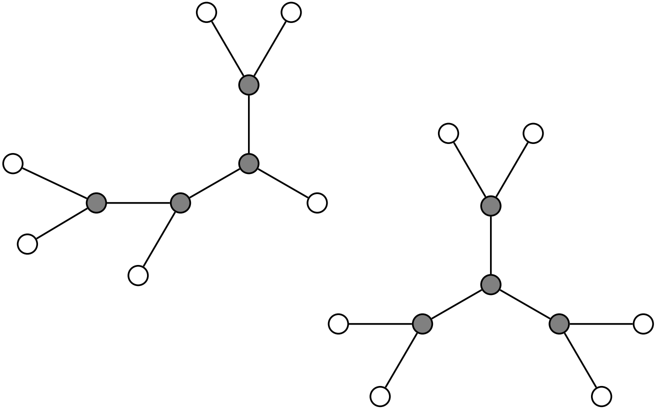
Cluster number density:
Consider a cluster of size s. Define the perimeter t of cluster:
$t =$ no. of unoccupied nearest-neighbours of cluster.

$$n(s,p) = \sum_{t=1}^{\infty} g(s,t)\,(1-p)^t\,p^s.$$

Clusters might not have unique geometry or orientation:
$g(s,t) =$ no. of different s-clusters with perimeter t.

For $d = 1$ $\quad g(s,t) = \begin{cases} 1 & \text{for } t = 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow n(s,p) = (1-p)^2 p^s.$

In Bethe lattice $\exists$ a unique relationship between s & t:
$t = 2 + s(z - 2)$; For $z = 3$ we have $t = 2 + s$:

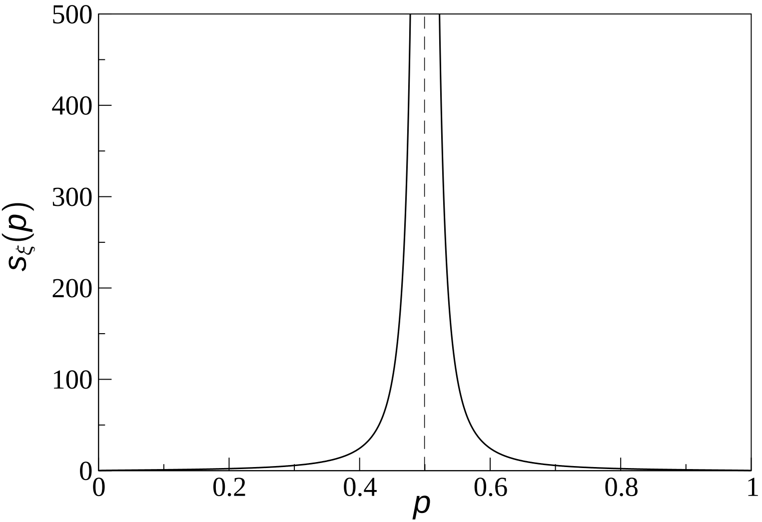


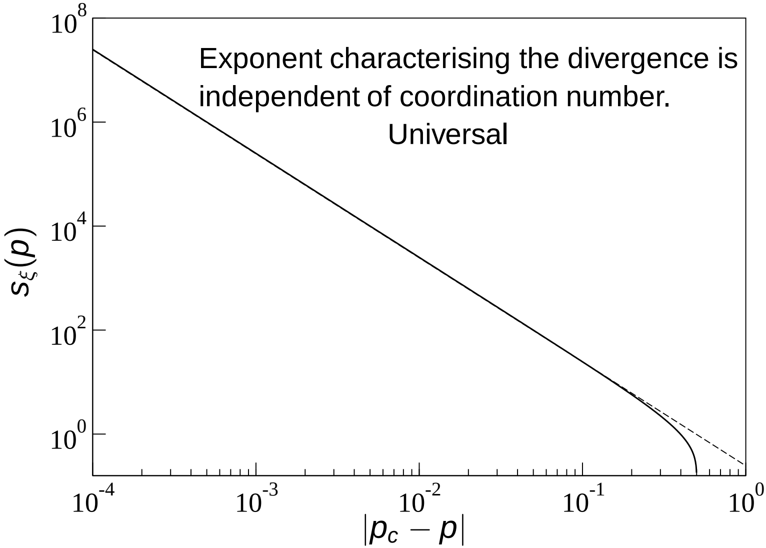
$n(s,p) = \sum_{t=1}^{\infty} g(s,t)\,(1-p)^t\,p^s = g(s,2+s)\,(1-p)^{2+s}\,p^s.$
Trick to avoid enumerating $g(s,2+s)$ by considering ratio:

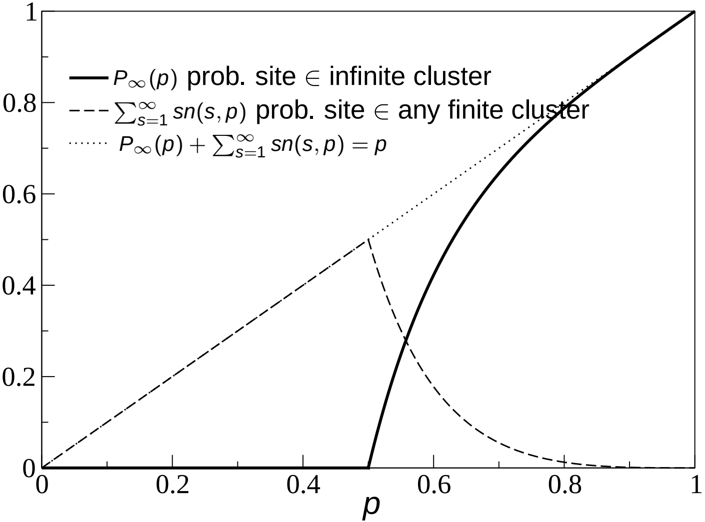
$$\frac{n(s,p)}{n(s,p_c)} = \left[\frac{1-p}{1-p_c}\right]^2 \left[\frac{(1-p\;)\,p}{(1-p_c)\,p_c}\right]^s$$
$$= \left[\frac{1-p}{1-p_c}\right]^2 \exp\left(s\ln\left[\frac{(1-p\;)\,p}{(1-p_c)\,p_c}\right]\right)$$
$$= \left[\frac{1-p}{1-p_c}\right]^2 \exp\left(-s/s_\xi\right),$$

where we have defined the characteristic cluster size

$$s_\xi(p) = \frac{-1}{\ln\left[\frac{(1-p\;)\,p}{(1-p_c)\,p_c}\right]} = \frac{-1}{\ln\left[1-4(p-p_c)^2\right]} \to \frac{1}{4}\,(p-p_c)^{-2} \text{ for } p \to p_c$$


500
400
300
200
100
0
0
0.2
0.4
0.6
0.8
1
$s_\xi(p)$
p


Exponent characterising the divergence is independent of coordination number.
Universal
10^8
10^6
10^4
10^2
10^0
$s_\xi(p)$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
$|p_c - p|$


$P_\infty(p)$ prob. site $\in$ infinite cluster
$\sum_{s=1}^{\infty} sn(s,p)$ prob. site $\in$ any finite cluster
$P_\infty(p) + \sum_{s=1}^{\infty} sn(s,p) = p$
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
p

The cluster no. density & the characteristic cluster size:

$$n(s,p) = \left[\frac{1-p}{1-p_c}\right]^2 n(s,p_c)\exp\left(-s/s_\xi\right);$$

$$s_\xi(p) = -\frac{1}{\ln\left[\frac{(1-p\ )\,p}{(1-p_c)\,p_c}\right]}.$$

$$\sum_{s=1}^{\infty} sn(s,p) = p - P_\infty(p)$$ finite for all p, also at $p = p_c$

$$\sum_{s=1}^{\infty} s^2 n(s,p) = \chi(p)\sum_{s=1}^{\infty} sn(s,p) \to \infty$$ for $p \to p_c$; diverges at $p = p_c$

Ansatz: $n(s, p_c) \propto s^{-\tau}$ for $s \gg 1$:

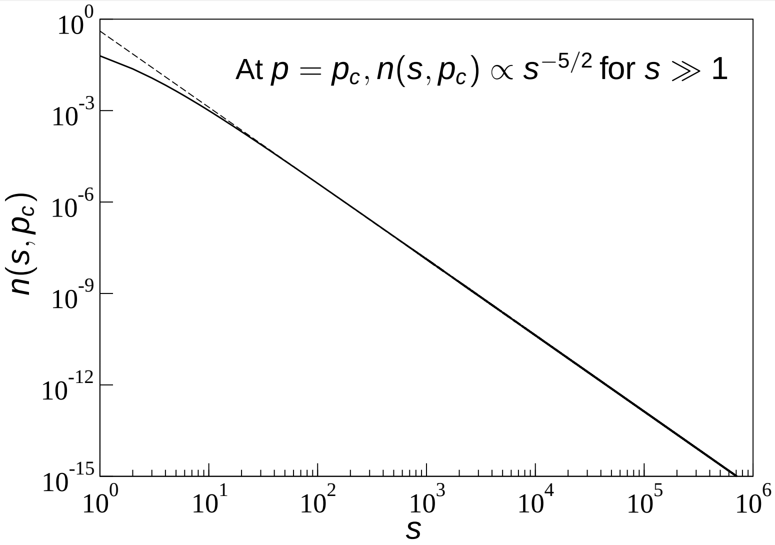
$$\sum_{s=1}^{\infty} s n(s, p_c) = \sum_{s=1}^{\infty} s^{1-\tau} \quad \text{finite} \Rightarrow \tau > 2$$

$$\sum_{s=1}^{\infty} s^2 n(s, p_c) = \sum_{s=1}^{\infty} s^{2-\tau} \quad \text{infinite} \Rightarrow \tau \leq 3$$

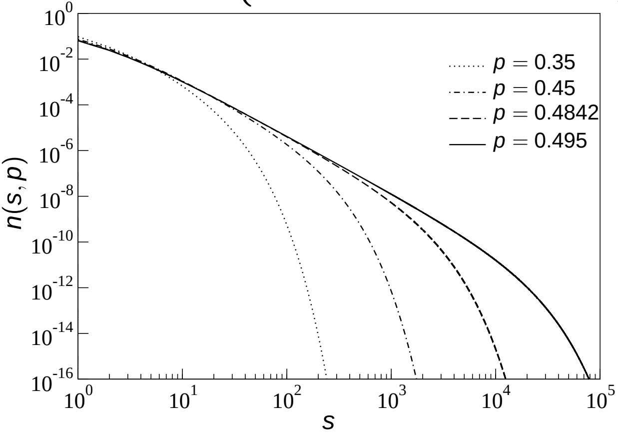
The Bethe lattice has a cluster number density

$$n(s, p) \propto s^{-5/2} \exp\left(-s/s_\xi\right) \qquad \text{for } s \gg 1, p \to p_c,$$

$$s_\xi(p) \propto (p - p_c)^{-2} \qquad \text{for } p \to p_c,$$


At $p = p_c, n(s, p_c) \propto s^{-5/2}$ for $s \gg 1$
10^{0}
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
$n(s, p_c)$
s

$$\text{For } p \neq p_c : n(s,p) = \begin{cases} s^{-\tau} & \text{for } 1 \ll s \ll s_\xi \\ \text{decays rapidly} & \text{for } s \gg s_\xi \end{cases}$$



General scaling ansatz for cluster no. density:

$$n(s,p) \propto s^{-\tau}\mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c, s \gg 1, \qquad (2)$$
$$s_\xi(p) \propto |p - p_c|^{-1/\sigma} \quad \text{for } p \to p_c,$$

Critical exponents: τ and σ.
Scaling function $\mathcal{G}$ with dimensionless argument s/s_ξ.

The scaling ansatz Eq. (2) allows a **data collapse** because

$$s^{\tau}n(s,p) \propto \mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c, s \gg 1.$$

Plotting the transformed cluster no. density $s^{\tau}n(s,p)$ vs. the re-scaled cluster size s/s_ξ, all the data fall onto the graph of the scaling function $\mathcal{G}$.

Scaling function & data collapse for Bethe lattice:

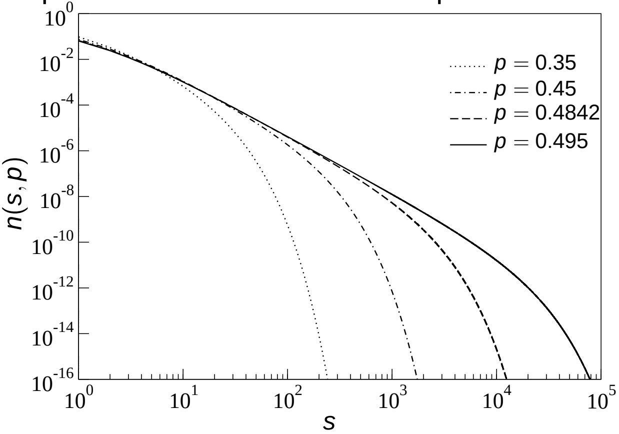
$$n(s,p) \propto s^{-5/2} \exp\left(-s/s_\xi\right) \quad \text{for } p \to p_c, s \gg 1 \tag{3}$$
$$s_\xi(p) \propto (p_c - p)^{-2} \quad \text{for } p \to p_c$$

The argument s/s_ξ in the scaling fct is a re-scaled cluster size:

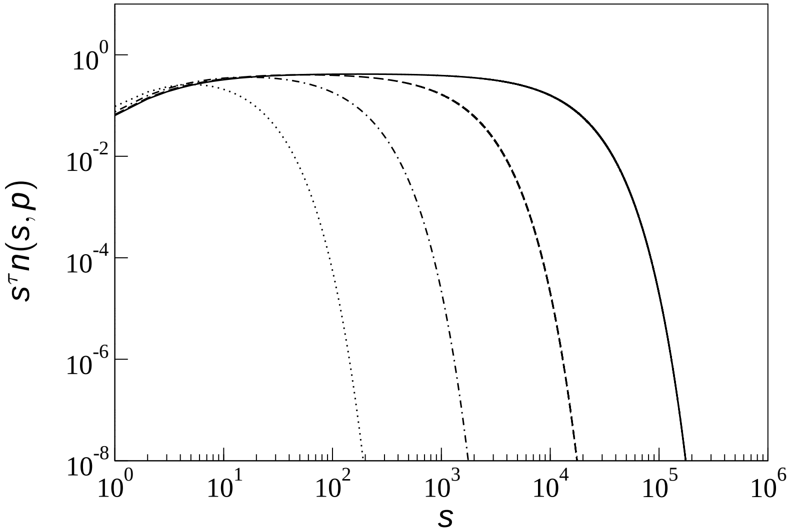
$$\mathcal{G}_{\text{Bethe}}\left(s/s_\xi\right) = \exp\left(-s/s_\xi\right),$$

Eq. (3) allows a data collapse: $s^{5/2} n(s,p) = \mathcal{G}_{\text{Bethe}}\left(s/s_\xi\right)$. Plotting $s^{5/2} n(s,p)$ vs. s/s_ξ the curves collapse onto the graph for the scaling function $\mathcal{G}_{\text{Bethe}}$. Another example of universality.

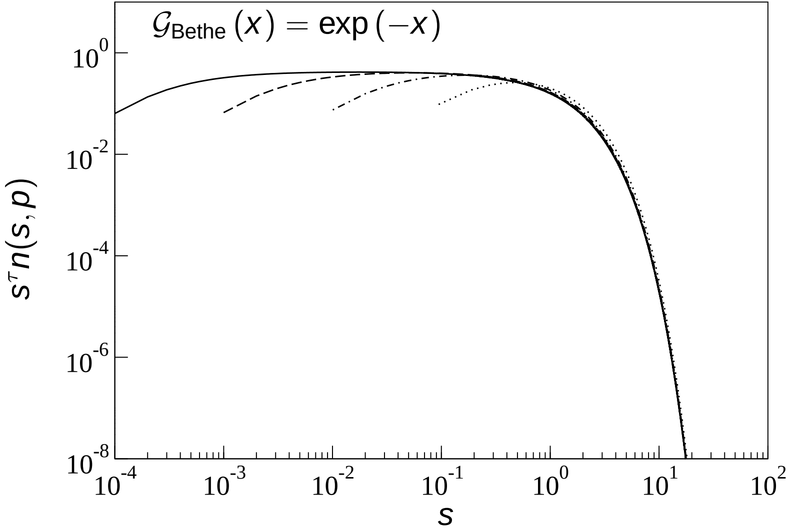
Data collapse of cluster no. densities for perc. on Bethe lattice:



Data collapse of cluster no. densities for perc. on Bethe lattice:



Data collapse of cluster no. densities for perc. on Bethe lattice:



Scaling function & data collapse for $d = 1$:

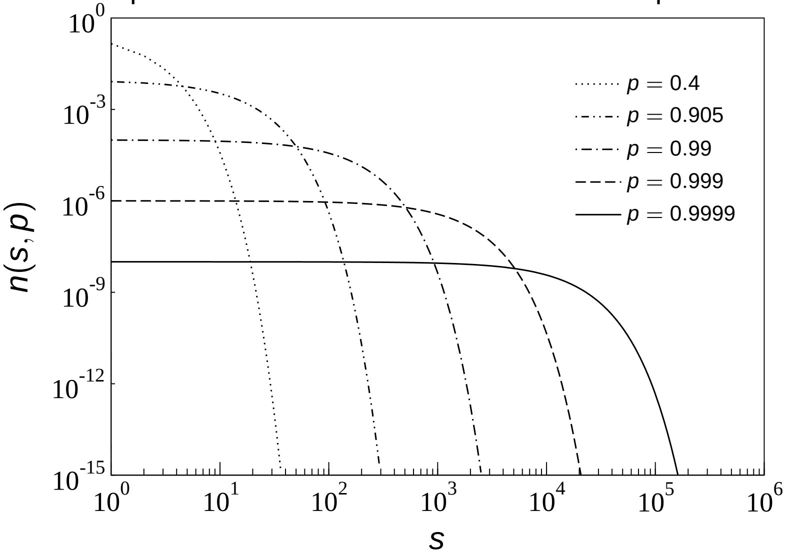
$$
\begin{aligned}
n(s,p) &= (p_c - p)^2 \exp(-s/s_\xi) \\
&= s^{-2}[s(p_c - p)]^2 \exp(-s/s_\xi) \\
&= s^{-2}(s/s_\xi)^2 \exp(-s/s_\xi) && \text{for } p \to p_c^- \\
&= s^{-2}\mathcal{G}_{1\mathrm{d}}(s/s_\xi) && \text{for } p \to p_c^- \qquad (4) \\
s_\xi(p) &= \frac{-1}{\ln p} \to (p_c - p)^{-1} && \text{for } p \to p_c-
\end{aligned}
$$

The argument s/s_ξ in the scaling fct is a re-scaled cluster size:

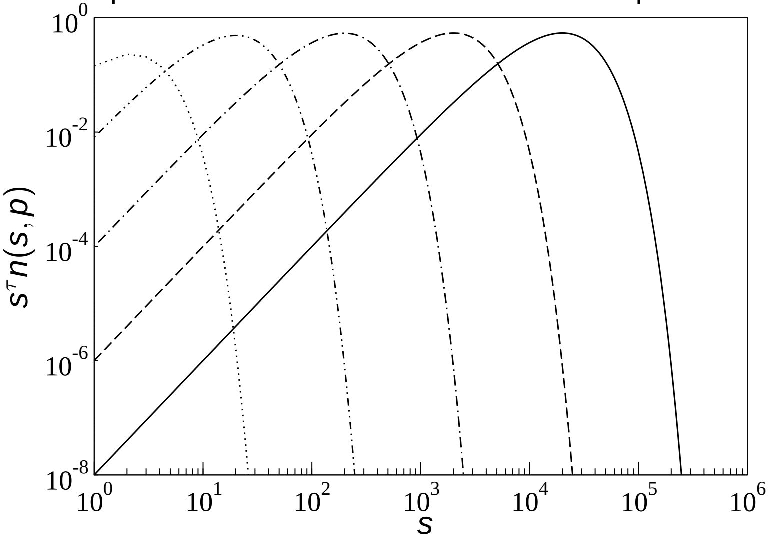
$$
\mathcal{G}_{1\mathrm{d}}(s/s_\xi) = (s/s_\xi)^2 \exp(-s/s_\xi),
$$

Eq. (4) allows a **data collapse** because $s^2 n(s,p) = \mathcal{G}_{1\mathrm{d}}(s/s_\xi)$. Plotting $s^2 n(s,p)$ vs. s/s_ξ the curves collapse onto the graph for the scaling function $\mathcal{G}_{1\mathrm{d}}$. Another example of universality.

Data collapse of the cluster no. densities for $d = 1$ percolation:



Data collapse of the cluster no. densities for $d = 1$ percolation:



Data collapse of the cluster no. densities for $d = 1$ percolation: