Newton’s Equations in Spaces of Constant Curvature

Florin Diacu
University of Victoria, Canada

We consider a natural extension of Newton’s equations of the N-body problem to spaces of constant curvature. We first present some qualitative results regarding the motion of the bodies, focusing on relative equilibria and rotopulsators, which generalize the notion of homographic orbits from Euclidean to curved space. Then we write the equations in intrinsic coordinates and discuss the advantages and disadvantages of this approach. Finally we come up with a new and simple form of the equations that brings together the Euclidean case (of Gaussian curvature $k=0$), the hyperbolic case (of Gaussian curvature $k<0$), and the elliptic case (of Gaussian curvature $k>0$). Thus Newton’s classical equations can be regarded in a broader context, namely that in which the motion of the bodies takes place in spaces of constant curvature. The equations of motion depend on the curvature k, and the Euclidean case is recovered when $k=0$. This conclusion could not be drawn from previously known forms of the equations of motion in curved space since taking $k \to 0$, for both $k>0$ and $k<0$, led to undetermined limits. This new form of the equations of motion allows the study of the classical case, $k=0$, in a larger framework and will help us better understand Newton’s original approach.